THH vs. crystalline cohomology and AΩ.

Joint work in progress with Bhatt, Morrow.

Assume everything in sight is p-complete.

A cyclotomic spectrum is X a spectrum, with an S^1-action and a "Frobenius" map $\varphi = \varphi_p : X \to X^{tC_p} = \mathrm{cone}(X_{hC_p} \overset{\mathrm{Norm}}{\to} X^{hC_p})$, which is $S^1 \cong S^1/C_p$-equivariant.

$$\mathrm{TC}(X) := \mathrm{eq}(X^{hS^1} \overset{\to}{\to} (X^{tC_p})^{h(S^1/C_p)}).$$

One map is $\varphi_p^{hS^1}$. The other is the canonical map to $\mathrm{cone}(\mathrm{Norm} : (X_{hC_p})^{hS^1/C_p} \to (X^{hC_p})^{hS^1/C_p}) = X^{hS^1})$.

Lemma: (Tate orbit lemma) If X is bounded below, $(X^{tC_p})^{hS^1/C_p} \overset{\sim}{\leftarrow} X^{tS^1} = \mathrm{cone}(\mathrm{Norm} : \Sigma X_{hS^1} \to X^{hS^1})$.

If R is an E_∞-ring spectrum, $\mathrm{TC}(R) := \mathrm{TC}(\mathrm{THH}(R))$. Goal: make these things (i.e., $\mathrm{THH}(R)^{hS^1}, \mathrm{THH}(R)^{tS^1}, \varphi$, etc.) explicit for smooth k-algebras k perfect, and for smooth $\mathcal{O}_C$-algebras ($C/\mathbb{Q}_p$ algebraically closed, complete).

The base: $\mathrm{THH}(k)$. By Bökstedt, $\pi_*\mathrm{THH}(k) = k[u]$ with $|u| = 2$. This implies $\pi_*\mathrm{THH}(k)^{hS^1} = W(k)[u,v]/(uv - p)$ with $|u| = |v| = 2$.

(draw the spectral sequence here)

Similarly, $\pi_*\mathrm{THH}(k)^{tS^1} = W(k)[v^{\pm 1}]$, and the canonical map sends $v \mapsto v$ (and thus $u \mapsto pv^{-1}$). On the other hand, $\varphi : \pi_*\mathrm{THH}(k)^{hS^1} \to \pi_*\mathrm{THH}(k)^{tS^1}$ sends $u \mapsto v^{-1}, v \mapsto pv$.

Remark: these rings appear in Fontaine–Jannsen theory of gauges.

Now let R be a smooth k-algebra. Want to find the following structures in THH: action of φ on $\mathbb{R}\Gamma_{\mathrm{crys}}(R/W(k)) \cong W\Omega_{R/k}$; "inside" this, the Nygaard filtration $\mathrm{Fil}^i R\Gamma_{\mathrm{crys}}(R/W(k))$ and the map φ/p^i from it back to $R\Gamma_{\mathrm{crys}}(R/W(k))$; the ith graded of this filtration is $\tau^{\leq i}\Omega_{R/k}$.

If $\tilde{R}/W(k)$ is a smooth lift, $\tilde{\varphi}$ a lift of φ, can write

$$R\Gamma_{\mathrm{crys}} : 0 \to \tilde{R} \overset{\nabla}{\to} \Omega^1_{\tilde{R}/W(k)} \to \cdots \to \Omega^d_{\tilde{R}/W(k)} \to 0.$$

Inside,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & p^i\tilde{R} & \longrightarrow & p^{i-1}\Omega^1_{\tilde{R}/W(k)} & \longrightarrow \cdots \longrightarrow & p^{\max(i-d,0)}\Omega^d_{\tilde{R}/W(k)} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \tilde{\varphi}/p^i} & & \downarrow{\scriptstyle \tilde{\varphi}/p^i} & & \downarrow{\scriptstyle \tilde{\varphi}/p^i} & & \\
0 & \longrightarrow & \tilde{R} & \longrightarrow & \Omega^1_{\tilde{R}/W(k)} & \longrightarrow \cdots \longrightarrow & \Omega^d_{\tilde{R}/W(k)} & \longrightarrow & 0
\end{array}$$

Again, we want to find these structures in THH.

Step 1: Define descending, separated multiplicative filtration on THH, $\mathrm{Fil}^i\mathrm{THH}(R) \to \mathrm{THH}(R)$ with associated graded $(\tau^{\leq i}\Omega^\bullet_{R/k})[2i]$.

Theorem (Hesselholt): R smooth/k. $\pi_*\mathrm{THH}(R)$ looks like this:

$$\begin{array}{lccccccc}
0 & R & & & & & & \\
1 & & \Omega^1_{R/k} & & & & & \\
2 & R \cdot u & \oplus & \Omega^2_{R/k} & & & & \\
3 & & \Omega^1_{R/k} \cdot u & \oplus & \Omega^3_{R/k} & & & \\
4 & R \cdot u^2 & \oplus & \Omega^2_{R/k} \cdot u & \oplus & \Omega^4_{R/k} & &
\end{array}$$

Construction: use Theorem (Bhatt): THH (for ordinary rings) satisfies fpqc descent.

Try do define filtration fpqc-locally. More carefully, (pro-)syntomic locally. Want to descend along $R \to R_{\mathrm{perf}}$; get

$$\mathrm{THH}(R) \cong \varprojlim(\mathrm{THH}(R_{\mathrm{perf}}) \overset{\to}{\to} \mathrm{THH}(R_{\mathrm{perf}} \otimes_R R_{\mathrm{perf}}) \cdots).$$

The ring $R_{\mathrm{perf}} \otimes_R R_{\mathrm{perf}}$ is "regular semiperfect". Zariski-locally it is a quotient of a perfect ring by a regular sequence. (Model case: $R = \mathbb{F}_p[T]$, get $\mathbb{F}_p[T_1^{1/p^\infty}, T_2^{1/p^\infty}]/(T_1 - T_2)$.)

Lemma: the cotangent complex $\mathbb{L}_{S/\mathbb{F}_p}$ is finite projective module in (homological) degree 1. Thus, $\wedge^i \mathbb{L}_{S/\mathbb{F}_p}$ is a finite projective module in degree i.

Now for S regular semiperfect, the homotopy groups $\pi_* \mathrm{THH}(S)$ have the following contributions:

0	S		
1			
2	$S \cdot u$	$\mathbb{L}_{S/\mathbb{F}_p}[-1]$	
3			
4	$S \cdot u^2$	$\mathbb{L}_{S/\mathbb{F}_p}[-1] \cdot u$	$(\wedge^2 \mathbb{L}_{S/\mathbb{F}_p})[-2]$

Thus, the desired graded pieces all sit in one homotopical degree, and we can define the desired filtration $\mathrm{Fil}^i \mathrm{THH}(S) = \tau_{\geq 2i} \mathrm{THH}(S)$. Extend to R

Theorem (BMS): for R smooth over k, $\mathrm{gr}^i \mathrm{THH}(R) \cong (\tau^{\leq i} \Omega^\bullet_{R/k})[2i]$, as desired.

Theorem (BMS): The same (pro-)syntomic descent works to define filtrations $\mathrm{THH}(R)^{hS^1} \overset{\to}{\to} \mathrm{THH}R^{tS^1}$ compatible with the canonical map and φ. Moreover,

$$\mathrm{gr}^0 \mathrm{THH}(R)^{hS^1} = \mathrm{gr}^0 \mathrm{THH}(R)^{tS^1} = R\Gamma_{\mathrm{crys}}(R/W(k))$$

and for $i \geq 0$

$$\begin{array}{ccc} \mathrm{gr}^i \mathrm{THH}(R)^{hS^1} & = & \mathrm{Fil}^i R\Gamma_{\mathrm{crys}}(R/W(k))[2i] \\ \downarrow{\scriptstyle \varphi_{\mathrm{THH}}} & & \downarrow{\scriptstyle \varphi/p^i} \\ \mathrm{gr}^i \mathrm{THH}(R)^{tS^1} & = & R\Gamma_{\mathrm{crys}}(R/W(k))[2i] \end{array}$$

(draw another spectral sequence here)

$A\Omega$ version:

R smooth over $\mathcal{O}_C$. Choose local coordinates via étale map $\mathcal{O}_C\langle T_1^{\pm 1}, \ldots, T_d^{\pm 1}\rangle \to R$. Get a unique étale def. $\mathbf{A}_{\mathrm{inf}}\langle T_1^{\pm 1}, \ldots, T_d^{\pm 1}\rangle \to \tilde{R}$ with lift $\tilde{\varphi}$ of φ.

$q = [\epsilon] \in \mathbf{A}_{\mathrm{inf}}$.

$$A\Omega_R : 0 \to \tilde{R} \overset{\Delta_q}{\to} \Omega^1_{R/\mathbf{A}_{\mathrm{inf}}} \overset{\Delta_q}{\to} \cdots \overset{\Delta_q}{\to} \Omega^d_{R/\mathbf{A}_{\mathrm{inf}}} \to 0.$$

Here $\Delta_q(T_i^n) = [n]_q T_i^{n-1} dT_i$, where $[n]_q = (q^n - 1)/(q - 1)$.

Can define similar Nygaard filtration (according to divisibility by $[p]_q$), and map $\tilde{\varphi}/[p]_q^i$ from $\mathrm{Fil}^i A\Omega_R$ to $A\Omega_R$.

The filtrations on THH, THH^{hS^1} and THH^{tS^1} can be defined as before, by pro-syntomic descent to "regular semiperfectoid" rings, and one recovers these objects in the same way.

Application: $A\Omega = \mathrm{gr}^0 \mathrm{THH}$ exists in greater generality (without perfectoid base ring), can be used to get cohomological construction of Breuil–Kisin modules (descending from $\mathbf{A}_{\mathrm{inf}}$ to Breuil–Kisin's $\mathfrak{S}$).