LECTURE IX: THE ÉTALE COMPARISON

Fix a perfect prism (A, I) and a p-complete A/I-algebra R. The étale comparison theorem for prismatic cohomology is the following assertion.

Theorem 0.1 (The étale comparison)**.** *For each $n \geq 1$, there is a canonical identification*

$$R\Gamma_{et}(\mathrm{Spec}(R[1/p]), \mathbf{Z}/p^n) \simeq (\Delta_{R/A}[1/d]/p^n)^{\phi=1}.$$

where $d \in I$ is a generator (and Notation 1.1 explains the meaning of the right side).

In this lecture, we sketch[1] how to prove this statement R topologically finitely generated over A/I with bounded p^∞-torsion; the strategy is to reduce to the case where R is perfectoid. Using this result, we deduce the dimension inequality from [2] mentioned in Theorem I.2.2.

1. A continuity property

In this section, we explain how to pass from prismatic cohomology to its perfection without affecting Theorem 0.1. Since the argument is a relatively formal argument in p-linear algebra, let us formulate it in suitable generality.

Notation 1.1 (Frobenius fixed points)**.** Let B be an $\mathbf{F}_p$-algebra equipped with an element $t \in B$. Let $\mathcal{D}_{comp}(B)$ denote the t-complete derived ∞-category of B.

Let ϕ_B (or just ϕ if there is no confusion) denote the Frobenius endomorphism of B. Let $\mathcal{D}(B[F])$ be the derived ∞-category of Frobenius B-modules, i.e., pairs (M, ϕ) where $M \in \mathcal{D}(B)$ and $\phi : M \to \phi_* M$ is a map. Write $\mathcal{D}_{comp}(B[F]) \subset \mathcal{D}(B[F])$ for the full subcategory spanned by pairs (M, ϕ) with $M \in \mathcal{D}_{comp}(B)$. This ∞-category has all colimits, and the forgetful functor $\mathcal{D}_{comp}(B[F]) \to \mathcal{D}_{comp}(B)$ commutes with all colimits. (Thus, concretely, the underlying complex of the colimit is the t-completion of the colimit of the underlying complexes.)

Given $(M, \phi) \in \mathcal{D}_{comp}(B[F])$, if we view ϕ as a ϕ-semilinear map $M \to M$, then it makes sense to set $M^{\phi=1} := \mathrm{fib}(M \xrightarrow{\phi-1} M) \in \mathcal{D}(\mathbf{F}_p)$; equivalently, we can also say

$$M^{\phi=1} := \mathrm{RHom}_{\mathcal{D}(B[F])}((B, \phi), (M, \phi)).$$

We shall informally refer to this object as *Frobenius fixed points of M*.

We want to show that passage from prismatic cohomology to its perfection does not affect the ϕ-dixed points. As the perfection of prismatic cohomology is under control only in some completed sense, the crucial result we need is that the functor $(-)^{\phi=1}$ is insensitive to certain completions:

Proposition 1.2. *Let $\mathcal{B}$, t be as in Notation 1.1.*

1. *The functors $\mathcal{D}_{comp}(B[F]) \to \mathcal{D}(\mathbf{F}_p)$ given by $M \mapsto M^{\phi=1}$ and $M \mapsto (M[1/t])^{\phi=1}$ commute with colimits.*
2. *For any $(M, \phi) \in \mathcal{D}_{comp}(B[F])$ with (completed) perfection*

$$(N, \phi) := \mathrm{colim}((M, \phi) \xrightarrow{\phi} (M, \phi) \xrightarrow{\phi} (M, \phi) \to ...) \in \mathcal{D}_{comp}(B[F]),$$

we have $M^{\phi=1} \simeq N^{\phi=1}$ and $(M[1/t])^{\phi=1} \simeq (N[1/t])^{\phi=1}$ via the natural maps.

[1]This lecture needs more background from the theory of étale cohomology of adic spaces. Consequently, more ideas have been punted to the references when compared to the previous lectures.

Proof. In this proof, all colimits refer to colimits in the underlying category $\mathcal{D}(B)$ (unlike in the statement of (2) above).

For (1), let $\{(M_i, \phi_i)\}$ be a diagram in $\mathcal{D}_{comp}(B[F])$. The fiber F of the map $\mathrm{colim}_i M_i \to \widehat{\mathrm{colim}_i M_i}$ is uniquely t-divisible as this always hold true for the derived t-completion map $N \to \widehat{N}$ for any $N \in \mathcal{D}(B)$. Thus, F also identifies with the fiber of the $\mathrm{colim}_i M_i[1/t] \to (\widehat{\mathrm{colim}_i M_i})[1/t]$. As the lemma can be reformulated as the statement that $F^{\phi=1} = 0$, it suffices to prove the statement for the functor $M \mapsto M^{\phi=1}$. For this, we claim that for any $(N, \phi) \in \mathcal{D}_{comp}(B[F])$, we have $N^{\phi=1} \simeq (N/t)^{\phi=1}$. First, this makes sense because we have $\phi(t) = t^p \subset tB$, so for any $N \in \mathcal{D}(B[F])$, there is an induced ϕ-structure on N/t compatible with the one on N. Secondly, if N is derived t-complete, then the fibre of $N \to N/t$ is complete when endowed with the t-adic filtration, and the ϕ-action on the fibre is topologically nilpotent with respect to this filtration as $\phi(t) = t^p \subset t^2B$. In particular, the functor $(-)^{\phi=1}$ must vanish on the fibre, giving $N^{\phi=1} \simeq (N/t)^{\phi=1}$, as asserted. The lemma now follows because both functors in the composition

$$\mathcal{D}_{comp}(B[F]) \xrightarrow{N \mapsto N/t} \mathcal{D}(B[F]) \xrightarrow{(-)^{\phi=1}} \mathcal{D}(\mathbf{F}_p)$$

commute with all colimits.

For (2), using (1), it suffices to observe the each map $(M, \phi) \xrightarrow{\phi} (M, \phi)$ induces an isomorphism on applying either $(-)^{\phi=1}$ or $(-[1/t])^{\phi=1}$, both of which are clear. $\square$

The promised reduction is now straightforward.

Corollary 1.3. *Fix a perfect prism (A, I) and a p-complete A/I-algebra R. The natural maps give isomorphisms*

$$(\Delta_{R/A}/p)^{\phi=1} \simeq (\Delta_{R/A,\mathrm{perf}}/p)^{\phi=1}$$

and

$$(\Delta_{R/A}[1/d]/p)^{\phi=1} \simeq (\Delta_{R/A,\mathrm{perf}}[1/d]/p)^{\phi=1}.$$

Proof. This follows from Proposition 1.2 (2) applied to $B = A/p$, $t = d$ coming from a generator of I, and $M = \Delta_{R/A}/p$ or $\Delta_{R/A}[1/d]/p$. $\square$

In other words, while proving Theorem 0.1, we are allowed to replace the prismatic cohomology complex $\Delta_{R/A}$ with its perfection $\Delta_{R/A,\mathrm{perf}}$

2. Reduction to semiperfectoids

Fix a perfect prism (A, I) as well as a topologically finitely generated A/I-algebra R with bounded p^∞-torsion. In this section, we explain how to reduce the statement of Theorem 0.1 to certain semiperfectoid rings.

Construction 2.1 (A semiperfectoid Cech nerve)**.** Let T be the p-adic completion of $A/I[x_1, ..., x_n]$. Let T_∞ be the p-adic completion of $A/I[x_1^{1/p^\infty}, ..., x_n^{1/p^\infty}]$, so there is a p-completely faithfully flat map $T \to T_\infty$. let T_∞^* be the Cech nerve of $T \to T_\infty$. Note that T_∞ is perfectoid, and hence each T_∞^i is a quotient of the perfectoid ring $T_\infty^{\widehat{\otimes}_{A/I}(i+1)}$; in fact, using the formal smoothness of $A/I \to T$, it is not difficult to see that each T_∞^i is regular semiperfectoid (Example VII.4.4).

Assume now that R/p is generated over $A/(p, I)$ by n elements $f_1, ..., f_n \in R$. We then have a surjective map $T \to R$ defined by $x_i \mapsto f_i$. Write $R_\infty := T_\infty \widehat{\otimes}_T R$ be its base change to R, so $R \to R_\infty$ is p-completely faithfully flat by the boundedness assumption on R. Let R_∞^* be the Cech nerve of $R \to R_\infty$, so $R_\infty^* \simeq T_\infty^* \widehat{\otimes}_T R$. We thus have a commutative pushout diagram

$$\begin{tikzcd} T \arrow[r] \arrow[d] & T_\infty^* \arrow[d] \\ R \arrow[r] & R_\infty^* \end{tikzcd}$$

with surjective vertical maps. Each R_∞^i is semiperfectoid with bounded p^∞-torsion.

We shall show Theorem 0.1 for R follows from Theorem 0.1 for each R_∞^i. This amounts to observing that both prismatic cohomology and étale cohomology satisfy descent along the map $R \to R_\infty$ constructed above. We briefly sketch how this works next, beginning with étale cohomology.

Lemma 2.2 (Descent for étale cohomology)**.** *The map $R \to R_\infty^*$ constructed above induces an equivalence*

$$R\Gamma_{et}(\mathrm{Spec}(R[1/p]), \mathbf{F}_p) \simeq \lim R\Gamma_{et}(\mathrm{Spec}(R_\infty^i[1/p]), \mathbf{F}_p)$$

on étale cohomology.

Sketch of proof. This statement is best proven using rigid geometry. More precisely, Huber's affinoid comparison theorem [5, Corollary 3.2.2] identifies $R\Gamma_{et}(\mathrm{Spec}(R[1/p]), \mathbf{F}_p)$ with $R\Gamma_{et}(\mathrm{Spa}(R[1/p], R), \mathbf{F}_p)$, and similarly for each R_∞^i. Having translated to étale cohomology of the rigid generic fibres, the statement follows from cohomological descent for the étale cohomology of adic spaces ([8, §11]).

One can also package this argument slightly more elementarily using the arc topology from [4]; as there have been two talks this semester on the arc topology, we briefly indicate how this works. In this language, the map $\mathrm{Spec}(R_\infty^*) \to \mathrm{Spec}(R)$ is an easily seen to be the Cech nerve of an arc_p-cover ([4, Definition 6.13]): the map $\mathrm{Spec}(R_\infty) \to \mathrm{Spec}(R)$ is an arc_p-cover, and the p-adic completion map on a ring gives an isomorphism on arc_p-sheafification ([4, Example 6.20]). The statement now follows from arc_p-descent for the functor $R \mapsto R\Gamma_{et}(\mathrm{Spec}(R[1/p]), \mathbf{F}_p)$ ([4, Corollary 6.17]). □

Next, we state the result for prismatic cohomology.

Lemma 2.3 (Descent for prismatic cohomology)**.** *The map $R \to R_\infty^*$ constructed above induces an equivalence*

$$\Delta_{R/A}/p \simeq \lim \Delta_{R_\infty^*/A}/p$$

and similarly for the functors $\Delta_{(-)/A}[1/d]/p$, $\Delta_{(-)/A,\mathrm{perf}}/p$ and $\Delta_{(-)/A,\mathrm{perf}}[1/d]/p$.

Sketch of proof. At least for R formally smooth, the statement for $\Delta_{(-)/A}$ can be proven using the Hodge-Tate comparison (Proposition VII.4.2) and flat descent for the cotangent complex ([3, Theorem 3.1]) as in [3, Example 5.11]. However, this does not suffice for the other functors as we run into the problem of commuting a colimit (such as inverting d or passing to the perfection) against a limit. To circumvent this problem, one proves a stronger statement for $\Delta_{(-)/A}$ itself: one shows that the map $\Delta_{R/A} \to \Delta_{R_\infty/A}$ is a descendable map of commutative algebras in $\mathcal{D}_{comp}(A)$ (in the sense of [6, Definition 3.18]; see also [1, §11.2]). This statement is proven by reduction to the case $R = T$, where it is eventually reduced to the following assertion: for a commutative ring k, if we write $\Omega := \Omega^*_{k[x]/k}$ for the de Rham complex of the polynomial ring in 1 variable over k, then the fibre $F \in D(\Omega)$ of the canonical map $\Omega \to k[x]$ satisfies $\mathrm{RHom}_\Omega(F^{\otimes 2}, \Omega) \in D^{<0}$. □

As a consequence, we have reduced the proof of Theorem 0.1 for topologically finitely generated A/I-algebras R with bounded p^∞-torsion to the case where R is semiperfectoid A/I-algebra.

3. Reduction to perfectoids

Fix a perfect prism (A, I), and let R be a semiperfectoid A/I-algebra. Our goal is to prove Theorem 0.1 in this setting. We shall explain how this reduces to the case of perfectoid rings. Thus, let $R \to R_{\mathrm{perfd}}$ be the perfectoidization (Definition VIII.2.2). We first show that the prismatic side of Theorem 0.1 does not change under this map.

Lemma 3.1 (Invariance under perfectoidization: prismatic side)**.** *The map $R \to R_{\mathrm{perfd}}$ induces equivalences*

$$(\Delta_{R/A}/p)^{\phi=1} \simeq (\Delta_{R_{\mathrm{perfd}}/A}/p)^{\phi=1} \quad \textit{and} \quad (\Delta_{R/A}[1/d]/p)^{\phi=1} \simeq (\Delta_{R_{\mathrm{perfd}}/A}[1/d]/p)^{\phi=1}.$$

Proof. Using Proposition 1.2, it is enough to prove the analogous statements after replacing $\Delta_{(-)/A}$ with $\Delta_{(-)/A,\mathrm{perf}}$. In this case, we shall prove the stronger statement that the map $R \to R_{\mathrm{perfd}}$ induces isomorphisms

$$\Delta_{R/A,\mathrm{perf}} \simeq \Delta_{R_{\mathrm{perfd}}/A,\mathrm{perf}}.$$

For this, note that by Example VIII.2.3 (2), we have $\Delta_{R_{\mathrm{perfd}}/A} \simeq A_{\mathrm{inf}}(R_{\mathrm{perfd}})$, which is already perfect. We are thus reduced to checking that the natural map

$$\Delta_{R/A,\mathrm{perf}} \to \Delta_{R_{\mathrm{perfd}}/A,\mathrm{perf}} \simeq A_{\mathrm{inf}}(R)$$

is an isomorphism. But we have defined R_{perfd} as $\Delta_{R/A,\mathrm{perf}} \otimes_A^L A/I$, and we have $A_{\mathrm{inf}}(R_{\mathrm{perfd}}) \otimes_A^L A/I \simeq R_{\mathrm{perfd}}$. Unwindining definitions shows that these are compatible, i.e., that the displayed map above gives an isomorphism after applying $- \otimes_A^L A/I$, and must thus be an isomorphism by derived Nakayama. $\square$

Next, we observe the analogous invariance of etale cohomology under perfectoidization.

Lemma 3.2 (Invariance under perfectoidization: étale side)**.** *The map $R \to R_{\mathrm{perfd}}$ induces an equivalence*

$$R\Gamma_{et}(\mathrm{Spec}(R[1/p]), \mathbf{F}_p) \simeq R\Gamma_{et}(\mathrm{Spec}(R_{\mathrm{perfd}})[1/p], \mathbf{F}_p).$$

Sketch of proof. Just like Lemma 2.2, there are two (closely related) approaches to understanding this result. If one is willing to use rigid geometry, one could simply pass to the étale cohomology of adic spaces [5] and observe that $\mathrm{Spa}(R_{\mathrm{perfd}}[1/p], R_{\mathrm{perfd}}) \to \mathrm{Spa}(R[1/p], R)$ is the "uniformification" map and thus induces an isomorphism on étale cohomology.

Alternately, one could use the arc_p-descent from [4, Corollary 6.17] to reduce to checking that $R \to R_{\mathrm{perfd}}$ gives an equivalence of arc_p-sheaves; the latter follows as the universal property (Remark VIII.2.5 (2)) of $R \to R_{\mathrm{perfd}}$ gives $\mathrm{Hom}(R, V) \simeq \mathrm{Hom}(R_{\mathrm{perfd}}, V)$ for any complete rank 1 valuation ring V with algebraically closed fraction field and $0 < |p| < 1$ (noting that any such V is perfectoid). $\square$

4. The proof of Theorem 0.1

Thanks to the reductions in the previous sections, to prove Theorem 0.1 for topologically finitely generated A/I-algebras R with bounded p^∞-torsion, it suffices to prove the following statement:

Lemma 4.1 (Étale cohomology of perfectoids via Artin-Schreier theory)**.** *Fix a perfect prism (A, I), and let R be a perfectoid A/I-algebra. Then there is a natural identification*

$$R\Gamma_{et}(\mathrm{Spec}(R[1/p]), \mathbf{F}_p) \simeq (R^\flat[1/d])^{\phi=1}.$$

Sketch of proof. This statement is best proven using rigid geometry. Let $X := \mathrm{Spa}(R[1/p], R)$ be the adic generic fibre, so $R\Gamma_{et}(X, \mathbf{F}_p) \simeq R\Gamma_{et}(\mathrm{Spec}(R[1/p], R), \mathbf{F}_p)$ by Huber's affinoid comparison theorem [5, Corollary 3.2.2]. By the theory of the pro-étale site from [7], we also have $R\Gamma_{et}(X, \mathbf{F}_p) \simeq R\Gamma_{proet}(X, \mathbf{F}_p)$ ([7, Lemma 3.16]). On X_{proet}, we have an Artin-Schreier short exact sequence

$$0 \to \mathbf{F}_p \to \mathcal{O}_X^{+,\flat}[1/d] \xrightarrow{\phi-1} \mathcal{O}_X^{+,\flat}[1/d] \to 0$$

of sheaves. We are thus reduced to showing that $R\Gamma(X_{proet}, \mathcal{O}_X^{+,\flat}[1/d]) \simeq R^\flat[1/d]$. This follows from the stronger statement that we have an almost isomorphism $R^\flat \to R\Gamma(X_{proet}, \mathcal{O}_X^{+,\flat})$ by [7, Lemma 4.10]. □

Remark 4.2 (Étale cohomology of the special fibre)**.** The following (easier) variant of Theorem 0.1 also holds true: for a perfect prism (A, I) and a p-complete A/I-algebra R, there is a natural identification

$$R\Gamma_{et}(\mathrm{Spec}(R/p), \mathbf{Z}/p^n) \simeq (\Delta_{R/A}/p^n)^{\phi=1}$$

for each $n \geq 1$. To prove this, one first reduces to the case $I = (p)$, and then argues using the comparison with crystalline cohomology (Theorem VI.3.2) and the usual Artin-Schreier sequence for étale cohomology; details omitted.

Remark 4.3 (A canonical isomorphism)**.** Strictly speaking, the isomorphism

$$R\Gamma_{et}(\mathrm{Spec}(R[1/p]), \mathbf{Z}/p^n) \simeq (\Delta_{R/A}[1/d]/p^n)^{\phi=1}$$

proving Theorem 0.1 constructed above depends on the choice of the surjection $T \to R$ from Construction 2.1; this surjection was determined by the set $S := \{f_1, ..., f_n\} \subset R$ of topological generators, so write η_S for the isomorphism above. A relatively formal way to get rid of this choice and obtain a canonical isomorphism is to consider the colimit $\eta := \mathrm{colim}\, \eta_S$ indexed by the category I of all sufficiently large finite subsets of R. As I is filtered, the colimit η gives an isomorphism

$$R\Gamma_{et}(\mathrm{Spec}(R[1/p]), \mathbf{Z}/p^n) \simeq (\Delta_{R/A}[1/d]/p^n)^{\phi=1}$$

as wanted.

5. Global statements

Let us use all our work so far to prove the dimension inequality from Theorem I.2.2.

Theorem 5.1. *Let $C/\mathbf{Q}_p$ be a complete and algebraically closed field. Let $\mathcal{O}_C \subset C$ be the valuation ring, and let k be the residue field. Let X be a proper smooth formal $\mathcal{O}_C$-scheme with generic fibre X_η/C and special fibre X_k/k. Then we have*

$$\dim_{\mathbf{F}_p} H^i_{et}(X_\eta, \mathbf{F}_p) \leq \dim_k H^i_{dR}(X_k/k). \tag{1}$$

Proof. Let (A, I) be the perfect prism corresponding to the perfectoid ring $\mathcal{O}_C$, and let $(W, (p))$ be the perfect prism corresponding to k, so $W \simeq W(k)$. The map $\mathcal{O}_C \to k$ lifts uniquely to a map $(A, I) \to (W, (p))$ of prisms (Theorem IV.2.3). Fix a generator $d \in I$.

Next, we introduce the relevant prismatic complexes. Note that the formal scheme X and the scheme X_k have the same underlying topological space. We can thus view sheaves on X as sheaves on X_k and vice versa. Let $\Delta_{X/A} \in D(X, A)$ be the prismatic cohomology complex of sheaves on X obtained by glueing prismatic cohomology locally as in Corollary VI.4.1; similarly, let $\Delta_{X_k/W}$ be the corresponding objects for X_k. Using the Hodge-Tate comparison (Theorem VI.0.1) and its base change compatibility, one checks that $\Delta_{X/A} \widehat{\otimes}^L_A W \simeq \Delta_{X_k/W}$, where the completion is p-adic.

Passing to cohomology, set $R\Gamma_A(X) := R\Gamma(X, \Delta_{X/A}) \in D(A)$. As $R\Gamma(X, -)$ preserves limits, $R\Gamma_A(X)$ is a (p, I)-complete object of $D(A)$. Moreover, by the Hodge-Tate comparison and finiteness of coherent cohomology, we know that $R\Gamma_A(X) \otimes^L_A A/I$ is a perfect A/I-complex. It follows by completeness that $R\Gamma_A(X)$ is also a perfect A-complex. (This is the fundamental source of finiteness in this theory.)

To prove (1), we use semicontinuity over the ring $V = A/p \simeq \mathcal{O}_C^\flat$, so V is a perfect rank 1 valuation ring with fraction field $C^\flat = V[1/d]$ and residue field k. Applying semicontinuity to the perfect V-complex $R\Gamma_A(X) \otimes_A^L V$ (or, more simply, by the structure of finitely presented modules over the valuation ring V), we obtain the valuation ring $V := A/p$, we have

$$\dim_{C^\flat} H^i(R\Gamma_A(X) \otimes_A^L C^\flat) \leq \dim_k H^i(R\Gamma_A(X) \otimes_A^L k). \tag{2}$$

We shall deduce (1) from (2).

Let us begin on the de Rham side, where we shall show

$$\dim_k H^i(R\Gamma_A(X) \otimes_A^L k) = \dim_k H^i_{dR}(X_k/k).$$

In fact, we shall prove the stronger statement that $R\Gamma_A(X) \otimes_A^L k \simeq \phi_* R\Gamma_{dR}(X_k/k)$ (where ϕ_* indicates a Frobenius twist that can be ignored for the purposes of computing dimensions over the perfect field k). We have already seen above that $\Delta_{X/A} \widehat{\otimes}_A^L W \simeq \Delta_{X_k/W}$, which gives $R\Gamma_A(X) \widehat{\otimes}_A^L W \simeq R\Gamma_W(X_k)$ (as $R\Gamma(X, -)$ commutes with limits and colimits) Reducing modulo p gets rid of the completion, giving $R\Gamma_A(X) \otimes_A^L k \simeq R\Gamma_W(X_k) \otimes_W^L k$. But now the crystalline comparison (Theorem VI.3.2) yields $R\Gamma_W(X_k) \simeq \phi_* R\Gamma_{crys}(X_k/W)$, and thus an identification $R\Gamma_W(X_k) \otimes_W^L k \simeq \phi_* R\Gamma_{dR}(X_k/k)$. Combining the two gives the desired identification $R\Gamma_A(X) \otimes_A^L k \simeq \phi_* R\Gamma_{dR}(X_k/k)$.

On the étale side, it suffices to show

$$\dim_{\mathbf{F}_p} H^i_{et}(X_\eta, \mathbf{F}_p) \leq \dim_{C^\flat} H^i(R\Gamma_A(X) \otimes_A^L C^\flat).$$

For this, it is enough construct an injective $C^\flat$-linear map $H^i(X_\eta, \mathbf{F}_p) \otimes_{\mathbf{F}_p} C^\flat \to H^i(R\Gamma_A(X) \otimes_A^L C^\flat)$. Glueing together the output of Theorem 0.1 over elements of an affine open cover of X (and noting that $R\Gamma_A(X) \otimes_A^L C^\flat \simeq R\Gamma(X, \Delta_{X/A}[1/d]/p)$ as $R\Gamma(X, -)$ commutes with filtered colimits), we obtain an identification

$$(R\Gamma_A(X) \otimes_A^L C^\flat)^{\phi=1} \simeq R\Gamma_{et}(X_\eta, \mathbf{F}_p).$$

We are now done thanks to a general and well-known lemma in p-linear algebra (Lemma 5.2). $\square$

Lemma 5.2. *Let K be an algebraically closed field of characteristic p. Let $(M, \phi) \in D(K[F])$ be pair with M perfect as a K-complex (see Proposition 1.2 for notation). Then for each integer i, the natural map*

$$H^i(M^{\phi=1}) \otimes_{\mathbf{F}_p} K \to H^i(M)$$

is injective. Moreover, this map is bijective if and only if $\phi : H^i(M) \to H^i(M)$ is bijective.

Remark 5.3. Using the q-de Rham complex, we shall show in a later lecture that the map $\phi_{X/A} : \Delta_{X/A} \to \Delta_{X/A}$ appearing in Theorem 5.1 is a d-isogeny, i.e., multiplication by $d^{\dim(X)}$ on both the source and target of $\phi_{X/A}$ factors over $\phi_{X/A}$. It follows that the theory $R\Gamma_A(X) \otimes_A^L C^\flat$ considered in the proof of Theorem 5.1 has a bijective Frobenius, and consequently we have

$$\dim_{\mathbf{F}_p} H^i_{et}(X_\eta, \mathbf{F}_p) = \dim_{C^\flat} H^i(R\Gamma_A(X) \otimes_A^L C^\flat)$$

in the proof.

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