A FLAT MAP THAT IS NOT A DIRECTED LIMIT OF FINITELY PRESENTED FLAT MAPS

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The goal of this note is to show:

Proposition 0.1. *There exists a commutative ring A and a flat A-algebra B which cannot be written as a filtered colimit of finitely presented flat A-algebras. In fact, we may choose A to be a finite type $\mathbf{Z}$-algebra.*

For the construction, fix a prime p, and let $A = \mathbf{F}_p[x_1, \dots, x_n]$. Choose an absolute integral closure A^+ of A, i.e., A^+ is the normalization of A in an algebraic closure of its fraction field. Recall the following theorem [HH92, §6.7]:

Theorem 0.2 (Hochster-Huneke)**.** *The map $A \to A^+$ is flat.*

To prove Proposition 0.1, it is enough to show:

Proposition 0.3. *The A-algebra A^+ is not a filtered colimit of finitely presented flat A-algebras if $n \geq 3$.*

Proof. We give an argument in the case $n = 3$, leaving the (obvious) generalization to the reader. It is enough to prove the analogous statement for the map $R \to R^+$, where R is the strict henselization of A at the origin (and is consequently a henselian regular local ring with residue field $\overline{\mathbf{F}_p}$), and R^+ is its absolute integral closure.

Now choose an ordinary abelian surface X over $\overline{\mathbf{F}_p}$ and a very ample line bundle L on X. The section ring $\Gamma_*(X, L) := \oplus_n H^0(X, L^n)$ is the co-ordinate ring of the affine cone over X with respect to L, and is normal for L sufficiently positive. Let S denote the henselization of $\Gamma_*(X, L)$ at vertex of the cone. Then S is a henselian noetherian normal domain of dimension 3. Thus, we can find some finite injective map $R \to S$ realizing a Noether normalization of S. As R^+ is an absolute integral closure of R, we can also fix an embedding $S \to R^+$ realizing R^+ as the absolute integral closure of S. To show R^+ is not a filtered colimit of flat R-algebras, it suffices to show:

(a) If there exists a factorisation $S \to P \to R^+$ with P flat over R, then there exists a factorisation $S \to T \to R^+$ with T finite flat over R.
(b) For any factorisation $S \to T \to R^+$ with $S \to T$ finite, the ring T is not R-flat.

Indeed, since S is finitely presented over R, if one could write $R^+ = \mathrm{colim}_i P_i$ as a filtered colimit of fintiely presented flat R-algebras P_i, then $S \to R^+$ would factor as $S \to P_i \to R^+$ for $i \gg 0$, which contradicts the above pair of assertions. To prove these, observe that (a) follows immediately by a standard slicing argument (see [Sta14, Tag 0571]). Part (b) was proven in [Bha12]; for the convenience of the reader, we recall the relevant argument.

Let $U \subset \mathrm{Spec}(S)$ be the punctured spectrum, so there are natural maps $X \leftarrow U \subset \mathrm{Spec}(S)$. The first map gives an identification $H^1(U, \mathcal{O}_U) \simeq H^1(X, \mathcal{O}_X)$; by passing to the Witt vectors of the perfection and using the Artin-Schreier sequences, this gives an identification $H^1_{\text{ét}}(U, \mathbf{Z}_p) \simeq H^1_{\text{ét}}(X, \mathbf{Z}_p)$. In particular, this group is a finite free $\mathbf{Z}_p$-module of rank 2 (since X is ordinary). Now assume that there exists some T as in (b) above. Let $V \subset \mathrm{Spec}(T)$ denote the preimage of U, and write $f : V \to U$ for the induced finite surjective map. Since U is normal, there is a trace map $f_*\mathbf{Z}_p \to \mathbf{Z}_p$ on $U_{\text{ét}}$ whose composition with the pullback $\mathbf{Z}_p \to f_*\mathbf{Z}_p$ is multiplication by $d = \deg(f)$. Passing to cohomology, and using that $H^1_{\text{ét}}(U, \mathbf{Z}_p)$ is non-torsion, then shows that $H^1_{\text{ét}}(V, \mathbf{Z}_p)$ is non-zero. Since $H^1_{\text{ét}}(V, \mathbf{Z}_p) \simeq \lim H^1_{\text{ét}}(V, \mathbf{Z}/p^n)$ (as there is no $\lim^1$ interference), the group $H^1(V_{\text{ét}}, \mathbf{Z}/p)$ must be non-zero. The Artin-Schreier sequence then shows $H^1(V, \mathcal{O}_V) \neq 0$. By excision, this gives $H^2_{\mathfrak{m}}(T) \neq 0$, where $\mathfrak{m} \subset R$ is the maximal ideal. Thus, T cannot be finite flat as an R-module since $H^2_{\mathfrak{m}}(R) = 0$, proving (b). ☐

References

[Bha12] Bhargav Bhatt. On the non-existence of small Cohen-Macaulay algebras. 2012. Available at http://arxiv.org/abs/1207.5413.

[HH92] Melvin Hochster and Craig Huneke. Infinite integral extensions and big Cohen-Macaulay algebras. *Ann. of Math. (2)*, 135(1):53–89, 1992.

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