In the series of 5 talks, we considered $A\Omega_{\mathfrak{X}}$, $A\Omega_R$. What about with coefficients?

k perfect field of char $p > 0$. $W = W(k)$, $K = \mathrm{Frac}W$, $C = \widehat{\overline{K}}$, $\mathcal{O}_{\overline{K}} \subset \overline{K}$, $\mathcal{O}_C \subset C$. $\mathrm{Spf}(R) \to \mathrm{Spf}(W)$ smooth and connected. Assume there exist coordinates $t_1, \ldots, t_d \in R^\times$.

$\mathcal{K} = \mathrm{Frac}R$, $\overline{\mathcal{K}}$, $\overline{R}$ = integral closure of R in the maximal unramified extension of $R[\frac{1}{p}]$ in $\overline{\mathcal{K}}$. $G_R = \mathrm{Aut}(\overline{R}/R)$. $\Delta_R = \mathrm{Aut}(\overline{R}/R\mathcal{O}_{\overline{K}}) \subset G_R$.

We consider objects $(M, \mathrm{Fil}^r M, \nabla, \Phi)$:

- M a finite free R-module;
- $\mathrm{Fil}^r M \subset M$ a decreasing filtration such that $\mathrm{Fil}^0 M = M$, $\mathrm{Fil}^{p-1} M = 0$. There exist a basis $e_1, \ldots, e_N \in M$ and $r_1, \ldots, r_N \in \mathbb{N} \cap [0, p-2]$ such that $\mathrm{Fil}^r M = \bigoplus_{r_i \geq r} Re_i$.
- $\nabla : M \to M \otimes_R \Omega^1_R$ topologically quasi-nilpotent integrable connection such that $\nabla(\mathrm{Fil}^r M) \subset \mathrm{Fil}^{r-1} M \otimes \Omega_R$.
- $\varphi : R \to R$ lifting of Frobenius. $\Phi : (\varphi^* M, \varphi^* \nabla) \to (M, \nabla)$ horizontal R-linear map, where $\varphi^*(\bullet) = (\bullet) \otimes_{R,\varphi} R$, such that $\Phi(\varphi^*(\mathrm{Fil}^r M)) \subset p^r M$ and $\sum_{0 \leq r \leq p-2} p^{-r} \Phi(\varphi^*(\mathrm{Fil}^r M)) = M$. (This condition does not depend on the lifting!)

Category $\mathrm{MF}^\nabla_{[0,p-2],\mathrm{free}}(R, \Phi)$. There exists $T_{\mathrm{crys}} : \mathrm{MF}^\nabla_{[0,p-2],\mathrm{free}}(R, \Phi) \to \mathrm{Rep}_{\mathbb{Z}_p,\mathrm{free}}(G_R)$ fully faithful. Put $A_{\mathrm{inf}}(\overline{R}) = W(\varprojlim_{\mathrm{Frob}} \overline{R}/p\overline{R})$ inside $A_{\mathrm{crys}}(\overline{R})$, on which G_R and φ act. Put $\epsilon = (\zeta_{p^n} \mod p)$ and $\mu = [\epsilon] - 1 \in A_{\mathrm{inf}}(\mathcal{O}_{\overline{K}}) = A_{\mathrm{inf}}$.

$\xi = \mu/\varphi^{-1}(\mu)$, $\tilde{\xi} = \varphi(\mu)/\mu = [p]_q$.

Category $M_{[0,p-2],\mathrm{free}}(A_{\mathrm{inf}}(\overline{R}), G_R, \varphi)$. Object is a finite free $A_{\mathrm{inf}}(\overline{R})$-module $\mathbb{M}$ plus semilinear continuous action of G_R plus G_R-equivariant φ-semilinear map $\varphi : \mathbb{M} \to \mathbb{M}$, such that there exist $e_1, \ldots, e_n \in \mathbb{M}$ a basis, $r_1, \ldots, r_n \in \mathbb{N} \cap [0, p-2]$ such that $\tilde{\xi}^{-r_i} \varphi(e_i)$ form a basis of $\mathbb{M}$.

Theorem 1:

(1) There exists a canonical fully faithful functor

$$TA_{\mathrm{inf}} : \mathrm{MF}^\nabla_{[0,p-2],\mathrm{free}}(R, \Phi) \to M_{[0,p-2],\mathrm{free}}(A_{\mathrm{inf}}(\overline{R}), G_R, \Phi).$$

taking M to $\mathbb{M}$. Write $T = T_{\mathrm{crys}}(M) \in \mathrm{Rep}_{\mathbb{Z}_p}(G_R)$.

(2) $T^* \cong \mathrm{Hom}_{A_{\mathrm{inf}}(\overline{R}),\varphi}(\mathbb{M}, A_{\mathrm{inf}}(\overline{R}))$ (G_R-equivariant).

(3) The previous line induces $\mathbb{M} \hookrightarrow T \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}(\overline{R}) \subset \mu^{-(p-2)}\mathbb{M}$.

(4) $\mathbb{M}/\mu\mathbb{M} = (A_{\mathrm{inf}}(\overline{R})/\mu)^N$ as G_R-representation.

(5) Given another $M', \mathbb{M}', T'$, and an $A_{\mathrm{inf}}(\overline{R})[\frac{1}{\mu}]$-linear map

$$f : T \otimes A_{\mathrm{inf}}(\overline{R})[\frac{1}{\mu}] \to T' \otimes A_{\mathrm{inf}}(\overline{R})[\frac{1}{\mu}]$$

G_R-equivariant, compatible with φ, we have $f(\mathbb{M}) \subset \mathbb{M}'$.

Remarks:

- (5) and (1) imply that T_{crys} is fully faithful.
- (5) follows from (4) and $(\mu^r A_{\mathrm{inf}}(\overline{R})/\mu^{r+1} A_{\mathrm{inf}}(\overline{R}))^{G_R} = 0$ for $r \neq 0$.

Construction of TA_{inf}:

$$\mathrm{MF}^\nabla_{[0,p-2],\mathrm{free}}(R, \Phi) \to \mathrm{MF}_{[0,p-2],\mathrm{free}}(A_{\mathrm{crys}}(\overline{R}), \Phi, G_R)$$

is fully faithful. This is roughly "evaluation on $\mathrm{Spec}A_{\mathrm{crys}}(\overline{R})/p^n$."

The category at right is equivalent to

$$M_{[0,p-2],\mathrm{free}}(A_{\mathrm{inf}}(\overline{R}), \Phi, G_R)$$

via Fontaine-Wach.

$A\Omega_R(M) := L\eta_\mu R\Gamma(\Delta_R, TA_{\mathrm{inf}}(M))$. $\mathrm{DR}(M) = M \otimes_R \Omega_R^*$.

Theorem 2:

- $p \geq 5$. There exists a canonical isomorphism

$$A\Omega_R(M) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/\mu \cong A_{\mathrm{inf}}/\mu \widehat{\otimes}_W \mathrm{DR}(M)$$

  compatible with G_K-action, φ. This yields

$$A\Omega_R(M) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} \mathcal{O}_C \cong \mathcal{O}_C \widehat{\otimes}_W \mathrm{DR}(M)$$

$$A\Omega_R(M) \widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}} W \cong \mathrm{DR}(M).$$

- There exists a canonical morphism

$$A_{\mathrm{crys}} \widehat{\otimes}_W \mathrm{DR}(M) \to A_{\mathrm{crys}} \widehat{\otimes}_{A_{\mathrm{inf}}} A\Omega(M).$$

  (But we don't know that this is an isomorphism. There is an isomorphism constructed another way, but can't check that they match except in special cases.)

Computation: a noncanonical injection $A^{\square}_{\mathrm{inf}}(R) = R\widehat{\otimes}_W A_{\mathrm{inf}} \hookrightarrow A_{\mathrm{inf}}(R_\infty)$, $t_i \mapsto [t_i^\flat]$. (Here we probably mean to use the Frobenius lift $t_i \mapsto t_i^p$.) Here $R_\infty = R[t_i^{p^{-\infty}}]\mathcal{O}_{\overline{K}}$. $\gamma_1, \ldots, \gamma_d \in \Gamma = \mathrm{Aut}(R_\infty/R\mathcal{O}_{\overline{K}}) \cong \mathbb{Z}_p(1)^d$.

There exists a descent $TA^{\square}_{\mathrm{inf}}(M)$ of $TA_{\mathrm{inf}}(M)$ carrying action of Γ, φ. $A\Omega_R(M) = K^\bullet(TA^{\square}_{\mathrm{inf}}(M), \frac{\gamma_1 - 1}{\mu}, \ldots$ (Koszul complex). $A_{\mathrm{crys}} \otimes_{A_{\mathrm{inf}}} TA^{\square}_{\mathrm{inf}}(M) \cong M \widehat{\otimes}_W A_{\mathrm{crys}}$.

$\gamma_i(m \otimes 1) = \sum_{n \in \mathbb{N}} t_i^n \nabla(\frac{\partial}{\partial t_i})^n(m) \otimes \mu^{[n]}$.