(joint with X. Zhu)

1. Introduction

Let k be a finite extension of $\mathbb{Q}_p$. Let X be a connected, smooth algebraic variety over k. Let $\mathbb{L}$ be a $\mathbb{Q}_p$-local system on X_{et}.

(L.-Zhu, 2016) de Rham rigidity: If $\mathbb{L}_x$ is de Rham for some closed point $x \in X$, then so is $\mathbb{L}_y$ for all closed points $y \in X$.

One proves more: in fact, $\mathbb{L}^{\text{an}}$ is a de Rham local system on $X_{\text{et}}^{\text{an}}$, which is to say there exists a filtered vector bundle M on $X_{\text{et}}^{\text{an}}$ with integrable connection (plus transversality) such that $\mathbb{L}^{\text{an}} \otimes \mathcal{OB}_{\text{dR}} \cong M \otimes \mathcal{OB}_{\text{dR}}$ on $X_{\text{proet}}^{\text{an}}$.

Interesting case: X is a non-abelian type Shimura variety.

Theorem **A**: in fact $M = D_{\text{dR}}(\mathbb{L}^{\text{an}})$ can be algebraized in some canonical way.

2. Log adic spaces

(aprés H. Diao and F. Tan)

Definition: for X a noetherian adic space over k, a pre-log structure is a homomorphism $\alpha : M \to \mathcal{O}_{X_{\text{et}}}$ for some sheaf of monoids M. It is a log structure if $\alpha^{-1}(\mathcal{O}_{X_{\text{et}}}^{\times}) \to \mathcal{O}_{X_{\text{et}}}^{\times}$ is an isomorphism. As usual, there is an adjunction from pre-log-structures to log-structures

$$\begin{array}{ccc} \mathcal{O}_X^{\times} & \longrightarrow & M^a \\ \uparrow & & \uparrow \\ \alpha^{-1}(\mathcal{O}_X^{\times}) & \longrightarrow & M \end{array}$$

Usual example: the closed unit ball $\mathbb{A}^r = (\text{Spa}(k\langle T_1, \dots, T_r\rangle, \mathcal{O}_k\langle T_1, \dots, T_r\rangle), \mathbb{N}^r)$.

Consider *fine, separated* (fs) log structures as usual.

Definition: $X \to Y$ of fs log adic spaces is *Kummer-étale* if étale locally there is a chart $P \to Q$ of F such that:

- $P \to Q$ is Kummer;
- $X \to Y \times_{\text{Spa}(k[P], \mathcal{O}_k[P])} \text{Spa}(k[Q], \mathcal{O}_k[Q])$ is étale.

Define "finite Kummer-étale covers" similarly. Get $X_{\text{ket}}, X_{\text{proket}}$.

3. Construction of $\mathcal{OB}_{\text{dR}}^{\log}$

Morally, "dlog $T_i = \frac{dT_i}{T_i} = \text{dlog } \frac{T_i}{[T_i^\flat]}$".

Let $U = \varprojlim_i U_i = (\text{Spa}(R_i, R_i^+), M_i, \alpha_i)$ be an affinoid perfectoid in X_{proet}, $\widehat{U} = \text{Spa}(R, R^+)$.

Put $(M, \alpha) =$ (associated log structure of) $\varinjlim (M_i, \alpha_i)$, $M^\flat = \varprojlim_{\times p} M$. $\alpha^\flat : M^\flat \to R^\flat$ takes $(m_0, m_1, \dots)$ to $(\alpha(m_0), \alpha(m_1), \dots)$. Consider

$$\theta_i : S_i := (R_i^+ \widehat{\otimes}_{W(\kappa)} \mathbf{A}_{\text{inf}}(R, R^+))[\frac{1}{p}, \{\frac{\theta([\alpha^\flat(m)])}{[\alpha^\flat(m)]}\}_{m_0 \in R_i^+}] \to R.$$

Set $\mathcal{OB}_{\text{dR}}^{+,\log}$ to be the sheafication of the presheaf $U \mapsto \varinjlim \widehat{S}_i$ (completion with respect to $\ker(\theta_i)$).

Example:

$$f : X \to \widetilde{\mathbb{A}}^r = \varprojlim \mathbb{A}^r_n = (\mathrm{Spa}(k_n\langle T_1^{1/n}, \dots, T_r^{1/n}\rangle, \mathcal{O}_{k_n}\langle T_1^{1/n}, \dots, T_r^{1/n}\rangle), (\frac{1}{n}\mathbb{N})^r)$$

$$\tilde{X} := X \times_{\mathbb{A}^r} \widetilde{\mathbb{A}}^r.$$

Then

$$\mathcal{O}\mathbb{B}^{+,\log}_{\mathrm{dR}}|_{\tilde{X}} \cong \mathbb{B}_{\mathrm{dR}}(\tilde{X})[[X_1, \dots, X_r]]$$

with $T_i/[T_i^\flat] - 1$ corresponding to X_i. It follows that

$$dX_i = (1 + X_i)\mathrm{dlog}T_i.$$

Note that X_i is invertible, thus the Poincaré lemma holds. Also,

$$\mathrm{gr}^i\mathcal{O}\mathbb{B}_{\mathrm{dR}} \cong t^i\widehat{\mathcal{O}}_X[\{t^{-1}\log\frac{T_i}{[T_i^\flat]}_{1\le i\le r}]$$

4. Logarithmic Riemann-Hilbert correspondence

Now let $\mathbb{L}$ be a d-dimensional $\mathbb{Q}_p$-local system on X_{ket}, and let $K = \widehat{k_\infty}$. Consider

$$\begin{array}{ccc} X_{\mathrm{proket}} & \longleftarrow & X_{\mathrm{proket}}/X_K \\ \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \nu'} \\ X_{\mathrm{ket}} & \xleftarrow{\mu} & (X_K)_{\mathrm{ket}} \end{array}$$

Set $\mathrm{RH}^{\log}(\mathbb{L}) := R\nu'_*(\mathbb{L} \otimes \mathcal{O}\mathbb{B}^{\log}_{\mathrm{dR}})$.

Theorem **B**:

- It turns out that $\mathrm{RH}^{\log}(\mathbb{L}) = \nu'_*(\mathbb{L} \otimes \mathcal{O}\mathbb{B}^{\log}_{\mathrm{dR}})$ is a d-dimensional filtered vector bundle on $X_{\mathbb{B}_{\mathrm{dR}}}$ with integrable log connection, satisfying transversality, plus semi-linear $\mathrm{Gal}(K/k)$-action.
- $D^{\log}_{\mathrm{dR}}(\mathbb{L}) := \mu_*\mathrm{RH}^{\log}(\mathbb{L})$ is a filtered coherent sheaf with integrable log connection, satisfying transversality. Beware: it is no longer forced to be a vector bundle by the presence of the connection.

Here $X_{\mathbb{B}_{\mathrm{dR}}}$ means the ringed space $(X_K, O_X\widehat{\otimes}\mathbb{B}_{\mathrm{dR}})$ (i.e., tensor with t-power truncations, take inverse limit, invert t).

Now specialize this to the case of interest: $X = \overline{X} \setminus D$ quasi-projective variety, D normal crossings divisor. Put $Y = (\overline{X}, D)$ the associated log scheme, and $i : X_{\mathrm{et}} \to Y_{\mathrm{ket}}$. By Theorem **B**, $D^{\log}_{\mathrm{dR}}(i^{\mathrm{an}}_*\mathbb{L}^{\mathrm{an}})$ is a coherent sheaf on $X^{\mathrm{an}}_{\mathrm{et}}$. Apply GAGA, get $D_{\mathrm{dR}}(\mathbb{L}^{\mathrm{an}})$ algebraic. Denote by $D_{\mathrm{dR}}(L)$ the corresponding vector bundle on X.

In progress:

- $D^{\log}_{\mathrm{dR}}(i^{\mathrm{an}}_*\mathbb{L}^{\mathrm{an}})$ is a vector bundle, not just a coherent sheaf, and $\mu^*D^{\log}_{\mathrm{dR}}(i^{\mathrm{an}}_*\mathbb{L}^{\mathrm{an}}) = \mathrm{RH}^{\log}(i^{\mathrm{an}}_*\mathbb{L}^{\mathrm{an}})$;
- residues of $\mathrm{RH}^{\log}(i^{\mathrm{an}}_*\mathbb{L}^{\mathrm{an}})$ along divisors belong to k;
- extend $D_{\mathrm{dR}}(\mathbb{L})$ to $\overline{X}$ using this choice of residues.