to enable us to carry out the numerical integration of Eq. (28). These were twofold: The ground state wave function of the deuteron was approximated by a sum of two Gauss functions, and the integrals were replaced by sums over finite intervals. The first of these approximations can at most account for a few percent of the discrepancy because the assumed wave function for the deuteron deviates measurably from the true wave function only for large values of r/a, and it is just for these values of r/a that the contributions to the integrals are negligible. As we have already seen, the second approximation is not serious because of the rapid convergence of the integrals.

Since we have neglected polarization in this paper, it may well be that taking it into account will get rid of most of the discrepancy, provided an interaction energy of type (2) is adequate for the process we are considering. Although a calculation taking polarization into account would be exceedingly difficult, its undertaking at the present time seems warranted.

We wish to thank Mr. Jerome Rothstein of Columbia University for aiding in the numerical work.

JULY 1, 1940 PHYSICAL REVIEW VOLUME 58

Multiple Scattering of Electrons. II

S. Goudsmit and J. L. Saunderson*
University of Michigan, Ann Arbor, Michigan

(Received April 22, 1940)

The series developed in a previous paper representing the distribution for the multiple scattering of electrons has been evaluated numerically for a large number of cases; the results are given in Table I. An approximate expression is found for the value of sin θ averaged over the distribution per unit solid angle, $f(\theta)$. This expression, which agrees within a few percent with the exact computation, is

$$w\langle\sin\theta\rangle_{Av}\sim 1.76A^{\frac{1}{2}}(5.60-\tfrac{1}{3}\log Z+\tfrac{1}{2}\log A)^{\frac{1}{2}}, \quad (18)$$

in which w is the energy in units mc^2 and $A=24.8\times10^{-26}Z^2Nt$. For the scattering intensity per unit solid angle at 0°, that is $f(0)$, an approximate relation is

$$4\pi f(0)/w^2\sim 0.43/A(5.60-\tfrac{1}{3}\log Z+\tfrac{1}{2}\log A). \quad (19)$$

The accurate calculations show also that $f(\theta)/w^2$ is almost independent of the energy. A series formula is derived for the projected scattering distribution as observed in a cloud chamber. The averages of $w\sin\alpha$, α being the projected angle, are given in Table VI. These averages are smaller than the values computed by Williams and show a variation with energy. It is believed that the largest inaccuracy remaining in the results given is due to uncertainties in the single scattering law.

1. Introduction

In a previous paper[1] we have treated the statistical problem of multiple electron scattering by thin foils. The principal purpose of the present article is to bring the results of that paper into a form which can be more easily compared with experimental data.

We consider an electron of total energy w (in units mc^2) which has traveled a path length t through scattering material of atomic number Z containing N atoms per cc. The normalized probability that the electron will be deflected into the angle between θ and $\theta+d\theta$ is given by the following series in Legendre polynomials

$$2\pi f(\theta)\sin\theta d\theta = \tfrac{1}{2}\sum(2l+1)G_lP_l(\cos\theta)\sin\theta d\theta. \quad (1)$$

The coefficients G_l depend only upon two param-

* Now at the Dow Chemical Company.
[1] S. Goudsmit and J. L. Saunderson, Phys. Rev. **57**, 24 (1940).

eters, μ and ξ,

$$G_l = \exp\{-2\mu l(l+1) \times [\log \xi - (\tfrac{1}{2}+\tfrac{1}{3}+\cdots 1/l)]\}. \quad (2)$$

The parameter μ is proportional to the path length,

$$\mu = \pi\kappa^2 Nt = \pi\{Ze^2 w/mc^2(w^2-1)\}^2 Nt = 24.8\times 10^{-26} Z^2 Ntw^2/(w^2-1)^2 = 0.151(\sigma Z^2/M)w^2/(w^2-1)^2. \quad (3)$$

In the last expression which is the most convenient one to use, σ is the superficial density of the foil in grams per cm² and M the atomic mass. The path length should, however, not be taken equal to the thickness of the foil but as a first approximation, may be taken as the thickness divided by the average of cos θ, that is, G_1. If this correction amounts to more than a few percent, the present theory is no longer sufficiently accurate.

The other parameter ξ depends upon the deviation of the true single scattering probability from the Rutherford law as a result of screening by orbital electrons. For a Thomas-Fermi atom, using the Born approximation, we find[2]

$$\xi = 150(w^2-1)^{\frac{1}{2}}/Z^{\frac{1}{3}}. \quad (4)$$

For heavy elements the Born approximation may not be sufficient and for light elements the Thomas-Fermi atom may give inaccurate results. This causes some uncertainty in the numerical factor. Fortunately this has little influence upon the final results, but may well explain eventual small discrepancies between theory and experimental data.

2. The Scattering per Unit Solid Angle

The scattering per unit solid angle in the direction θ is $f(\theta)$, for which we have

$$4\pi f(\theta) = \sum (2l+1) G_l P_l(\cos\theta). \quad (5)$$

Table I shows the results of computations of $4\pi f(\theta)$ for twenty-one cases, for some of which as many as forty-eight terms in the series were used. In order to have a series which converges sufficiently rapidly, μ must be large enough that G_l becomes negligibly small when $l \ll \xi$. When this condition is not fulfilled the scattering is very nearly single. In such a case the exponent of Eq. (2) is no longer sufficiently accurate for the high terms in the series.

The table also gives the Rutherford single scattering, $\mu/\sin^4(\theta/2)$ for $\theta = 45°$. With the exception of possibly eight cases, single scattering has evidently not yet been approached at that angle. It must be remarked, however, that the frequent change of sign of the P_l for the larger angles makes the results of the series computation less accurate than for the smaller angles.

3. The Average Sin θ

A quantity which is most easily computed and observed is the value of sin θ averaged over the scattering distribution per unit solid angle. This is defined as

$$\langle \sin\theta \rangle_{\mathrm{Av}} = \int_0^\pi f(\theta)\sin\theta d\theta \Big/ \int_0^\pi f(\theta) d\theta. \quad (6)$$

Substituting for $f(\theta)$ the series of Eq. (5), we see at once that all terms in the numerator except

Table I.* *Values of $4\pi f(\theta)$.*

	Appr.† 0°	0°	6°	12°	18°	24°	30°	45°	Ruth. 45°
log ξ = 4									
μ = 0.0025	107	120.9	72.6	26.3	8.3	2.7	1.1	0.1	0.12
0.0050	44.1	45.7	38.6	24.3	12.5	5.8	2.7	0.5	0.23
0.010	18.7	19.1	17.9	14.8	11.0	7.4	4.7	1.3	0.46
0.015	11.5	11.8	11.4	10.2	8.5	6.6	4.9	1.9	0.70
log ξ = 5									
μ = 0.0015	122	122.4	81.2	27.9	7.1	1.9	0.6	0.1	0.07
0.0025	66.0	65.8	53.1	28.9	11.8	4.2	1.5	0.2	0.12
0.0050	29.2	29.2	26.6	20.3	13.1	7.5	3.9	0.7	0.23
0.010	13.1	13.3	12.7	11.3	9.4	7.1	5.1	1.8	0.46
log ξ = 6									
μ = 0.0005	307	303.2	118.8	13.6	1.5	0.6	0.3		0.02
0.0015	85.6	85.3	65.3	30.6	9.8	2.7	0.9		0.07
0.0025	47.6	47.5	41.1	26.8	13.7	5.8	2.2	0.3	0.12
0.0050	21.8	21.9	20.5	16.9	12.3	8.0	4.7	1.0	0.23
0.010	10.2	10.3	10.0	9.2	7.9	6.5	5.0	2.2	0.46
log ξ = 7									
μ = 0.0005	226	224.3	113.9	19.0	2.2	0.5	0.3		0.02
0.0015	66.0	66.1	54.2	30.4	12.3	4.0	1.2	0.1	0.07
0.0025	37.3	37.7	34.7	24.9	14.2	7.0	3.0	0.4	0.12
0.0050	17.4	17.8	17.0	14.5	11.3	8.3	5.2	1.3	0.23
log ξ = 8									
μ = 0.0005	179	179.2	104.8	24.1	2.8	0.5	0.2		0.02
0.0015	53.6	54.3	46.3	28.9	13.6	5.1	1.1	0.1	0.07
0.0025	30.7	31.3	28.6	21.8	14.0	9.6	3.7	0.5	0.12
0.0050	14.5	15.0	14.4	13.0	10.2	7.7	5.3	1.6	0.23

* The normalizing factor is given by Eq. (8).
† The values in this column were obtained from the approximate Eq. (15).

[2] For a Wentzel potential the numerical factor is 160. Compare Eqs. (I 36) and (I 35). We shall denote equations in the first paper (reference 1) by a Roman I.

TABLE II.* *Average values of sin θ.*

LOG ξ = 4		5	6	7	8
μ = 0.00050			0.0667 (0.0656)	0.0768 (0.0764)	0.0853 (0.0859)
0.00075		0.0702 (0.0679)	0.0835 (0.0832)	0.0960 (0.0960)	0.1063 (0.1073)
0.0010	0.059 (0.059)	0.0826 (0.0816)	0.0995 (0.0981)	0.113 (0.113)	0.124 (0.127)
0.0015	0.078 (0.079)	0.106 (0.104)	0.125 (0.124)	0.141 (0.142)	0.154 (0.157)
0.0025	0.111 (0.111)	0.143 (0.142)	0.166 (0.166)	0.184 (0.188)	0.202 (0.208)
0.0050	0.174 (0.173)	0.212 (0.213)	0.242 (0.246)	0.266 (0.276)	0.288 (0.303)
0.010	0.262 (0.266)	0.307 (0.319)	0.342 (0.364)	0.374 (0.404)	0.402 (0.440)
0.015	0.327 (0.340)				

* The values given in parenthesis were found using the approximate Eq. (14).

the first vanish and that the denominator can be obtained from the development of cosec θ in a series of Legendre polynomials. Thus

$$\langle\sin\theta\rangle_{\text{Av}}=2/\pi\sum_{\text{even } l}(2l+1)G_l\left(\frac{1\cdot3\cdot5\cdots l-1}{2\cdot4\cdot6\cdots l}\right)^2$$

$$=1/2(0.786+0.983G_2+0.995G_4+G_6+\cdots).\quad(7)$$

The coefficients in Eq. (7) approach a common value very rapidly. The values of $\langle\sin\theta\rangle_{\text{Av}}$ computed in this way are entered in Table II.

The scattering per unit solid angle, $f(\theta)$, is not normalized to unity. The normalizing factor for the entries of Table I is given by

$$4\pi\int_0^\pi f(\theta)d\theta=4\pi/\langle\sin\theta\rangle_{\text{Av}}.\quad(8)$$

These values can be computed with the help of Table II.

4. Approximate Formulas

If it were permissible to replace the partial harmonic series in the exponent of G_l by some constant average value, s, it would not be difficult to obtain an approximate value for the sum[3] of Eq. (7), for instance, by replacing it by an integral. Abbreviating

$$\gamma^2=2\mu[\log\xi-s],\quad(9)$$

we find that, approximately,

$$\langle\sin\theta\rangle_{\text{Av}}\sim1\Big/\int_0^\infty e^{-\gamma^2l^2}dl=(4\gamma^2/\pi)^{\frac{1}{2}}.\quad(10)$$

Similarly, for $\theta=0$ we find from Eq. (5)

$$4\pi f(0)\sim\sum(2l+1)e^{-\gamma^2l(l+1)}\sim1/\gamma^2.\quad(11)$$

It is also possible to show that with the same degree of approximation the distribution may be represented by a Gaussian expression in

$$y=\sin\theta/2,$$

namely,

$$4\pi f(\theta)\sim e^{-y^2/\gamma^2}/\gamma^2.\quad(12)$$

This can be verified, for example, by expanding Eq. (12) in a series of Legendre polynomials.

Although these expressions are only approximate, they help in finding more accurate expressions for $\langle\sin\theta\rangle_{\text{Av}}$ and $4\pi f(0)$ which is done by adjusting s and the numerical factor in front until the results agree best with the direct numerical calculations. In this manner we found that a good fit is obtained when

$$-s=\tfrac{1}{2}\log\mu+0.60\quad(13)$$

and

$$\langle\sin\theta\rangle_{\text{Av}}\sim\{3.08\mu[\log\xi+\tfrac{1}{2}\log\mu+0.60]\}^{\frac{1}{2}},\quad(14)$$

$$4\pi f(0)\sim0.43/\mu[\log\xi+\tfrac{1}{2}\log\mu+0.60].\quad(15)$$

The results of these approximate expressions are also given in Tables I and II for comparison with the direct computations.

5. Energy Dependence

It has been pointed out by Williams[4] that by considering $f(\theta)$ as a function of $w\theta$, the distribution function is of the Gaussian type whose shape is independent of the energy and depends only upon the thickness and the material of the scatterer. This method greatly facilitates the interpretation of experimental data. We shall see that the present formulas allow the same simplification to a sufficient degree of approximation.

[3] Formulas for such sums are given by L. S. Kassel, J. Chem. Phys. **1**, 576 (1933).

[4] E. J. Williams, Proc. Roy. Soc. **A169**, 531 (1939).

We now characterize the scatterer by two new parameters A and η, defined as follows

$$A=\mu(w^2-1)^2/w^2=24.8\times 10^{-26}Z^2Nt$$
$$=0.151\sigma Z^2/M \quad (16)$$

and[5]

$$\eta=\xi/(w^2-1)^{\frac{1}{2}}=150/Z^{\frac{1}{3}}. \quad (17)$$

Substituting these parameters in the approximate Eq. (14) for $\langle\sin\theta\rangle_{\text{Av}}$ taking $w^2\gg 1$, we find that the right-hand side of the following approximate expression is indeed independent of the energy.[6] Thus

$$\langle w\sin\theta\rangle_{\text{Av}}\sim 1.76A^{\frac{1}{2}}(\log\eta+\tfrac{1}{2}\log A+0.60)^{\frac{1}{2}}$$
$$=1.76A^{\frac{1}{2}}(5.60+\tfrac{1}{2}\log A-\tfrac{1}{3}\log Z)^{\frac{1}{2}}. \quad (18)$$

Equation (18) is an approximate expression. Table III shows the values of $\langle w\sin\theta\rangle_{\text{Av}}$ computed from interpolations between the exact values of $\langle\sin\theta\rangle_{\text{Av}}$ for a few examples. It shows indeed a negligible dependence upon energy over a wide range of values, and demonstrates the sufficient accuracy of the approximate formula. This table can also be used for interpolations; it will be noticed that the entries are approximately proportional to $A^{\frac{1}{2}}$.

[5] The numerical coefficient is again based on the Thomas-Fermi atom but can of course be adjusted if required.

[6] In most cases $\sin\theta$ can be replaced by θ, giving $w\langle\theta\rangle_{\text{Av}}$. If expressed in Mev degrees, the numerical coefficient must be replaced by 50. When $H\rho\theta$ is recorded in gauss cm degrees, as is usual in cloud-chamber work, the coefficient is 1.70×10^5.

We can treat the approximate formula for $f(0)$ in the same manner, obtaining

$$4\pi f(0)/w^2\sim 0.43/A[5.60+\tfrac{1}{2}\log A-\tfrac{1}{3}\log Z]. \quad (19)$$

Table IV shows the same quantity as interpolated from the exact values of $4\pi f(0)$. The exact values show that the function $4\pi f(0)/w^2$ is still slightly dependent upon the energy, although the variation is only 7 percent, over the range of energies given, for the thinnest foil calculated. As the foil becomes thicker, the energy dependence is reduced. It can be verified that within the same degree of approximation the whole distribution function $f(\theta)/w^2$ taken as a function of $w\theta$ is independent of the energy. A Gaussian curve of the form

$$4\pi f(\theta)/w^2=Ce^{-\frac{1}{4}C(w\theta)^2} \quad (20)$$

with

$$C=4\pi f(0)/w^2 \quad (21)$$

gives a fair approximation.

The data given in Table IV show an interesting dependence of $4\pi f(0)$ upon Z for constant Z^2Nt. For the thinnest foil the value of $f(0)$ for $Z=90$ is about 25 percent greater than for $Z=6$. This variation is somewhat reduced for the thicker foils. For foils used to approximate single scattering conditions at moderate angles, the variation of the scattering distribution at $\theta=0$ would be even larger for a similar range of Z. This indi-

TABLE III. *Values of $w\langle\sin\theta\rangle_{\text{Av}}$. For $w\langle\theta\rangle_{\text{Av}}$ in Mev degrees multiply these numbers by 28.4, for $H\rho\langle\theta\rangle_{\text{Av}}$ in gauss cm degrees multiply by 0.97×10^5.*

	Appr. Eq. (18)	$w=5$	7	10	15	20	30	40	50	60
$A=0.20$, $Z^2Nt=0.81\times 10^{24}$										
$\log\eta=3.5$ $Z\sim 90$	1.43	1.44	1.44	1.44	1.44	1.44				
$=4.0$ ~ 24	1.53	1.53	1.53	1.53	1.53	1.54				
$=4.4$ ~ 6	1.62	1.60	1.60	1.60	1.60	1.60				
$A=0.40$, $Z^2Nt=1.61\times 10^{24}$										
$\log\eta=3.5$ $Z\sim 90$	2.09		2.09	2.11	2.12	2.12	2.12			
$=4.0$ ~ 24	2.26		2.19	2.22	2.24	2.26	2.26			
$=4.4$ ~ 6	2.37		2.27	2.32	2.33	2.34	2.34			
$A=0.80$, $Z^2Nt=3.22\times 10^{24}$										
$\log\eta=3.5$ $Z\sim 90$	3.16			3.05	3.08	3.10	3.14	3.16		
4.0 ~ 24	3.34			3.18	3.25	3.28	3.30	3.32		
4.4 ~ 6	3.49			3.28	3.36	3.40	3.42	3.44		
$A=1.60$, $Z^2Nt=6.45\times 10^{24}$										
$\log\eta=3.5$ $Z\sim 90$	4.63				4.43	4.52	4.55	4.60	4.58	4.58
4.0 ~ 24	4.90				4.65	4.74	4.78	4.84	4.81	4.81
4.4 ~ 6	5.10				4.78	4.90	4.96	5.00	4.89	4.88

TABLE IV. *Values of* $4\pi f(0)/w^2$.

	Appr. Eq. (19)	$w=5$	7	10	15	20	30	40	50	60
$A=0.20$										
$\log\eta=3.5$ $Z\sim90$	0.654	0.605	0.622	0.634	0.644	0.650				
4.0 ~24	0.566	0.527	0.541	0.554	0.560	0.560				
4.4 ~6	0.513	0.478	0.494	0.504	0.509	0.510				
$A=0.40$										
$\log\eta=3.5$ $Z\sim90$	0.295		0.287	0.290	0.294	0.295	0.295			
4.0 ~24	0.260		0.251	0.257	0.258	0.260	0.260			
4.4 ~6	0.237		0.230	0.234	0.236	0.237	0.238			
$A=0.80$										
$\log\eta=3.5$ $Z\sim90$	0.135			0.134	0.134	0.135	0.134	0.134		
4.0 ~24	0.120			0.120	0.120	0.121	0.120	0.119		
4.4 ~6	0.110			0.111	0.112	0.111	0.110	0.110		
$A=1.60$										
$\log\eta=3.5$ $Z\sim90$	0.062				0.063	0.063	0.062	0.062	0.062	0.061
4.0 ~24	0.057				0.057	0.056	0.056	0.056	0.056	0.056
4.4 ~6	0.051				0.053	0.053	0.052			

cates that, for foils of constant Z^2Nt, the scattering is actually more multiple for the foils of small Z.

6. Scattering in Mixtures or Compounds

From the derivation of the expressions for coefficients G_l as given in the first paper we find for a mixture of two elements a and b that

$$G_l=G_l(a)G_l(b). \tag{22}$$

Considering the exponent of G_l in Eq. (2) we write

$$\mu[\log\xi-s_l]=\mu_a[\log\xi_a-s_l]+\mu_b[\log\xi_b-s_l]. \tag{23}$$

Thus for a compound containing two elements we introduce

$$\mu=\mu_a+\mu_b, \tag{24}$$

so that

$$\log\xi=(\mu_a\log\xi_a+\mu_b\log\xi_b)/(\mu_a+\mu_b). \tag{25}$$

Similarly for A and η we obtain

$$A=A_a+A_b \tag{26}$$

and

$$\log\eta=(A_a\log\eta_a+A_b\log\eta_b)/(A_a+A_b). \tag{27}$$

7. Projected Single Scattering

Most cloud-chamber observations do not yield the scattering angle θ but its projection on the plane of the cloud chamber. It is possible to transform Eq. (1) so as to give directly the projected distribution. We shall, however, derive the projected scattering in a somewhat different way which is closely related to Williams' treatment of the problem and emphasizes the advantages of the present method.

We denote the projected angle for a single scattering by ϕ and after multiple scattering by α. In our previous paper[7] we indicated that after ν collisions we have

$$G_m=\langle\cos m\alpha\rangle_{Av}=\exp\left[-\nu(1-\langle\cos m\phi\rangle_{Av})\right]. \tag{28}$$

We must therefore first obtain the Fourier coefficients for the single scattering law in order to calculate the coefficients G_m for the multiple scattering distribution. The latter will be obtained in the form[8]

$$\pi f(\alpha)=\sum G_m\cos m\alpha. \tag{29}$$

Normalization to unity requires that

$$G_0=\tfrac{1}{2}. \tag{30}$$

We take the Wentzel law for single scattering[9] and denote the angle with the normal to the plane of the cloud chamber by ψ. The scattering

[7] Compare Eq. (I 13) and the footnote 4 on page 25.

[8] If only the quadratic term in ϕ is taken in the exponent of G_m and if the Fourier series is replaced by an integral the results for $f(\alpha)$ will be a Gaussian curve. This treatment is identical with the classical derivation of the Gaussian distribution by Laplace and Poisson. It is slightly simpler here because the number of collisions is not fixed but has a Poisson distribution with an average value ν.

[9] Compare Eqs. (I 16) and (I 22).

law becomes

$$I(\theta)\sin\theta d\theta d\varphi = k^2 y dy d\varphi/(y^2+y_0^2)^2$$
$$= k^2\sin\psi d\psi d\phi/(1+2y_0^2-\sin\psi\cos\phi)^2. \quad (31)$$

This transformation was obtained by using

$$\cos\theta = \sin\psi\cos\phi;$$
$$\sin\theta d\theta d\varphi = \sin\psi d\psi d\phi. \quad (32)$$

In the foil the vertical scattering is not limited and we must integrate Eq. (31) over ψ from 0 to π which gives

$$2k^2 d\phi\int_0^{\pi/2}\sin\psi d\psi/(1+2y_0^2-\sin\psi\cos\phi)^2$$
$$= \frac{2k^2 d\phi}{(1+y_0^2)^2}\left[\frac{1}{\sin^2\phi'}+\frac{(\pi-\phi')\cos\phi'}{\sin^3\phi'}\right] \quad (33)$$

with ϕ' defined by

$$\cos\phi' = \cos\phi/(1+2y_0^2). \quad (34)$$

A sufficient approximation, normalized to unity is given by

$$W(\phi)d\phi \sim 2y_0^2 d\phi/\phi'^3 \sim 2y_0^2 d\phi/(\phi^2+4y_0^2)^{3/2}. \quad (35)$$

8. The Average Value of Cos $m\phi$

We use the approximate scattering law and write

$$\langle 1-\cos m\phi\rangle_{Av}$$
$$= 4y_0^2\left[\int_0^{\phi_1}(1-\cos m\phi)d\phi/(\phi^2+4y_0^2)^{3/2}\right.$$
$$\left.+\int_{\phi_1}^{\infty}(1-\cos m\phi)d\phi/(\phi^2+4y_0^2)^{3/2}\right]. \quad (36)$$

The small angle ϕ_1 is so chosen that in the first integral we may use $\frac{1}{2}(m\phi)^2$ for $(1-\cos m\phi)$ and that in the second integral we may neglect $4y_0^2$ in the denominator. In this approximation the final result does not contain the arbitrary angle ϕ_1.[10] The second integral is extended to infinity, which does not influence the result.[11] The integrations give

$$\langle 1-\cos m\phi\rangle_{Av} = 2y_0^2 m^2\{-1+\log 2\phi_1-2y_0^2\}$$
$$+2y_0^2 m^2\{\tfrac{3}{2}-Ci(m\phi_1)\}$$
$$\sim 2y_0^2 m^2[-\log y_0+\tfrac{1}{2}-C-\log m]. \quad (37)$$

Here C stands for Euler's constant, 0.58. For Ci, the cosine integral, we took the first terms in the expansion.[12] Using Eqs. (I 23) and (I 35) we can write

$$\log\xi = -\log y_0-\tfrac{1}{2}, \quad (38)$$

which finally gives

$$\langle 1-\cos m\phi\rangle_{Av}$$
$$= 2y_0^2 m^2[\log\xi+1-C-\log m]. \quad (39)$$

It is obvious that the last terms in the brackets are equivalent to the partial harmonic series which was obtained for the case of Legendre polynomials.

9. Projected Multiple Scattering

For the Fourier coefficients for the projected multiple scattering distribution, using $\nu y_0^2=\mu$, we find

$$G_m = \exp\{-2\mu m^2[\log\xi+1-C-\log m]\}. \quad (40)$$

It is easy to determine again the average value of $\sin\alpha$, now given by

$$\langle\sin\alpha\rangle_{Av} = \int_0^{\pi}f(\alpha)\sin\alpha d\alpha\bigg/\int_0^{\pi}f(\alpha)d\alpha$$
$$= (4/\pi)[\tfrac{1}{2}-\sum_1^{\infty}G_{2n}/(4n^2-1)]. \quad (41)$$

This result is obtained by substituting for $f(\alpha)$

Table V. *Values of* $\langle\sin\alpha\rangle_{Av}$.

$\log\xi=4$		5	6	7	8
$\mu=0.00050$			0.076	0.084	0.092
0.0015	0.101	0.121	0.136	0.150	0.163
0.0025	0.134	0.158	0.176	0.195	0.210
0.0050	0.199	0.228	0.254	0.274	0.294
0.010	0.283	0.320	0.350	0.377	0.399
0.015	0.344				

[10] Williams' method consists essentially in making a special choice for ϕ_1 and omitting the second integral. The first integral alone gives a Gaussian curve for the multiple scattering distribution. Williams finally makes a correction to this distribution for scattering beyond ϕ_1.

[11] For larger angles Eq. (35) is no longer a very good approximation, but replacing it by a better formula made no difference in the results for Eq. (36).

[12] See for instance Jahnke-Emde, *Funktionentafeln.*

TABLE VI. *Values of* $w\langle\sin\alpha\rangle_{Av}$. *For* $w\langle\alpha\rangle_{Av}$ *in Mev degrees multiply these numbers by 28.4, for* $H\rho\langle\alpha\rangle_{Av}$ *in gauss cm degrees multiply by* 0.97×10^5.

	WILLIAMS	$w=5$	7	10	15	20	30	40	50	60
$A=0.20$, $Z^2Nt=0.81\times10^{24}$										
$\log\eta=3.5$ $Z\sim90$	1.69	1.51	1.53	1.56	1.58	1.60				
$=4.0$ ~24	1.75	1.57	1.60	1.64	1.66	1.68				
$=4.4$ ~6	1.80	1.64	1.67	1.71	1.73	1.74				
$A=0.40$, $Z^2Nt=1.61\times10^{24}$										
$\log\eta=3.5$ $Z\sim90$	2.46		2.16	2.23	2.28	2.31	2.34			
$=4.0$ ~24	2.54		2.27	2.35	2.40	2.43	2.45			
$=4.4$ ~6	2.62		2.33	2.41	2.48	2.52	2.54			
$A=0.80$, $Z^2Nt=3.22\times10^{24}$										
$\log\eta=3.5$ $Z\sim90$	3.55			3.12	3.24	3.31	3.39	3.42		
$=4.0$ ~24	3.66			3.26	3.40	3.40	3.56	3.58		
$=4.4$ ~6	3.77			3.38	3.53	3.60	3.68	3.71		
$A=1.60$, $Z^2Nt=6.45\times10^{24}$										
$\log\eta=3.5$ $Z\sim90$	5.18				4.60	4.74	4.86	4.93	4.97	4.98
$=4.0$ ~24	5.34				4.78	4.93	5.07	5.14	5.18	5.20
$=4.4$ ~6	5.50				4.89	5.07	5.23	5.30	5.35	5.38

the Fourier series of Eq. (29). Many other averages can be obtained in the same way. Eq. (41) converges rapidly, and numerical results are given in Tables V and VI for $\langle\sin\alpha\rangle_{Av}$ and $w\langle\sin\alpha\rangle_{Av}$. The latter can be compared with Williams' formula which in our notation[13] is

$$\langle\alpha\rangle_{Av}=1.76A^{\frac{1}{2}}[2.51-0.052\log Z+0.078\log A]. \quad (42)$$

Williams' results are also given in Table VI. It is interesting to note that $w\langle\sin\alpha\rangle_{Av}$ has a larger variation with energy than $w\langle\sin\theta\rangle_{Av}$.

10. CONCLUSION

The results derived in the present paper can still be refined in various ways. A small correction may arise from replacing the Thomas-Fermi field by the Hartree field. Consecutive impacts are not quite independent[14] as was assumed in the present treatment, with the result that the scattering may be decreased somewhat. Preliminary computations show that the relativistic terms in the Mott formula increase the average angle for lead by several percent, but the slow convergence of the relativistic correction terms makes this result uncertain. In cloud-chamber observations the vertical angle is limited by the depth of the illuminated layer, so that only a part of the projected scattering distribution is measured. A reliable correction for this cut off is rather complicated but fortunately $w\langle\sin\alpha\rangle_{Av}$ does not differ much from $w\langle\sin\theta\rangle_{Av}$.

All such corrections are not very useful at present because the experimental data are not sufficiently accurate. Moreover, the statistical complications of the multiple scattering problem make it rather unsuitable for testing the single scattering law, which is after all the ultimate purpose of the experiments.

This investigation was made in connection with work supported by the Horace H. Rackham Fund.

[13] This is Eq. (37) of Williams' paper.
[14] J. A. Wheeler, Phys. Rev. **57**, 352 (1940).