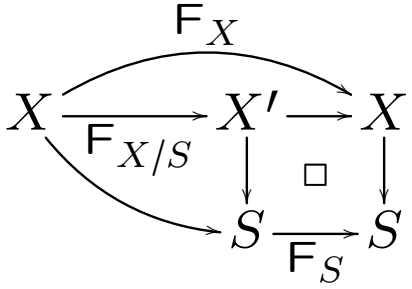
Simpson correspondence in characteristic $p > 0$ and splittings of the algebra of PD-differential operators

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$\widetilde{S}$ a flat scheme over $\mathbb{Z}/p^2\mathbb{Z}$, $S = \widetilde{S} \otimes_{\mathbb{Z}_p} \mathbb{F}_p$,
X a smooth S-scheme,



Theorem 1 (Deligne-Illusie)
(i) *Given smooth $\widetilde{S}$-schemes $\widetilde{X}$ and $\widetilde{X}'$ and an $\widetilde{S}$-morphism $\widetilde{\mathsf{F}}\colon \widetilde{X} \to \widetilde{X}'$ lifting $\mathsf{F}_{X/S}$, there exists a canonical quasi-isomorphism*

$$\oplus_{i\geq 0}\Omega^i_{X'/S}[-i] \to \mathsf{F}_{X/S*}(\Omega^\bullet_{X/S}),$$

inducing the Cartier operator C^{-1}.

(ii) *If we are given only a smooth lifting* $\widetilde{X}'$ *of* X' *over* $\widetilde{S}$*, there exists a canonical isomorphism in* $\mathbf{D}^b(\mathscr{O}_{X'})$

$$\oplus_{i=0}^{p-1}\Omega^i_{X'/S}[-i] \xrightarrow{\sim} \tau_{<p}\mathsf{F}_{X/S*}(\Omega^\bullet_{X/S}).$$

$X \to S$ a smooth *separated* morphism,
$\mathscr{I} \subset \mathscr{O}_{X\times_S X}$ the ideal of the diagonal,

$$\mathscr{P}^n_{X/S} = \mathscr{O}_{X\times_S X}/\mathscr{I}^{n+1} \quad (n \geq 0);$$

$$\mathscr{D}_{X/S,n} = \mathscr{H}om_{\mathscr{O}_X}(\mathscr{P}^n_{X/S}, \mathscr{O}_X)$$

$\mathscr{P}_{X/S,(0)}$ the PD-envelop of $\mathscr{I}$,
$\overline{\mathscr{I}} \subset \mathscr{P}_{X/S,(0)}$ the PD-ideal generated by $\mathscr{I}$,

$$\mathscr{P}^n_{X/S,(0)} = \mathscr{P}_{X/S,(0)}/\overline{\mathscr{I}}^{[n+1]} \quad (n \geq 0).$$

$$\mathscr{D}^{(0)}_{X/S,n} = \mathscr{H}om_{\mathscr{O}_X}(\mathscr{P}^n_{X/S,(0)}, \mathscr{O}_X)$$

[$\mathscr{O}_X$-algebras via the first projection.]

The $\mathscr{O}_X$-modules

$$\begin{aligned} \mathscr{D}_{X/S} &= \bigcup_{n\geq 0} \mathscr{D}_{X/S,n} \\ \mathscr{D}^{(0)}_{X/S} &= \bigcup_{n\geq 0} \mathscr{D}^{(0)}_{X/S,n}, \end{aligned}$$

are naturally equipped with ring structures.

$\mathscr{D}^{(0)}_{X/S}$ is the sheaf of PD-differential operators.

Assume in the following that $p\mathscr{O}_S = 0$ and $X \to S$ smooth.

Proposition 2 *The image of the composed homomorphism*

$$\mathscr{D}^{(0)}_{X/S} \to \mathscr{D}_{X/S} \hookrightarrow \mathscr{E}nd_{\mathscr{O}_S}(\mathscr{O}_X)$$

is the ring of $\mathscr{O}_{X'}$-linear endomorphisms of $\mathscr{O}_X$, and its kernel is the bilateral ideal $\mathscr{K}$ of $\mathscr{D}^{(0)}_{X/S}$ locally generated by the operators $\partial_i^{[p]}$ (for $1 \leq i \leq d := \dim(X/S)$); the latter are elements of the center $\mathrm{Z}\mathscr{D}^{(0)}_{X/S}$ of $\mathscr{D}^{(0)}_{X/S}$.

$$\begin{aligned} \overline{\mathscr{D}}^{(0)}_{X/S} &= \mathscr{D}^{(0)}_{X/S}/\mathscr{K}, \\ \widehat{\mathscr{D}}^{(0)}_{X/S} &= \varprojlim_{n\geq 0} \mathscr{D}^{(0)}_{X/S}/\mathscr{K}^n. \end{aligned}$$

$\Rightarrow \exists$ a canonical isomorphism of $\mathscr{O}_{X'}$-algebras

$$\mathsf{F}_{X/S*}(\overline{\mathscr{D}}^{(0)}_{X/S}) \overset{\sim}{\to} \mathscr{E}nd_{\mathscr{O}_{X'}}(\mathsf{F}_{X/S*}\mathscr{O}_X).$$

General result of linear algebra (example of *Morita equivalence*)

$\mathscr{A}$ a ring, $\mathscr{M}$ a non-trivial locally free $\mathscr{A}$-module of finite type, $\mathscr{B} = \mathscr{E}nd_{\mathscr{A}}(\mathscr{M})$ The functors:

$$\begin{aligned} \psi : \mathbf{Mod}(\mathscr{B}) &\to \mathbf{Mod}(\mathscr{A}), \\ \mathscr{E} &\mapsto \mathscr{H}om_{\mathscr{B}}(\mathscr{M}, \mathscr{E}), \\ \phi : \mathbf{Mod}(\mathscr{A}) &\to \mathbf{Mod}(\mathscr{B}), \\ \mathscr{F} &\mapsto \mathscr{M} \otimes_{\mathscr{A}} \mathscr{F}, \end{aligned}$$

are equivalences of categories quasi-inverse to each other ($\phi \circ \psi \xrightarrow[\text{ev}]{\sim} \mathrm{Id}$, $\mathrm{Id} \xrightarrow[\text{can}]{\sim} \psi \circ \phi$).

By *Morita equivalence*, the functors:

$$\begin{aligned}
\mathbf{Mod}(\overline{\mathscr{D}}^{(0)}_{X/S}) &\to \mathbf{Mod}(\mathscr{O}_{X'}),\\
\mathscr{E} &\mapsto \mathsf{F}_{X/S*}(\mathscr{H}om_{\overline{\mathscr{D}}^{(0)}_{X/S}}(\mathscr{O}_X,\mathscr{E})),\\
\mathbf{Mod}(\mathscr{O}_{X'}) &\to \mathbf{Mod}(\overline{\mathscr{D}}^{(0)}_{X/S}),\\
\mathscr{F} &\mapsto \mathsf{F}^*_{X/S}(\mathscr{F}) = \mathscr{F}\otimes_{\mathscr{O}_{X'}}\mathscr{O}_X,
\end{aligned}$$

are equivalences of categories quasi-inverse to each other (Cartier's *Frobenius descent*).

$\exists$ an $\mathscr{O}_X$-linear morphism

$$c\colon \mathsf{F}^*_X(\mathsf{T}_{X/S}) \to \mathsf{Z}\mathscr{D}^{(0)}_{X/S}, \qquad D \mapsto D^p - D^{(p)}.$$

inducing an isomorphism of $\mathscr{O}_{X'}$-algebras

$$\mathsf{S}(\mathsf{T}_{X'/S}) \xrightarrow[c]{\sim} \mathsf{F}_{X/S*}(\mathsf{Z}\mathscr{D}^{(0)}_{X/S}).$$

A left $\mathscr{D}^{(0)}_{X/S}$-module $\Leftrightarrow$ an $\mathscr{O}_X$-module $\mathscr{E}$ with an integrable connection $\nabla\colon \mathscr{E} \to \mathscr{E} \otimes_{\mathscr{O}_X} \Omega^1_{X/S}$.

Then, c induces an $\mathscr{O}_X$-linear morphism

$$\psi\colon \mathscr{E} \to \mathscr{E} \otimes_{\mathscr{O}_X} \mathsf{F}^*_{X/S}(\Omega^1_{X'/S}),$$

satisfying $\psi \wedge \psi = 0$, called the *p-curvature*.

- ψ vanishes $\Leftrightarrow$ the action of $\mathscr{D}^{(0)}_{X/S}$ factors through $\overline{\mathscr{D}}^{(0)}_{X/S}$.
- ψ is *nilpotent* $\Leftrightarrow$ the action of $\mathscr{D}^{(0)}_{X/S}$ factors through $\mathscr{D}^{(0)}_{X/S}/\mathscr{K}^N$ for $N \gg 0$.
- ψ is *quasi-nilpotent* $\Leftrightarrow$ $\forall x \in \mathscr{E}$, $\exists N \geq 0$ such that $\mathscr{K}^N \cdot x = 0$ ($\Rightarrow$ the action of $\mathscr{D}^{(0)}_{X/S}$ on $\mathscr{E}$ extends to an action of $\widehat{\mathscr{D}}^{(0)}_{X/S}$).

Want to extend Cartier's Frobenius descent to quasi-nilpotent objects.

Theorem 3 (Berthelot, Ogus-Vologodsky) *Given smooth $\widetilde{S}$-schemes $\widetilde{X}$ and $\widetilde{X}'$ and an $\widetilde{S}$-morphism $\widetilde{\mathsf{F}}\colon \widetilde{X} \to \widetilde{X}'$ lifting $\mathsf{F}_{X/S}$, there exists a canonical isomorphism of $\widehat{\mathsf{S}}(\mathsf{T}_{X'/S})$-algebras*

$$\lambda_{\widetilde{\mathsf{F}}}\colon \mathsf{F}_{X/S*}(\widehat{\mathscr{D}}^{(0)}_{X/S}) \xrightarrow{\sim} \mathsf{F}_{X/S*}(\overline{\mathscr{D}}^{(0)}_{X/S}) \otimes_{\mathscr{O}_{X'}} \widehat{\mathsf{S}}(\mathsf{T}_{X'/S})$$

compatible with the natural augmentations to $\mathsf{F}_{X/S*}(\overline{\mathscr{D}}^{(0)}_{X/S})$.

We deduce an isomorphism

$$\widehat{\mathscr{D}}^{(0)}_{X/S} \xrightarrow{\sim} \mathscr{E}nd_{\widehat{\mathsf{S}}(\mathsf{T}_{X'/S})}(\mathscr{O}_X \otimes_{\mathscr{O}_{X'}} \widehat{\mathsf{S}}(\mathsf{T}_{X'/S})).$$

The lifting $\widetilde{\mathsf{F}}$ equip $\mathsf{F}^*_{X/S}(\widehat{\mathsf{S}}(\mathsf{T}_{X'/S}))$ with a left $\widehat{\mathscr{D}}^{(0)}_{X/S}$-module structure and induces equiv. of cat. quasi-inverse to each other :

$$\mathbf{Mod}(\widehat{\mathscr{D}}^{(0)}_{X/S}) \to \mathbf{Mod}(\widehat{\mathsf{S}}(\mathsf{T}_{X'/S})),$$
$$\mathscr{E} \mapsto \mathscr{H}om_{\widehat{\mathscr{D}}^{(0)}_{X/S}}(\mathsf{F}^*_{X/S}(\widehat{\mathsf{S}}(\mathsf{T}_{X'/S})), \mathscr{E}),$$
$$\mathbf{Mod}(\widehat{\mathsf{S}}(\mathsf{T}_{X'/S})) \to \mathbf{Mod}(\widehat{\mathscr{D}}^{(0)}_{X/S}),$$
$$\mathscr{F} \mapsto \mathsf{F}^*_{X/S}(\mathscr{F}) = \mathscr{F} \otimes_{\widehat{\mathsf{S}}(\mathsf{T}_{X'/S})} \mathsf{F}^*_{X/S}(\widehat{\mathsf{S}}(\mathsf{T}_{X'/S})).$$

$\mathscr{F}$ an $\widehat{\mathsf{S}}(\mathsf{T}_{X'/S})$-module $\Rightarrow$ $\mathsf{T}_{X'/S} \otimes_{\mathscr{O}_{X'}} \mathscr{F} \to \mathscr{F}$ $\Rightarrow$ $\theta\colon \mathscr{F} \to \mathscr{F} \otimes_{\mathscr{O}_{X'}} \Omega^1_{X'/S}$ $\mathscr{O}_{X'}$-linear; $\theta \wedge \theta = 0$,
[θ is a called a *Higgs field*].

This is *Simpson correspondance in characteristic* $p > 0$.

Proposition 4 *The direct image by* $\mathsf{F}_{X/S*}$ *of the de Rham complex of* $\mathscr{E}$ *is quasi-isomorphic to the Higgs complex of* $\mathscr{F}$*.*

Sketch of proof : Koszul resolution of $\mathscr{O}_{X'}$:

$$[... \longrightarrow \mathsf{S}(\mathsf{T}_{X'/S}) \otimes_{\mathscr{O}_{X'}} \wedge^2 \mathsf{T}_{X'/S} \longrightarrow \mathsf{S}(\mathsf{T}_{X'/S}) \otimes_{\mathscr{O}_{X'}} \mathsf{T}_{X'/S} \longrightarrow \mathsf{S}(\mathsf{T}_{X'/S})] \longrightarrow \mathscr{O}_{X'} \to 0$$

$$\begin{aligned} \mathscr{F} \otimes_{\mathscr{O}_{X'}} \Omega^{\bullet}_{X'} &\tilde{\to} \mathscr{H}om_{\widehat{\mathsf{S}}(\mathsf{T}_{X'/S})}(\widehat{\mathsf{S}}(\mathsf{T}_{X'/S}) \otimes_{\mathscr{O}_{X'}} \wedge^{\bullet} \mathsf{T}_{X'/S}, \mathscr{F}) \\ &\tilde{\to} \mathscr{H}om_{\widehat{\mathscr{D}}^{(0)}_{X/S}}(\mathsf{F}^*_{X/S}(\widehat{\mathsf{S}}(\mathsf{T}_{X'/S}) \otimes_{\mathscr{O}_{X'}} \wedge^{\bullet} \mathsf{T}_{X'/S}), \mathscr{E}) \end{aligned}$$

Spencer resolution of $\mathscr{O}_X$:

$$[\ldots \longrightarrow \mathscr{D}^{(0)}_{X/S} \otimes_{\mathscr{O}_X} \wedge^2 \mathsf{T}_{X/S} \longrightarrow \mathscr{D}^{(0)}_{X/S} \otimes_{\mathscr{O}_X} \mathsf{T}_{X/S} \longrightarrow \mathscr{D}^{(0)}_{X/S}] \longrightarrow \mathscr{O}_X \longrightarrow 0$$

Reduced to compare two resolutions of $\mathscr{O}_X$.

$$\begin{array}{ccccccc}
\cdots \longrightarrow & \widehat{\mathscr{D}}^{(0)}_{X/S} \otimes_{\mathscr{O}_X} \mathsf{T}_{X/S} & \longrightarrow & \widehat{\mathscr{D}}^{(0)}_{X/S}] & \longrightarrow & \mathscr{O}_X \\
& \downarrow & & \downarrow{\scriptstyle P \mapsto P.1} & & \| \\
\cdots \longrightarrow & \mathsf{F}^*_{X/S}\widehat{\mathsf{S}}(\mathsf{T}_{X'/S}) \otimes_{\mathscr{O}_X} \mathsf{F}^*_{X/S}\mathsf{T}_{X'/S} & \longrightarrow & \mathsf{F}^*_{X/S}\widehat{\mathsf{S}}(\mathsf{T}_{X'/S})] & \longrightarrow & \mathscr{O}_X
\end{array}$$

commutes with left vertical arrow induced by the dual of $\frac{1}{p!}d(\widetilde{F}) : \mathsf{F}^*_{X/S}\Omega^1_{X'/S} \to \Omega^1_{X/S}$.

Sketch of proof of thm. 3

Will proceed by duality using

Lemma 5 *The ring* $\widehat{\mathscr{D}}^{(0)}_{X/S}$ *is isomorphic to the ring (of hyper-PD-differential operators)* $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{P}_{X/S,(0)}, \mathscr{O}_X)$.

Local coordinates $t_1, ..., t_d$ on X, $\tau_i = 1 \otimes t_i - t_i \otimes 1 \in \mathscr{I}$. Locally $\mathscr{P}_{X/S,(0)} = \oplus_{\underline{k}} \mathscr{O}_{X \times_S X}/(\tau_1^p, ..., \tau_d^p).\underline{\tau}^{[p.\underline{k}]}$ is filtered increasingly and exhaustively by

$\mathrm{Fil}^n \mathscr{P}_{X/S,(0)} = \oplus_{|\underline{k}| \leq n} \mathscr{O}_{X \times_S X}/(\tau_1^p, ..., \tau_d^p).\underline{\tau}^{[p.\underline{k}]}$

on which $\mathscr{K}^{n+1}$ acts trivially, hence

$\mathscr{H}om_{\mathscr{O}_X}(\mathrm{Fil}^n \mathscr{P}_{X/S,(0)} \mathscr{P}_{X/S,(0)}, \mathscr{O}_X) \tilde{\to} \mathscr{D}^{(0)}_{X/S}/\mathscr{K}^{n+1}$.

Lemma 6 *The map* $\frac{1}{p!}(\mathsf{F}^* \times \mathsf{F}^*) : \mathscr{I} \to \mathscr{P}_{X/S,(0)}$ *;* $f \to f^{[p]}$ *composed with the projection* $\mathscr{P}_{X/S,(0)} \to \mathscr{P}_{X/S,(0)}/\mathscr{I}.\mathscr{P}_{X/S,(0)}$ *is an* F^*_X*-linear map that is zero on* $\mathscr{I}^2$*.*

Linearization gives an $\mathscr{O}_X$-linear map :

$$\mathsf{F}^*_{X/S}\Omega^1_{X'/S} \to \mathscr{P}_{X/S,(0)}/\mathscr{I}.\mathscr{P}_{X/S,(0)}$$

called *divided Frobenius*.

Let $\Gamma(\Omega^1_{X'/S})$ the PD-algebra of the $\mathscr{O}_{X'}$-module $\Omega^1_{X'/S}$.

Proposition 7 *The divided Frobenius map extends uniquely to an isomorphism of PD-*$\mathscr{O}_X$*-algebras :*

$$\mathsf{F}^*_{X/S}\Gamma(\Omega^1_{X'/S}) \tilde{\to} \mathscr{P}_{X/S,(0)}/\mathscr{I}.\mathscr{P}_{X/S,(0)}.$$

Description in local coordinates : $t'_1, ..., t'_d$ the pull-back of the t_i's.

$$\begin{array}{rcl} \mathscr{O}_X < dt'_1, ..., dt'_d > & \to & \mathscr{O}_X < \tau_i, ..., \tau_d > /(\tau_1, ..., \tau_d) \\ dt'_i & \mapsto & \tau_i^{[p]} \end{array}$$

Remark : The composite

$$\mathsf{S}(\mathsf{T}_{X'/S}) \overset{\sim}{\underset{c}{\to}} \mathsf{F}_{X/S*}(\mathsf{Z}\mathscr{D}^{(0)}_{X/S}) \hookrightarrow \mathsf{F}_{X/S*}\mathscr{D}^{(0)}_{X/S}.$$

can be obtained by duality from the composite $\mathscr{P}_{X/S,(0)} \to \mathscr{P}_{X/S,(0)}/\mathscr{I}.\mathscr{P}_{X/S,(0)} \overset{\sim}{\leftarrow} \mathsf{F}^*_{X/S}\Gamma(\Omega^1_{X'/S})$.

Next step : show that the data of thm. 3 (we fix such smooth $\widetilde{S}$-schemes $\widetilde{X}$ and $\widetilde{X}'$ and an $\widetilde{S}$-morphism $\widetilde{\mathsf{F}}\colon \widetilde{X} \to \widetilde{X}'$ lifting $\mathsf{F}_{X/S}$ in the sequel) allows to canonically lift the latter isomorphism

$$\mathsf{F}^*_{X/S}\Gamma(\Omega^1_{X'/S}) \xrightarrow{\sim} \mathscr{P}_{X/S,(0)}/\mathscr{I}.\mathscr{P}_{X/S,(0)}$$

to a morphism

$$\mathsf{F}^*_{X/S}\Gamma(\Omega^1_{X'/S}) \to \mathscr{P}_{X/S,(0)}.$$

Proposition 8 *There exists a well-defined map*

$$\begin{array}{lll} \frac{1}{p!}(\widetilde{\mathsf{F}}^* \times \widetilde{\mathsf{F}}^*) : \tilde{\mathscr{J}}' \to p\mathscr{P}_{\widetilde{X}/\widetilde{S},(0)} & \underset{.p!}{\overset{\sim}{\leftarrow}} & \mathscr{P}_{\widetilde{X}/\widetilde{S},(0)}/p\mathscr{P}_{\widetilde{X}/\widetilde{S},(0)} \\ & \overset{\sim}{\to} & \mathscr{P}_{X/S,(0)} \end{array}$$

that factors through $\Omega^1_{X'/S}$ *such that the induced map* $\Omega^1_{X'/S} \to \mathscr{P}_{X/S,(0)}$ *is a lifting of the divided Frobenius :*

$$\begin{array}{ccc} \Omega^1_{X'/S} & \longrightarrow & \mathscr{P}_{X/S,(0)} \\ \| & & \downarrow \\ \Omega^1_{X'/S} & \longrightarrow & \mathscr{P}_{X/S,(0)}/\mathscr{I}.\mathscr{P}_{X/S,(0)} \end{array}$$

commutes.

Sketch of proof : want to understand $(\widetilde{\mathsf{F}} \times \widetilde{\mathsf{F}})^* : \tilde{\mathscr{I}}' \to \tilde{\mathscr{I}}$.
Take $x \in \mathscr{O}_X$, $x' := 1 \otimes x \in \mathscr{O}_{X'}$ with lifts $\tilde{x} \in \mathscr{O}_{\tilde{X}}$, $\tilde{x}' \in \mathscr{O}_{\tilde{X}'}$, $\widetilde{\mathsf{F}}^*(\tilde{x}') = \tilde{x}^p + py$.

$\tilde{\xi} := 1 \otimes \tilde{x} - \tilde{x} \otimes 1$, $\tilde{\xi}' := 1 \otimes \tilde{x}' - \tilde{x}' \otimes 1$

$$\begin{aligned}
(\widetilde{\mathsf{F}} \times \widetilde{\mathsf{F}})^*(\tilde{\xi}') &= 1 \otimes \tilde{x}^p - \tilde{x}^p \otimes 1 + p.(1 \otimes y - y \otimes 1) \\
&= \tilde{\xi}^p + \sum_{i=1}^{p-1} \binom{p}{i} (\tilde{x}^{p-i} \otimes 1)\tilde{\xi}^i + p(1 \otimes y - y \otimes 1) \\
&\equiv \tilde{\xi}^p \bmod p\tilde{\mathscr{I}}
\end{aligned}$$

Hence

$$(\widetilde{\mathsf{F}} \times \widetilde{\mathsf{F}})^*(\tilde{\xi}') = p!.\tilde{\xi}^{[p]} + p\zeta \in \mathscr{P}_{\widetilde{X}/\widetilde{S},(0)}$$

with $\zeta \in \tilde{\mathscr{I}}$.

Then divides by $p!$.

Proposition 9 . *The divided Frobenius extends canonically to a morphism*

$$\mathsf{F}^{*}_{X/S}\Gamma(\Omega^{1}_{X'/S}) \to \mathscr{P}_{X/S,(0)}$$

that, by duality, induces a morphism of $\mathscr{O}_X$-modules :

$$\Phi_{\widetilde{F}} : \widehat{\mathscr{D}}^{(0)}_{X/S} \to \mathsf{F}^{*}_{X/S}(\widehat{\mathsf{S}}(\mathsf{T}_{X'/S})) \hookrightarrow \widehat{\mathscr{D}}^{(0)}_{X/S}$$

Warning : $\Phi_{\widetilde{F}}$ is not a morphism of rings but its restriction to the center is one.

Explicit formulas

Proposition 10 . *Given local coordinates* $t_1, ..., t_d$ *on* X, *then*

$$\Phi_{\widetilde{F}}(\underline{\partial}^{\underline{n}}) = \begin{cases} 1 & \text{if} \quad |\underline{n}| = 0\,, \\ 0 & \text{if} \quad 0 < |\underline{n}| < p\,, \\ \frac{1}{p!} \sum_{j=1}^{d} \partial_i(\tilde{F}^*_{\widetilde{X}}(\tilde{t}'_j))\partial_j^p & \text{if} \quad \underline{n} = p.\mathbf{1}_i. \end{cases}$$

If $\tilde{F}^*_{\widetilde{X}}(\tilde{t}'_j) = \tilde{t}'^p_j + p\tilde{g}_j$, *the last expression can be rewritten*

$$-t_i^{p-1}\partial_i^p - \sum_{j=1}^{d} \partial_i(g_j)\partial_j^p$$

Theorem 11 . *The divided Frobenius map* $\frac{1}{p!}(\widetilde{\mathsf{F}}^* \times \widetilde{\mathsf{F}}^*)$ *extends canonically to a morphism*

$$\sigma_{\widetilde{\mathsf{F}}} : \mathscr{O}_{X\times_{X'}X} \otimes_{\mathscr{O}_{X'}} \Gamma(\Omega^1_{X'/S}) \tilde{\to} \mathscr{P}_{X/S,(0)}$$

which is a PD-isomorphism of $\mathscr{O}_{X\times_{X'}X}$*-algebras.*

Theorem 3 follow by duality. Note that $\mathscr{O}_{X\times_{X'}X} = \mathscr{O}_{X\times_S X}/\mathscr{I}^{(p)}$.

The verification is the local : $\mathscr{P}_{X/S,(0)}$ is filtered by the $\mathrm{Fil}^n\mathscr{P}_{X/S,(0)}$'s and $\Gamma(\Omega^1_{X'/S})$ is graded. The application $\sigma_{\widetilde{\mathsf{F}}}$ is a PD-morphism and $\mathrm{gr}(\sigma_{\widetilde{\mathsf{F}}})$ send $1\otimes dt'_i$ to $\tau_i^{[p]}$.

The $\widehat{\mathsf{S}}(\mathsf{T}_{X'/S})$-algebra nature of the morphism $\lambda_{\widetilde{\mathsf{F}}}$ comes from the commutative diagram

$$\begin{array}{ccc} \mathscr{O}_{X\times_{X'}X} \otimes_{\mathscr{O}_{X'}} \Gamma(\Omega^1_{X'/S}) & \xrightarrow[dt'_i\mapsto\tau_i^{[p]}+\zeta_i]{} & \mathscr{P}_{X/S,(0)} \\ \downarrow & & \downarrow \\ \mathscr{O}_X \otimes_{\mathscr{O}_{X'}} \Gamma(\Omega^1_{X'/S}) & \xrightarrow[dt'_i\mapsto\tau_i^{[p]}]{} & \mathscr{P}_{X/S,(0)}/\mathscr{I}.\mathscr{P}_{X/S,(0)} \end{array}$$

and the compatibility of $\sigma_{\widetilde{\mathsf{F}}}$ with comultiplications :

$$\Gamma(\Omega^1_{X'/S}) \xrightarrow[\Delta]{} \Gamma(\Omega^1_{X'/S}\oplus\Omega^1_{X'/S}) \xrightarrow{\sim} \Gamma(\Omega^1_{X'/S}) \otimes_{\mathscr{O}_{X'}} \Gamma(\Omega^1_{X'/S})$$

$$\begin{array}{rcl} \mathscr{O}_{X\times_{X'}X} & \to & \mathscr{O}_{X\times_{X'}X} \otimes_{\mathscr{O}_X} \mathscr{O}_{X\times_{X'}X} \\ a\otimes b & \mapsto & a\otimes 1\otimes b \end{array}$$

$$\begin{array}{rcl} \mathscr{P}_{X/S,(0)} & \to & \mathscr{P}_{X/S,(0)} \otimes_{\mathscr{O}_X} \mathscr{P}_{X/S,(0)} \\ a\otimes b & \mapsto & a\otimes 1\otimes b \end{array}$$

Example

$\widetilde{S} = \mathrm{Spec}\,(\mathbb{Z}/p^2\mathbb{Z})$, $\widetilde{X} = \mathrm{Spec}\,(\mathbb{Z}/p^2\mathbb{Z}[t])$, $\widetilde{F}(t) = t^p$, $(1, t, ..., t^{p-1})$ basis of $\mathscr{O}_X \otimes_{\mathscr{O}_{X'}} \widehat{\mathsf{S}}(\mathsf{T}_{X'/S})$ $(= \mathbb{Z}/p\mathbb{Z}[t, \partial^p])$ over $\widehat{\mathsf{S}}(\mathsf{T}_{X'/S})$ $(= \mathbb{Z}/p\mathbb{Z}[t^p, \partial^p])$

$$t \to \begin{pmatrix} 0 & 0 & \cdots & \cdots & t^p \\ 1 & 0 & \ddots & & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix},$$

$$\partial \to \begin{pmatrix} 0 & 1 - t^p\partial^p & 0 & \cdots & 0 \\ 0 & 0 & 2 - t^p\partial^p & \ddots & 0 \\ 0 & 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & p - 1 - t^p\partial^p \\ -\partial^p & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

The Cartier transform of Ogus-Vologodsky

Given (only) a smooth $\widetilde{S}$-lifting $\widetilde{X}'$ of X', Ogus and Vologodsky "glue" in a sense the isomorphisms $\lambda_{\widetilde{F}}$ for various liftings $\widetilde{\mathsf{F}}$ of $\mathsf{F}_{X/S}$. The Morita equivalence of the isomorphism they obtain is the *Cartier transform*.

$\Gamma(\mathsf{T}_{X'/S})$ the PD-algebra of the $\mathscr{O}_{X'}$-module $\mathsf{T}_{X'/S}$ and $J = \oplus_{n\geq 1}\Gamma_n(\mathsf{T}_{X'/S})$ its PD-ideal.
$\widehat{\Gamma}(\mathsf{T}_{X'/S})$ the completion of $\Gamma(\mathsf{T}_{X'/S})$ with respect to the PD-filtration $(J^{[n]})_{n\geq 1}$.

$\mathbf{HIG}_\gamma(X'/S)$ the category of $\widehat{\Gamma}(\mathsf{T}_{X'/S})$-modules [*PD-Higgs modules* over X'/S].

$$\mathscr{D}^{\gamma}_{X/S} = \mathscr{D}^{(0)}_{X/S} \otimes_{\mathsf{S}(\mathsf{T}_{X'/S})} \widehat{\Gamma}(\mathsf{T}_{X'/S}).$$

$\mathbf{MIC}_\gamma(X/S)$ the category of left $\mathscr{D}^\gamma_{X/S}$-modules.

A left $\mathscr{D}^\gamma_{X/S}$-modules $\Leftrightarrow$ an $\mathscr{O}_X$-module with an integrable connection (E,∇) equipped with a homomorphism

$$\widehat{\Gamma}(\mathsf{T}_{X'/S}) \to \mathsf{F}_{X/S*}(\mathscr{E}nd_{\mathscr{O}_X}(E,\nabla))$$

which extends its p-curvature

$$\mathsf{S}(\mathsf{T}_{X'/S}) \to \mathsf{F}_{X/S*}(\mathscr{E}nd_{\mathscr{O}_X}(E,\nabla)).$$

Denote by ι the automorphism of $\widehat{\Gamma}(\mathsf{T}_{X'/S})$ induced by $-\mathrm{id}_{\mathsf{T}_{X'/k}}$

Theorem 12 (Ogus-Vologodsky) *Assume given a smooth $\widetilde{S}$-lifting $\widetilde{X}'$ of X'.*

(i) *There exists a can. left $\mathscr{D}^{\gamma}_{X/k}$-module $\mathsf{B}_{\widetilde{X}'}$ splitting the Azumaya algebra $\mathsf{F}_{X/S*}(\mathscr{D}^{\gamma}_{X/S})$ over $\widehat{\Gamma}(\mathsf{T}_{X'/S})$.*

(ii) *There exist canonical equiv. of cat. quasi-inverse to each other*

$$\begin{aligned}
\mathsf{C}_{\widetilde{X}'}\colon \mathbf{MIC}_{\gamma}(X/S) &\xrightarrow{\sim} \mathbf{HIG}_{\gamma}(X'/S),\\
E &\mapsto \iota^{*}(\mathscr{H}om_{\mathscr{D}^{\gamma}_{X/S}}(\mathsf{B}_{\widetilde{X}'}, E)),\\
\mathsf{C}^{-1}_{\widetilde{X}'}\colon \mathbf{HIG}_{\gamma}(X'/S) &\xrightarrow{\sim} \mathbf{MIC}_{\gamma}(X/S),\\
E' &\mapsto \mathsf{B}_{\widetilde{X}'} \otimes_{\widehat{\Gamma}(\mathsf{T}_{X'/S})} \iota^{*}(E').
\end{aligned}$$

Furthermore, they induce equiv. between the subcat. of quasi-nilpotent objects.

(iii) *Let* $(E, \nabla, \psi) \in \mathbf{MIC}_{\gamma}(X/S)$ *and* $(E', \theta') = \mathsf{C}_{\widetilde{X}'}(E, \nabla, \psi)$. *An* $\widetilde{S}$*-lifting* $\widetilde{\mathsf{F}}\colon \widetilde{X} \to \widetilde{X}'$ *of* $\mathsf{F}_{X/S}$ *induces a canonical functorial isomorphism*

$$\eta_{\widetilde{\mathsf{F}}}\colon (E, \psi) \xrightarrow{\sim} \mathsf{F}^{*}_{X/S}(E', -\theta').$$

(iv) $\forall 0 \leq \ell \leq p-1$, $\mathsf{C}^{-1}_{\widetilde{X}'}$ *induces an equiv. of cat.*

$$\mathsf{C}^{-1}_{\widetilde{X}'}\colon \mathbf{HIG}_{\ell}(X'/S) \xrightarrow{\sim} \mathbf{MIC}_{\ell}(X/S).$$

$\mathbf{HIG}_{\ell}(X'/S)$ *is the cat. of* Higgs modules (M', θ') *(*i.e., M' *is an* $\mathscr{O}_{X'}$*-mod.,* $\theta'\colon M' \to M' \otimes_{\mathscr{O}_{X'}} \Omega^{1}_{X'/S}$ *is an* $\mathscr{O}_{X'}$*-linear map such that* $\theta' \wedge \theta' = 0$*) which are* nilpotent of level $\leq \ell$ *(*i.e., *there exists an increasing* θ*-stable filtration* $M'_{\bullet}$ *of* M' *of length* $\leq \ell + 1$ *such that* $\mathrm{Gr}^{M'}_{\bullet}(\theta') = 0$*).*

$\mathbf{MIC}_\ell(X/S)$ *is the cat. of* $\mathscr{O}_X$*-modules with integrable connection which are nilpotent of level* $\leq \ell$ (i.e., *the associated* p*-curvature is a nilpotent* F*-Higgs field of level* $\leq \ell$*).*

(v) *For* $(M', \theta') \in \mathbf{HIG}_\ell(X'/S)$ *and* $(M, \nabla) = \mathsf{C}^{-1}_{\widetilde{X}'}(M', \theta')$, $\exists$ *a canonical isom in* $\mathbf{D}^b(\mathscr{O}_{X'})$

$$\tau_{<p-\ell}(M' \otimes \Omega^\bullet_{X'/S}) \xrightarrow{\sim} \tau_{<p-\ell}(\mathsf{F}_{X/S*}(M \otimes \Omega^\bullet_{X/S})).$$

Hint for the construction of $\mathsf{B}_{\widetilde{X}'}$:

$\widetilde{\mathsf{F}}_1, \widetilde{\mathsf{F}}_2 : \widetilde{X} \to \widetilde{X}'$

$\frac{1}{p!}(\widetilde{\mathsf{F}}_2^* - \widetilde{\mathsf{F}}_1^*) : \mathscr{O}_{X'} \to \mathsf{F}_{X/S*}\mathscr{O}_X$ is a derivation, i.e. a linear form $u : \Omega^1_{X'/S} \to \mathsf{F}_{X/S*}\mathscr{O}_X$

$\sigma^{-1}_{\widetilde{\mathsf{F}}_2} \circ \sigma_{\widetilde{\mathsf{F}}_1}$ induces an isomorphism

$$\mathscr{O}_{X\times_{X'}X} \otimes_{\mathscr{O}_{X'}} \mathsf{S}(\Omega^1_{X'/S}) \tilde{\to} \mathscr{O}_{X\times_{X'}X} \otimes_{\mathscr{O}_{X'}} \mathsf{S}(\Omega^1_{X'/S})$$

(not $- \otimes_{\mathscr{O}_{X'}} \Gamma(\Omega^1_{X'/S})$! : $\sigma_{\widetilde{\mathsf{F}}_i}(1 \otimes dt'_i) = \tau_i^{[p]} + \ldots$)

$\sigma^{-1}_{\widetilde{\mathsf{F}}_2} \circ \sigma_{\widetilde{\mathsf{F}}_1}$ gives the glueing.

Even more explicitly,

$$\sigma_{\widetilde{\mathsf{F}}_2}^{-1} \circ \sigma_{\widetilde{\mathsf{F}}_1}(1 \otimes \omega') = 1 \otimes \omega' - (1 \otimes u(\omega') - u(\omega') \otimes 1) \otimes 1$$

i.e. a translation on $\mathsf{S}(\mathscr{O}_{X \times_{X'} X} \otimes_{\mathscr{O}_{X'}} \Omega^1_{X'/S})$ whose dual on $\widehat{\Gamma}((\mathscr{O}_{X \times_{X'} X} \otimes_{\mathscr{O}_{X'}} \Omega^1_{X'/S})^\vee)$ is multiplication by exp(u) (Taylor's formula).

On $U_\alpha \cap U_\beta$, one glue λ_α and λ_β using conjugation by $\exp(\mathsf{u}_{\alpha\beta})$.