In the previous lectures, $A/k \mapsto \mathrm{HH}(A/k)$. When A/k smooth, $\mathrm{HH}_*(A/k) \cong \Omega^*_{A/k}$ with a circle action giving rise to the Connes differential.

For X/k a smooth scheme, de Rham cohomology is defined as hypercohomology of $\Omega^*_{A/k}$, so get a Hodge-de Rham spectral sequence

$$E_1^{i,j} = H^j(X, \Omega^i_{X/k}) \Rightarrow H^{i+j}_{\mathrm{dR}}(X/k).$$

Meanwhile,

$$E^2_{i,j} = H^{-i}(X, \Omega^j_{X/k}) \Rightarrow \mathrm{HH}_{i+j}(X/k).$$

(These pictures form a "B/A" pair in the sense of mirror symmetry, whatever that means.)

Circle action $\mathbb{T}$ on $\mathrm{HH}(X/k)$: gives rise to d on $\mathrm{HH}_*(X/k)$. Per Connes, don't take homotopy groups first; use the direct circle action. To this end, form a triangle

$$\begin{array}{ccc}
\mathrm{HP}(X/k) & = & \mathrm{HH}(X/k)^{t\mathbb{T}} \\
\uparrow & & \uparrow \\
\mathrm{HC}^-(X/k) & = & \mathrm{HH}(X/k)^{h\mathbb{T}} \\
\uparrow & & \uparrow{\scriptstyle N} \\
\Sigma\mathrm{HC}(X/k) & = & \Sigma\mathrm{HH}(X/k)_{h\mathbb{T}}
\end{array}$$

Can do likewise with the base ring k replaced by the sphere spectrum $\mathbb{S}$. By definition, $\mathrm{THH}(X/\mathbb{S})$ is just $\mathrm{HH}(X/\mathbb{S})$ by the same formula as before; again it carries a circle action. Likewise, define $\mathrm{TP}(X) := \mathrm{HP}(X/\mathbb{S})$.

$$\begin{array}{ccc}
\mathrm{HH}_*(\mathbb{F}_p/\mathbb{Z}) & = & \Gamma_{\mathbb{F}_p}\{\delta\} \\
\uparrow & & \uparrow \\
\mathrm{HH}_*(\mathbb{F}_p/\mathbb{S}) & = & \Gamma_{\mathbb{S}_p}\{\delta\}
\end{array}$$

where the bottom row is Bökstedt periodicity. Novelty here: descending to $\mathbb{S}$ introduces Frobenius!

$$E^2_{i,j} = H^{j-i}(X, \mathbb{Z}(j)) \Rightarrow K^{-i-j}(X)$$

where the twist (j) means that the action of the kth Adams operator ψ^k being multiplied by k^j. We want to dualize the right side to $K_{i+j}(X)$; by Voevodsky, we have this when X is smooth, using motivic cohomology; but also expect this in the presence of nilpotents.

Based on examples, expect to fill in the bottom row of this diagram:

$$\begin{array}{ccc}
E^2_{i,j} = H^{j-i}(X, \mathbb{Z}(j)) & \Longrightarrow & K_{i+j}(X) \\
 & & \Big\downarrow \text{Chern char.} \\
E^2_{i,j} = H^{j-i}(X, W(j)) & \Longrightarrow & \mathrm{TP}_{i+j}(X)
\end{array}$$

but the bottom right is not homotopy invariant! This is done by BMS in some cases.

- If $X/\mathbb{F}_p$ smooth, then $W(J) \cong W\Omega^\bullet_X$.
- If $X/\mathcal{O}_C$ smooth, then $W(j) \cong A\Omega^\bullet_X$.

By analogy with the universal map $A \to \Omega^\bullet_{A/k}$ of graded k-algebras (but not complexes), get a map $A \to \mathrm{THH}(A)$ of E_∞-rings, where the target has a $\mathbb{T}$-action and is universal for this property. New feature: the Tate diagonal of spectra

$$X \xrightarrow{\Delta_p} ((X \otimes_{\mathbb{S}} \cdots \otimes_{\mathbb{S}} X)^{tC_p}$$

where p is prime and there are p factors in the product (and $C_p \subseteq \mathbb{T}$). Need p prime in order to preserve cofiber sequences.

Universal property gives

$$\begin{array}{ccc} A & \longrightarrow & \mathrm{THH}(A) \\ \downarrow{\scriptstyle \Delta_p} & & \downarrow{\scriptstyle \exists!} \\ (A \otimes_{\mathbb{S}} \cdots \otimes_{\mathbb{S}} A)^{tC_p} & \longrightarrow & \mathrm{THH}(A)^{tC_p} \end{array}$$

Everything in this diagram can be represented by chain complexes except for the vertical maps.

Take X now to be a spectrum with C_{p^2}-action. Nikolaus-Scholze:

$$\begin{array}{ccccccc} (X_{hC_p})_{h(C_{p^2}/C_p)} & = & X_{hC_{p^2}} & \xrightarrow{N_{C_{p^2}}} & X^{hC_{p^2}} & \longrightarrow & X^{tC_{p^2}} \\ & \searrow & & & \| & & \\ & & (X_{hC_p})^{h(C_{p^2}/C_p)} & \longrightarrow & (X^{hC_p})^{h(C_{p^2}/C_p)} & \longrightarrow & (X^{tC_p})^{h(C_{p^2}/C_p)} \\ & & & \searrow & & & \\ & & & & (X_{hC_p})^{t(C_{p^2}/C_p)} & & \end{array}$$

and this thing on the bottom equals 0 if X is bounded below.

$$\begin{array}{ccc} \vdots & & \mathrm{TR}^3_0(A;p) = W_3(A) \\ {\scriptstyle R}\downarrow\uparrow{\scriptstyle F} & & {\scriptstyle R}\downarrow\uparrow{\scriptstyle F} \quad {\scriptstyle R}\downarrow\uparrow{\scriptstyle F} \\ \mathrm{TR}^3(X;p) & & \mathrm{TR}^2_0(A;p) = W_2(A) \\ {\scriptstyle R}\downarrow\uparrow{\scriptstyle F} & & {\scriptstyle R}\downarrow\uparrow{\scriptstyle F} \quad {\scriptstyle R}\downarrow\uparrow{\scriptstyle F} \\ \mathrm{TR}^2(X;p) & & \mathrm{TR}^1_0(A;p) = \mathrm{THH}(X) = A \\ {\scriptstyle R}\downarrow\uparrow{\scriptstyle F} & & \\ \mathrm{TR}^1(X;p) = \mathrm{THH}(X) & & \end{array}$$

To define TR:

$$\begin{array}{ccccc}
 & & \overset{F}{\frown} & & \\
\mathrm{TR}^2(X;p) & \longrightarrow & \mathrm{THH}(X)^{hC_p} & \longrightarrow & \mathrm{THH}(X) \\
\downarrow{\scriptstyle R} & & \downarrow & & \| \\
\mathrm{TR}^1(X;p) := \mathrm{THH}(X) & \xrightarrow{\psi_p} & \mathrm{THH}(X)^{tC_p} & & \mathrm{TR}^1(X;p)
\end{array}$$

with the square on the left is Cartesian. Next stage:

$$\begin{array}{ccccc}
\mathrm{TR}^3(X;p) & & \longrightarrow & & \mathrm{THH}(X)^{hC_{p^2}} \\
\downarrow{\scriptstyle R} & & & & \downarrow \\
 & & & & \mathrm{THH}(X)^{tC_{p^2}} \\
 & & & & \| \\
\mathrm{TR}^2(X;p) & \xrightarrow{f} & \mathrm{THH}(X)^{hC_p} & \xrightarrow{\varphi_p^{hC_p}} & (\mathrm{THH}(X)^{tC_p})^{h(C_{p^2}/C_p)}
\end{array}$$

and so on.