Recap from Scholze: let k be a commutative ring and A an associative K-algebra. Last time, we introduced Hochschild homology $\mathrm{HH}^k_*(A) = \mathrm{Tor}_*^{A\otimes_k A^{\mathrm{op}}}(A, A)$. We'll mostly be interested in the case where A is commutative, so $A = A^{\mathrm{op}}$.

If A is smooth over k, $\mathrm{HH}^k_*(A) = \Omega^{-*}_{A/k}$.

Now say $k = \mathbb{F}_p$. Question: can the crystalline cohomology of A be recovered from Hochschild homology?

Motivation: think of crystalline cohomology as arising by viewing A as an algebra over $W(k)$ via $W(k)/k$. Changing the previous definition by tensoring $A \otimes A$ over $W(k)$, rather than k, would not literally change, but...

$\mathrm{CH}^k_*(A) = A \otimes^{\mathbb{L}}_{A\otimes_K A} A$. What about $A \otimes^{\mathbb{L}}_{A\otimes^{\mathbb{L}}_K A} A$, where we derive both tensor products?

Example: $A = \mathbb{F}_p$, but $k = \mathbb{Z}_p$. Replace $0 \to \mathbb{F}_p$ with $\mathbb{Z}_p \xrightarrow{\times p} \mathbb{Z}_p$. Now tensor over $\mathbb{Z}_p$ with $\mathbb{F}_p$; the differential becomes 0.

But how to tensor over this? View $\mathbb{Z}_p \to \mathbb{Z}_p$ as $\mathbb{Z}_p[\epsilon]/(\epsilon^2)$ with differential $d\epsilon = p$. Now tensoring, we get $\mathbb{F}_p[\epsilon]/(\epsilon^2)$ with $d\epsilon = 0$.

Now we compute $\mathbb{F}_p \otimes_{\mathbb{F}_p[\epsilon]} \mathbb{F}_p$. Again, we want to projective-resolve $\mathbb{F}_p$ remembering an algebra structure. Resolve $\mathbb{F}_p$ as the DGA $\mathbb{F}_p[\epsilon, \frac{\delta^n}{n!}]$ with $\deg(\epsilon) = 1$, $\deg(\delta) = 2$, $d\frac{\delta^n}{n!} = \frac{\delta^{n-1}}{(n-1)!}\epsilon$.

Finally, get

$$\mathbb{F}_p[\epsilon, \frac{\delta^n}{n!}] \otimes_{\mathbb{F}_p[\epsilon]/(\epsilon^2)} \mathbb{F}_p = \mathbb{F}_p[\frac{\delta^n}{n!}].$$

Objects of the crystalline site are algebras A plus augmentation $\epsilon : A \to \mathbb{F}_p$ (kernel is PD-nilpotent) and divided powers on $\ker(\epsilon)$ (fine print: which are compatible with usual divided powers on $(p) \subseteq \mathbb{Z}_p$). With fine print, taking inverse limit gives $\mathbb{Z}_p$. Without, get $\mathbb{Z}_p[(\text{divided powers on } p)]$ suitably completed. The latter is what is related to Hochschild homology; the former will be related to *topological Hochschild homology*.

Recall: if X is a topological space, we have singular cohomology $H^*(X; A)$ with the following properties.

(1) H^* is a functor from the opposite category of topological spaces to graded abelian groups.
(2) The construction is homotopy invariant.
(3) Disjoint unions go to products of cohomology groups.
(4) $H^*_{\mathrm{red}} \cong H^{*+1}_{\mathrm{red}}(\Sigma X)$ (the suspension).
(5) For suitable inclusions $Y \subseteq X$ (e.g., an inclusion of cell complexes), $H^*_{\mathrm{red}}(X/Y) \to H^*(X) \to H^*(Y)$.
(6) Dimension axiom: $H^*(pt)$ is A in degree 0, 0 elsewhere.

These axioms characterize $H^*(X; A)$, but there are many more examples if you drop the dimension axiom.

Eilenberg–Steenrod: a *cohomology theory* is a functor E^* from topological spaces to graded abelian groups plus isomorphisms $E^*_{\mathrm{red}}(X) \cong E^{*+1}_{\mathrm{red}}(\Sigma X)$ satisfying conditions 1-5 above.

Theorem (Brown): If E^* is a cohomology theory, then each E^n is representable in the homotopy category of (nice) topological spaces by some $E(n)$. That is, $E^n(X) \cong [X, E(n)]_*$ (continuous maps mod homotopy). The suspension axiom: $[X, E(n)]_* \cong [\Sigma X, E(n+1)]_*$, so $E(n) \cong \Omega E(n+1)$ (the loop space: $p : [0, 1] \to E(n+1)$ such that $p(0) = p(1) = *$).

Definition: a *spectrum* is a collection of pointed spaces $E(n)$ for $n \in \mathbb{Z}$, plus a collection of homotopy equivalences $E(n) \cong \Omega E(n+1)$. Thus any spectrum determines a cohomology theory and vice versa.

Now form the homotopy category Ho(Sp) of spectra with homotopy equivalences inverted. The functor $A \to H^*(\bullet; A)$ defines a fully faithful functor $\mathrm{Ab} \to \mathrm{Ho(Sp)}$ where A maps to $HA = \{K(A,n)\}_{n\in\mathbb{Z}}$ (with the convention that $K(A,n)$ is contractible when n is negative). Using general nonsense, extend this functor to the derived category $\mathcal{D}(\mathrm{Ab})$, but not fully faithful anymore! Namely:

$$\mathrm{Ext}^n_{\mathrm{Ab}}(M,N) \cong \mathrm{Hom}_{\mathcal{D}(\mathrm{Ab})}(M, N[n]) \to \mathrm{Hom}_{\mathrm{Ho(Sp)}}(HM, HN[n])$$

is an isomorphism for $n \leq 1$, but generally not for $n \geq 2$ (for $n \geq 2$, the left hand side vanishes but the right hand side does not).

In abelian groups, we have the universal bilinear map $A \times B \to A \otimes B$. What about in cohomology theories? $A \times B \to C$ bilinear would mean $A^i(X) \times B^j(X) \to C^{i+j}(X)$ suitably compatible with suspensions (and some signs). Associated map $A(i) \times B(j) \to C(i+j)$ should be null-homotopic on basepoint times the other factor, which can be guaranteed by requiring that it factors through a map $A(i) \wedge B(j) \to C(i+j)$. Compatibility with suspension should guarantee the commutativity of the diagram

$$\begin{array}{ccccc}
A(k)\wedge B(k) & \longrightarrow & \Omega(A(k)\wedge B(k+1)) & \longrightarrow & \Omega^2(A(k+1)\wedge B(k+1)) \\
\downarrow & & \downarrow & & \downarrow \\
C(2k) & \xrightarrow{\sim} & \Omega C(2k+1) & \longrightarrow & \Omega^2 C(2k+2)
\end{array}$$

Beware that the natural map: $A(k) \wedge B(k) \to \Omega(A(k) \wedge B(k+1))$ is not a homotopy equivalence! The sequence of spaces $\{\begin{cases} A(k)\wedge B(k) & \text{if } n = 2k \\ A(k)\wedge B(k+1) & \text{if } n = 2k+1 \end{cases}\}_{n\in\mathbb{Z}}$ is instead an example of a *prespectrum*: that is, a sequence of spaces $\{E(n)\}_{n\in\mathbb{Z}}$ with maps $E(n) \to \Omega E(n+1)$ which are not required to be homotopy equivalences. Any prespectrum E has an associated spectrum, which can be constructed by replacing $E(n)$ with $\varinjlim \Omega^k E(n+k)$. Applying this to the prespectrum $\{\begin{cases} A(k)\wedge B(k) & \text{if } n = 2k \\ A(k)\wedge B(k+1) & \text{if } n = 2k+1 \end{cases}\}_{n\in\mathbb{Z}}$ yields a spectrum which is denoted by $A \wedge B$, called the *smash product* of A and B. Upshot: Ho(Sp) becomes a symmetric monoidal category under $\wedge$. For many applications, this isn't good enough: need some fancier definitions to get closer to true commutativity and associativity (not just at the homotopy level).

Let M and N be abelian groups (or chain complexes thereof). Now compare $HM \wedge HN \to H(M \otimes N)$; this factors through $H(M \otimes^{\mathbb{L}} N)$, but even that map is not an equivalence. Nonetheless, get

$$HR \wedge HR \to H(R \otimes R) \xrightarrow{\mu} HR.$$

That is, any (commutative) ring may be viewed as a (commutative) ring spectrum.

Definition: Let A be an associative (A_∞) ring spectrum.

$$\mathrm{THH}(A) := A \wedge_{A\wedge A^{\mathrm{op}}} A.$$

To make sense of this, one needs to define smash product relative to a base (which requires working in a more refined setting, as alluded to above). Alternatively, it can be computed as the geometric realization of a simplicial spectrum (given by the cyclic bar construction on A, as in Scholze's lecture, with $\otimes$ replaced by $\wedge$.

Let A be a commutative ring. Then

$$\mathrm{THH}(A) := \mathrm{THH}(HA) = HA \wedge_{HA \wedge HA} HA$$

maps to $H(A \otimes^{\mathbb{L}}_{A \otimes^{\mathbb{L}} A} A)$.

How to get abelian groups out of this? For E a spectrum, write

$$\pi_n(E) := E^{-n}(\mathrm{point}) = \pi_{n+m} E(m)$$

once this last expression makes sense. For example, if M is a chain complex, then $\pi_* HM \cong H_*(M)$.

Now get a map

$$\pi_* \mathrm{THH}(A) \to \mathrm{HH}_* \mathbb{Z}(A).$$

In the previous example, when $A = \mathbb{F}_p$,

$$\pi_* : \mathrm{THH}(\mathbb{F}_p) \to \mathrm{HH}^{\mathbb{Z}}_*(\mathbb{F}_p)$$

and the target was a divided power algebra over $\mathbb{F}_p$ on one generator δ. Theorem of B okstedt: the source is the ordinary polynomial algebra $\mathbb{F}_p[\delta]$.