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Considerations on the Random Traversal of Convex Bodies and Solutions for General Cylinders[1]

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The basic concepts applicable to the random traversal of convex bodies are discussed, and a set of theorems relevant to the problem is presented. Formulae are derived for the chord length distribution in cylinders of arbitrary convex cross section in a uniform isotropic field of straight random tracks.

1. INTRODUCTION

Chord length distributions resulting from the random intersection of convex bodies by straight lines have been discussed in such different fields as acoustics (*1–3*), reactor design (*4–6*), ecology (*7*), and microscopy (*8*, *9*). In radiation physics chord length distributions are needed for the evaluation of pulse height spectra obtained with proportional counters (*10*, *11*); the distributions are also relevant to various other problems of microdosimetry (*12*, *13*), general dosimetry (*14*), and radiation shielding.

For spheroids chord length distributions have been given in analytical (*15*) and in numerical (*16*) form. For circular cylinders the problem is more complicated and several different ways have been chosen to derive chord length distributions. Schwed and Ray (*17*) have attempted analytical solutions. Wilson and Emery (*18*) have applied mixed analytical and numerical methods. Birkhoff *et al.* (*19*) have used Monte Carlo techniques; they have also given a useful survey of the topic and have pointed out that the results from the various methods are not in full agreement. Little work has been done for cylinders of noncircular cross section. But Coleman (*20*) has obtained the solution for a cube.

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In this paper a formula for chord length distributions in cylinders of arbitrary cross section will be given. The idea of the solution is that the problem is reduced to one in two dimensions. The chord length distribution in the cylinder is expressed in terms of the chord length distribution for the cross section of the cylinder.

Sections 2 and 3 contain considerations of general interest to problems involving the random traversal of convex bodies. Some of the theorems listed in section 3 are used in the solution for cylinders.

2. CONCEPTS

An excellent outline of the problems arising in the random traversal of convex bodies is given by Kendall and Moran in "Geometrical Probability" (*21*). Recently Kingman (*3*, *22*) and Coleman (*20*) have derived various results on random chords. Some of these results will be used in the following considerations. The original works should be used for reference, but the basic concepts will in brief and simple form be stated in the present paper.

Coleman defines five different kinds of randomness of the secants of a convex body. Three of these are relevant in the present context:

Mean free path randomness (*μ-randomness*). A chord of a convex body K is defined by a point in Euclidian space and a direction. The point and the direction are from independent uniform distributions. This randomness results if the convex body is exposed to a uniform, isotropic field of straight infinite tracks; it is the main object of the present paper.

Surface radiator randomness (S-randomness). A chord of a convex body K is defined by a point on its surface and a direction. The point and the direction are from independent uniform distributions. S-randomness results if the surface is a uniform, isotropic radiation source.

Interior radiator randomness (I-randomness). A chord is defined by a point in the interior of K and a direction. The point and the direction are from independent uniform distributions.

The distributions of chord length are different for the various types of randomness. This difference and the use of the word randomness in an unspecified sense are the basis of some of the so-called paradoxes of probability (*21*). In the following, indices will be used to identify the different kinds of randomness. The expectation values will be labeled in the same way. For example $f_\mu(l)$ is the probability density of chord length l under μ-randomness, while $\bar{l}_\mu$ and $\overline{l_\mu^2}$ are the mean and the second moment of the distribution.

Sum distributions will be symbolized by the capital letter corresponding to the small letter used for the differential distribution. For convenience and because this simplifies some of the formulae the sum distributions are summed from the tail of the distribution. Thus $F_\mu(l)$ is the probability that the chord length exceeds

l under μ-randomness:[2]

$$F_\mu(l) = \int_l^\infty f_\mu(x)\, dx. \tag{1}$$

Surface radiator randomness has been discussed in the context of radiation physics; specifically it has been shown that for a sphere of diameter d one obtains the constant density (*23*):

$$f_S(l) = d^{-1}, \qquad 0 < l \leqq d, \tag{2}$$

while for μ-randomness one has

$$f_\mu(l) = 2l/d^2, \qquad 0 < l \leqq d. \tag{3}$$

S-randomness will not be further discussed in this paper. It appears that there is no simple relation between the distributions $f_S(l)$ and $f_\mu(l)$. It is an open question whether there exists a unique relation between $f_S(l)$ and $f_\mu(l)$, and to what degree these distributions determine the shape of the body K.

Kingman (*3*) has shown that I-randomness is closely related to μ-randomness:

$$f_I(l) \propto l f_\mu(l). \tag{4}$$

From this relation one finds that the mean chord length $\bar{l}_I$ under I-randomness always exceeds the mean chord length $\bar{l}_\mu$ under μ-randomness,

$$\bar{l}_I = \overline{l_\mu^2}/\bar{l}_\mu = (V_\mu + 1)\, \bar{l}_\mu, \tag{5}$$

where V_μ is the fractional variance of the distribution $f_\mu(l)$:

$$V_\mu = \frac{\sigma_\mu{}^2}{\bar{l}_\mu{}^2} = \frac{\overline{l_\mu^2}}{\bar{l}_\mu{}^2} - 1. \tag{6}$$

For a sphere V_μ is equal to 0.125 as can readily be derived from relation (3). One may assume that all nonspherical convex bodies have a larger fractional variance of their chord length distribution. It seems, however, that a proof of this simple assertion has yet to be found.

One must note that interior radiator randomness is not what one obtains if the interior of a convex body K is a uniform source of straight particle tracks. The definition of I-randomness refers to *full* straight lines while an actual radiation source produces tracks which, even if they are assumed to be infinite, represent only *half* lines. If one generalizes the concept of a chord so far as to include the

[2] The upper limit of integration is given as ∞ in all cases where the integration extends to the maximal chord length. This is correct because the densities and the sum distributions are zero for $l > l_{max}$. One must, however, be careful to observe this condition in actual cases where the distributions may be given analytically.

segments of these half lines which are contained in the body K one may introduce the corresponding concept which shall be named *internal source randomness* and labeled by the small letter i.

For i-randomness one obtains the *density*,

$$f_i(l) = F_\mu(l)/\bar{l}_\mu, \tag{7}$$

where $F_\mu(l)$ is the *sum distribution* of l under μ-randomness according to the formula:

$$F_\mu(l) = \int_l^\infty f_\mu(x)\,dx. \tag{8}$$

Proof

Relation (4) has already been derived by Kingman. Here its validity will be demonstrated again in a consideration which can then be varied to yield relation (7).

$f_\mu(l)$ is the chord length distribution if K is exposed to an isotropic field. Consider an infinitesimal sphere of cross section Δa centered around a position distributed uniformly over the interior of K. The probability that a chord of length l intersects the infinitesimal sphere is $l\,\Delta a/V$, where V is the volume of the body K. If one samples the chords which intersect the infinitesimal sphere in all its possible positions one fulfills the conditions for I-randomness (see above), and one therefore obtains,

$$f_I(l) \propto \frac{\Delta a}{V}\, l f_\mu(l) \propto l f_\mu(l), \tag{9}$$

or with proper normalization,

$$f_I(l) = l f_\mu(l)\Big/\int_0^\infty x f_\mu(x)\,dx = l f_\mu(l)/\bar{l}_\mu. \tag{10}$$

In case of an internal source one counts the half tracks originating from the point of emission separately (i-randomness). In this case the probability that the chord of length l intersects the infinitesimal sphere is still $l\,\Delta a/V$, but the point of intersection on the chord must be considered. This point of intersection is distributed uniformly over the length of the chord. The chords of length l therefore give rise to an equidistribution of half segments s:

$$f_i(s) = 1/l, \qquad 0 < s \leqq l. \tag{11}$$

Integration over all chords occurring in I-randomness yields Eq. (7):

$$f_i(s) = \int_s^\infty \frac{l}{l} f_\mu(l)\,dl/\bar{l}_\mu = F_\mu(s)/\bar{l}_\mu. \tag{12}$$

From the preceding considerations it is apparent that the mean chord length for an internal source is half the mean chord length under I-randomness and is therefore related to the first and second moment of the chord length distribution under μ-randomness in the following way [see Eq. (5)]:

$$\bar{l}_i = \bar{l}_I/2 = \overline{l_\mu^2}/2\bar{l}_\mu = \frac{V_\mu + 1}{2}\,\bar{l}_\mu . \tag{13}$$

The relation can also be derived by partial integration from Eq. (12), and it can be generalized to

$$\overline{l_i^N} = \frac{\overline{l_\mu^{N+1}}}{(N+1)\bar{l}_\mu}, \; N \neq -1. \tag{14}$$

In section 4 results are given for chord length distributions in uniform, isotropic fields (μ-randomness). Relation (7) means that the results can also be applied to the case of interior sources.

One should note that all considerations in this section, with exception of relations (2) and (3) which refer to a sphere, equally apply to three-dimensional and two-dimensional Euclidian space.

3. AUXILIARY RELATIONS

This section contains theorems relevant to the problem of chord length distributions. Some of the relations will be applied in section 4. Theorems 3.1–3.7 have been derived earlier (*3*, *10*, *20*). For convenience proofs and some applications are included here.

3.1 Distribution of the Angle of Incidence to a Surface Element

Theorem

If a surface element is exposed to an isotropic field (μ-randomness) and if θ is the angle between the incident line and the normal of the surface element at the point of incidence, then θ is distributed according to the density $f_\mu(\theta)$:

$$f_\mu(\theta)\, d\theta = 2 \sin\theta \cos\theta \, d\theta, \quad 0 \leqq \theta < \frac{\pi}{2}, \tag{15}$$

while the sum distribution, i.e. the probability that the angle of incidence exceeds θ, is

$$F_\mu(\theta) = \cos^2\theta. \tag{16}$$

Proof

The distribution of θ on a surface element in an isotropic uniform field does not depend on the orientation of this surface element. One may therefore consider the surface of a sphere. For a sphere one can, however, assume a unidirectional field instead of an isotropic field. This is schematically represented in Fig. 1. The total

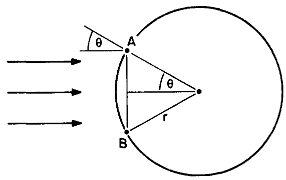

FIG. 1. Determination of the distribution of angles of incidence to a surface element.

cross section of the sphere of radius r is equal to $r^2\pi$. The disc whose projection in Fig. 1 is the line AB represents the cross section for angles of incidence equal to or less than θ. Its area is $r^2\pi \sin^2\theta$, and the fraction of tracks incident on the sphere with angles exceeding θ is therefore given by Eq. (16). By differentiation one obtains relation (15).

As an example of the application of relation (16) one can derive the chord length distribution in an infinite slab of thickness h. This distribution will be used in section 4. The chord length as a function of the angle of incidence is

$$l = h/\cos\theta. \tag{17}$$

Inserting this in (16) one has the sum distribution

$$F_\mu(l) = h^2/l^2, \qquad l \geqq h, \tag{18}$$

and therefore the density

$$f_\mu(l) = 2h^2/l^3. \tag{19}$$

3.2 Distribution of the Angle of Incidence to a Line Element

Theorem

In 2-space the angle θ of incidence to a line element in an isotropic field has the sum distribution

$$F_\mu(\theta) = 1 - \sin\theta, \qquad 0 \leqq \theta < \pi/2, \tag{20}$$

and therefore the density[3]

$$f_\mu(\theta) = \cos\theta. \tag{21}$$

The angle of incidence θ is the angle between the random track and the normal to the line element.

The proof can be omitted. It is analogous to the proof in 3.1.

[3] A frequent source of confusion in considerations on chord length distributions is failure to distinguish between μ-randomness and S-randomness. For a surface source in three dimensions the angle θ of entrance into the body K is distributed according to the density: $f_s(\theta) = \sin\theta$; while the density in the two-dimensional case is constant: $f_s(\theta) = 2/\pi$.

One can use Eq. (20) to derive the chord length distribution in a circle, which will be used in section 4. For a circle of diameter d one has:

$$l = d \cos \theta. \tag{22}$$

By substitution in (20) and transition to the complementary probability in order to account for the inverse variation of θ with l one obtains the sum distribution

$$F_\mu(l) = \sqrt{1 - l^2/d^2}, \qquad l \leqq d, \tag{23}$$

and the density

$$f_\mu(l) = \frac{l}{d\sqrt{d^2 - l^2}}. \tag{24}$$

Another example is an infinite strip of height h. In this case one has

$$\sin \theta = \sqrt{1 - h^2/l^2}, \tag{25}$$

and therefore

$$F_\mu(l) = 1 - \sqrt{1 - h^2/l^2}, \qquad l \geqq h, \tag{26}$$

and

$$f_\mu(l) = \frac{h^2}{l^2\sqrt{l^2 - h^2}}. \tag{27}$$

3.3 Total Track Length in a Volume

Theorem

In a uniform field of fluence Φ the expected[4] total track length per unit volume element is equal to Φ.

Proof

The fluence is defined as the expectation value of the number of traversals of a sphere of unit cross section. The total track length per unit volume is a function of Φ only. In order to obtain the numerical relation one may consider a cylinder of unit cross section and height h and a uniform unidirectional field of fluence Φ parallel to the axis of the cylinder. The expected number of traversals is Φ, the expected total track length is Φh, and the expected total track length per unit volume is therefore Φ.

[4] Here and in the following subsections the term *expected* is used as an abbreviation for *expectation value of*. The term *mean* is not used in order to avoid confusion with the mean of the chord length distributions.

3.4 Number of Traversals Through a Surface Element

Theorem

In an isotropic uniform field of fluence Φ the expected number of traversals through a unit surface element is $\Phi/2$. The expected number of chords through a convex body of surface S is therefore $S\Phi/4$.

Proof

As discussed in 3.1 one can consider a sphere exposed to a unidirectional field of fluence Φ. The expected number of chords is $r^2\pi\Phi$. Each chord intersects the surface of the sphere twice. The expected number of intersections per unit surface element is therefore equal to $\Phi/2$; the expected number of chords is equal to $S\Phi/4$.

3.5 Total Track Length and Number of Traversals in Two Dimensions

Theorem

In two-dimensional space fluence Φ is defined as the expected number of traversals of a circle of unit diameter. The expected total track length per area element in an isotropic uniform field is Φ, the expected number of intersections with a unit line element is $2\Phi/\pi$.

The proof is analogous to the proof of 3.3 and 3.4.

3.6 Mean Chord Length in Three Dimensions

From 3.3 and 3.4 one obtains a relation which is sometimes called the Cauchy theorem. The relation, implicity deduced by Cauchy (*24*) in 1850, seems to have first been stated by Czuber (*25*).

Theorem

If a convex body has the volume V and the surface S, then the mean chord length $\bar{l}_\mu$ is equal to

$$\bar{l}_\mu = 4V/S. \tag{28}$$

Proof

The theorem follows from the fact (see 3.3 and 3.4) that the mean chord length is equal to the total track length ΦV in the body divided by the number $S\Phi/4$ of chords through the body.

3.7 Mean Chord Length in Two Dimensions

Theorem

For a two-dimensional convex figure of area A and circumference C one has

$$\bar{l}_\mu = \pi A/C. \tag{29}$$

Proof

This theorem which has also been implicitly deduced by Cauchy follows from the fact (see 3.5) that the mean chord length is equal to the total track length $A\Phi$ in the figure divided by the number $\Phi C/\pi$ of chords through the figure.

From the proofs of 3.6 and 3.7 one concludes that, with the interpretation of a track segment as a simply connected piece of the track within the region, the Cauchy theorem is valid for a uniform isotropic field even if the random tracks are not straight (*13*). The generalization applies equally to two and three dimensions. One can furthermore show that the tracks may be branched; but they must not be closed curves. The present discussion will, however, be confined to straight random tracks. The generalization of the Cauchy theorem to the case of finite straight tracks is given in the following subsection.

A similar simple relation exists for the fourth moment of the chord length distribution (*21*, *26*). But the case is more complicated for the second and third moment, and such characteristics of the distribution as fractional variance or skewness must therefore be evaluated numerically in each particular case.

3.8 Mean Segment Length in the Case of Finite Tracks

The following theorem deals with the mean segment length which results in a convex body when exposed to random tracks of finite length. It can be considered as the generalized form of the Cauchy theorem. All considerations in this and the following section refer to μ-randomness. The index μ is therefore omitted. Throughout this paper the letter F will be used for distributions referring to infinite random tracks, while P will be used for distributions which result if the random tracks are of finite length.

Theorem

Assume that the finite random tracks have an orientation. A random track is then defined by a starting point, a direction, and a length. The point and the direction are from independent uniform distributions, the length is distributed according to $r(u)$. This is the generalization of μ-randomness to finite tracks. $r(u)$ is called the range distribution.

If a convex body of mean chord length $\bar{l} = 4V/S$ is exposed to a uniform, isotropic field of random tracks of mean range $\bar{u}$, then the resulting mean segment length $\bar{s}$ in the body is given by the relation

$$\frac{1}{\bar{s}} = \frac{1}{\bar{u}} + \frac{1}{\bar{l}}. \tag{30}$$

Proof

The total track length in the body is $V\Phi$ (see 3.3), and the number of end points plus starting points of tracks in the body is therefore $2V\Phi/\bar{u}$, the number of traversal points on the surface of the body is $S\Phi/2$ (see 3.4). Each track segment has two *terminal* points which are either starting points or end points of tracks, or traversal points. The mean segment length $\bar{s}$ is therefore equal to the total track length divided by half the number of terminal points:

$$\bar{s} = \frac{V\Phi}{\Phi V/\bar{u} + S\Phi/4} = (\bar{u}^{-1} + S/4V)^{-1}. \tag{31}$$

One can readily show that Eq. (30) holds also in the two-dimensional case.

3.9 Chord Length Distributions in the Case of Finite Random Tracks

Theorem

Assume that the convex body K has the sum distribution $F(l)$ of chord length under μ-randomness and that it is exposed to an isotropic uniform field of straight random tracks whose length u has the sum distribution $R(u)$. Then the resulting sum distribution of segment length in K is

$$P(s) = k\left(F(s)\int_s^\infty R(x)\,dx + R(s)\int_s^\infty F(x)\,dx\right). \tag{32}$$

The normalization constant has the value $k = (\bar{u} + \bar{l})^{-1}$, where $\bar{l}$ is the mean chord length $4V/S$, and $\bar{u}$ is the mean range of the tracks.

In the special case where all random tracks are of a fixed length u, the formula reduces to

$$P(s) = \frac{1}{\bar{l} + u}\left((u - s)F(s) + \int_s^\infty F(x)\,dx\right), \qquad s \leqq u. \tag{33}$$

Note that the sum distributions are summed from the right, i.e., equal to the probability that the random variable exceeds a given value.

Proof

First an auxiliary relation has to be given: If two intervals of length u and l on a straight line overlap randomly, then the length s of the overlap has the sum distribution:

$$H(s) = (u + l - 2s)/(u + l), \qquad s \leqq u, l. \tag{34}$$

The relation follows from the geometrical consideration indicated in Fig. 2. The solid line represents the interval of length u, the dotted line those positions of the left border of the other interval which correspond to overlaps. The positions which

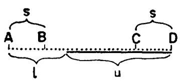

Fig. 2. Determination of the overlap probability of two intervals.

belong to an overlap greater than s are inside $\overline{BC}$. The ratio of the interval length $\overline{BC}$ to $\overline{AD}$ is equal to $H(s)$.

Assume a sphere S which is centered around the body K and has a diameter large as compared to the dimensions of K and the maximum length of the random tracks. Consider a track of length u inside S. The probability that the straight-line extension of the track intersects K is independent of u. If an intersection occurs the probability for a chord length l in K is proportional to $f(l)$. Given the values u and l, the probability that the track itself intercepts K is proportional to $u + l$.

The resulting sum distribution of segment length is therefore

$$\begin{aligned} P(s) &= k \int_s^\infty \int_s^\infty f(l) r(u)(u + l) H(s)\, du\, dl \\ &= k \int_s^\infty \int_s^\infty (u + l - 2s) f(l) r(u)\, du\, dl \qquad (35) \\ &= k \left(F(s) \left[\int_s^\infty u r(u)\, du - sR(s) \right] + R(s) \left[\int_s^\infty l f(l)\, dl - sF(s) \right] \right). \end{aligned}$$

By another partial integration one obtains Eq. (32). The value of k results from the condition $P(0) = 1$. This ends the proof.

By differentiation of Eq. (32) one obtains the probability density

$$p(s) = k \left(f(s) \int_s^\infty R(x)\, dx + r(s) \int_s^\infty F(x)\, dx + 2F(s)R(s) \right), \qquad (36)$$

and in the special case of a fixed track length u,

$$p(s) = k \left(f(s)(u - s) + \delta(s - u) \int_u^\infty F(x)\, dx + 2F(s) \right), \qquad s \leqq u. \qquad (37)$$

One can show that the three terms in Eqs. (36) and (37) correspond to those segments which Caswell (*12*) categorizes as crossers, insiders, and as starters and stoppers.

In Eq. (32) the first term represents the contribution of the interior source (i-randomness) and the second term the random tracks originated outside K. If Eq. (32) is split up into the contribution of crossers, insiders, and starters and stoppers it takes the form

$$\begin{aligned} P(s) = k \Bigg(& F(s) \int_s^\infty R(x)\, dx - \int_s^\infty F(x) R(x)\, dx \\ & + R(s) \int_s^\infty F(x)\, dx - \int_s^\infty F(x) R(x)\, dx \qquad (38) \\ & + 2 \int_s^\infty F(x) R(x)\, dx \Bigg) \end{aligned}$$

or in the case of a fixed track length u,

$$P(s) = k\left((u - s)F(s) - \int_s^u F(x)\,dx + \int_u^\infty F(x)\,dx + 2\int_s^u F(x)\,dx\right), \qquad s \leqq u. \tag{39}$$

These formulae are important for microdosimetric calculations where one deals with finite charged particle tracks incident on a sensitive volume of cylindrical or spherical shape. In the case of a sphere which has been treated by Caswell one has to insert the functions

$$f(s) = 2s/d^2, \qquad F(s) = 1 - s^2/d^2, \qquad \int_s^\infty F(x)\,dx = \frac{2}{3}d - s + \frac{s^3}{3d^2}, \qquad s \leqq d. \tag{40}$$

For cylinders the corresponding functions will be derived in the next section.

Theorems (32) and (33) have been derived here for 3-space. Going through the arguments in the proof one finds that they equally hold in 2-space. This will be used in section 4.

4. SOLUTION FOR GENERAL CYLINDERS

This section deals with the derivation of chord length distributions for right cylinders in an isotropic, uniform field (μ-randomness). The principle underlying the solution is to reduce the problem to one in two dimensions. All distributions and expectation values are understood to refer to μ-randomness, if not otherwise stated. The index μ is therefore omitted.

4.1 Derivation of the Chord Length Distributions

Assume that the cylinder has height h and an arbitrary convex cross section, and consider the orthogonal projection of the random tracks and of the cross section of the cylinder onto a plane parallel to the faces of the cylinder. Figure 3 represents this projection. The heavy line segments are the projections of those segments of the random tracks which lie inside the infinite slab formed by the two planes through the faces of the cylinder. The length u of the projection is related to the length v of the actual segment in the slab by

$$v^2 = h^2 + u^2. \tag{41}$$

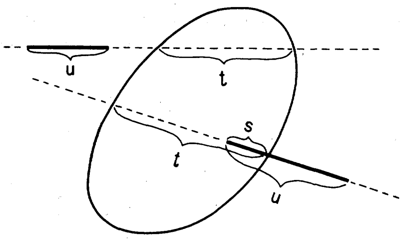

FIG. 3. Projection of a cylinder exposed to random tracks.

With this relation and Eq. (18) one obtains the sum distribution of the projection length:

$$G(u) = h^2/v^2 = 1/(1 + u^2/h^2). \tag{42}$$

The projections are uniform and isotropic in the plane. The straight lines through these projections form the chord length distribution in 2-space, $f(t)$, in intercepting the cross section of the cylinder. The distribution $f(t)$ is known for a circle and for a rectangle [see Eqs. (24) and (61)]; for more complicated convex figures it can be derived numerically. The mean chord length in the cross section is $\bar{t}$.

When the intervals u and t overlap, then the segment s of overlap is the projection of the actual chord in the cylinder. The problem thus reduces to a determination of the distribution of these segments. This problem has been solved in 3.9. For a fixed projection length u, one obtains the following sum distribution of segments s in the projection of the cylinder in accordance with Eq. (33):

$$C(s) = \frac{1}{\bar{t} + u}\left((u - s)F(s) + \int_s^\infty F(x)\, dx\right), \qquad s \leqq u. \tag{43}$$

The true chord length l in the cylinder is proportional to s:

$$l = s\sqrt{1 + h^2/u^2}. \tag{44}$$

With this relation and with the abbreviation

$$F^*(s) = \int_s^\infty F(x)\, dx - sF(s), \tag{45}$$

one obtains the chord length distribution for a given angle of incidence of random tracks relative to the axis of the cylinder:

$$C(l) = \frac{1}{\bar{t} + u}\left(uF\left(\frac{l}{\sqrt{1 + h^2/u^2}}\right) + F^*\left(\frac{l}{\sqrt{1 + h^2/u^2}}\right)\right), \quad l \leqq \sqrt{u^2 + h^2}. \tag{46}$$

Except for a stretching by the factor $\sqrt{1 + h^2/u^2}$ the distribution is equal to the distribution (43) which results when the cross section of the cylinder is exposed to random tracks of length u.

Integration over the distribution of u yields the full solution. As in Eq. (35) and for the same reasons one has to include the weight factor $(\bar{l} + u)$ in the integral

$$C(l) = -k \int_a^\infty \left(uF\left(\frac{l}{\sqrt{1 + h^2/u^2}}\right) + F^*\left(\frac{l}{\sqrt{1 + h^2/u^2}}\right)\right) dG(u), \quad (47)$$

where the lower limit a of integration is 0 for $l \leqq h$ and $\sqrt{l^2 - h^2}$ for $l > h$. The chord length distribution in the cylinder is designated by the letter C and not F because F is used for the 2-space chord length distribution of the cross section of the cylinder. With the substitution

$$x = \frac{1}{\sqrt{1 + h^2/u^2}}, \qquad u = \frac{hx}{\sqrt{1 - x^2}}, \quad (48)$$

and with $G(u)$ from Eq. (42),

$$G(u) = 1 - x^2, \quad (49)$$

one obtains the general formula for the chord length distribution in right cylinders:

$$C(l) = 2k \int_a^1 \left(\frac{hx^2}{\sqrt{1 - x^2}} F(lx) + xF^*(lx) \right) dx,$$

$$a = \begin{cases} 0 & \text{for } l \leqq h \\ \sqrt{1 - h^2/l^2} & \text{for } l > h. \end{cases} \quad (50)$$

4.2 Results

In this section the result of 4.1 will be restated, and the formulae for the chord length density will be added. The auxiliary function $f(t)$ will be given for the special cases of a circle and a square.

The chord length distribution in a right cylinder exposed to a uniform isotropic field of straight infinite tracks (μ-randomness) can be expressed in terms of the chord length distribution of the two-dimensional figure which is the cross section of the cylinder.

Let $f(t)$ and $\bar{t}$ be the differential distribution and the mean value of chord length t for the cross section. The sum distribution of t is

$$F(t) = \int_t^\infty f(x)\, dx, \quad (51)$$

and $F^*(t)$ is an auxiliary function:

$$F^*(t) = \int_t^\infty F(x)\, dx - tF(t) = \int_t^\infty xf(x)\, dx - 2tF(t). \quad (52)$$

Then the sum distribution of chord length in the cylinder, i.e., the probability that a chord exceeds length l, is given by the formula:

$$C(l) = 2k \int_a^1 \left(\frac{hx^2}{\sqrt{1-x^2}} F(lx) + xF^*(lx) \right) dx. \tag{53}$$

h is the height of the cylinder. The lower limit a of the integration is 0 for $l \leqq h$ and $\sqrt{1 - h^2/l^2}$ for $l > h$. The normalization constant is

$$k = \frac{1}{\bar{t} + \bar{u}} = \frac{1}{\pi(A/C + h/2)}, \tag{54}$$

as follows from 3.7 and from Eq. (42). A is the area and C the perimeter of the cross section of the cylinder. By differentiation of Eq. (53) one obtains

$$\begin{aligned} c(l) = -\frac{d\,C(l)}{dl} = -2k \int_a^1 \left(-\frac{hx^3}{\sqrt{1-x^2}} f(lx) + x^3 l f(lx) - 2x^2 F(lx) \right) dx \\ + 2k \frac{da}{dl} \left(\frac{ha^2}{\sqrt{1-a^2}} F(la) + aF^*(la) \right). \end{aligned} \tag{55}$$

The probability density of chord length is therefore equal to

$$\begin{aligned} c(l) = 2k \int_a^1 \left(\left(\frac{hx^3}{\sqrt{1-x^2}} - x^3 l \right) f(lx) + 2x^2 F(lx) \right) dx \\ + \frac{2kh^2}{l^3} \int_{\sqrt{l^2 - h^2}}^{\infty} F(x)\, dx, \end{aligned} \tag{56}$$

where the second integral applies only to values $l \geqq h$.

Equations (53) and (56) are the formulae for chord length distributions of cylinders of arbitrary convex cross section. In the general case the chord length distribution $f(t)$ for the cross section has to be evaluated numerically. For circular and square cross sections one can, however, give $f(t)$, $F(t)$, and $F^*(t)$ in analytical form.

For a circle of diameter d one obtains according to Eqs. (23), (24) and (52),

$$f(t) = \frac{t}{d\sqrt{d^2 - t^2}}, \qquad F(t) = \sqrt{1 - t^2/d^2} \tag{57}$$

$$\int_t^d F(x)\, dx = \frac{d}{2} \arccos \frac{t}{d} - \frac{t}{2} \sqrt{1 - t^2/d^2} \tag{58}$$

$$F^*(t) = \frac{d}{2} \arccos \frac{t}{d} - \frac{3t}{2} \sqrt{1 - t^2/d^2}. \tag{59}$$

All functions are zero for $t > d$.

Coleman (*20*) has derived the chord length distribution $f(t)$ for a rectangle One.

may note that one can obtain this function from Eq. (47). This equation is valid in a space of arbitrary dimensions, and a rectangle can be considered as a cylinder in two dimensions. In the special case of a square of side length c one inserts the distribution $G(u)$ for an infinite strip of height c [see Eq. (26)] and a delta function $\delta(t - c)$ for $f(t)$. Equation (47) with the substitution (48) then takes the form:

$$F(t) = k \int_a^b \left(\frac{xc}{\sqrt{1 - x^2}} + c - 2tx \right) dx \qquad \begin{matrix} a = 0 \\ b = 1 \end{matrix} \ \text{for } t \leqq c, \qquad \begin{matrix} a = \sqrt{1 - c^2/t^2} \\ b = c/t \end{matrix} \ \text{for } c < t \leqq \sqrt{2}c$$

$$= \begin{cases} 1 - t/2c & \text{for } t \leqq c \\ t/2c - \sqrt{1 - c^2/t^2} & \text{for } c < t \leqq \sqrt{2}c \end{cases} \tag{60}$$

By differentiation one obtains the chord length density for the square:

$$f(t) = \begin{cases} 1/2c & \text{for } t \leqq c \\ \dfrac{c^2}{t^2\sqrt{t^2 - c^2}} - 1/2c & \text{for } c < t \leqq \sqrt{2}c \end{cases} \tag{61}$$

Evaluation of the integrals in (53) and (56) for the special cases of circular or square cross section, analysis of the moments of the distribution, and discussion of the limiting cases of very long and very flat cylinders will be the object of a forthcoming paper. In general it is practical to evaluate the integrals numerically. Numerical results for circular cylinders have already been given (27).

In work with cylindrical proportional counters the case of a unidirectional radiation field is of some interest. One must then use Eq. (46) instead of Eq. (53). The value of u is equal to $h\ tg\theta$, where θ is the angle between the field and the axis of the cylinder. In the general case of an anisotropic field one has to use Eq. (47) and the distribution $G(u)$ which belongs to the angular distribution of the field.

If one deals with finite tracks of a range distribution $R(u)$ or of the fixed range u, one can first derive the chord length distribution $C(l)$ for infinite tracks and then apply the theorem (32) or (33). The distribution C takes the place of F in these equations.

Finally one should notice that except for a normalization constant Eq. (53) gives the probability density of segments created in a cylinder by an internal source. This follows from the considerations given in section 2.

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