Cohomology of p-adic curves (joint with G. Dospinescu, W. Nizioł)

Put $C := \widehat{\overline{\mathbb{Q}_p}}$, let L be a finite extension of $\mathbb{Q}_p$, and let $(\mathcal{M}_n)_{n\in\mathbb{N}}$ be the Drinfeld tower of coverings of $\mathbb{P}^1_C \setminus \mathbb{P}^1(\mathbb{Q}_p)$.

Theorem 1. (CDN)*: for V an irreducible L-representation of $G_{\mathbb{Q}_p}$, $\dim(V) \geq 2$,*

$$\mathrm{Hom}_{L[W_{\mathbb{Q}_p}]}(V, \varinjlim_n H^1_{\mathrm{et}}(\mathcal{M}_n, L(1)) = ??$$

where ?? is $\mathrm{JL}(V) \otimes \Pi(V)^*$ if V is de Rham, 2-dimensional with Hodge-Tate weights 0 and 1, when $D_{\mathrm{pst}}(V)$ is irreducible as a representation of $\mathrm{WD}_{\mathbb{Q}_p}$; and 0 otherwise.

Remark 2. (i) $\mathrm{JL}(V)$ comes from classical local Langlands plus Jacquet-Langlands. $\Pi(V)$ comes from Colmez's p-adic local Langlands.

(ii) Proof uses:

- Can recover H^1_{et} inside H^1_{proet};
- describe H^1_{proet} in terms of H^1_{HK} (Hyodo-Kato) plus de Rham complex;
- description of de Rham complex in terms of p-adic local Langlands (Dospinescu-Le Bras);
- $H^1_{\mathrm{HK}} \cong H^1_{\mathrm{proet}}(\bullet, \mathbb{Q}_\ell)$ as $\mathrm{GL}_2(\mathbb{Q}_p) \times D^* \times W_{\mathbb{Q}_p}$-representations;
- description of $H^1_{\mathrm{proet}}(\bullet, \mathbb{Q}_\ell)$ (Carayol, Faltings–Fargues).

(iii) If we look at the completed Drinfeld tower $\widehat{\mathcal{M}}_\infty$, one can hope that

$$\mathrm{Hom}_{L[W_{\mathbb{Q}_p}]}(V, H^1_{\mathrm{et}}(\widehat{\mathcal{M}}_\infty, L(1))) = \mathrm{JL}_p(\Pi(V)) \otimes \Pi(V)^*$$

for *any* 2-dimensional, absolutely irreducible V, where $\mathrm{JL}_p(\Pi(V))$ is a p-adic local Jacquet-Langlands correspondence coming out of Scholze's functor.

Theorem 3. *Let X be a rigid analytic curve over C (smooth, connected, separated).*

(i) *X proper,*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1_{\mathrm{proet}}(X, \mathbb{Q}_p(1)) & \longrightarrow & (B^+_{\mathrm{st}} \otimes H^1_{\mathrm{HK}}(X))^{N=0,\varphi=p} & \longrightarrow & H^1(X, \mathcal{O}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle dlog} & & \downarrow{\scriptstyle \theta\otimes\iota_{\mathrm{HK}}} & & \| & & \\ 0 & \longrightarrow & \Omega^1(X) & \longrightarrow & H^1_{\mathrm{dR}}(X) & \longrightarrow & H^1(X, \mathcal{O}) & \longrightarrow & 0 \end{array}$$

(ii) *X Stein (so $H^1(X, \mathcal{O}) = 0$),*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}(X)/C & \longrightarrow & H^1_{\mathrm{proet}} & \longrightarrow & (B^+_{\mathrm{st}} \otimes H^1_{\mathrm{HK}})^{N=0,\varphi=p} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}(X)/C & \longrightarrow & \Omega^1(X) & \longrightarrow & H^1_{\mathrm{dR}}(X) & \longrightarrow & 0 \end{array}$$

(iii) *similarly if X is an overconvergent affinoid.*

(iv) *if X is an affinoid,*

$$0 \to \mathcal{O}(X)/C \to H^1_{\mathrm{proet}} \to (B^+_{\mathrm{st}} \otimes H^1_{\mathrm{HK}}(X))^{N=0,\varphi=p} \oplus [\mathbb{Q}_p \widehat{\otimes} \mathcal{O}^{**}(X)/\mathbb{Q}_p \otimes \mathcal{O}^{**}(X)] \to 0.$$

with an ad hoc definition of $H^1_{\mathrm{HK}}(X)$.

Remark 4. (i) In (i), (iii), (iv), H^1_{HK} is a finite-dimensional $\breve{\mathbb{Q}}_p$-Banach space (in the usual sense). Therefore $(B^+_{\mathrm{st}} \otimes H^1_{\mathrm{HK}})^{N=0,\varphi=1}$ is a finite-dimensional Banach-Colmez space. In (ii), H^1_{HK} is a countable product of finite-dimensional $\breve{\mathbb{Q}}_p$-Banach spaces.

(ii) H^1_{proet} is always separated (but the inclusion of $\mathcal{O}(X)$ is not strict in the affinoid case).

(iii) H^1_{HK} can be described using the analytic skeleton of the Berkovich curve attached to X.

(iv) There are different possible approaches to prove the theorem: proétale cohomology of relative Fontaine rings, comparison with syntomic cohomology, symbols and p-adic integration on curves.