NON-LIFTABILITY OF VECTOR BUNDLES TO THE WITT VECTORS

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We give an example of a vector bundle E on a smooth projective variety X over an algebraically closed field of characteristic p such that E does not extend across the nilpotent thickening $X \hookrightarrow W_2(X)$ defined by the Witt vector construction; this goes against an earlier claim in the literature.

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Notation 0.1. Fix a prime number p. Let k be an algebraically closed field of characteristic p. For an $\mathbf{F}_p$-scheme X, write $W_m(X)$ for the scheme $(X, W_m(\mathcal{O}_X))$, so we have nilpotent closed immersions $X \simeq W_1(X) \subset W_2(X) \subset W_3(X) \subset \cdots$. We write $X_{perf} \to X$ for the perfection of X; we shall identify X_{perf} with the scheme $(X, \mathcal{O}_{X_{perf}})$, where $\mathcal{O}_{X_{perf}} := \varinjlim_\phi \mathcal{O}_X$. For any sheaf of rings Λ on X_{et}, write $\mathrm{Loc}_\Lambda(X_{et})$ for the category of local systems of Λ-modules, i.e., sheaves of Λ-modules that are locally constant with finite projective values on X_{et}; we shall mainly use this when $\Lambda = W_n(k)$ for some $n \geq 1$. Write $\mathrm{Vect}(X)$ for the category of vector bundles on X (or equivalently on X_{et}).

The goal of this note is to record the following.

Theorem 0.2. *There exists a smooth projective surface X/k and a rank 2 vector bundle E on X that does not extend across the nilpotent closed immersion $X \hookrightarrow W_2(X)$. Moreover, we can arrange that X lifts to $W(k)$ and that E comes from an $\mathbf{F}_p$-local system (and thus admits a flat connection with p-curvature 0).*

Following [CFA], this also yields a counterexample on Grassmannians. More precisely, since line bundles always lift (by applying the Teichmuller map), and because a sufficiently positive twist of any vector bundle on a projective scheme is globally generated (and thus pulled back from a Grassmannian), we deduce the following result (that we did not attempt to make effective):

Corollary 0.3. *For some $n \gg 0$, the tautological bundle on $\mathrm{Gr}(2,n)$ does not extend to $W_2(\mathrm{Gr}(2,n))$.*

To prove Theorem 0.2, we first construct a surface X equipped with a k-local system L that does not lift to a $W_2(k)$-local system using non-liftable representations of finite groups; the local system L gives a vector bundle $E := L \otimes_k \mathcal{O}_X$ on X, and we show that this vector bundle does not extend to $W_2(X)$. To implement this strategy, we need to compare the deformation theories of L and E. The relevant comparison will be a consequence of the following independently interesting observation that came as a surprise to the author.

Proposition 0.4. *Let X/k be a proper k-variety. For an integer $m \geq 1$, consider the functor*

$$\mathrm{RH} : \mathrm{Loc}_{W_m(k)}(X_{et}) \to \mathrm{Vect}(W_m(X_{perf})) \quad \text{defined by} \quad L \mapsto L \otimes_{W_m(k)} W_m(\mathcal{O}_{X_{perf}}).$$

This functor commutes with tensor products and duals. Moreover, for any $L \in \mathrm{Loc}_{W_m(k)}(X_{et})$, this functor induces an isomorphism

$$H^*(X_{et}, L) \simeq H^*(W_m(X_{perf}), \mathrm{RH}(L)).$$

In particular, $\mathrm{RH}(-)$ is fully faithful (even at the derived level).

Proof. The compatibility with tensor products and duality is clear. For the comparison of cohomology, by devissage, we may assume $m = 1$. Fix a k-local system L on X_{et}. Assume first that $L = \underline{k}_X$ is the trivial local system, so $\mathrm{RH}(L) \simeq \mathcal{O}_{X_{perf}}$. Then $H^i(X_{et}, L) \simeq H^i(X_{et}, \mathbf{F}_p) \otimes_{\mathbf{F}_p} k$, while $H^i(X_{perf}, \mathcal{O}_{X_{perf}}) \simeq H^i(X, \mathcal{O}_X)_{perf}$; the claim now follows from Lemma 0.5 applied to $V := H^i(X, \mathcal{O}_X)$ with ϕ being induced by the Frobenius on X (and using the Artin-Schreier sequence to identify $V^{\phi=1} \simeq H^i(X_{et}, \mathbf{F}_p)$). In general, there is a finite étale Galois cover $\pi : Y \to X$ with Galois group G such that $\pi^* L \simeq \underline{k}_Y$ is the trivial local system on Y. By Galois descent, we have $R\Gamma(X_{et}, L) \simeq R\Gamma(G, R\Gamma(Y, \pi^* L))$ and $R\Gamma(X_{perf}, \mathrm{RH}(L)) \simeq R\Gamma(G, R\Gamma(Y_{perf}, \pi_{perf}^* \mathrm{RH}(L)))$, so the claim follows from the previous special case. ☐

The following standard lemma was used above; see [CL, §III, Lemma 3.3] for a detailed exposition.

Lemma 0.5. *Let V be a finite dimensional k-vector space V equipped with a p-linear endomorphism ϕ. Then $V_{perf} := \varinjlim_{\phi} V$ identifies with $V^{\phi=1} \otimes_{\mathbf{F}_p} k$. Moreover, the map $\phi - 1 : V \to V$ is surjective.*

Proof. It is well-known that we can decompose V in a ϕ-equivariant fashion as $V_{nilp} \oplus (V^{\phi=1} \otimes_{\mathbf{F}_p} k)$, where $V_{nilp} \subset V$ is the set of vectors annihilated by a sufficiently high power of ϕ. Passage to the perfection kills the first summand while leaving the second summand unchanged, so the first part follows. The surjectivity of $\phi - 1$ is also clear: this map is surjective on V_{nilp} (as ϕ acts locally nilpotently) and $V^{\phi=1} \otimes_{\mathbf{F}_p} k$ (since k is algebraically closed). $\square$

The relevance of the previous lemma to the liftability discussion is the following:

Corollary 0.6. *Let X/k be a proper k-variety. Let L be a k-local system on X that does not lift to a $W_m(k)$-local system for some $m \geq 2$. Then the vector bundle $E := L \otimes_k \mathcal{O}_X$ on X does not lift to $W_m(X)$.*

Proof. Consider the pullback E_{perf} of E to X_{perf}; under the identification $X_{perf} = (X, \mathcal{O}_{X_{perf}})$, we have $E_{perf} := E \otimes_{\mathcal{O}_X} \mathcal{O}_{X_{perf}} \simeq L \otimes_k \mathcal{O}_{X_{perf}} \simeq \mathrm{RH}(L)$, with notation as in Proposition 0.4. Now recall that deformations of L are governed by $H^*(X_{et}, \mathrm{ad}(L))$, while those of E_{perf} are governed by $H^*(X_{perf}, \mathrm{ad}(E_{perf}))$. More precisely, the obstruction to lifting a given $W_n(k)$-local system lifting L to a $W_{n+1}(k)$-local system is given by a class in $H^2(X_{et}, \mathrm{ad}(L))$, while the set of all lifts is a torsor for $H^1(X_{et}, \mathrm{ad}(L))$; similar descriptions also apply to deformations of E_{perf}, and deformation/obstruction classes are compatible with each other under the natural map $H^*(X_{et}, \mathrm{ad}(L)) \to H^*(X_{perf}, \mathrm{ad}(E_{perf}))$. As this map is an isomorphism by Proposition 0.4, it follows from our assumption on L that E_{perf} does not lift to $W_m(X_{perf})$. But then E cannot lift to $W_m(X)$ either: if it did, then the pullback of any lift along the map $W_m(X_{perf}) \to W_m(X)$ (defined by functoriality of $W_m(-)$) would give a lift of E_{perf} to $W_m(X_{perf})$, which is impossible. $\square$

We record an example of non-liftability to $W_2(k)$ coming from group theory.

Example 0.7 (Serre [Se2, §IV.3.4, Lemma 3])**.** Assume $p \geq 5$. Write $\epsilon = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in M_2(\mathbf{Z})$ for simplicity. Let $\rho : \mathbf{Z}/p \to \mathrm{GL}_2(k)$ be the representation defined by $1 \mapsto 1 + \epsilon \in \mathrm{GL}_2(k)$. We shall check that ρ does not lift to $W_2(k)$, i.e., there is no lift $\mathbf{Z}/p \to \mathrm{GL}_2(W_2(k))$ of ρ. It is enough to show that for any $g \in \mathrm{GL}_2(W_2(k))$ that lifts $1 + \epsilon \in \mathrm{GL}_2(k)$, we have $g^p \neq 1 \in \mathrm{GL}_2(W_2(k))$. In fact, we shall check that $g^p \equiv 1 + p\epsilon \in \mathrm{GL}_2(W_2(k))$ for such g. Write $g = 1 + \epsilon + ph$ for some $h \in M_2(W_2(k))$. Using $\epsilon^2 = 0$, it is easy to see that $p \cdot (\epsilon + ph)^i = 0$ if $i \geq 2$. As the identity matrix is central, this gives

$$g^p = (1 + (\epsilon + ph))^p = 1 + p\epsilon + (\epsilon + ph)^p \in M_2(W_2(k)).$$

It is thus enough to show that $(\epsilon + ph)^p = 0 \in M_2(W_2(k))$. Note that since $\epsilon^2 = p^2h^2 = 0$, we also have $(\epsilon + ph)^2 = p(\epsilon h + h\epsilon) \in M_2(W_2(k))$. As this quantity is killed by p, multiplying repeatedly with $\epsilon + ph$ then gives the desired equality

$$(\epsilon + ph)^p = (\epsilon + ph)^{p-2} p(\epsilon h + h\epsilon) = \epsilon^{p-2} p(\epsilon h + h\epsilon) = 0,$$

where the last equality follows as $\epsilon^{p-2} = 0$ since $p - 2 \geq 2$ by our assumption $p \geq 5$.

Combining the previous example with the Godeaux-Serre construction [Se, §20] gives our theorem.

Proof of Theorem 0.2. Let G be a finite group equipped with a representation $\rho : G \to \mathrm{GL}_2(k)$ that does not lift to $\mathrm{GL}_2(W_2(k))$; an example (that even has coefficients in $\mathbf{F}_p$) is recorded in Example 0.7. By the Godeaux-Serre construction, we may choose a smooth projective surface X/k with $\pi_1(X) \simeq G$. The representation ρ gives a k-local system L of rank 2 on X_{et} that does not lift to a $W_2(k)$-local system. Corollary 0.6 then ensures that the rank 2 vector bundle $E := L \otimes_k \mathcal{O}_X$ on X does not extend to $W_2(X)$. $\square$

References

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