DIVISIBLE CONVEX SETS WITH PROPERLY EMBEDDED CONES

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ABSTRACT. In this article we construct many examples of properly convex irreducible domains divided by Zariski dense relatively hyperbolic groups in every dimension at least 3. This answers a question of Benoist. Relative hyperbolicity and non-strict convexity are captured by a family of properly embedded cones (convex hulls of points and ellipsoids) in the domain. Our construction is most flexible in dimension 3 where we give a purely topological criterion for the existence of a large deformation space of geometrically controlled convex projective structures with totally geodesic boundary on a compact 3-manifold.

CONTENTS

1. INTRODUCTION

In this article we construct properly convex open subsets $\Omega \subset \mathbb{RP}^d$ divided by subgroups $\Gamma < \mathrm{PSL}_{d+1}\mathbb{R}$ with special geometric properties in arbitrary dimension $d \geq 3$.

A *properly convex* subset $\Omega \subset \mathbb{RP}^d$ is an open subset contained in some affine chart of $\mathbb{R}^d$ where it is convex and bounded. It comes with a group of projective symmetries

$$\mathrm{Aut}(\Omega) := \{A \in \mathrm{PSL}_{d+1}(\mathbb{R}) \mid A(\Omega) = \Omega\}$$

and an $\mathrm{Aut}(\Omega)$-invariant Finsler metric called the *Hilbert metric*.

Discrete and torsion-free subgroups $\Gamma < \mathrm{Aut}(\Omega)$ give rise to quotient projective d-manifolds

$$M := \Omega/\Gamma.$$

Among these groups Γ, of special interest are those such that $M = \Omega/\Gamma$ is compact. If such discrete subgroup exists, then the convex set Ω is said *divisible* and the group Γ *divides* Ω. Divisible convex sets are a rich source of geometry, dynamics, and group theory (see [13]).

Basic examples are

- The *hyperbolic space*, the symmetric space of $\mathrm{PO}_0(d, 1)$

$$\mathbb{H}^d = \{[x] \in \mathbb{RP}^d \,\big|\, x_1^2 + \cdots + x_d^2 - x_{d+1}^2 < 0\}.$$

- The projective model of the symmetric space of $\mathrm{SL}_n\mathbb{R}$

$$\mathrm{SL}_n\mathbb{R}/\mathrm{SO}(n) = \mathrm{PSym}_n^+ = \{[S] \in \mathbb{P}(\mathrm{Sym}_n(\mathbb{R})) \,|\, S \text{ is positive definite}\},$$

  where $\mathrm{Sym}_n(\mathbb{R})$ is the space of symmetric square matrices of size n.
- A projective model of the symmetric space $\mathbb{R}^d$

$$\Delta^d = \left\{[x] \in \mathbb{RP}^d \,|\, x_1, \ldots, x_{d+1} > 0\right\}.$$

(The Hilbert metrics do not necessarily agree with the Riemannian ones.)

The study of divisible convex sets, initiated by Benzécri [14] and vastly expanded by Benoist [7–10], branches into two major subjects:

- *Existence*: Construct examples beyond symmetric ones (see Table 1).
- *Classification*: Describe the structure of divisible convex sets, of the groups dividing them, and of the quotient projective manifolds.

We are going to come back to this in Section 1.1.

The purpose of this article is to contribute to the first goal by exhibiting new examples of divisible convex sets with novel geometric and group theoretic features. First we answer a question of Benoist [11, Prob. 10] (see also Marquis [46, Open Q. 3]).

Theorem 1.1. *For every $d \geq 3$ there exists a divisible convex set $\Omega \subset \mathbb{RP}^d$ divided by a* Zariski dense *group $\Gamma < \mathrm{SL}_{d+1}(\mathbb{R})$ such that Ω is* not strictly convex*, i.e. $\partial\Omega$ contains non-trivial line segments.*

We will give a more precise version of Theorem 1.1 later on (see Theorem 1.3), which gives a description similar to a result of Benoist [10, Th. 1.1] on the structure of divisible convex sets in dimension 3 (see also [34,35,67]). Let us remark the following.

- Recall that a subgroup $\Gamma < \mathrm{PSL}_{d+1}(\mathbb{R})$ is *Zariski dense* if every polynomial in the matrix entries that vanishes on Γ also vanishes on $\mathrm{PSL}_{d+1}(\mathbb{R})$.

- Non-strict convexity comes from *properly embedded cones* in Ω, i.e. convex hulls in Ω of a $(d-2)$-dimensional ellipsoid $H \subset \partial\Omega$ (the *base* of the cone) and a point $p \in \partial\Omega$ (the *vertex*), whose relative boundary is contained in $\partial\Omega$. In dimension at least 4, the presence of these cones is also a novelty.
- Our construction allows a good amount of flexibility. For instance, if $d \geq 5$, then we can produce infinitely many different *commensurability classes* of projective manifolds with the properties stated in the theorem. Recall that two groups Γ, Γ' are *commensurable* if they admit isomorphic finite index subgroups $G < \Gamma, G' < \Gamma'$.

    In particular, for each $d \geq 5$ there exist infinitely many different non-strictly convex divisible convex sets divided by a Zariski dense group.
- Ω will be constructed by gluing compact convex projective manifolds *with totally geodesic boundary*, like the ones in the next result.

Our second result is specific to dimension 3, where geometry and topology are strictly tied by breakthroughs of Thurston [59]. We provide a purely topological condition for the existence of (deformation spaces of) convex projective structures with totally geodesic boundary on a large class of compact 3-manifolds with toroidal boundary, addressing a question of Ballas, Danciger, and Lee [2] (see Question 1.6).

Theorem 1.2. *Consider:*

- *A compact, orientable, irreducible and atoroidal 3-manifold M with non-empty connected boundary.*
- *A doubly incompressible separating simple closed curve $\alpha \subset \partial M$.*

Then for all $a > 1$ close enough to 1, $b > 3$ and $1 < c < -2 + b^2/2$, the complement $N = DM \smallsetminus U$ of an open tubular neighborhood U of α in the double DM admits a convex projective structure with totally geodesic boundary isomorphic to a projective torus with holonomy generated by $A = \mathrm{diag}(a, a^{-1}, 1, 1)$ and $B = \mathrm{diag}(b^{-1}, b^{-1}, bc, bc^{-1})$.

We will later state Theorem 1.7, a more precise version of Theorem 1.2, which allows more boundary components in ∂M. We will then recall classical results on the topology of 3-manifolds.

- Recall that M irreducible (resp. atoroidal) means that it does not contain essential spheres (resp. tori), and α doubly incompressible means that it intersects every properly embedded essential disk, Möbius band, and annulus.
- Curves $\alpha \subset \partial M$ that satisfy the assumptions of Theorem 1.2 exist and are abundant in a precise sense (by work of Lecuire [42]).
- Given $(M, \alpha), (M', \alpha')$ as in Theorem 1.2, the control on the boundary holonomy gives us a homeomorphism $f : \partial N \to \partial N'$ such that

the closed manifold $N \cup_f N'$ can be endowed with a convex projective structure. In particular, the double DN has a convex projective structure.

The remainder of the introduction is structured as follows:

(a) In Section 1.1 we put our work into context by briefly surveying the classification of divisible convex sets and the known methods to produce examples.
(b) In Section 1.3 we give and comment a more precise statement (Theorem 1.3) of Theorem 1.1. In particular we give more details on properly embedded cones and their stabilizers. Theorem 1.3 will be deduced from Theorem 1.5, stated in Section 1.3, which gives compact convex projective manifolds with totally geodesic boundary (which are convex-cocompact in the sense of [30]).
(c) In Section 1.4 we give a more precise version (Theorem 1.7) of Theorem 1.2, and we put this result into context by briefly surveying classical result in 3-dimensional topology.

1.1. **Classification and known examples.** Before describing our new examples and their novel features, let us briefly survey the classes of examples known so far in order to provide an adequate context.

To this purpose, it is convenient to briefly recall the general classification scheme (see Table 1 and Figure 1) of divisible convex sets.

symmetric		non-symmetric	
$\mathrm{Aut}(\Omega)$ semisimple Lie group		$\mathrm{Aut}(\Omega) < \mathrm{PSL}_{d+1}(\mathbb{R})$ Zariski dense	
$\mathrm{Aut}(\Omega) \curvearrowright \Omega$ transitive		$\mathrm{Aut}(\Omega) \curvearrowright \Omega$ properly discontinuous	
[40, 63, 64]		[8]	
$\mathbb{R}$-rank ≥ 2	$\mathbb{R}$-rank $= 1$	strictly convex	non-strictly convex
$\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$	$\mathbb{H}^d$	Ω	Ω
[22]	[55]	[26, 36, 37, 39]	[2, 10, 27, 44]

TABLE 1. Classification of irreducible divisible convex sets Ω. The last row surveys some of the known examples.

Classification. Let us comment on some features in Table 1:

Reducible and irreducible. If Γ_j divides $\Omega_j \subset \mathbb{RP}^{d_j}$ for $j = 1, 2$, then $\Gamma_1 \times \Gamma_2 \times \mathbb{Z}$ divides the convex hull of the two

$$\mathcal{CH}(\Omega_1, \Omega_2) \subset \mathbb{RP}^d$$

with $d = d_1 + d_2 + 1$ and where $\mathbb{RP}^d = \mathbb{P}(\mathbb{R}^{d_1+1} \oplus \mathbb{R}^{d_2+1})$. Such examples are called *reducible.* If a divisible convex domain is not of this form, it

$\mathbb{H}^d$ $\mathcal{CH}(\mathbb{H}^{d-1}, \star)$
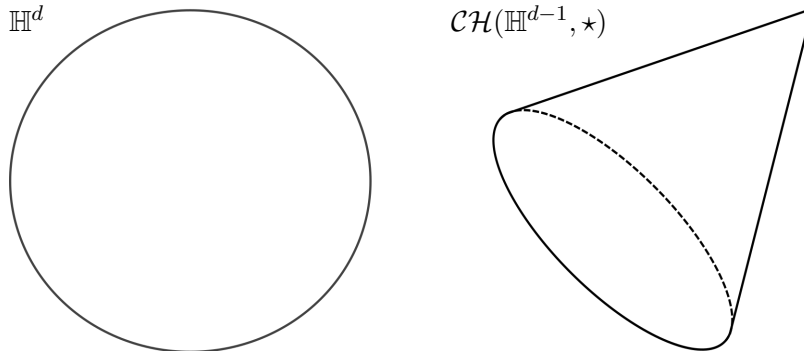

FIGURE 1. Irreducible and reducible examples.

is *irreducible*. Vey proved [62] that if Γ divides Ω then Ω is irreducible if and only if Γ is *strongly irreducible* (does not preserve any finite union of subspaces of $\mathbb{R}^{d+1}$). As we saw in Theorem 1.1, reducible divisible domains come up naturally in our constructions in the form of *cones*, that is, convex hulls of ellipsoids and points.

Symmetric and non-symmetric. As hinted in Table 1, the dichotomy between the symmetric and non-symmetric cases is the result of the combination of the works [8, 40, 63, 64].

Regularity of $\partial\Omega$ and geometry. Strict convexity can be seen as an avatar of negative curvature: Benoist proves in [7, Th. 1.1] that if Γ divides Ω then the following are equivalent:

- Ω is strictly convex.
- Γ is *Gromov hyperbolic*.
- $\partial\Omega$ is $\mathcal{C}^1$.

Recall that Gromov hyperbolicity is a group theoretic abstraction of the geometric properties of fundamental groups of closed hyperbolic manifolds and free groups (see Definition 6.7 and Remark 6.8).

In the same direction, Islam [33, Th. 1.6] showed that a larger class of divisible convex sets possesses a weak form of negative curvature: If Γ divides a non-symmetric irreducible Ω, then Γ is *acylindrically hyperbolic* (in the sense of Osin [51]).

Dimensions 2 and 3. As a consequence of Benzécri's work [14], every divisible convex set of dimension 2 is either a triangle (hence reducible) or strictly convex. Benoist gave in [10] a beautiful geometric description of *all* irreducible non-symmetric non-strictly divisible convex domains of dimension 3, similar to that in Theorem 1.3. He also gave examples, as mentioned later on.

Geometric rank-one and Higher rank rigidity. As in the non-positively curved Riemannian setting there is a notion of *geometric rank-one* of a convex projective manifold $M = \Omega/\Gamma$ introduced by Islam [33]. Zimmer proved

in [70] a higher rank rigidity result for projective manifolds (analogous to Ballmann [4] and Burns and Spatzier [24]): If Ω is irreducible and not symmetric then it has geometric rank-one.

Examples. We are now ready to describe the classes of examples known so far according to the methods used to produce them.

Polyhedral tilings in dimension $d \leq 6$ and local-to-global convexity: One way of constructing groups dividing properly convex domains is to consider *Coxeter groups* Γ generated by the reflections along the codimension 1 faces of a polyhedron $P \subset \mathbb{RP}^d$. Under suitable combinatorial and geometric assumptions on P, a local-to-global convexity argument asserts that the domain $\Omega = \Gamma \cdot P$ is properly convex and tiled by copies of P. This method, that goes back to Poincaré, has been extensively used by Kac and Vinberg [37], Benoist [10], and Marquis [44] to produce irreducible divisible domains in dimension $d \leq 6$ which are non-symmetric.

Arithmetic methods. A classical theorem of Borel [22] shows that every semisimple Lie group G admits a so-called *uniform arithmetic lattice* Γ, which is a discrete subgroup $\Gamma < G$ acting cocompactly on the symmetric space G/K. This implies that every irreducible symmetric properly convex domain is divisible. In the case of the hyperbolic space $\mathbb{H}^d$, the first examples are due to Siegel [55].

Bending and Bulging. A subgroup $\Gamma = \Gamma_1 *_\Lambda \Gamma_2 < \mathrm{PSL}_{d+1}(\mathbb{R})$ can be algebraically deformed by performing a very general procedure called *bending*: Let $(B_t)_t \subset \mathrm{PSL}_{d+1}(\mathbb{R})$ be a path of elements starting at the identity, that commute with Λ. One can deform the inclusion of $\iota : \Gamma \to \mathrm{PSL}_{d+1}(\mathbb{R})$ to representations ρ_t which are the identity on Γ_1 and send $\gamma_2 \in \Gamma_2$ to $B_t \gamma_2 B_t^{-1}$. By work of Koszul [41] and Benoist [9], if Γ divides some Ω, then the representations ρ_t are injective and $\rho_t(\Gamma)$ divides a properly convex domain Ω_t. Furthermore, by the Ehresmann–Thurston principle [58, Ch. 3], the quotients $\Omega_t/\rho_t(\Gamma)$ are diffeomorphic to Ω/Γ.

The construction applies in particular to hyperbolic d-manifolds $M = \mathbb{H}^d/\Gamma$ containing a codimension 1 submanifold $\Sigma = \mathbb{H}^{d-1}/\Lambda$ separating M into two connected components $M \smallsetminus \Sigma = M_1 \sqcup M_2$ (such manifolds can be constructed using the arithmetic techniques mentioned above). By Seifert–van Kampen Theorem, we can write $\Gamma = \Gamma_1 *_\Lambda \Gamma_2$ where $\Gamma_j = \pi_1(M_j)$. The bending obtained using elements B that fix $\mathbb{H}^{d-1}$ and its orthogonal for the Lorentzian form is called *bulging*. These deformations have been extensively studied by Johnson and Millson [36]. They allow to produce irreducible non-symmetric strictly divisible convex domains in arbitrary dimension (see [13, §8.2]).

Note that, by Margulis' Superrigidity (see [49, Ch. 16]) divisible symmetric properly convex domains with $\mathbb{R}$-rank at least 2 cannot be deformed.

Using a bulging construction, Kapovich [39] showed that certain manifolds constructed by Gromov and Thurston [32] admit convex projective

structures. These manifolds have the property that they admit metrics of pinched negative curvature, but no purely hyperbolic one (i.e. with constant curvature). In particular, the examples are strictly convex and not deformations of symmetric ones.

Other works that used a bulging construction in a convex projective context include [3, 20, 45].

Surfaces. Generalizing classical Fenchel–Nielsen coordinates for hyperbolic surfaces, Choi and Goldman [26] developed a gluing construction of convex projective surfaces starting from projective pairs of pants with totally geodesic boundary. They showed that their construction parametrizes all *Hitchin representations* $\rho : \pi_1(S_g) \to \mathrm{PSL}_3(\mathbb{R})$, which are continuous deformations of the holonomy $\rho_0 : \pi_1(S_g) \to \mathrm{PSO}_0(2,1)$ of a hyperbolic structure on a closed orientable surface S_g of genus $g \geq 2$.

3-Manifolds and Dehn fillings. Thanks to Thurston's breakthroughs, in dimension 3 we have several techniques to produce closed hyperbolic 3-manifolds. Among those there is the celebrated Hyperbolic Dehn Filling Theorem (see [58, Ch. 4]), which allows to deform a non-compact complete finite volume hyperbolic 3-manifold into many closed hyperbolic ones. Inspired by ideas from Thurston's theory, Ballas, Danciger, and Lee [2] gave many examples of convex projective 3-manifolds with a non-trivial JSJ decomposition. In a different direction, in combination with some Coxeter theory, Choi, Lee, and Marquis [27] produced examples of irreducible nonsymmetric domains in dimension $d \leq 6$ divided by groups which are relatively hyperbolic with respect to a collection of $\mathbb{Z}^2$ subgroups. (See also the work of Lee, Marquis, and Riolo [43].)

Note that there is actually a lot of interactions between these methods. Likewise, our construction owes to these previous works. Indeed, to prove Theorem 1.1, we will:

- Construct via arithmetic methods a particular hyperbolic manifold with codimension 2 cone-singularities, which is the double of a hyperbolic manifold with totally geodesic boundary and corners.
- Deform it into a *projective* cone-manifold M via bulging along totally geodesic hypersurfaces adjacent to the singularities.
- Use ideas from polyhedral tilings to show that the complement of the singularities in M is properly convex.
- Blow up the singularities of M to totally geodesic boundaries.

The closed convex projective examples of Theorem 1.1 are actually doubles DM of the above construction.

Let us now state refined versions of our main theorems.

1.2. **Divisible convex sets with properly embedded cones.**

Theorem 1.3. *For every $d \geq 3$ there exists a divisible convex set $\Omega \subset \mathbb{RP}^d$ divided by a discrete and Zariski dense subgroup $\Gamma < \mathrm{PSL}_{d+1}(\mathbb{R})$ with the following properties:*

(1) Ω contains a Γ-invariant family $\mathcal{C}$ of properly embedded cones with pairwise disjoint closures such that $\mathcal{C}/\Gamma$ is finite.

(2) The stabilizer Γ_C of each cone $C \in \mathcal{C}$ acts cocompactly on it, and has the form $\mathbb{Z} \times \Theta_C$, where Θ_C acts properly cocompactly on the base H_C of C while the $\mathbb{Z}$ factor acts trivially on it, so that C/Γ_C is diffeomorphic to $H_C/\Theta_C \times \mathbb{S}^1$.

(3) The group Γ is relatively hyperbolic *with respect to $\{\Gamma_C\}_{C \in \mathcal{C}}$, and its* Bowditch boundary *naturally identifies with $\partial\Omega/_\sim$, where $p \sim q$ if $p, q \in \bar{C}$ for some $C \in \mathcal{C}$.*

(4) Every point of $\partial\Omega \smallsetminus \bigsqcup_{C \in \mathcal{C}} \bar{C}$ is extremal and $\mathcal{C}^1$.

Let us remark the following.

- *Relative hyperbolicity* is a group theoretic abstraction of geometric properties of fundamental groups of finite volume hyperbolic manifolds. It comes with a notion of *Bowditch boundary* (see Definition 6.7).

- Note that $\Gamma_C = \mathbb{Z} \times \Theta_C$ is abelian only for $d = 3$, since Θ_C is conjugate to a uniform lattice of $\mathrm{SO}(d-2,1)$. The existence of Ω (non-strictly convex) divided by a group which is hyperbolic relative to non-abelian subgroups is a novelty.

  The answer to the following is still unknown:

  Question 1.4. *Is there for every $d > 6$ an irreducible divisible convex set $\Omega \subset \mathbb{RP}^d$ which is non-symmetric and non-strictly convex and such that Γ is relatively hyperbolic with respect to a collection of abelian subgroups?*

- Given the description of the structure of the properly convex set Ω and its properly embedded cones, parts (3) and (4) of the above result can be deduced from [67, Th. 1.16] or [35, Th. 1.3-6]. We will give a similar but different proof that applies to a different set of examples (see Theorem 1.5).

- Each component of $\left(\Omega \smallsetminus \bigcup_{C \in \mathcal{C}} C\right)/\Gamma$ admits an incomplete hyperbolic metric whose completion is a hyperbolic cone-manifold.

Let us now state formally the construction of the manifold whose double is the non-strictly convex closed convex projective manifold Ω/Γ in Theorem 1.3.

1.3. Convex-cocompact manifolds with totally geodesic boundary.

The notion of convex-cocompactness in projective spaces has been introduced by Danciger, Guéritaud, and Kassel [30]. It is inspired by (and generalizes) the corresponding definition of convex-cocompactness for Kleinian

groups (see for example [25, Ch 3]) and it is linked to the concept of Anosov subgroups of higher rank Lie groups (see for example [25, Ch. 7]).

Let $\Omega \subset \mathbb{RP}^d$ be open and properly convex. Let $\Gamma < \mathrm{Aut}(\Omega)$ be a discrete subgroup. Consider the *full orbital limit set* $\Lambda \subset \partial\Omega$ consisting of the accumulation points on $\partial\Gamma$ of every orbit $\Gamma \cdot o$ with $o \in \Omega$ (i.e. $\Lambda = \cup_{o \in \Omega}\overline{\Gamma o} \cap \partial\Omega$) and let $\mathcal{CH}(\Lambda)$ be the convex hull in Ω of Λ. We say that Γ acts *convex-cocompactly* on Ω if $\mathcal{CH}(\Lambda)/\Gamma$ (the *convex core*) is compact.

Compared to divisibility, convex-cocompactness is far more relaxed and flexible, and examples are abundant (see for example [30, 31]). Their study is an area of very active research (see for example [17, 19, 34, 35, 67]).

As mentioned, we will construct the closed convex projective manifolds of Theorem 1.1 by gluing compact convex projective manifolds with totally geodesic boundary given by the following.

Theorem 1.5. *For every $d \geq 3$ there exist properly convex open sets $\Omega_1, \Omega_2 \subset \mathbb{RP}^d$ and discrete groups $\Gamma_1, \Gamma_2 < \mathrm{PSL}_{d+1}(\mathbb{R})$, preserving respectively Ω_1, Ω_2, with the following properties. Let $\Lambda_j \subset \partial\Omega_j$ be the full orbital limit set with convex hull $\mathcal{CH}_j$ in Ω_j. Then*

(1) *Γ_j acts properly and cocompactly on $\overline{\mathcal{CH}_j} \smallsetminus \Lambda_j$.*

(2) *The connected components of $\partial\mathcal{CH}_j \smallsetminus \Lambda_j = \bigsqcup_{C \in \mathcal{C}_j} C$ form a collection $\mathcal{C}_j$ of cones (convex hulls of ellipsoids and points) with pairwise disjoint closures.*

(3) *The group Γ_j is relatively hyperbolic with respect to the family $\{\Gamma_C\}_{C \in \mathcal{C}_j}$ of stabilizers of cones $C \in \mathcal{C}_j$, with Bowditch boundary $\partial\mathcal{CH}_j/\sim$, where $p \sim q$ if $p, q \in \bar{C}$ for some $C \in \mathcal{C}_j$.*

(4) *Every point of $\Lambda_j \smallsetminus \bigsqcup_{C \in \mathcal{C}_j} \bar{C}$ is an extremal and $\mathcal{C}^1$ point of $\partial\Omega_j$.*

(5) *Γ_1 acts convex-cocompactly on Ω_1 and $\mathcal{C}_1$ is a family of properly embedded cones.*

(6) *Γ_2 does not act convex-cocompactly on any properly convex open set, and $\mathcal{C}_2$ is the family of codimension 1 faces of $\partial\Omega_2$ (and $\Omega_2 = \mathcal{CH}_2$).*

Some remarks:

- We have $\Gamma_C = \mathbb{Z} \times \Theta_C$, where Θ_C acts properly cocompactly on the base of the cone while the $\mathbb{Z}$ factor acts trivially on it.
- The quotients $M_j = (\overline{\mathcal{CH}_j} \smallsetminus \Lambda_j)/\Gamma_j$ are compact convex projective manifolds with totally geodesic boundary

$$\partial M_j = (\partial\mathcal{CH}_j \smallsetminus \Lambda_j)/\Gamma_j = \bigsqcup_{[C] \in \mathcal{C}_j/\Gamma_j} C/\Gamma_C.$$

- The double DM_1 is a closed convex projective manifold. Its universal cover $\Omega_1 \subset \mathbb{RP}^d$ is the properly divisible convex set of Theorem 1.1.
- As in Theorem 1.3, one can use work of Weisman [67] and Islam and Zimmer [35] to simplify our proof of (3) and (4) *in the convex-cocompact case ($j = 1$).*

- In the non-convex-cocompact case ($j = 2$): If $d = 3$ then $\text{int}(M_2) = \Omega_2/\Gamma_2$ is a 3-manifold with *generalized cusps* of *type 2* in the sense of [1]. If $d > 3$, then $\text{int}(M_2) = \Omega_2/\Gamma_2$ is not a manifold with generalized cusps (in the sense of [1]), although it is the interior of a compact d-manifold with boundary. The fundamental group of the boundary components have similar features to generalized cusp groups of type $d - 1$ but are not solvable.
- The inclusions $\Gamma_j < \text{PSL}_{d+1}(\mathbb{R})$ are *extended geometrically finite* on the (partial) flag variety of points and hyperplanes in the sense of Weisman [68, Def. 1.3]. For $j = 1$ this is due to Weisman [68, Th. 1.12], for $j = 2$ this is proved in Section 6.6.

1.4. **Geometrization in dimension 3.** Theorem 1.2 is a geometrization theorem that turns topological information into geometric structure. Before we state a refined version of it, let us describe a few structural results from 3-dimensional topology.

By Alexander's Theorem (see [47, Th 9.2.10]) every convex projective 3-manifold $M = \Omega/\Gamma$ is *irreducible* that is, it does not contain essential spheres. From the work of Jaco, Shalen, and Johannson (see [47, Th 11.5.1]), M can be split along an essentially canonical collection $S \subset M$ of tori and Klein bottles into pieces

$$M \smallsetminus S = M_1 \sqcup \cdots \sqcup M_r$$

that are either *atoroidal* (not containing essential tori and Klein bottles), or *Seifert fibered* (fibrations in circles over 2-dimensional orbifolds). The decomposition is called the *JSJ decomposition* of M.

Thanks to the major breakthough given by Thurston's Geometrization Conjecture [59], whose proof has been completed by Perelman, we know that every piece of the JSJ decomposition of M admits a homogeneous Riemannian metric locally modeled on one of 8 geometries (see [47, Ch. 12]), among which hyperbolic geometry occupies a prominent role. By Thurston's Hyperbolization for Haken manifolds (see [38, Th. 1.43]), each atoroidal piece N_j with $\partial N_j \neq \emptyset$ admits a complete finite volume hyperbolic metric.

Convex projective geometry fits well into this picture: Benoist proved [10, Prop. 3.2] that the JSJ decomposition of M only contains atoroidal pieces and it is realized geometrically, meaning that $S \subset M$ can be chosen to be *totally geodesic* (the lifts to Ω are the intersection of Ω with projective 2-planes). Therefore, each piece of $M \smallsetminus S$ is the interior of a convex projective manifold with totally geodesic boundary.

Question 1.6 (Ballas, Danciger, and Lee [2]). *Suppose a closed orientable irreducible 3-manifold has a JSJ decomposition containing only atoroidal pieces. Does it also admit a convex projective structure?*

In the positive direction, Ballas, Danciger, and Lee [2] provide an infinite family of examples. Our second contribution consists of a large flexible

class of controlled deformation families of convex projective structures on 3-manifolds with totally geodesic boundary.

Theorem 1.7. *Let M be a compact orientable irreducible atoroidal 3-manifold with non-empty boundary $\partial M = \Sigma_1 \sqcup \cdots \sqcup \Sigma_n$. Let $\alpha := \alpha_1 \cup \cdots \cup \alpha_n$ be a doubly incompressible multicurve where α_j is a separating simple closed curve of Σ_j. Consider the manifold*

$$N := DM \smallsetminus U_1 \cup \cdots \cup U_n$$

obtained by removing from the double DM of M an open tubular neighborhood $U_1 \cup \cdots \cup U_n$ of $\alpha_1 \cup \cdots \cup \alpha_n$.

Then there exists $\epsilon = \epsilon(M, \alpha) > 0$ such that for every (a_j, b_j, c_j) in

$$\mathcal{P} := \{(a, b, c) \in (1, 1 + \epsilon) \times (3, \infty) \times (1, \infty) \,\big|\, c \leq -2 + b^2/2\}$$

there exists a convex projective structure with totally geodesic boundary on N, such that the boundary ∂U_j is isomorphic to $\Delta/\mathbb{Z}A_j \oplus \mathbb{Z}B_j$ where $A_j = \operatorname{diag}(a_j, a_j^{-1}, 1, 1), B_j = \operatorname{diag}(b_j^{-1}, b_j^{-1}, b_j c_j, b_j c_j^{-1})$ and $\Delta = \Delta(e_1, e_2, e_3) \subset \{x_4 = 0\}$ is the standard simplex.

Double incompressibility was introduced by Thurston (see [60]) in the study of deformations of geometrically finite hyperbolic structures on $\operatorname{int}(M)$ with prescribed cusps.

Let us remark that:

- These convex projective structures on N are convex-cocompact in the sense of [30] (See Section 1.3).
- The convex projective structures on N vary continuously with the parameters $\{(a_j, b_j, c_j) \in \mathcal{P}\}_{j \leq n}$.
- Multicurves $\alpha \subset \partial M$ that satisfy the assumptions of Theorem 1.2 exist and are abundant in a precise sense (by work of Lecuire [42]).
- Given $(M, \alpha), (M', \alpha')$ as in Theorem 1.2 and a pairing $\{S_j \subset \partial N \leftrightarrow S_j' \subset \partial N'\}_{j \leq r}$ of boundary components, the control on the boundary holonomy gives us homeomorphisms $f_j : S_j \to S_j'$ such that the gluing $N \cup_{f_1 \sqcup \cdots \sqcup f_r} N'$, can be endowed with a convex projective structure with totally geodesic boundary (which is empty if the pairing involves all connected components of ∂N and $\partial N'$).
- In particular, the double DN of N is a closed orientable 3-manifold with a convex projective structure.
- The manifold N admits on its interior $\operatorname{int}(N)$ a complete finite volume hyperbolic metric which is itself a double of a complete finite volume hyperbolic metric with totally geodesic boundary on $M \smallsetminus \alpha$. In particular, it contains $\partial M \smallsetminus \alpha$ as a totally geodesic embedded subsurface.
- The class that we construct is transverse to the one of Ballas, Danciger, and Lee [2]: Since the hyperbolic manifold N contains a totally

geodesic subsurface $\partial M \smallsetminus \alpha$ it is not *infinitesimally projectively rigid relative to the boundary* as they need in their work.

- N is a projective manifold with generalized cusps of type 3 in the sense of [1]. Allowing $c_i = 1$, one obtains projective manifolds with generalized cusps of type 2 (the formulae for the holonomy would change though). One should also be able to allow $a_i = 1$ to obtain cusps of type 1 and 0 (the finite volume hyperbolic metric on N), but the above parametrization might not be well-suited for that. In fact one should retrieve constructions in [3, 20, 45].

1.5. **Hyperbolic manifolds with totally geodesic boundary and corners.** As mentioned at the end of Section 1.1, the building blocks for our constructions of the examples of all the previous theorems are convex compact hyperbolic manifolds with totally geodesic boundary and corners. If $d = 3$ then the manifolds we need are provided by work of Bonahon and Otal [21]. For the general case we construct arithmetic manifolds similar to the ones considered by Gromov and Thurston [32] and Kapovich [39]. More precisely we prove:

Theorem 1.8. *For every $k \geq 3$ there exist:*

(a) *A closed orientable hyperbolic d-manifold M with closed connected totally geodesic hypersurfaces $N_1, \cdots, N_k \subset M$ intersecting along a connected totally geodesic $(d-2)$-submanifold $C(= N_i \cap N_j$ for $i \neq j)$ with angles $\angle_C N_j N_{j+1} = \pi/k$. Moreover, C is fixed by a cyclic isometry ρ of M with $\rho(N_j) = N_{j+1}$.*

(b) *A compact convex hyperbolic d-manifold M' with totally geodesic boundary and corners such that each corner has angle π/k and the graph of the boundary is bipartite.*

The vertices of the graph of the boundary are the totally geodesic pieces of the boundary, and its edges are the corners. For our bulging construction to work, it is crucial that this graph is bipartite, so that we can "alternate between huge and small bulging parameters" (see Section 2). Corner angles less than $\pi/4$ would allow one to work with a tripartite assumption instead.

The existence of (b) follows from (a) as the completion of every connected component of $M \smallsetminus (N_1 \cup \cdots \cup N_k)$ satisfies all the requirements of (b).

Outline. The article is organized as follows.

- Section 2: *Ingredients of the proofs.* We discuss the ideas and ingredients of the proofs first in dimension 2 and then in general.
- Section 3: *Preliminaries.* We recall classical notions from convex projective geometry.
- Section 4: *Tubes, cone-manifolds, and totally geodesic blowup.* We define and classify tubes, define cone-manifolds, and describe the totally geodesic blowup of cone-manifolds whose singularities satisfy (2.1).

- Section 5: *Tessellations of convex domains.* We expose the local-to-global convexity argument that guarantees that a given collection of convex sets tiles a convex domain Ω, and then describe the regularity of $\partial \Omega$.
- Section 6: *Convex-cocompactness and relative hyperbolicity.* We work with compact convex projective manifolds M with totally geodesic boundary. We give sufficient conditions for M to be convex-cocompact, for $\pi_1(M)$ to be hyperbolic relatively to its boundary components, and for $\pi_1(M)$ to be extended geometrically finite in the sense of Weisman [68].
- Section 7: *Projective gluings.* We define projective gluings of projective manifolds with totally geodesic boundary and corners, and explain how to bulge those gluings with respect to polars. We check that projective gluings are cone-manifolds. Then we translate into the language of projective gluings the results obtained in Sections 5 and 6.
- Section 8: *Hyperbolic building blocks.* We construct the convex compact hyperbolic manifolds with totally geodesic boundary and corners which will be projectively glued and bulged to prove the main theorems. In particular, we prove Theorem 1.8.
- Section 9: *Hyperbolic doubles.* We apply the results of Section 7 to the particular case of the bulged double gluing of a single convex compact hyperbolic manifolds with totally geodesic boundary and corners, and then prove Theorems 1.1, 1.2, 1.3, 1.5 and 1.7.

2. Ingredients of the proofs

In this section we discuss the ideas and ingredients of the proofs of this paper. In Section 2.1 we investigate the special case where the dimension d equals 2, before turning to the general case in Section 2.2.

The proofs involve projective cone-manifolds obtained by gluing projective manifolds with totally geodesic and corners. We explain how the singularities of certain projective cone-manifolds can be blown up to a totally

geodesic boundary, and how to describe the geometry of certain gluings of projective manifolds with totally geodesic and corners.

Finally, in Section 2.3 we discuss the hyperbolic building blocks needed for our construction: In dimension 3, such objects are classified by work of Bonahon and Otal [21]. In higher dimension, we construct them using arithmetic techniques (Theorem 1.8).

2.1. **A guiding example in dimension** 2. As mentioned before, the compact convex projective manifolds with totally geodesic boundary in Theorems 1.5 and 1.7 are obtained by bulging enough a hyperbolic cone-manifold along totally geodesic hypersurfaces adjacent to the cone-singularities. This hyperbolic cone-manifold is in fact obtained by taking the double of a convex compact hyperbolic manifold N with totally geodesic boundaries $\Sigma, \Sigma' \subset \partial N$ and *codimension 2* corners $\mathfrak{C} \subset \partial N$, see Figure 2. ($\Sigma, \Sigma', \mathfrak{C}$ are not necessarily connected.)
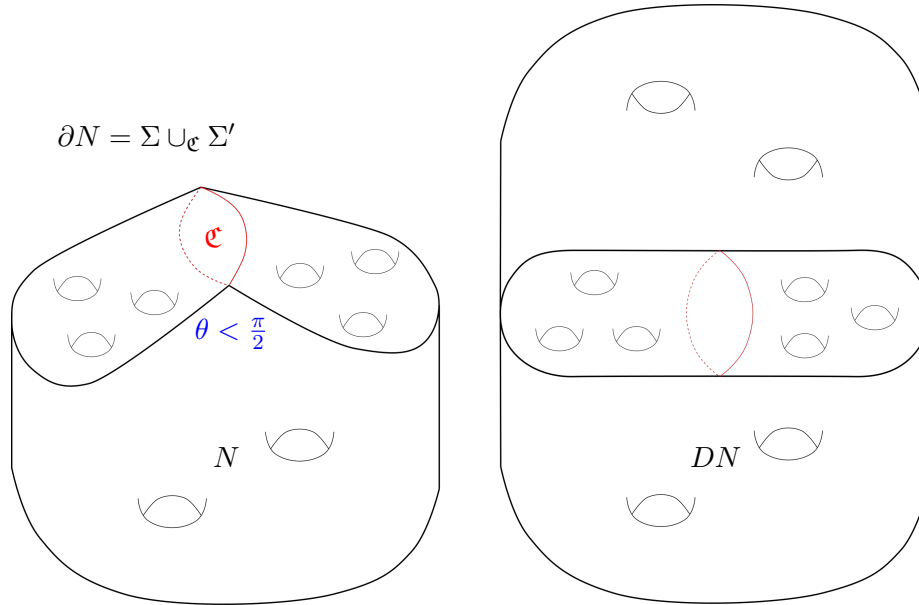

FIGURE 2. Double of a hyperbolic manifold with corners.

In this section we explain this bulging construction in the easier but already interesting case where $d = 2$. In this case one can pick for N a compact quadrilateral of the hyperbolic plane with vertex angles less than $\pi/2$ (in dimension at least 3 the construction of a suitable N is not as obvious).

We discuss two prototypical examples in order to explain two different problems of the proof of Theorem 1.5, namely *totally geodesic blowup* and *global convexity*.

2.1.1. *Local model.* The first example is a hyperbolic sector $\Sigma \subset \mathbb{H}^2$ of angle $0 < \theta < \pi/2$ bounded by two geodesic rays ℓ_1, ℓ_2 issuing from the vertex v.

The double $S = D\Sigma$ is a hyperbolic cone with vertex v and angle 2θ. The complement of the cone point $S' = S - \{v\}$ is a punctured plane with an incomplete hyperbolic metric.
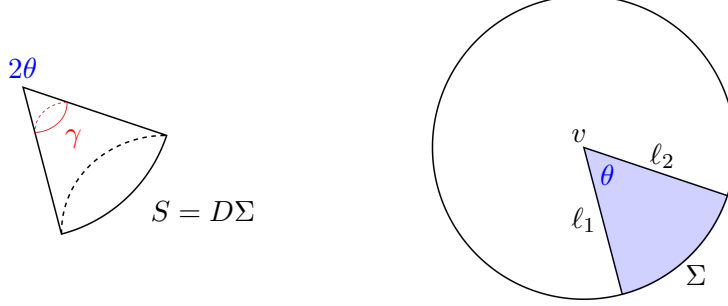

FIGURE 3. Hyperbolic cone: Double of a sector.

Let us see the universal cover $\widetilde{S'}$ as a fan of sectors $\{\Sigma_j\}_{j\in\mathbb{Z}}$ lifting the sectors of S', and consider the hyperbolic developing map $\mathrm{dev}_0 : \widetilde{S'} \to \mathbb{H}^2$ sending Σ_0 onto Σ. If γ is the simple curve that winds once clockwise around the vertex, then its holonomy is given by the rotation $R_{2\theta} \in \mathrm{SO}(2,1)$ of angle 2θ around the vertex v. Moreover, $\mathrm{dev}_0(\Sigma_j) = R_{j\theta}\Sigma$ for every j.

We deform projectively the hyperbolic structure on S' by performing independent bendings along each of the sides $\ell_1, \ell_2 \subset S'$. Recall that these bendings are bulgings, i.e. of the following form. Every line $\ell \subset \mathbb{H}^2$ has a dual point $\ell^* \in \mathbb{RP}^2 - \mathbb{H}^2$. For every $\mu > 0$ the "bulging" transformation $B_{\ell,\mu} \in \mathrm{SL}_3(\mathbb{R})$ is the homothety by $1/\mu$ on the 2-plane $L \subset \mathbb{R}^3$ representing ℓ and is the homothety by μ^2 on the line $L^\perp \subset \mathbb{R}^3$ representing ℓ^*.

We describe the (bulged) projective structure on S' via the developing map dev that sends Σ_0 onto Σ. We have $\mathrm{dev}(\Sigma_1) = B_{\ell_2,\mu_2} \mathrm{dev}_0(\Sigma_1)$ and $\mathrm{dev}(\Sigma_{-1}) = B_{\ell_1,\mu_1} \mathrm{dev}_0(\Sigma_{-1})$, the bulged holonomy of γ (which sends $\mathrm{dev}(\Sigma_{-1})$ on $\mathrm{dev}(\Sigma_1)$) changes as

$$R_{2\theta} \rightsquigarrow \rho := B_{\ell_2,\mu_2} R_{2\theta} B_{\ell_1,\mu_1}^{-1},$$

and $\mathrm{dev}(\Sigma_{2j}) = \rho^j \Sigma$ and $\mathrm{dev}(\Sigma_{2j+1}) = \rho^j B_{\ell_2,\mu_2} R_\theta \Sigma$ for every j.

The idea is that, under suitable geometric and dynamical assumptions on ρ the projective structure on S' is convex and one can "blow up" the singularity v of the projective cone-manifold S into a totally geodesic circle, a boundary for S'.

The heuristic picture is the following (see Figure 4). Note that v is an eigenvector of ρ with eigenvalue μ_1/μ_2. Assume that

$$(2.1) \qquad \rho \text{ is diagonalizable with eigenvalues } \mu_1/\mu_2 < \mu < \mu',$$
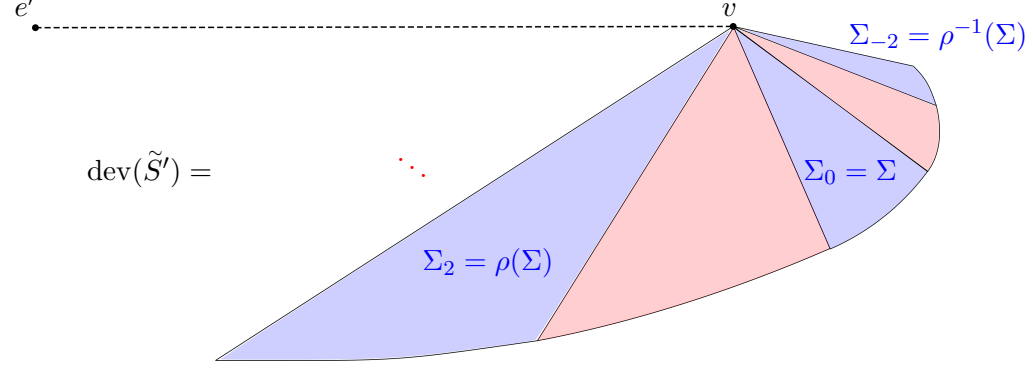
FIGURE 4. Image of the new developing map dev.

and denote by e, e' the eigenlines of μ, μ'. (This holds e.g. if $\theta < \pi/2$ and μ_2 and μ_1^{-1} are large.)

The image $\Omega = \mathrm{dev}(\tilde{S}')$ does not intersect the two lines through v that contain respectively e and e'. The sectors $\mathrm{dev}(\Sigma_j)$ accumulate onto the segment $[v, e']$ for $j \to \infty$ and to the vertex v for $j \to -\infty$. The properly convex subset $\Omega \cup (v, e')$ is a ρ-invariant convex set where ρ acts properly discontinuously. This gives the desired totally geodesic blowup

$$\bar{S} := S' \cup ((v, e')/\rho) = (\Omega \cup (v, e'))/\rho$$

of the deformed cone $S' = \Omega/\rho$.

The non-convex-cocompact case of Theorem 1.5 corresponds to $\mu = \mu'$ and ρ acting as a parabolic transformation on the dual line to v. Then convexity and totally geodesic blowup work the same way, except that $\ell = \ell'$ and v is a $\mathcal{C}^1$ point of $\partial\Omega$.

2.1.2. *Global convexity.* The second prototypical example is a hyperbolic convex quadrilateral $Q = Q(a, b, c, d) \subset \mathbb{H}^2$ with angles $0 < \alpha, \beta, \gamma, \delta < \pi/2$ at the vertices a, b, c, d.

The double $S = DQ$ is a sphere with a singular hyperbolic metric with four cone points of angle $2\alpha, 2\beta, 2\gamma, 2\delta$ corresponding to the vertices a, b, c, d of Q. The complement of the cone points $S' = S \smallsetminus \{a, b, c, d\}$ is an incomplete hyperbolic sphere with four punctures.

Let us bulge S' along each of the sides $\ell_{ab}, \ell_{bc}, \ell_{cd}, \ell_{da}$ of the quadrilateral with parameters $\mu_{ab}, \mu_{bc}, \mu_{cd}, \mu_{da}$. Suppose $\mu_{ab}, \mu_{bc}^{-1}, \mu_{cd}, \mu_{da}^{-1}$ are large, so that the holonomies $\rho_a, \rho_b, \rho_c, \rho_d$ of the curves around a, b, c, d satisfy (2.1).

By Section 2.1.1, S admits a projective totally geodesic blowup

$$\bar{S} = S' \cup (\mathbb{S}_a^1 \cup \mathbb{S}_b^1 \cup \mathbb{S}_c^1 \cup \mathbb{S}_d^1)$$

where each cone point $v \in \{a, b, c, d\}$ is replaced by a totally geodesic boundary circle $\mathbb{S}_v^1$. Let us now explain why $\bar{S}$ is convex.
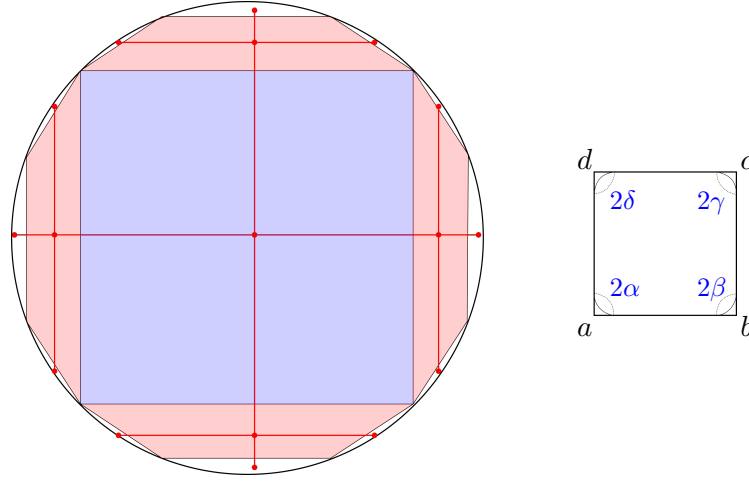
FIGURE 5. Universal covering of S'.

One way to describe combinatorially the universal cover $\widetilde{S}'$ of S' is to identify it with $\mathbb{H}^2$ via *another* hyperbolic metric on S' which is complete finite volume. The two copies of $Q \smallsetminus \{a, b, c, d\}$ lift to an ideal tessellation $\mathcal{F}$ of $\mathbb{H}^2$ whose dual graph is a regular 4-valent tree (see Figure 5). We now describe the image Ω of the (bulged) developing map

$$\mathrm{dev} : \widetilde{S}' \to \mathbb{RP}^2.$$

By the local model, for each vertex at infinity of the tessellation of $\widetilde{S}'$, the fan of quadrilaterals adjacent to it develops into an open convex subset of a quadrant of $\mathbb{RP}^2 - (\ell \cup \ell')$ where ℓ, ℓ' are suitable lines. A local-to-global argument inspired by work of Vinberg [65] and revisited by Benoist [12] (see Proposition 5.1) guarantees that the image $\Omega = \mathrm{dev}(\widetilde{S}')$ is a properly convex domain tiled by the images of the quadrilaterals in the tessellation.

2.2. **A generalization of the guiding example.** The ideas presented in Section 2.1 apply more generally to gluings of the hyperbolic manifolds with totally geodesic boundary and corners. We will in fact develop these ideas in a much more general framework, which includes:

- Totally geodesic blowups of projective cone-manifolds.
- A local-to-global convexity result for general unions of convex sets.

The reason why we work in such a general setting is threefold: First, as we use gluings twice with different building blocks, it is convenient to work in an abstract setup that encompasses both constructions. Second, we think that within that general framework some of the arguments are natural. Last, we hope that our techniques could be well-suited to produce many new interesting projective manifolds (for example, as the ones in Questions 1.4 and 1.6).
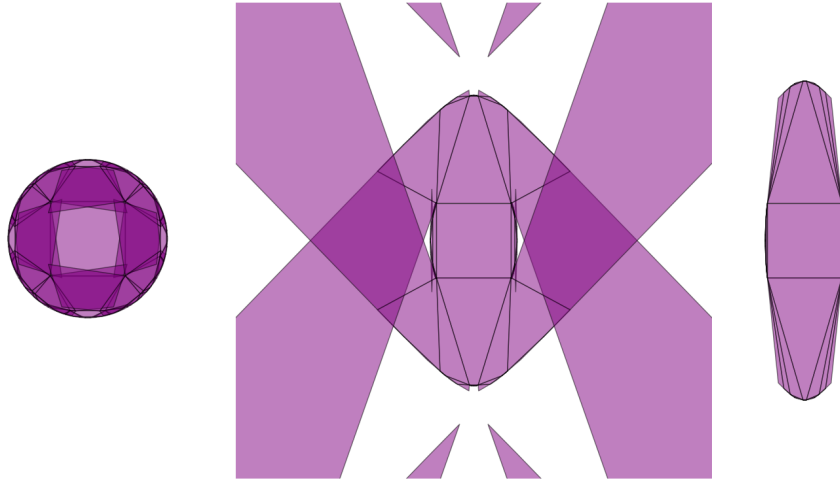
(A) No bulging.   (B) Moderate bulging.   (C) Large bulging.

FIGURE 6. Developing 52 tiles with different bulging parameters. The last picture is convex.

We briefly describe the main steps.

Tubes and cone-manifolds. We define a suitable class of *projective manifolds with cone-singularities* $(M, \mathcal{C})$, slightly more general than the previous notions in [5, 29, 43]. The most important feature is that every singularity $\mathfrak{C} \in \mathcal{C}$ has a neighborhood U locally modeled on a *tube*.

We work in $\mathbb{S}^d$ the sphere of $\mathbb{R}^d$ (a double cover of $\mathbb{RP}^d$). Consider a *sector* $\Sigma \subset \mathbb{S}^d$ bounded by two half-spheres H_1, H_2 containing $\mathbb{S}^{d-2}$. Glue H_1 and H_2 via $\alpha \in G$ the group of transformations of $\mathbb{S}^d$ fixing $\mathbb{S}^{d-2}$. The quotient $S = \Sigma / \alpha$ is a projective manifold with a singularity $\mathbb{S}^{d-2} \subset S$. We call it *tube* and think of it as a *projective cone* with *angle* $\alpha \in G$, analogous to a *hyperbolic cone* of angle *smaller* than 2π. As in the hyperbolic setting, we will allow angles larger than 2π by modifying the above picture, taking $\alpha \in \widetilde{G}$ in the universal cover of G.

Generalizing Section 2.1.1, the *totally geodesic blowup* that we introduce is a local construction for standard tubes whose angle satisfies (2.1) (and "have angle less than π"). It replaces each singularity $\mathfrak{C} \subset M$ with a totally geodesic boundary component of the (topological) form $\mathfrak{C} \times \mathbb{S}^1$.

Periodic tessellations of convex domains. As in Section 2.1.2, consider a collection of convex sets (tiles) that satisfies a local convexity property (ensured among other things by considerations like in Section 2.1.1).

A local-to-global convexity argument ensures that this collection tessellates a properly convex domain Ω. Then we describe under additional assumptions the boundary of Ω and its regularity, with roughly three kinds of points.

The codimension 1 faces of the tiles that lie inside Ω are called *walls*. First, for any tile D, the complement of closures of walls in ∂D is contained in $\partial \Omega$, and its points have the same regularity in ∂D and in $\partial \Omega$.

The remaining part of the boundary $\partial \Omega$ consists of *cells at infinity*, Hausdorff limit of sequences of tiles. By *periodicity* of the tiling (a finite number of tiles up to isometry), there will be essentially two cases to consider:

(1) A *telescope* of consecutive tiles $\{D_n\}_{n\in\mathbb{N}}$ with empty global intersection. The corresponding cell at infinity is reduced to an extremal point of $\partial \Omega$.

(2) A *fan* of tiles $\{D_n\}_{n\in\mathbb{Z}}$ sharing a codimension 2 face $H \subset \partial\Omega$, with $\gamma D_j = D_{j+k}$ for some $\gamma \in \mathrm{Aut}(\Omega)$ satisfying (2.1). The *two* corresponding cells at infinity are H and a codimension 1 face $\mathcal{CH}(H, \gamma_+) \subset \partial\Omega$ where $\gamma^+ \in \partial\Omega \smallsetminus H$ is the attracting fixed point of γ (as in Figure 4).

In addition to these ideas of totally geodesic blowup and periodic tessellations (which generalize Section 2.1) and to the constructions presented in Section 2.3, the proof of the main theorems will use classical arguments from the theory of convergence actions and geometrically finite actions to prove the statements about relative hyperbolicity, as in [34, 35, 67].

We now discuss how to produce hyperbolic manifolds with totally geodesic boundary and pleated along codimension 2 corners in arbitrary dimension.

2.3. Hyperbolic building blocks.
Convex compact hyperbolic manifolds with totally geodesic boundary and corners that look like the one in Figure 2 exist in every dimension.

2.3.1. *Dimension $d = 3$.* This is a special case, where we have the following complete topological classification thanks to work of Bonahon and Otal [21]. Simple closed curves $\alpha_1 \cup \cdots \cup \alpha_k \subset \partial M$ and angles $0 < \theta_1, \cdots, \theta_k < \pi$ appear as pleating (or bending) locus and pleating angles of the boundary of a hyperbolic metric on M with totally geodesic boundary and corners if and only if

- $\sum_{j\leq k} \theta_j i(\alpha_j, \partial A) > 0$ for every properly embedded essential annulus or Möbius band $(A, \partial A) \subset (M, \partial M)$.
- $\sum_{j\leq k} \theta_j i(\alpha_j, \partial D) > 2\pi$ for every properly embedded essential disk $(D, \partial D) \subset (M, \partial M)$.

Such collections α_j, θ_j are abundant (for example by [42]). Note that the θ_i's are *pleating angles* and not *corner angles*, which are the $\pi - \theta_i$'s.

Furthermore, we also have control on the lengths $\ell(\alpha_j)$ of the geodesic corners $\alpha_j \subset \partial M$ by work of Choi and Series [28].

Fact 2.1 (Bonahon–Otal [21, Th. 2-3], Choi–Series [28, Th. A]). *Let M be a compact orientable irreducible atoroidal 3-manifold with non-empty boundary $\partial M = \Sigma_1 \sqcup \cdots \sqcup \Sigma_n$. Let $\alpha := \alpha_1 \cup \cdots \cup \alpha_n$ be a doubly incompressible multicurve where α_j is a separating simple closed curve of Σ_j. Then for $\epsilon > 0$ small enough, to any lengths $\ell_1, \ldots, \ell_n < \epsilon$ can be associated a hyperbolic metric on M with totally geodesic boundary and corners, whose corners are the α_i's, with length ℓ_i and angle at most $\pi/4$.*

See Section 8.1 for more details.

2.3.2. *Dimension $d \geq 4$.* In the general case our hyperbolic building blocks are provided by Theorem 1.8, which is proved via arithmetic methods. Let us discuss the construction of the manifold M of the point (a) of this theorem in the case $k = 4$ (where M contains 4 totally geodesic closed hypersurfaces intersecting along a single codimension 2 closed submanifold).

The manifold M is $\mathbb{H}^d/G$ where G is a suitable finite index subgroup of

$$G_0 := \mathrm{SO}\left(x_1^2 + \cdots + x_d^2 - \sqrt{2}x_{d+1}^2\right) \cap \mathrm{SL}_{d+1}\left(\mathbb{Z}[\sqrt{2}]\right)$$

By classical facts, G_0 is discrete and $\mathbb{H}^d/G_0$ is compact. Consider the hyperplane $H = \{x_1 = 0\}$, the codimension 2 plane $V = \{x_1 = x_2 = 0\}$, and the $\pi/4$-rotation around V

$$\rho = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & \\ \sqrt{2}/2 & \sqrt{2}/2 & \\ & & \mathbb{I}_{d-1} \end{pmatrix} \in \mathrm{SL}_{d+1}\left(\mathbb{Q}(\sqrt{2})\right).$$

We exploit a ping-pong system adapted to the configuration of four hyperplanes $H_j = \rho^j H$ and their stabilizers in G_0 for $j \leq 4$ to find a subgroup $Q < G_0$ with the following properties:

(i) Q is convex-cocompact.
(ii) The hyperplanes H_j project to embedded compact hypersurfaces $N_j \subset \mathbb{H}^d/Q$ that pairwise intersect only along the projection of V which is a codimension 2 totally geodesic embedded submanifold.

Given (i) and (ii), strong subgroups separability properties of G (relatively to Q) due to Bergeron, Haglund, and Wise [15] allow us to embed $N_1 \cup N_2 \cup N_3 \cup N_4$ in a finite cover $M = \mathbb{H}^d/G \to \mathbb{H}^d/G_0$.

3. CONVEX PROJECTIVE GEOMETRY

In this section we review two notions that will be needed throughout the paper: We first recall the basic structure of convex sets in projective spaces. Then we discuss geometric structure, or (G, X)-structure, introducing (convex) projective manifolds with totally geodesic boundary and corners.

3.1. **Convex domains.** We start with some basic notions and terminology about convex subsets of $\mathbb{S}^d$, the sphere of rays in $\mathbb{R}^{d+1}$.

Definition 3.1 (Properly Convex). A *convex* subset $K \subset \mathbb{S}^d$ is the image of a convex cone of $\mathbb{R}^{d+1}$. The convex set K is *properly convex* if its closure does not contain two antipodal points, that is, the closure of K is contained in an affine chart $\mathbb{R}^d \subset \mathbb{S}^d$.

Note that intersections of (properly) convex subsets is again (properly) convex. Thus one can always define the smallest closed convex subset and the smallest linear subspace containing a given properly convex set.

Definition 3.2 (Convex Hull). If $A \subset \mathbb{S}^d$ is contained in some properly convex subset $K \subset \mathbb{S}^d$, then we can define the *convex hull* $\mathcal{CH}(A)$ of A in K as the intersection of all the closed convex subsets of K containing A.

Definition 3.3 (Span and Dimension). The *span* $\mathbb{S}(K) \subset \mathbb{S}^d$ of the convex set K is the smallest linear subspace containing K. We call $\dim(\mathbb{S}(K))$ the *dimension* of K. One-dimensional convex subsets of $\mathbb{S}^d$ are called *segments*.

Observe that any two non-antipodal points $x, y \in \mathbb{S}^d$ are contained in a unique minimal (properly convex) segment denoted by $[x, y]$.

3.1.1. *Topology.* The span $\mathbb{S}(K)$ of the convex set $K \subset \mathbb{S}^d$ has the property that K has non-empty interior in it.

Definition 3.4 (Interior and Boundary). We call $\mathrm{int}_{\mathbb{S}(K)}(K)$ and $\partial_{\mathbb{S}(K)}K$ the *relative interior* and *relative boundary* of K respectively.

Topologically, the pair $(\mathrm{int}_{\mathbb{S}(K)}(K) \cup \partial_{\mathbb{S}(K)}K, \partial_{\mathbb{S}(K)}K)$ is always homeomorphic to $(D, \partial D)$ where D is an Euclidean disk of dimension $\dim(K)$.

3.1.2. *Structure of the boundary.* The relative boundary of a properly convex set K has a stratified structure.

Definition 3.5 (Supporting Hyperplanes). A *supporting hyperplane* of a convex subset $K \subset \mathbb{S}^d$ with non-empty interior is a hyperplane which intersects the closure of K only in ∂K.

For every point $\xi \in \partial K$ there always exists a supporting hyperplane passing through ξ, however it is not necessarily unique.

Definition 3.6 ($\mathcal{C}^1$-point). A point $\xi \in \partial K$ admitting a unique supporting hyperplane, denoted by $T_\xi \partial K$, is called a $\mathcal{C}^1$-*point*.

The following criterion will be sometimes useful.

Fact 3.7. *Suppose K spans $\mathbb{S}^d$. Consider two subspaces $\mathbb{S}(V), \mathbb{S}(W) \subset \mathbb{S}^d$ whose union spans $\mathbb{S}^d$ such that $\mathbb{S}(V) \cap \mathbb{S}(W)$ intersects the interior of K, and $x \in \mathbb{S}(V) \cap \mathbb{S}(W) \cap \partial K$. Then x is $\mathcal{C}^1$ in ∂K if and only if it is $\mathcal{C}^1$ in both $\partial K \cap \mathbb{S}(V)$ and $\partial K \cap \mathbb{S}(W)$.*

Definition 3.8 (Faces). Let $K \subset \mathbb{S}^d$ be a convex set with non-empty interior and let $\xi \in \partial K$ be a boundary point. The convex subset

$$F_K(\xi) := \overline{K} \cap \bigcap_{H \text{ supporting at } \xi} H$$

is the *closed face* of ξ in $\overline{K}$ (its relative interior is the *open face* of ξ). Note that it is contained in ∂K. Every face has a *dimension* $\dim(F_K(\xi))$ and a *codimension* $\mathrm{codim}(F_K(\xi)) = \dim(\mathrm{K}) - \dim(F_K(\xi))$.

Definition 3.9 (Extremal and Strictly Convex). If $F_K(\xi)$ is reduced to a point, then ξ is said to be *extremal*. If every boundary point is extremal, or, equivalently, there are no non-trivial segments in ∂K, then K is said to be *strictly convex*.

3.1.3. *Symmetries.* The group $\mathrm{SL}_{d+1}(\mathbb{R})$ acts on the sphere of rays $\mathbb{S}^d$ by linear isomorphisms. Given a properly convex subset $K \subset \mathbb{S}^d$, we define the following natural group of symmetries:

Definition 3.10 (Automorphisms). Let $K \subset \mathbb{S}^d$ be an open properly convex subset. The group of *projective automorphisms* of K is

$$\mathrm{Aut}(K) := \{A \in \mathrm{SL}_{d+1}(\mathbb{R}) \,|\, A(K) = K\}.$$

We will measure the size of $\mathrm{Aut}(\Omega)$ in terms of its Zariski closure:

Definition 3.11 (Zariski Dense). The *Zariski topology* on $\mathrm{SL}_{d+1}(\mathbb{R})$ is the topology whose closed subsets are the intersections of zero loci of polynomials in the matrix entries.

In this topology a subset $\Gamma < \mathrm{SL}_{d+1}(\mathbb{R})$ is *Zariski dense* if every polynomial in the matrix entries that vanishes on the elements of Γ vanishes on all matrices in $\mathrm{SL}_{d+1}(\mathbb{R})$.

Notice that the Zariski closure of a subgroup $\Gamma < \mathrm{SL}_{d+1}(\mathbb{R})$ is always a Lie subgroup. Similar to the concept of lattices in Lie groups, we have the following:

Definition 3.12 (Divisible). An open properly convex subset $K \subset \mathbb{S}^d$ is *divisible* if $\mathrm{Aut}(K)$ contains a subgroup Γ acting properly discontinuously, freely, and cocompactly on K.

Our goal is to construct examples of domains K with special geometric properties where K is divisible and Γ, the group dividing it, is as large as possible in the sense that it is Zariski dense.

3.1.4. *Hilbert metric.* Every properly convex set $K \subset \mathbb{S}^d$ comes equipped with a natural $\mathrm{Aut}(K)$-invariant metric, called the Hilbert metric, whose geometric features are strictly tied to the regularity of the boundary ∂K.

Definition 3.13 (Hilbert Metric). The *Hilbert metric* on K is defined as follows: Consider $x, y \in K$. Let a, b be the intersections of the projective line spanned by x, y with ∂K where a lies on the side of x and b lies on the side of y. One sets

$$d_K(x, y) := \frac{1}{2} \log([a, x, y, b]),$$

where $[a, x, y, b]$ denotes the *cross-ratio* on the projective line spanned by x, y normalized so that $[0, 1, t, \infty] = t$.

It is a classical fact that $d_K(\bullet, \bullet)$ defines a metric inducing the standard topology and such that metric balls are compact. As a consequence, as the group of projective automorphisms preserves the Hilbert metric, the action $\mathrm{Aut}(K) \curvearrowright K$ is proper.

There are a couple of useful general properties of the Hilbert metric that we will exploit several times. First, notice that $d_K(\bullet, \bullet)$ is monotone under inclusion, that is, if $K \subset K'$, then $d_K(x, y) \geq d_{K'}(x, y)$ for all $x, y \in K \subset K'$. Secondly, d_K can always be extended to a lower semi-continuous function $d_{\bar{K}}(\bullet, \bullet)$ on $\bar{K} \times \bar{K}$ by setting

- $d_{\bar{K}}(x, y) = \infty$ if x and y are in different faces.
- $d_{\bar{K}}(x, y) = d_F(x, y)$ if x and y share the same open face F.

3.2. Geometric structures.

We now introduce the second crucial notion of the paper, namely the one of geometric structure and, in particular, projective structures with totally geodesic boundary and corners.

Let G be a Lie group. Let $G \curvearrowright X$ be a *transitive* and *effective* action on a manifold X. Effective means that if $g, g' \in G$ agree on some open set of X, then they are equal.

Definition 3.14 ((G, X)-structure). A (G, X)-*structure* on a topological space M is an atlas of charts

$$\mathcal{A} = \{\phi_j : U_j \subset M \to V_j \subset X\}_{j \in J}$$

such that every change of charts $\phi_i \phi_j^{-1}$ is a restrictions of a transformation $g_{ij} \in G$.

Via a standard process of analytic continuation of the local charts, every (G, X)-structure on M has an associated *developing map*

$$\mathrm{dev} : \widetilde{M} \to X,$$

where $\widetilde{M}$ is the universal cover of M, and a *holonomy representation*

$$\rho : \pi_1(M) \to G.$$

The developing map is a local homeomorphism which is ρ-equivariant with respect to the deck group action $\pi_1(M) \curvearrowright \hat{M}$.

Definition 3.15 (Uniformisability and Completeness). A (G, X)-manifold M is *uniformisable* in X if the developing map dev : $\tilde{M} \to X$ is an embedding. In this case, we identify M with the quotient $\mathrm{dev}(\tilde{M})/\rho(\pi_1(M))$, where $\rho : \pi_1(M) \to G$ is the holonomy representation.

If M is uniformisable and $\mathrm{dev}(\tilde{M}) = X$ we say that M is *complete*.

We refer to [58, Ch. 3] for more on the subject.

3.2.1. *Projective structures.* Projective structures correspond to the pair $(\mathrm{SL}_{d+1}(\mathbb{R}), \mathbb{S}^d)$. We will consider also projective structures locally modeled on pieces of $\mathbb{S}^d$ like hemispheres and half-hemispheres. They are defined as follows.

Definition 3.16 (Projective structures). A *projective structure* with *totally geodesic boundary* and *corners* on M is a maximal atlas of charts into $\mathbb{S}^d \cap \{x_1, x_2 \geq 0\}$ with change of charts induced by restrictions of elements of $\mathrm{SL}_{d+1}(\mathbb{R})$.

The atlas of charts induces a compatible structure of d-manifold with boundary and corners on M where

- The interior $\mathrm{int}(M)$ consists of those point mapped to $\mathbb{S}^d \cap \{x_1, x_2 > 0\}$ by some (hence every) local chart.
- The boundary ∂M is the set of points mapped to $\mathbb{S}^d \cap \{x_1 = 0\} \cup \mathbb{S}^d \cap \{x_2 = 0\}$ by some (hence every) local chart.
- The union of corners $C \subset \partial M$ is given by the set of points mapped to $\mathbb{S}^d \cap \{x_1 = x_2 = 0\}$ by some (hence every) local chart.
- A *wall* is a connected component of $\partial M \smallsetminus C$ and a *corner* is a connected component of C.

In order to ease the terminology, throughout the paper we will write *b-manifold* (resp. *bc-manifold*) instead of projective manifold with totally geodesic boundary (resp. projective manifold with totally geodesic boundary and corners).

Let dev : $\tilde{M} \to \mathbb{S}^d$ be the developing map of a bc-manifold M.

Definition 3.17 (Convex Projective Manifolds). A bc-manifold is said to be *convex* (resp. *properly convex*) if the developing map is an embedding (uniformisable in $\mathbb{S}^d$) and its image is convex (resp. properly convex) in $\mathbb{S}^d$.

In Section 7, we will explain a procedure to assemble several convex bc-manifolds in a single singular object, a projective manifold with cone-singularities. We formally define and analyze such objects in the next section (Section 4) where we also explain why they are very useful for our purposes: Under suitable (local) geometric conditions, one can remove from a projective cone-manifold the singularities and replace them with totally geodesic boundaries.

3.2.2. *Hyperbolic structures.* A particularly flexible and rich class of projective structures is provided by hyperbolic manifolds. We will use them in the proofs of our main theorems as building blocks for our constructions.

Definition 3.18 (Hyperbolic Space)**.** Let $\langle \bullet, \bullet \rangle$ denote the quadratic form

$$\langle \bullet, \bullet \rangle := x_1^2 + \ldots + x_d^2 - x_{d+1}^2$$

on $\mathbb{R}^{d+1}$. The hyperbolic d-space $\mathbb{H}^d$ can be described as

$$\mathbb{H}^d = \{ x \in \mathbb{S}^d \,|\, \langle x, x \rangle < 0, x_{d+1} > 0 \}.$$

Note that $\mathbb{H}^d$ is a properly convex subset of $\mathbb{S}^d$.

Its symmetry group is $\mathrm{SO}_0(d, 1) < \mathrm{SL}_{d+1}(\mathbb{R})$, the (identity component of the) group of linear isometries of the quadratic form. Its Hilbert metric comes from a complete $\mathrm{SO}_0(d, 1)$-invariant Riemannian metric defined at a point $x \in \mathbb{H}^d$ by the (positive definite) restriction of the quadratic form $\langle \bullet, \bullet \rangle$ to the tangent space $T_x \mathbb{H}^d = x^\perp$.

Totally geodesic subspaces of $\mathbb{H}^d$ are exactly those of the form $\mathbb{S}(V) \cap \mathbb{H}^d$ where $V \subset \mathbb{R}^d$ is a linear subspace. Thus, in $\mathbb{H}^d$ the notion of being geodesically convex with respect to the Riemannian metric coincides with the notion of being convex in $\mathbb{S}^d$.

Definition 3.19 (Convex Hyperbolic Manifolds)**.** A *convex hyperbolic manifold* is a quotient $M = \mathcal{C}/\Gamma$ of a convex subset $\mathcal{C} \subset \mathbb{H}^d$ by a discrete torsion free subgroup $\Gamma < \mathrm{SO}_0(d, 1)$ preserving $\mathcal{C}$.

We will also exploit the following useful criterion:

Fact 3.20. *Let $\Gamma < \mathrm{SO}(d, 1)$ be a finitely generated discrete subgroup. The following are equivalent:*

- *The orbit Γo of a point $o \in \mathbb{H}^d$ is a* quasi-convex *subset of $\mathbb{H}^d$.*
- *There exists a Γ-invariant convex subset $\mathcal{C} \subset \mathbb{H}^d$ such that $\mathcal{C}/\Gamma$ is compact.*

We recall that a subset $S \subset \mathbb{H}^d$ is quasi-convex if there exists a constant $R > 0$ such that for every $x, y \in S$ the geodesic segment $[x, y] \subset \mathbb{H}^d$ is contained in the R-neighborhood of S.

Definition 3.21 (Totally Geodesic Boundary and Corners)**.** A convex hyperbolic manifold $M = \mathcal{C}/\Gamma$ has *totally geodesic boundary and corners* if the boundary ∂M decomposes as a union of totally geodesic codimension 1 submanifolds intersecting along totally geodesic codimension 2 submanifolds.

Note that a convex hyperbolic manifold with totally geodesic boundary and corners is also a convex projective manifold with boundary and corners according to Definition 3.16.

4. Tubes, cone-manifolds, and totally geodesic blowup

The goal of this section is to generalize the notion of cone-manifolds (with singularities of codimension 2 only) from the hyperbolic setting to the projective setting. Projective cone-manifolds have been considered in [5,29,43,53]. Note that Riolo-Seppi [53, Def. 5.3] allow for singularities of any codimension, although they have other kinds of restrictions.

Our presentation is most similar to that of [43, Def. 2.1], but we will be slightly more general. Indeed, while in [43] projective singularities are (essentially) determined by a projective structures on a circle (equivalently, a conjugacy class in the universal cover of $\mathrm{SL}_2\mathbb{R}$), in our case they can be (roughly) parametrized by conjugacy classes in the universal cover of the fixator of $\mathbb{S}^{d-2}$ in $\mathbb{S}^d$.

Depending on the type of this conjugacy class, we show that the corresponding singularity admits a totally geodesic blowup. This is the main result of this section.

4.1. **Tubes and cone-manifolds.**

We start by studying the local model of a singularity of a projective cone-manifold, which is what we call a tube.

We identify $\mathbb{R}^{d-1}$ with a subspace of $\mathbb{R}^{d+1}$, $\mathbb{S}^{d-2}$ with the corresponding subspace of $\mathbb{S}^d$, and $\mathbb{S}^1$ with $\mathbb{S}(\mathbb{R}^{d+1}/\mathbb{R}^{d-1})$ the set of half-hyperspheres that contain $\mathbb{S}^{d-2}$. Note that we have a natural map

$$\mathbb{S}^d \smallsetminus \mathbb{S}^{d-2} \to \mathbb{S}^1 = \mathbb{S}(\mathbb{R}^{d+1}/\mathbb{R}^{d-1})$$

sending a point $x \in \mathbb{S}^d \smallsetminus \mathbb{S}^{d-2}$ to the unique half-hypersphere passing through x and $\mathbb{S}^{d-2}$.

Definition 4.1 (Universal Branched Cover). Consider

$$\mathbf{B} = \{(H, x) \in \widetilde{\mathbb{S}}^1 \times (\mathbb{S}^d \smallsetminus \mathbb{S}^{d-2}) : \ x \in \pi(H)\},$$

where $\pi : \widetilde{\mathbb{S}}^1 \to \mathbb{S}^1$ is the universal cover of $\mathbb{S}^1$. The *universal branched cover of $\mathbb{S}^d$ along $\mathbb{S}^{d-2}$* is the space (endowed with the quotient topology)

$$\mathbf{B} \sqcup \mathbb{S}^{d-2} = \{(H, x) \in \widetilde{\mathbb{S}}^1 \times \mathbb{S}^d : \ x \in \pi(H)\}/_\sim,$$

where $(H, x) \sim (H', x)$ for all $H, H' \in \widetilde{\mathbb{S}}^1$ and $x \in \mathbb{S}^{d-2}$.

Note that the natural projection $\mathbf{B} \to \mathbb{S}^d \smallsetminus \mathbb{S}^{d-2}$ is a (universal) covering.

Definition 4.2 (Structure Group). There is natural group $\mathrm{Aut}(\mathbf{B})$ acting on $\mathbf{B}$ and $\mathbf{B} \sqcup \mathbb{S}^{d-2}$, consisting of transformations of the form

$$g = \begin{pmatrix} \mu^{-1}A & C \\ 0 & \mu^{(d-1)/2}\widetilde{B} \end{pmatrix}$$

where:

- $\mu > 0$.
- $\widetilde{B} \in \widetilde{\mathrm{SL}}_2^\pm(\mathbb{R})$.

- $A \in \mathrm{SL}_{d-1}^{\pm}(\mathbb{R})$.
- $C \in \mathrm{Mat}_{2,d-1}(\mathbb{R})$.

We use the convention that $C' \cdot \tilde{B}' := C' \cdot B'$ for all $\tilde{B}' \in \tilde{\mathrm{SL}}_2^{\pm}(\mathbb{R})$ and $C' \in \mathrm{Mat}_{2,d-1}(\mathbb{R})$ where $B' \in \mathrm{SL}_2^{\pm}(\mathbb{R})$ is the projection of $\tilde{B}'$.

Observe that the projection $\mathbf{B} \to \tilde{\mathbb{S}}^1$ is equivariant with respect to the actions of $\mathrm{Aut}(\mathbf{B})$ and $\tilde{\mathrm{SL}}_2(\mathbb{R})$ and the projection $\mathrm{Aut}(\mathbf{B}) \to \tilde{\mathrm{SL}}_2(\mathbb{R})$.

Using this projection we define sectors and walls:

Definition 4.3 (Sector and Wall). A *sector* of $\mathbf{B}$ is the preimage by $\mathbf{B} \to \tilde{\mathbb{S}}^1$ of an interval. A sector of $\mathbf{B} \sqcup \mathbb{S}^{d-2}$ is the union of $\mathbb{S}^{d-2}$ with a sector of $\mathbf{B}$. The *walls* of a sector are the preimages of the endpoints of the corresponding interval of $\tilde{\mathbb{S}}^1$.

We are now ready to define tubes:

Definition 4.4 (Tubes). A *tube* is a space obtained by identifying the walls of a compact sector of $\mathbf{B} \sqcup \mathbb{S}^{d-2}$ via an element of $\mathrm{Aut}(\mathbf{B})$ of the form

$$g = \begin{pmatrix} \mu^{-1}\mathbb{I}_{d-1} & C \\ 0 & \mu^{\frac{d-1}{2}}\tilde{B} \end{pmatrix}.$$

The *singular locus* of the tube is the image of $\mathbb{S}^{d-2}$ in the quotient, and the *smooth locus* is the complement of the singular locus.

A *meridian* is a generator of the fundamental group of the smooth locus of the tube. Note that its holonomy is g or g^{-1}.

The $\tilde{\mathrm{SL}}_2$-*angle* of the tube is the $\tilde{\mathrm{GL}}_2$-conjugacy class of $\tilde{B}$.

Notice that the smooth locus of a tube T naturally fibers over a projective circle

$$T \to C,$$

whose holonomy is given by the $\tilde{\mathrm{SL}}_2(\mathbb{R})$-angle of the tube. The fibers of this bundle are the hyperplanes passing through the singular locus.

We have the following natural notion of equivalence between tubes:

Definition 4.5 (Isomorphisms of Tubes). Let $T \cup \mathbb{S}^{d-2}$ and $T' \cup \mathbb{S}^{d-2}$ be tubes with corresponding fibrations of the smooth loci $T \to N$ and $T' \to N'$. An *isomorphism* between the tubes is a homeomorphism $T \cup \mathbb{S}^{d-2} \to T' \cup \mathbb{S}^{d-2}$ that induces a diagram

$$\begin{array}{ccc} T & \longrightarrow & T' \\ \downarrow & & \downarrow \\ N & \longrightarrow & N' \end{array}$$

where the horizontal arrows are projective isomorphisms.

An *automorphism* of a tube $T \cup \mathbb{S}^{d-2}$ is an isomorphism $T \cup \mathbb{S}^{d-2} \to T \cup \mathbb{S}^{d-2}$. We denote the group of automorphisms by $\mathrm{Aut}(T)$.

We are now ready to give the definition of projective cone-manifolds. We will see examples in Section 7.1.2, where we show that any gluing of projective manifold with totally geodesic boundary and corners is a projective cone-manifold.

Definition 4.6 (Projective Cone Manifold)**.** A *cone-manifold* consists of the following data:

- A topological manifold M.
- A codimension 2 submanifold $\mathfrak{C} \subset M$ (the *singular locus*).
- A choice of a tube T_C for each connected component $C \subset \mathfrak{C}$.
- An atlas of charts $\{\phi_x\}_{x \in M}$ such that if x is in the smooth locus $M \smallsetminus \mathfrak{C}$ then $\phi_x : U_x \subset M \smallsetminus \mathfrak{C} \hookrightarrow \mathbb{S}^d$, and if $x \in C \subset \mathfrak{C}$ then $\phi_x : U_x \hookrightarrow T_C$ such that $U_x \subset M \smallsetminus (\mathfrak{C} \smallsetminus C)$ and $\phi_x^{-1}(\mathbb{S}^{d-2}) = U_x \cap C$.
- If $U_x \cap U_y \neq \emptyset$, then either $x, y \in C \subset \mathfrak{C}$ for some C and the transition map $\phi_y \circ \phi_x^{-1}$ is the restriction of an automorphism of T_C, or $U_x \cap U_y \subset M \smallsetminus \mathfrak{C}$ and $\phi_y \circ \phi_x^{-1}$ is projective.

Note that the singular locus of a cone-manifold has a natural projective structure.

We now classify tubes in terms of their holonomy.

4.2. **Uniformisability and completeness.** Note that in general the projective structure of the smooth locus of a tube is not uniformisable in $\mathbb{S}^d$. However it is uniformisable in the universal cover of $\mathbb{S}^d \smallsetminus \mathbb{S}^{d-2}$. This is a consequence of the fact that closed projective manifolds of dimension 1 are uniformisable in the universal cover of $\mathbb{S}^1$, as explained in [5, §3.3.1], where all projective structures on $\mathbb{S}^1$ are described.

Proposition 4.7. *Let* $T \sqcup \mathbb{S}^{d-2}$ *be a tube, such that the smooth locus* T *fibers over the projective circle* N. *Then:*

(1) T and N are uniformisable in $\mathbf{B}$ and $\widetilde{\mathbb{S}}^1$ respectively. Choose developing maps with images $\widetilde{T} \subset \mathbf{B}$ and $\widetilde{N} \subset \widetilde{\mathbb{S}}^1$ and holonomy generated by $g \in \mathrm{Aut}(\mathbf{B})$ fixing $\mathbb{S}^{d-2}$. Then $T \sqcup \mathbb{S}^{d-2}$ is the quotient by $g\mathbb{Z}$ of the sector $\widetilde{T} \sqcup \mathbb{S}^{d-2}$ of $\mathbf{B} \sqcup \mathbb{S}^{d-2}$.

(2) We have a one-to-one correspondence

$$\{\text{Tubes}\}/\text{isom.} \quad \simeq \quad \left\{ \begin{array}{c} \mathbb{Z} < \mathrm{Fix}_{\mathrm{Aut}(\mathbf{B})}(\mathbb{S}^{d-2}) \\ \text{with non-trivial angle} \end{array} \right\} / \mathrm{Aut}(\mathbf{B})\text{-conj.}$$

$$[T] \quad \mapsto \quad [g\mathbb{Z}].$$

Moreover, there are two disjoint cases.

- *T is complete in $\mathbf{B}$: $\widetilde{N} = \widetilde{\mathbb{S}}^1$ and $\widetilde{T} = \mathbf{B}$.*
- *T is uniformisable in $\mathbb{S}^d \smallsetminus \mathbb{S}^{d-2}$: $\widetilde{N} \subset \widetilde{\mathbb{S}}^1$ projects injectively onto an interval $I \subset \mathbb{S}^1$ between two fixed points of the non-elliptic SL_2-angle*

$h \in \mathrm{SL}_2(\mathbb{R})$ *(that has positive trace). If h is hyperbolic then I is properly convex. If h is parabolic then it is a half-circle.*

Remark 4.8 (Barbot–Bonsante–Schlenker [5, §3.3.1]). In the setting of Proposition 4.7, let $\widetilde{h} \in \widetilde{\mathrm{SL}}_2(\mathbb{R})$ be the $\widetilde{\mathrm{SL}}_2$-angle of g with projection $h \in \mathrm{SL}_2(\mathbb{R})$. Then the following is classical.

- If h is elliptic, i.e. $-2 < \mathrm{tr}(h) < 2$ or $h = \pm\mathbb{I}_2$, then $\widetilde{h}$ acts freely, properly, and cocompactly on $\widetilde{S}^1$, hence T is complete in $\mathbf{B}$.
- If $\mathrm{tr}(h) \leq -2$ with $h \neq -\mathbb{I}_2$, then $\widetilde{h}$ acts freely, properly, and cocompactly on $\widetilde{S}^1$, hence T is complete in $\mathbf{B}$.
- If $\mathrm{tr}(h) \geq 2$ with $h \neq \mathbb{I}_2$, then it has exactly one lift $\widetilde{h}_0 \in \widetilde{\mathrm{SL}}_2(\mathbb{R})$ with fixed points in $\widetilde{\mathbb{S}}^1$.
  - If $\widetilde{h} = \widetilde{h}_0$ then $\widetilde{h}$ acts freely, properly, and cocompactly on any interval of $\widetilde{\mathbb{S}}^1$ between two fixed points, which projects injectively onto an interval of $\mathbb{S}^1$, so T is uniformisable in $\mathbb{S}^d \smallsetminus \mathbb{S}^{d-2}$.
  - Otherwise $\widetilde{h}$ acts freely, properly, and cocompactly on $\widetilde{\mathbb{S}}^1$, hence T is complete in $\mathbf{B}$.

Proof of Proposition 4.7. **Property (1)**. The developing maps of $\widetilde{T}$ and $\widetilde{N}$ fit into the following diagram, which is a morphism of bundles.

$$
\begin{array}{ccc}
\widetilde{T} & \longrightarrow & \mathbf{B} \\
\downarrow & & \downarrow \\
\widetilde{N} & \longrightarrow & \widetilde{\mathbb{S}}^1
\end{array}
$$

In other words, the map $\widetilde{T} \to \mathbf{B}$ is an embedding on each fiber (which is a half-hypersphere of $\mathbb{S}^d$). Thus, uniformisability or completeness of $\widetilde{T}$ in $\mathbf{B}$ follows from that of $\widetilde{N}$ in $\widetilde{\mathbb{S}}^1$. For the same reasons, uniformisability or completeness of $\widetilde{T}$ in $\mathbb{S}^d \smallsetminus \mathbb{S}^{d-2}$ follows from that of $\widetilde{N}$ in $\mathbb{S}^1$. The latter is always uniformisable in $\widetilde{\mathbb{S}}^1$ since any local homeomorphism from $\mathbb{R}$ to $\mathbb{R}$ is injective.

The dichotomy at the end of Proposition 4.7 is an immediate consequence of Remark 4.8 and the fact that uniformisability or completeness of $\widetilde{T}$ follows from that of $\widetilde{N}$.

Property (2). Finally, let us check the one-to-one correspondence.

Claim 1. The map Ψ that associates to the isomorphism class of T the conjugacy class of $g\mathbb{Z}$ is well-defined.

Proof of the claim. By definition, if two tubes $T \cup \mathbb{S}^{d-2}, T' \cup \mathbb{S}^{d-2}$ are equivalent, then the isomorphism between them determines projective isomorphisms $\widetilde{T} \subset \mathbf{B} \to \widetilde{T}' \subset \mathbf{B}$ and $\widetilde{N} \subset \widetilde{\mathbb{S}}^1 \to \widetilde{N}' \subset \widetilde{\mathbb{S}}^1$ that fit into the

commutative diagram

$$\begin{array}{ccc} \widetilde{T} & \longrightarrow & \widetilde{T}' \\ \downarrow & & \downarrow \\ \widetilde{N} & \longrightarrow & \widetilde{N}'. \end{array}$$

Such isomorphism extends to an automorphism of the bundle $\mathbf{B} \to \widetilde{\mathbb{S}}^1$: Let us work locally. On a sufficiently small sector, the projective isomorphism $\phi : \widetilde{T} \to \widetilde{T}'$ descends to a projective isomorphism between sectors of $\mathbb{S}^d$ centered at $\mathbb{S}^{d-2}$. On $\mathbb{S}^d$ any projective isomorphism between open subsets is the restriction of a global projective transformation. In our case, the unique extension of the projective isomorphism between sectors has to fix $\mathbb{S}^{d-2}$. Hence it lifts to an automorphism of $\mathbf{B}$ that agrees with ϕ on the small sector. By analytic continuation, they agree everywhere. This implies that the holonomies of T, T' are conjugate in $\mathrm{Aut}(\mathbf{B})$. $\qquad\square$

Claim 2. The map Ψ is bijective.

Proof of the claim. We construct an inverse. Let $g\mathbb{Z} < \mathrm{Fix}_{\mathrm{Aut}(\mathbf{B})}(\mathbb{S}^{d-2})$ be an infinite cyclic subgroup generated by an element g with non-trivial angle $\widetilde{h} \in \widetilde{\mathrm{SL}}_2(\mathbb{R})$.

Consider the interval $I = [x, \widetilde{h}x] \subset \widetilde{\mathbb{S}}^1$ where x is not a fixed point of $\widetilde{h}$ (see Remark 4.8). Let $S \subset \mathbf{B}$ be the corresponding sector. Then the tube $T = S/g$ is mapped to $g\mathbb{Z}$ by Ψ. It is not hard to check that the isomorphism class of T does not depend on the choice of the representative of $\mathbb{Z}$ in the conjugacy class, the generator $g \in \mathbb{Z}$, and the point x. $\qquad\square$

This concludes the proof of the proposition. $\qquad\square$

We will need later on (proof of Theorem 9.1, Section 9.2.2) the following technical characterization of tubes that are uniformisable in $\mathbb{S}^d \smallsetminus \mathbb{S}^{d-2}$.

Lemma 4.9. *In the setting of Proposition 4.7 and Remark 4.8, T is uniformizable in $\mathbb{S}^d \smallsetminus \mathbb{S}^{d-2}$ if and only if $h \neq \mathbb{I}_2$, $\mathrm{tr}(h) \geq 2$, and $[x, \widetilde{h}x] \subset \widetilde{N}$ projects onto a properly convex segment of $\mathbb{S}^1$ for some $x \in \widetilde{N}$.*

Proof. Remark 4.8 implies that if T is uniformisable in $\mathbb{S}^d \smallsetminus \mathbb{S}^{d-2}$ then $\mathrm{tr}(h) \geq 2$ and $h \neq \mathbb{I}_2$ and any segment of $\widetilde{N}$ projects onto a properly convex segment of $\mathbb{S}^1$.

Conversely, suppose $\mathrm{tr}(h) \geq 2$ and $h \neq \mathbb{I}_2$. Identify $\widetilde{\mathbb{S}}^1$ with $\mathbb{R}$ so that the preimage of a rotation of $\mathrm{SL}_2(\mathbb{R})$ of angle θ acts on $\mathbb{R}$ as a translation of the form $\tau^{\theta+2n\pi}(x) = x + \theta + 2n\pi$ for some integer n. Then $\widetilde{h} = \tau^{2n\pi}\widetilde{h}_0$ with n integer and $\widetilde{h}_0$ from Remark 4.8.

Up to conjugation, we may assume $\widetilde{h}_0$ fixes every multiple of π in $\mathbb{R}$ (if h is hyperbolic then $\widetilde{h}_0$ has more fixed points), and preserves every interval in between two consecutive multiples of π.

If T is not uniformisable in $\mathbb{S}^d \smallsetminus \mathbb{S}^{d-2}$, then $n \neq 0$. Say $n \geq 1$. Consider $x \in \mathbb{R}$, say in $[m\pi, (m+1)\pi]$. Then $\tilde{h}(x) \geq \tilde{h}(m\pi) = (m+2n)\pi \geq x + \pi$ so the projection in $\mathbb{S}^1$ of $[x, \tilde{h}x]$ is not properly convex. $\qquad\square$

4.3. **Totally geodesic blowup.** Our goal is to replace the singularities of a projective cone-manifold with totally geodesic boundary components. This is not always possible. We describe a family of tubes for which this process can be carried out.

Definition 4.10 (Special Tubes). Consider a tube with holonomy in $\mathrm{Aut}(\mathbf{B})$ generated by

$$g = \begin{pmatrix} \mu^{-1}\mathbb{I}_{d-1} & C \\ 0 & \mu^{(d-1)/2}\tilde{B} \end{pmatrix}$$

with $\mu \geq 1$. Let $B \in \mathrm{SL}_2(\mathbb{R})$ be the projection of $\tilde{B} \in \tilde{\mathrm{SL}}_2(\mathbb{R})$.

The tube is called *special* if

- It is uniformisable in $\mathbb{S}^d$ (which implies B is hyperbolic or parabolic).
- All eigenvalues of $\mu^{(d-1)/2}B$ are greater than μ^{-1}.

The latter is equivalent to

(4.1) $$\mu^{d+1} - \mu^{(d+1)/2}\operatorname{tr} B + 1 > 0.$$

Note that the above implies that $\mu > 1$. Hence, among the two generators of the holonomy of the tube only one satisfies this property. We call such generator the *special generator*.

Before going on, let us make the following simple observation.

Remark 4.11. Notice that for any $g \in \mathrm{Aut}(\mathbf{B})$, if g preserve a sector S of $\mathbf{B}$, then it preserves *each* of its walls. In other words, the endpoints of the interval I in $\tilde{\mathbb{S}}^1$ defining the sector are fixed points of g.

If the sector S is small enough, more precisely, if I projects (injectively) to a convex proper interval of $\mathbb{S}^1$ (we allow half-circles), then g acts properly discontinuously on the interior of I, hence one of the endpoints H^+ is *attracting in I* ($g^n H \to H^+$ for any $H \in I$ and $n \to \infty$) and the other is *repelling in I* ($g^n H \to H^-$ for any $H \in I$ and $n \to -\infty$). Note that if I is a half-circle then H^+ is not an attracting fixed point in $\mathbb{S}^1$.

We use the same terminology for the corresponding walls.

The next proposition describes the blowup of a special tube.

Proposition 4.12. *Consider a special tube $T \sqcup \mathbb{S}^{d-2}$ with smooth locus T and special generator g, with the notation from Definition 4.10. By Proposition 4.7, $T \sqcup \mathbb{S}^{d-2}$ is a quotient by $g\mathbb{Z}$ of a sector $\tilde{T} \sqcup \mathbb{S}^{d-2} \subset \mathbb{S}^d$.*

Let H^+, H^- be the g-invariant attracting and repelling walls of $\tilde{T}$ (see Remark 4.11). Note that H^+ contains a fixed point $x^+ \in \mathbb{S}^d$ of g, corresponding to the highest eigenvalue of g. Then:

(1) $g\mathbb{Z}$ acts freely and properly discontinuously on

$$\tilde{T} \sqcup \left(H^+ \smallsetminus (\mathbb{S}^{d-2} \cup \{x^+\}) \right).$$

(2) The quotient $\bar{T}$ is a compact projective manifold with totally geodesic boundary with interior isomorphic to the smooth locus T, such that the isomorphism extends to a continuous onto map $\bar{T} \to T \sqcup \mathbb{S}^{d-2}$.

(3) Every isomorphism of tubes $T \sqcup \mathbb{S}^{d-2} \to T' \sqcup \mathbb{S}^{d-2}$ extends to an isomorphism of projective manifolds with totally geodesic boundary $\bar{T} \to \bar{T}'$.

Proof. **Property (1)**. Set

$$\bar{S} := \tilde{T} \sqcup \left(H^+ \smallsetminus (\mathbb{S}^{d-2} \cup \{x^+\}) \right).$$

Notice that the assumption on the holonomy of a special tube implies that there is a change of basis fixing $\mathbb{R}^{d-1}$ where g is represented by a matrix of the form

$$g = \begin{pmatrix} \mu^{-1}\mathbb{I}_{d-1} & & \\ & \mu^{(d-1)/2}\lambda & \\ & & \mu^{(d-1)/2}/\lambda \end{pmatrix} \text{ or } \begin{pmatrix} \mu^{-1}\mathbb{I}_{d-1} & & \\ & \mu^{(d-1)/2} & 1 \\ & & \mu^{(d-1)/2} \end{pmatrix}$$

where $\lambda > 1/\lambda > 1/\mu$ or $\mu > 1$. The first case occurs when B is hyperbolic while the second case occurs when B is parabolic. Note that $x^+ = [e_d]$ and H^+ is a half-hypersphere in $\text{Span}\{e_1, \cdots, e_d\}$.

We now show that the action is properly discontinuous on $\bar{S}$.

In the case B is hyperbolic, it is an elementary computation to show that the action of g on the space $\mathbb{S}^d \smallsetminus (\mathbb{S}(\text{Span}\{e_1, \cdots, e_{d-1}, e_{d+1}\}) \cup \{\pm[e_d]\})$ is properly discontinuous.

We give a more general argument that works also in the parabolic case: Consider a sequence $x_n \in \tilde{T}$ converging to $x \in \bar{S}$ and a sequence of integers k_n going to $k = \pm\infty$. We have to prove that $g^{k_n}x_n$ leaves every compact set of $\bar{S}$. There are four cases, depending whether $k = +\infty$ or $-\infty$, and whether x lies in $\tilde{T}$ or in H^+.

Let $H_n \in \mathbb{S}^1$ be the projection of x_n, seen as half-hyperspheres containing $\mathbb{S}^{d-2}$, that converge to H. Let $C \subset \mathbb{S}^d$ be the g-invariant 1-sphere complementary to $\mathbb{S}^{d-2}$.

(1) If $k = \infty$ and $x \in H^+$ then one checks that $g^{k_n}H_n$ converges to H^+ and $g^{k_n}x_n$ accumulates on $H^+ \cap C = \{x^+\}$.

(2) If $k = \infty$ and $x \in \tilde{T}$ then $g^{k_n}x_n$ converges to x^+.

(3) If $k = -\infty$ and $x \in H^+$ then $g^{k_n}x_n$ converges to the stereographic projection of x on $\mathbb{S}^{d-2}$ seen from x^+.

(4) If $k = -\infty$ and $x \in \tilde{T}$ then $g^{k_n}H_n$ converges to H^- and $g^{k_n}x_n$ accumulates on H^-.

This ends the proof of properness of the action.

Property (2). The projection $\bar{T} \to T \sqcup \mathbb{S}^{d-2}$ is constructed by descending the map $\bar{S} \to \tilde{T} \sqcup \mathbb{S}^{d-2}$ that sends every $x \in H^+$ to its stereographic projection on $\mathbb{S}^{d-2}$ seen from x^+. To see that the resulting map is continuous, one may check that for every sequence $x_n \in \tilde{T}$ converging to $x \in H^+ \smallsetminus (\mathbb{S}^{d-2} \cup \{x^+\})$, with projection $H_n \in \mathbb{S}^1$, the sequence of integers k_n such that g^{k_n} brings back x_n in a compact fundamental domain of $\tilde{T} \sqcup \mathbb{S}^{d-2}$ converges to $-\infty$, and then apply point (3) above.

Property (3). Finally, consider an isomorphism of tubes $T \sqcup \mathbb{S}^{d-2} \to T' \sqcup \mathbb{S}^{d-2}$. As we saw in the proof of Proposition 4.7, the induced projective isomorphism $\tilde{T} \subset \mathbf{B} \to \tilde{T}' \subset \mathbf{B}$ is the restriction of an element $\phi \in \mathrm{Aut}(\mathbf{B})$. To conclude, it is not difficult to check that ϕ maps the attracting wall of $\tilde{T}$ to the attracting wall of $\tilde{T}'$. $\qquad\square$

Definition 4.13 (Totally Geodesic Blowup)**.** Using the notation from Proposition 4.12, $\pi : \bar{T} \to T \cup \mathbb{S}^{d-2}$ is called the *totally geodesic blowup* of $T \cup \mathbb{S}^{d-2}$.

The *totally geodesic blowup* of a cone-manifold with special singularities is obtained by locally blowing-up each component of the singular locus in charts in tubes such that the change of charts are automorphisms of tubes. This process is independent of the chart by invariance of the blowup under automorphisms in Proposition 4.12.

5. Tessellations of convex domains

The key result of this section is Proposition 5.1. It provides conditions which imply (uniformisability and) convexity for a union of convex tiles.

We will see in Section 7.1 that this result can be used to establish uniformisability and convexity for gluings of properly convex projective manifolds. The reason is that the universal cover of a gluing of properly convex projective manifolds can be described as a gluing of universal covers of these convex projective manifolds.

This phenomenon is illustrated in Section 2 and Figures 5 and 6.

5.1. **A local-to-global convexity statement.** Let us now state the key result of this section, which is based on an idea of Vinberg [65, §3], revisited by Benoist [12, §1.5].

Proposition 5.1. *Let $\mathcal{G}$ be a connected graph with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$. Suppose that to each vertex $v \in \mathcal{V}$ is associated a closed convex subset $D_v \subset \mathbb{S}^d$ with non-empty interior. Assume that:*

(1) $F_e := D_v \cap D_w$ is a codimension 1 face of D_v and D_w for any edge $e = [v, w] \in \mathcal{E}$, such that every point of ∂F_e is contained in the boundary of a half-space of $\mathbb{S}^d$ which contains $D_v \cup D_w$.

(2) *For any path $p = (v_1, v_2, \ldots, v_n) \subset \mathcal{G}$ such that $F_p := D_{v_1} \cap \cdots \cap D_{v_n}$ is a codimension 2 face of D_{v_i} for every i, there exists a half-space of $\mathbb{S}^d$ which contains $D_{v_1} \sqcup \cdots \sqcup D_{v_n}$ and whose boundary contains F_p.*

Let $\sim$ be the smallest equivalence relation on $\bigsqcup_{v \in \mathcal{V}} D_v \times \{v\}$ such that $(x, v) \sim (y, w)$ whenever v and w are adjacent and $x = y$. Denote by X the quotient space. Then the natural projection map

$$\pi : X \to \mathbb{S}^d$$

is injective with convex image.

The proof of Proposition 5.1 is postponed to Section 5.2.

Corollary 5.2. *In the setting of Proposition 5.1, the graph $\mathcal{G}$ is a tree.*

Proof. Suppose by contradiction that there is a non-trivial loop $(v_1, \ldots, v_n) \subset \mathcal{G}$ with $v_n = v_1$. Let $\mathcal{G}'$ be the graph with vertices $1, \ldots, n$ and with an edge from i to $i + 1$ for every $1 \leq i < n$. Associate to every vertex i the convex set $D_i := D_{v_i}$. One checks that $\mathcal{G}'$ satisfies again the assumptions of Proposition 5.1. However, $\bigsqcup_{1 \leq i \leq n} D_i \times \{i\}/_\sim \to \mathbb{S}^d$ is clearly not injective since since any x in $\mathrm{int}(D_1) = \mathrm{int}(D_n)$ is the image of both $(x, 1)$ and (x, n), which are not equivalent for $\sim$. $\square$

Definition 5.3 (Gluing Kit)**.** The data of $(\mathcal{G}, (D_v)_{v \in \mathcal{V}})$ satisfying Conditions 1 and 2 of Proposition 5.1 is called a *gluing kit.* To ease the notation, we will identify $X = \bigsqcup_{v \in \mathcal{V}} D_v \times \{v\}/_\sim$ with its image under π. Moreover, Proposition 5.1 implies that the gluing kit is determined by the family of convex sets $\{D_v\}_{v \in \mathcal{V}}$. As a consequence, we will identifies each vertex $v \in \mathcal{V}$ with its associated convex set D_v.

Definition 5.4 (Cells, Walls, Strata)**.** The convex sets of the form D_v with $v \in \mathcal{V}$ (resp. $D_v \cap D_w$ with $v, w \in \mathcal{V}$ adjacent, resp. $\cap_{v \in A} D_v$ with $A \subset \mathcal{V}$ with size at least 2) are called *cells* (resp. *walls*, resp. *strata*). A stratum with codimension 2 is called a *corner.*

Remark 5.5. In the proof of Proposition 5.1, it will be proved that for any geodesic $(D_1, \ldots, D_n) \subset \mathcal{G}$ and any pair $(x, y) \in D_1 \times D_n$, there is a segment $[x, y]$ that *goes successively through $D_1, \ldots, D_n$,* in the sense that there exists $x_1, \ldots, x_{n-1} \in [x, y]$ in this order such that $x_i \in D_i \cap D_{i+1}$ for any $1 \leq i \leq n$.

Remark 5.6. Note that Proposition 5.1 has the following corollary. Consider two closed convex sets $D, D' \subset \mathbb{S}^d$ that intersect on a codimension 1 face $F = D \cap D'$. Then $D \cup D'$ is convex if and only if every point of ∂F is contained in the boundary of a half-space of $\mathbb{S}^d$ which contains $D \cup D'$.

5.2. **The proof of Proposition 5.1.** In this section we explain the argument of Vinberg that, informally speaking, local convexity along codimension 1 and codimension 2 faces are enough to show the convexity of a union of convex tiles.

Proof of Proposition 5.1. Let us call *segment* a subset $s \subset X$ such that the restriction of π to s is injective and $\pi(s)$ is a segment of $\mathbb{S}^d$. In order to prove that π is injective with convex image, it is enough to show that every pair of points of X can be joined by a segment.

For all $v \in \mathcal{V}$ and $x \in D_v$ (resp. $A \subset D_v$), let us denote by $x|v$ (resp. $A|v$) the projection in X of (x, v) (resp. $A \times \{v\}$).

Let $X' \subset X$ be the projection of $\bigsqcup_{v \in \mathcal{V}} \text{int}(D_v) \times \{v\} \sqcup \bigsqcup_{e \in \mathcal{E}} \text{int}(F_e) \times e$. If X is endowed with the quotient topology, then

Fact 5.7. π *is a continuous map, X' is an open subset, and the restriction of π to X' is a local homeomorphism.*

Denote $F_p := D_{v_1} \cap \cdots \cap D_{v_n}$ for any path $p = (v_1, v_2, \ldots, v_n) \subset \mathcal{G}$.

Fix a path $p = (v_1, v_2, \ldots, v_n) \subset \mathcal{G}$ and set $D_i := D_{v_i}$ for every i. Denote by S_p the set of pairs of points $(x, y) \in D_1 \times D_n$ such that $x|v_1$ and $y|v_n$ can be joined by a segment of X going successively through $D_1|v_1, D_2|v_2, \ldots, D_n|v_n$ (see Remark 5.5).

Fact 5.8. S_p *is a closed subset of $D_1 \times D_n$.*

Denote by G_p the set of pairs of points $(x, y) \in \text{int}(D_1) \times \text{int}(D_n)$ such that $x \neq -y$ and $\text{Span}(x, y) \cap F_q = \emptyset$ for any subpath $q \subset p$ such that F_q has codimension at least 3.

Fact 5.9. G_p *is an open, dense and connected subset of $\text{int}(D_1) \times \text{int}(D_n)$.*

Now comes the main ingredient, which makes use of the assumptions (1) and (2). Denote $F_i = D_i \cap D_{i+1}$ for any $1 \leq i < n$.

Lemma 5.10. *For any $(x, y) \in G_p$, any segment of X from $x|v_1$ to $y|v_n$ and going successively through $D_1|v_1, \ldots, D_n|v_n$ is contained in*

$$\text{int}(D_1)|v_1 \cup \text{int}(F_1)|v_1 \cup \text{int}(D_2)|v_2 \cup \cdots \cup \text{int}(F_{n-1})|v_{n-1} \cup \text{int}(D_n)|v_n.$$

Proof. Let s be a segment from $x|v_1$ to $y|v_n$ and going successively through $D_1|v_1, \ldots, D_n|v_n$. Pick $z_1|v_1, \ldots, z_{n-1}|v_{n-1} \in s$ in this order such that $z_i|v_i = z_i|v_{i+1} \in F_i|v_i$ for any $1 \leq i < n$. It is enough to show that $z_i \in \text{int}(F_i)$ for any $1 \leq i < n$. Let us assume by contradiction that $z_i \in \partial F_i$ for some i. We may assume that $z_j \in \text{int}(F_j)$ for any $1 \leq j < i$. Pick $a \in]z_{i-1}, z_i[\subset \text{int}(D_i)$, where we may need the notation $z_0 := x$. Set also $z_n := y$.

If $z_i \neq z_{i+1}$, then by Assumption (1) we may find a closed half-space $H \subset \mathbb{S}^d$ which contains $D_i \cup D_{i+1}$ and whose boundary contains z_i. We have $z_{i+1} \in H$ and $a \in \text{int}(H)$ since $a \in \text{int}(D_i)$. Hence $z_i \in]a, z_{i+1}[\subset \text{int}(H)$, which is a contradiction.

Therefore $z_i = z_{i+1}$. Let $i + 1 < j \leq n$ be such that $z_i = z_{i+1} = \cdots = z_{j-1} \neq z_j$. Set $q = (v_i, v_{i+1}, \ldots, v_{j-1}) \subset p$. By definition, $z_i \in F_q$, which has codimension 2 since $(x, y) \in G_p$. By Assumption (2), there exists a closed

half-space $H \subset \mathbb{S}^d$ which contains $D_i \cup \cdots \cup D_{j-1}$ and whose boundary contains z_i. We have $z_j \in D_{j-1} \subset H$ and $a \in \text{int}(D_i) \subset \text{int}(H)$, hence $z_i \in]a, z_j[\subset \text{int}(H)$, which is a contradiction. $\qquad\square$

Facts 5.7, 5.8 and 5.9, and Lemma 5.10 have the following consequence.

Corollary 5.11. *$G_p \cap S_p$ is clopen in G_p. Thus, if it is non-empty then $S_p = D_1 \times D_n$.*

Let us prove that if $G_p \cap S_p$ is non-empty, then so is $G_q \cap S_q$ for any extension $q = (p, v_{n+1})$ such that $v_{n+1} \neq v_{n-1}$.

Set $F_n := D_n \cap D_{n+1}$. For dimensions reasons, we can find $(x, y) \in \text{int}(D_1) \times \text{int}(F_n)$ such that $x \neq -y$ and $\text{Span}(x, y) \cap F_{q'} = \emptyset$ for any subpath $q' \subset q$ such that $F_{q'}$ has codimension at least 3, and $\text{Span}(x, y)$ is transverse to F_n. Then any segment from $x|v_1$ to $y|v_n$ going successively through $D_1|v_1, \ldots, D_n|v_n$ (it exists by Corollary 5.11) may be extended to a point $y'|v_{n+1} \in \text{int}(D_{n+1})|v_{n+1}$ such that $x \neq -y'$. One checks that $(x, y') \in S_q \cap G_q$, which is therefore non-empty.

$\qquad\square$

5.3. The geometry of X.

Let us fix for the whole section a gluing kit $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $X = \bigcup_{D \in \mathcal{V}} D$.

We now describe the geometry of the convex set X. In particular, we relate the stratification of the boundary to the shape of the cells. As it turns out, we have three distinct types of faces:

(i) Some faces of the tiles are also faces of X.
(ii) Limits of *fans* of tiles sharing a common codimension 2 face.
(iii) Limits of *telescopes* of tiles containing infinitely many pairwise disjoint walls.

We will define them later on in this section and we will analyze in detail the last two types under additional assumptions of periodicity, satisfied for example if the tiling is preserved by a group Γ acting cocompactly on X.

5.3.1. *The interior of X.*

Let us start with the interior of X.

Proposition 5.12. *The interior of X is*

$$\bigcup_{v \in \mathcal{V}} \text{int}(D_v) \cup \bigcup_{e \in \mathcal{E}} \text{int}(F_e).$$

Proof. It is clear if $\mathcal{G}$ is finite. In the general case write $\mathcal{G}$ as the union of an increasing sequence of finite subgraphs $\mathcal{G}_n$, and set $X_n := \bigcup_{v \in \mathcal{V}_n} D_v$ for each n. As $\text{int}(X) = \bigcup_n \text{int}(X_n)$, this finishes the proof. $\qquad\square$

Corollary 5.13. *Any segment of $\bar{X}$ that intersects non-trivially a wall has to be contained in it.*

Proof. Consider a segment s intersecting non-trivially a closed wall W. It is clear that $s \subset \mathrm{Span}(W)$. By Proposition 5.12, $\partial W \subset \partial X$ so $\mathrm{Span}(W) \cap \bar{X} = W$, hence $s \subset W$. $\qquad\square$

We now turn to the boundary ∂X.

5.3.2. *The boundary of the cells in ∂X.* We first show that some of the faces of X correspond to faces of the tiles.

Lemma 5.14. *Consider a cell $D \in \mathcal{V}$. Then*

- $F \cap D$ *is a closed face of D for any closed face F of $\bar{X}$.*
- *If F is a closed face of D that does not intersect any closed wall, then it is also a closed face of $\bar{X}$.*

Proof. The first statement is clear.

Consider a closed face F of D that does not intersect any closed wall. Let F' be the closed face of $\bar{X}$ containing F. Note that $F' \cap D = F$. Suppose by contradiction that there is $x \in F' \smallsetminus F$. Pick $p \in \mathrm{int}(D)$. By Proposition 5.12, $[p, x[$ meets ∂D in the relative interior of some closed wall W of D. The non-empty subset $A \subset F'$ consisting of points y such that $[p, y]$ meets W is clearly closed and disjoint from F. By connectedness of F', in order to finish the proof it is enough to check that A is open. In particular, it suffices to check that $[p, y]$ meets $\mathrm{int}(W)$ for any $y \in A$. If this was not the case then $[p, y]$ would meet ∂W at y itself by Proposition 5.12, but then y would lie in D and, hence, in $W \cap F$ which is absurd. $\qquad\square$

Corollary 5.15. *Consider two distinct adjacent cells $D, D' \in \mathcal{V}$.*

- *If x is an extremal point of ∂D outside closed walls then it is also extremal in ∂X.*
- *If $x \in D \cap D'$ is an extremal point of $D \cup D'$ outside closed strata then it is also extremal in ∂X.*

Proof. The first statement is a direct application of Lemma 5.14.

The second statement is an application of the first one. Indeed the tree $\mathcal{G}'$ obtained as a quotient of $\mathcal{G}$ by replacing D and D' by the single vertex $D \cup D'$ also yields a gluing kit, for which $D \cap D'$ is not a wall. $\qquad\square$

Finally in order to detect $\mathcal{C}^1$ points of $\partial X \smallsetminus X$, we can use the following.

Fact 5.16. *Consider two distinct adjacent cells $D, D' \in \mathcal{V}$.*

- *If x is a $\mathcal{C}^1$ point of ∂D outside closed walls then it is also $\mathcal{C}^1$ in ∂X.*
- *If $x \in D \cap D'$ is a $\mathcal{C}^1$ point of $\partial(D \cup D')$ then it is also $\mathcal{C}^1$ in ∂X.*

Proof. This is an immediate consequence of the well-known observation that for any two nested properly convex open sets $\Omega \subset \Omega'$ and $x \in \partial\Omega \cap \partial\Omega'$ if x is $\mathcal{C}^1$ in $\partial\Omega$ then it is also $\mathcal{C}^1$ in $\partial\Omega'$. $\qquad\square$

5.3.3. *The complement in ∂X of the boundary of the cells.* Next, we consider the portion of the boundary that does not come from faces of the tiles. It naturally comes from accumulation points of sequences of tiles which we conveniently organize as points on the boundary at infinity of the graph $\mathcal{G}$.

Recall that $\mathcal{G}$ is a tree, so it has a natural boundary at infinity $\partial_\infty \mathcal{V}$: The set of rays up to equivalence, two rays being equivalent if they agree after some time. We set $\bar{\mathcal{V}} := \mathcal{V} \sqcup \partial_\infty \mathcal{V}$, a space on which we have a natural topology and notion of convexity.

Each point $\xi \in \partial_\infty \mathcal{V}$ corresponds to a *cell at infinity.*

Lemma 5.17. *Consider a ray $\xi = \{D_n\}_{n \in \mathbb{N}} \in \partial_\infty \mathcal{V}$. Then the sequences of compact convex sets D_n and $D_n \cap D_{n+1}$ both converge to the same closed convex subset $D(\xi) \subset \partial X$, called a* cell at infinity.

Proof. Let K be the set of accumulation points of sequences $x_n \in D_n$. We need to check that any point of K is also the limit of a sequence $y_n \in D_n \cap D_{n+1}$. Consider $x \in K$, limit of x_k where $x_k \in \text{int}(D_{n_k})$ with n_k increasing. Consider for each k the intersection points $y_{n_k}, \ldots, y_{n_{k+1}-1}$ of the segment $[x_k, x_{k+1}]$ with successively $D_{n_k} \cap D_{n_k+1}, \ldots, D_{n_{k+1}-1} \cap D_{n_{k+1}}$. The sequence y_n converges to x, as required. $\qquad\square$

Remark 5.18. As a warning:

- It is possible that $D(\xi) \subset X$.
- It is possible that $D(\xi) = D(\eta)$ for different $\xi, \eta \in \partial_\infty \mathcal{V}$.
- It is also possible that $D(\xi)$ is not a face of X.

Proposition 5.19. *We have:*

(1) *For every point $x \in \bar{X} \smallsetminus X$ there exists a unique $\xi \in \partial_\infty \mathcal{V}$ such that $x \in D(\xi)$.*
(2) *Consider a geodesic ray $\xi = \{D_n\} n \in \mathbb{N} \in \partial_\infty \mathcal{V}$, then there exists N such that*

$$D(\xi) \cap X = \bigcap_{n \geq N} D_n.$$

Moreover $D(\xi) \cap X$ is a closed face of $D(\xi)$.

Proof. **Property (1).** Fix $D \in \mathcal{V}$. Pick $x \in \bar{X} \smallsetminus X$ and a countable basis of (decreasing) properly convex neighborhoods U_n of x in $\mathbb{S}^d$. For each n let $\mathcal{V}_n \subset \mathcal{V}$ be the set of all vertices that intersect U_n. Since U_n is properly convex, $\mathcal{V}_n$ is a convex subset of $\mathcal{V}$. Let D_n be the shortest point projection in $\mathcal{G}$ of D on $\mathcal{V}_n$. Since $\mathcal{V}_n$ is non-increasing, D_n must lie on a geodesic ray of $\mathcal{G}$ starting at D. Choose $x_n \in D_n \cap U_n$ for each n. Then x_n tends to x. The fact that $x \notin X$ implies that D_n is not constant after some time and, hence, the geodesic ray containing D_n is infinite and represents some $\xi \in \partial_\infty \mathcal{V}$ whose cell $D(\xi)$ contains x. Moreover, if D'_k accumulates on x then $D'_k \in \mathcal{V}_{n_k}$ for some diverging sequence n_k, and D'_k converges to ξ.

Property (2). The sequence of sets $\bigcap_{n \geq N} D_n$ is non-decreasing for inclusion and constant after some time. Hence, it suffices to check that any $x \in D(\xi) \cap X$ lies in $\bigcap_{n \geq N} D_n$ for some N. Pick $D \in \mathcal{V}$ containing x and $x_n \in D_n$ converging to x. There is N such that D_m lies on the geodesic segment from D to D_n for every $n \geq m \geq N$. For all $n \geq m \geq N$ pick $y_{n,m} \in [x, x_n] \cap D_m$. Observe that $y_{n,m} \in D_m$ converges to x for every m, so $x \in D_m$.

Suppose by contradiction that $D(\xi) \cap X$ is not a face of $D(\xi)$, i.e. that there are $x, y \in D(\xi) \smallsetminus X$ and $z \in]x, y[\cap D_n$ for some n. Pick $p \in \operatorname{int}(D_n)$. By Remark 5.5, $[p, q]$ meets $D_n \cap D_{n+1}$ for any $q \in \bigcup_{k > n} D_k$. Passing to the limit, $[p, x]$ (resp. $[p, y]$) meets $D_n \cap D_{n+1}$ at $x' \neq x$ (resp. $y' \neq y$). By Proposition 5.12, these two intersection points lie in the relative interior of $D_n \cap D_{n+1}$. The span of $D_n \cap D_{n+1}$ intersects the plane spanned by p, x, y into a line which contains x', y', z. But these three points are not collinear, which is a contradiction. $\square$

As we have already observed, sequences of tiles can essentially be grouped into two different types: *Fans* of tiles and *telescopes* of tiles. We now analyze in detail both cases.

5.4. **Fan of tiles.** Informally speaking, a fan of tiles is an infinite sequence of consecutive tiles sharing a common codimension 2 face. We will work under the assumption that the sequence is periodic under some special projective transformation and describe for such a fan of tiles the corresponding cell at infinity. See Figure 4.

5.4.1. *Condition 2 revisited.* First, we discuss a convexity criterion for a periodic fan of tiles. In order to do so, we investigate the Condition (2) in Proposition 5.1 in this special setting.

Proposition 5.20. *Let* $(D_n)_{n \in \mathbb{Z}}$ *be a sequence of closed convex subsets of* $\mathbb{S}^d$ *with non-empty interior such that:*

(a) *They all share a codimension 2 face* E *spanning* $\mathbb{S}^{d-2} \subset \mathbb{S}^d$.
(b) *Any two consecutive domains* D_n *and* D_{n+1} *intersect along a codimension 1 face* F_n, *with* $F_n \neq F_{n-1}$.

Consider for every n *the projections* D'_n *of* D_n *to the circle* $\mathbb{S}(\mathbb{R}^{d+1}/\mathbb{R}^{d-1}) = \mathbb{S}^1$. *Then* $\mathbb{S}^{d-2}$ *is contained in the boundary of a half-space of* $\mathbb{S}^d$ *containing* $\bigcup_n D_n$ *if and only if* $\Omega := \bigcup_n D'_n$ *is contained in a half-circle.*

Assume now that there exists $g \in \operatorname{GL}_{d+1}(\mathbb{R})$ *and* $T > 0$ *such that* $gD_{n+T} = D_n$ *for every* n. *Let* $g' \in \operatorname{GL}(\mathbb{R}^{d+1}/\mathbb{R}^{d-1})$ *be the induced transformation on* $\mathbb{S}^1$. *Then* Ω *is contained in a half-circle if and only if*

(i) $|\det(g')|^{-1/2} g' \neq \mathbb{I}_2$ *and* $\operatorname{tr}(|\det(g')|^{-1/2} g') \geq 2$.
(ii) $D'_0 \cup \cdots \cup D'_{T-1}$ *does not intersect any eigenline of* g'.

Proof. The fact that $\mathbb{S}^{d-2}$ is contained in the boundary of a half-space of $\mathbb{S}^d$ containing $\bigcup_n D_n$ if and only if $\Omega' := \bigcup_n D'_n$ is contained in a half-circle is clear.

Assume that Ω' is contained in a half-circle (it is a proper, open and convex subset of $\mathbb{S}^1$).

Let us prove properties (i)+(ii). As Ω' is g'-invariant and no transformation $h \in \mathrm{SL}_2(\mathbb{R})$ with $\mathrm{tr}(h) < 2$ preserves a proper convex subset of $\mathbb{S}^1$, we must have $\mathrm{tr}(|\det(g')|^{-1/2}g') \geq 2$. As $g'D'_n = D'_{n+T}$, we also have $|\det(g')|^{-1/2}g' \neq \mathbb{I}_2$ and Ω' does not contain fixed the points of g', that is, its eigenline(s).

Assume now that properties (i),(ii) hold.

Let us prove that Ω' is contained in a half-circle. As $\mathrm{tr}(|\det(g')|^{-1/2}g') \geq 2$, the eigenlines of g' determine either a half-circle or a unique convex segment. As Ω' does not intersect any eigenline of g' and is connected, it must be contained in one the components of $\mathbb{S}^1$ minus the eigenline(s) of g'. $\qquad\square$

Remark 5.21. In the setting of Proposition 5.20, if Ω is contained in a half-circle then:

- If $\mathrm{tr}(|\det(g')|^{-1/2}g') = 2$, then Ω is one of the two half-circles delimited by the unique eigenline of g'.
- If $\mathrm{tr}(|\det(g')|^{-1/2}g') > 2$, then Ω is one of the four quadrants delimited by the two eigenlines of g'.

5.4.2. The cell at infinity of a fan of tiles. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a gluing kit with $X = \bigcup_{D \in \mathcal{V}} D$.

Pick $\mu > 1$ and $\lambda \geq 1$ and set, depending whether $\lambda > 1$ or $\lambda = 1$,

$$g := \begin{pmatrix} \mu^{-1}\mathbb{I}_{d-1} & 0 & 0 \\ 0 & \lambda\mu^{\frac{d-1}{2}} & 0 \\ 0 & 0 & \lambda^{-1}\mu^{\frac{d-1}{2}} \end{pmatrix} \text{ or } \begin{pmatrix} \mu^{-1}\mathbb{I}_{d-1} & 0 & 0 \\ 0 & \mu^{\frac{d-1}{2}} & 1 \\ 0 & 0 & \mu^{\frac{d-1}{2}} \end{pmatrix}.$$

The following describes the cell at infinity of a fan of tiles:

Proposition 5.22. *Let $\mathcal{S}$ be a corner. Suppose that:*

- *$\mathcal{S} \subset \mathbb{S}^{d-2}$.*
- *The set of $D \in \mathcal{V}$ such that $\mathcal{S} \subset D$ is a bi-infinite path $(D_n)_{n \in \mathbb{Z}}$.*
- *$gD_n = D_{n+m}$ for any n, for some $m > 0$.*

Set $\xi := \{D_n\}_{n \in \mathbb{N}} \in \partial_\infty \mathcal{V}$. Then there exists $\alpha, \beta \in \{\pm 1\}$ such that:

(1) $D(\xi)$ is the convex hull of $\mathcal{S}$ with $\alpha e_d\mathbb{R}_+$.
(2) If X is properly convex or $\lambda > 1$, then $D(\xi)$ is a closed face of X.
(3) If $\lambda > 1$ then X is contained in the convex hull of $D(\xi)$ and $\beta e_{d+1}\mathbb{R}_+$.

Proof. **Property (1).** For any generic $x \in \mathbb{S}^d$, the sequence $g^n x$ tends to $e_d\mathbb{R}_+$ or $-e_d\mathbb{R}_+$. We may assume that the limit is $e_d\mathbb{R}_+$ for some $x \in \mathrm{int}(D_0)$.

This implies that the convex hull of $\mathcal{S}$ with $e_d\mathbb{R}_+$ is contained in $D(\xi)$, and hence that $\mathbb{S}^{d-1}$ is a supporting hyperplane for X containing $D(\xi)$.

By Part (2) of Proposition 5.19, $\mathcal{S} \subset \mathbb{S}^{d-2}$ is a closed face of $D(\xi)$. Hence $\mathbb{S}^{d-2} \subset \mathbb{S}^{d-1}$ is a supporting hyperplane of $D(\xi)$, which lies in the closed half-space H of $\mathbb{S}^{d-1}$ delimited by $\mathbb{S}^{d-2}$ that contains $e_d\mathbb{R}_+$

For any $x \in D(\xi) \smallsetminus \{e_d\mathbb{R}_+\}$, the limit y of $g^{-n}x$ is in $\mathbb{S}^{d-2} \cap D(\xi) = \mathcal{S}$ (as $D(\xi)$ is clearly g-invariant), and $x \in [y, e_d\mathbb{R}_+]$. Therefore $D(\xi)$ is the convex hull of $\mathcal{S}$ with $e_d\mathbb{R}_+$.

Property (2). Suppose that there exists $x \in \mathbb{S}^{d-1} \smallsetminus D(\xi)$, lying in some cell $D(v)$ with $v \in \bar{\mathcal{V}} \smallsetminus \{\xi\}$. Let us show that X is not properly convex and $\lambda = 1$. Set $\eta = \{D_{-n}\}_{n\in\mathbb{N}} \in \partial_\infty\mathcal{V}$ and $X' := \bigcup_n D_n$, so that $\bar{X}' = X' \cup D(\xi) \cup D(\eta)$. Pick $y \in \text{int}(\mathcal{S})$.

We may assume that $x \in \bar{X}'$. Indeed, if it not the case then consider the shortest point projection D_k of v on $\{D_n\}_{n\in\mathbb{Z}}$ in $\bar{\mathcal{V}}$, and a geodesic path $(v_m)_{0\leq m\leq M}$ from D_k to v in $\mathcal{V}$, with $M \in \mathbb{N} \cup \{\infty\}$. Then the last point of $[y, x]$ in $\bar{D}_k$ lies in $D_k \cap D(v_1)$, which does not intersect $D(\xi)$. We may then consider this last point instead of x.

By Proposition 5.12, x does not lie in $\mathbb{S}^{d-2}$. Moreover, x does not lie in $\text{int}(H)$, otherwise $[x, y]$ would intersect $D(\xi) \smallsetminus X$, which is not possible since the complementary of $D(\xi) \smallsetminus X$ in $\bar{X}$ is convex.

Hence $g^n x$ tends to $-e_d\mathbb{R}_+$, which hence belongs to $\bar{X}'$ since this last set is g-invariant. Thus X is not properly convex. Moreover $\lambda = 1$ by Proposition 5.20.

Property (3). Finally, we may assume that $\bar{X}$ is contained in the half-space of $\mathbb{S}^d$ delimited by $\mathbb{S}^{d-1}$ and containing $e_{d+1}\mathbb{R}_+$. Let us prove that, under the condition that $\lambda > 1$, then $\bar{X}$ is contained in the convex hull K of $D(\xi)$ and $e_{d+1}\mathbb{R}_+$. Suppose by contradiction that there exists $x \in \bar{X} \smallsetminus K$. First note that $x \notin \bar{X}'$ by Lemma 5.23 below.

As before, consider $v \in \bar{\mathcal{V}}$ such that $x \in D(v)$, the shortest point projection D_k of v on $\{D_n\}_{n\in\mathbb{Z}}$ in $\bar{\mathcal{V}}$, and a geodesic path $(v_m)_{0\leq m\leq M}$ from D_k to v in $\mathcal{V}$, with $M \in \mathbb{N} \cup \{\infty\}$. One may find $y \in \partial D(\xi) \smallsetminus \partial\mathcal{S}$ such that $[x, y[$ does not intersect K. The last point z of $[y, x]$ in $\bar{X}'$ belongs to $D_k \cap D(v_1)$ which does not intersect $\partial D(\xi) \smallsetminus \partial\mathcal{S}$. Thus $z \neq y$, so $z \in]y, x]$, so $z \in \bar{X}' \smallsetminus K$ which is absurd. $\qquad\square$

Lemma 5.23. *Consider* $0 < \lambda_1 < \lambda_2 < \lambda_3$ *and*

$$g = \begin{pmatrix} \lambda_1 I_{d-1} & 0 & 0 \\ 0 & \lambda_3 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}.$$

Let $\Omega \subset \mathbb{S}^d$ *be a g-invariant properly convex open set. Then its projection on $\mathbb{S}^{d-1}$ parallel to $[e_{d+1}]$ is contained in $\overline{\Omega} \cap \mathbb{S}^{d-1}$, which is the convex hull of $\partial\Omega \cap \mathbb{S}^{d-2}$ and $[e_d]$.*

Proof. Consider $x = [x_1, \ldots, x_{d+1}] \in \overline{\Omega}$. If x lies in Ω or $\mathbb{S}^{d-1} \smallsetminus \{[e_d]\}$, then $x_i \neq 0$ for some $1 \leq i \leq d - 1$. Thus $g^{-n}x$ converges in this case to $[x_1, \ldots, x_{d-1}, 0, 0]$ when n tends to infinity, and this limit must belong to $\partial\Omega \cap \mathbb{S}^{d-2}$. This implies that $[x_1, \ldots, x_d, 0]$ lies in the convex hull of $\partial\Omega \cap \mathbb{S}^{d-2}$ and $[e_d]$.   □

5.5. **Telescope of tiles.** We now move to the second type of sequences which we call telescopes of tiles. Informally speaking, a telescope of tiles is a sequence of consecutive tiles D_n such that infinitely many of the walls $D_m \cap D_{m+1}$ are pairwise disjoint.

Again, we will assume some periodicity assumptions, namely, that, up to projective transformations, there are only finitely (or compactly) many configurations of four consecutive tiles $D_n, D_{n+1}, D_{n+2}, D_{n+3}$ where the walls $D_n \cap D_{n+1}$ and $D_{n+1} \cap D_{n+2}$ are disjoint.

In this case $(D_n \cap D_{n+1}, \Omega, D_{n+2} \cap D_{n+3})$ form what we call a visible triple.

Definition 5.24 (Visible Triple). A *visible triple* is a triple (L, K, M) of (non-empty) compact properly convex sets such that $L, M \subset K$, $L \cap M = \emptyset$, and any segment of K from L to M intersects the interior of K.

The analysis of the cell at infinity of a telescope of tiles rests on a contraction property of visible triples which we now discuss.

5.5.1. *Contraction for visible triples.* The set of visible triples is endowed with a Hausdorff topology. A sequence of compact subsets $C_n \subset \mathbb{S}^d$ converges to a compact subset $C \subset \mathbb{S}^d$ if the following two conditions are satisfied:

- If a sequence $x_{n_j} \in C_{n_j}$ converges to $x \in \mathbb{S}^d$, then $x \in C$.
- For every $x \in C$ there exists a sequence $x_n \in C_n$ converging to it.

This induces a Hausdorff topology on the space of triples in a natural way.

Lemma 5.25. *Let $\mathcal{K}$ be a compact set of visible triples. Then there exists $\lambda > 1$ such that the following holds. Consider a properly convex open set Ω, a point $x \in \Omega$, two rays r, r' of Ω starting at x, two hyperplanes H, H' and a closed convex subset $K \subset \bar{\Omega}$ such that:*

- *$(H \cap \bar{\Omega}, K, H' \cap \bar{\Omega}) \in \mathrm{GL}_{d+1}(\mathbb{R}) \cdot \mathcal{K}$.*
- *The ray r (resp. r') meets first $H \cap \Omega$ at y (resp. y') and then $H' \cap \Omega$ at z (resp. z').*

Then $d_\Omega(z, z') \geq \lambda d_\Omega(y, y')$.

Proof. We may assume that $\mathcal{K}$ is a very small neighborhood of a given triple (L, K, M). Fix an affine chart $\mathcal{A}$ containing K, with a Euclidean distance $d_{\mathcal{A}}$. We may assume that it contains K' for every $(L', K', M') \in \mathcal{K}$.
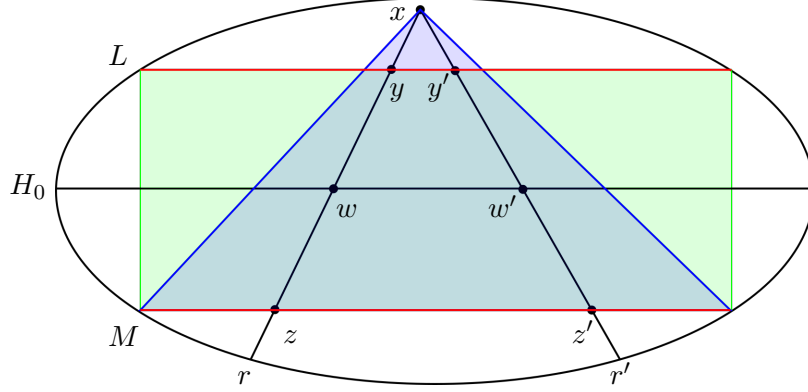
FIGURE 7. Contraction for visible triples.

Let $H_0 \subset \mathbb{S}^d$ by a hyperplane separating L and M. We may assume that it separates L' and M' for every $(L', K', M') \in \mathcal{K}$.

Let $\epsilon > 0$ be the minimal distance for $d_{\mathcal{A}}$ between a point of $H_0 \cap \mathrm{Conv}(L', M')$ and a point of $H_0 \cap \partial K'$ for $(L', K', M') \in \mathcal{K}$, and D the maximal possible $d_{\mathcal{A}}$-diameter for K'. By [16] (see also [18, Fact 2.1.3]), there exists $\lambda > 1$ depending on ϵ and D such that for any $(L', K', M') \in \mathcal{K}$, denoting by Ω_1 (resp. Ω_2) the relative interior of $H_0 \cap \mathrm{Conv}(L', M')$ (resp. $H_0 \cap K'$), for all $x, y \in \Omega_1$, we have

$$d_{\Omega_1}(x, y) \geq \lambda d_{\Omega_2}(x, y).$$

Consider a properly convex open set Ω, a point $x \in \Omega$, two rays r, r' of Ω starting at x, two hyperplanes H, H' and a closed convex subset $K' \subset \bar{\Omega}$ such that, denoting $L' := H \cap \bar{\Omega}$ and $M' := H' \cap \bar{\Omega}$

- $(L', K', M') \in \mathrm{GL}_{d+1}(\mathbb{R}) \cdot \mathcal{K}$.
- The ray r (resp. r') meets first $\mathrm{int}(L')$ at y (resp. y') and then $\mathrm{int}(M')$ at z (resp. z').

Denote by Ω_0 (resp. Ω_1, resp. Ω_2, resp. Ω_3) the relative interior of $H_0 \cap \mathrm{Conv}(x, M')$ (resp. $H_0 \cap \mathrm{Conv}(L', M')$, resp. $H_0 \cap K'$, resp. $H_0 \cap \Omega$). Observe that

$$\Omega_0 \subset \Omega_1 \subset \Omega_2 \subset \Omega_3.$$

Therefore, denoting by w and w' the stereographic projections in Ω_0 of z and z' seen from x, we obtain by Remark 3.13 that

$$d_\Omega(z, z') = d_{\Omega_0}(w, w') \geq d_{\Omega_1}(w, w') \geq \lambda d_{\Omega_2}(w, w')$$
$$\geq \lambda d_{\Omega_3}(w, w') \geq \lambda d_\Omega(x, x'). \qquad \square$$

5.5.2. *Visible triples in X.* Let $X = \bigcup_{v \in \mathcal{V}} D_v$ be a tessellated properly convex set as above. We now describe where to find visible triples in it.

Assume that each tile D_v has the following properties:

(1) Any intersection of (at least 2) walls is either empty or a corner.
(2) Any closed proper face touching a wall is contained in a (possibly different) wall.
(3) Any closed proper face with codimension at least 2 that touches a corner is contained in a corner.

We have the following:

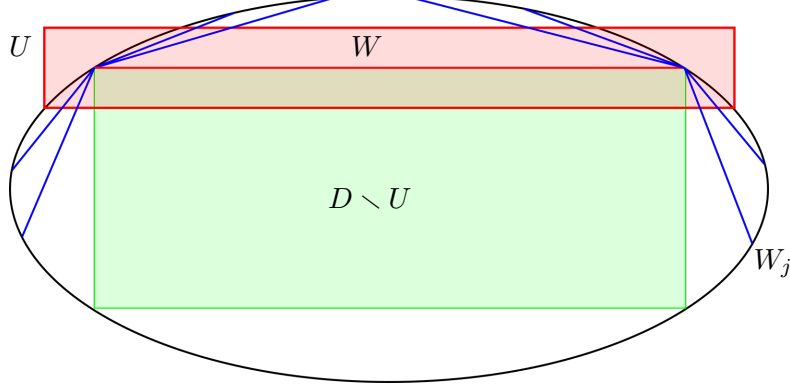FIGURE 8. Visible triples in X.

Lemma 5.26. *Let $D \subset X$ be a tile. Let W be a wall of D. Consider an open neighborhood U of W in $\mathbb{S}^d$ such that $D \smallsetminus U$ is convex. Then:*

- *There are only finitely many walls $W_1, \ldots, W_n$ adjacent to W and not contained in U.*
- *$(W, K, D \smallsetminus U)$ is a visible triple for every convex compact set $K \subset \bar{X}$ with $\mathrm{int}(W_j) \subset \mathrm{int}(K)$ for every $j \leq n$.*

Proof. Only a finite number of walls $W_1, \ldots, W_n$ are adjacent to W but not contained in U. Indeed, if there was an infinite sequence of them then they would accumulate on a closed codimension ≥ 2 face F touching W but not contained in U, which by assumption would be contained in a wall W' adjacent to W. The fact that F touches the corner $W \cap W'$ would then imply that it is contained in it, which is absurd.

To finish the proof we only need to prove that any segment s from W to $D \smallsetminus U$ intersects $\mathrm{int}(D)$ or $\mathrm{int}(W_i)$ for some i. If $s \subset \partial D$, then by assumption it is contained in a wall adjacent to W, which is not contained in U. So $s \subset W_i$ for some i. If, by contradiction, $s \subset \partial W_i$ then it is contained in a face of D of codimension at least 2 which touches $W \cap W_i$, hence is contained in it. This is absurd. $\square$

We will also need the following technical lemma.

Lemma 5.27. *For any neighborhood U of W there exists a smaller neighborhood U' such that $D \smallsetminus U'$ contains all walls not contained in U and non-adjacent to W.*

Proof. If by contradiction this was not the case, then there would exist an infinite sequence of walls limiting on a comdimension-≥ 2 face F intersecting W but not contained in U. This is absurd, as remarked in the proof of Lemma 5.26. $\qquad\square$

5.5.3. *The extremal point of a telescope of tiles.* Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a gluing kit with $X = \bigcup_{D \in \mathcal{V}} D$.

Proposition 5.28. *Suppose that all strata have codimension 2, and that there exists a compact set $\mathcal{K}$ of visible triples such that the following holds: For any path $\{A, B, C, D\} \subset \mathcal{V}$ such that $A \cap B$ and $C \cap D$ are disjoint, there exists a convex compact subset $K \subset \bar{X}$ such that*

$$(A \cap B, K, C \cap D) \in \mathrm{GL}_{d+1}(\mathbb{R}) \cdot \mathcal{K}.$$

Then any cell at infinity that does not intersect X is an extremal $\mathcal{C}^1$ point.

Proof. Consider the ray $\xi = \{D_n\}_{n \in \mathbb{N}} \in \partial_\infty \mathcal{V}$, such that $D(\xi) \cap X = \emptyset$.

Let us prove that $D(\xi)$ is reduced to a point.

Suppose by contradiction that it is not the case. Pick $x \in \mathrm{int}(D_0)$ and $p, p' \in \mathrm{int}(D(\xi))$, and denote by r and r' the rays of $\Omega := \mathrm{int}(X)$ starting at x and ending respectively at p and p'. Set $F_n^k := D_n \cap \cdots \cap D_{n+k}$ for all n, k. Let y_n (resp. y_n') be the successive intersection points of r (resp. r') with $\mathrm{int}(F_n^1)$.

Claim 1. The sequence $d_\Omega(y_n, y_n')$ is non-decreasing and bounded from above by $d_{\mathrm{int}(D(\xi))}(p, p') < \infty$.

Proof of the claim. Let us take care of monotonicity first: Consider the 2-plane P spanned by r, r' and the intersection $X \cap P$. Let $[a_n, a_n']$ be the intersection of the projective line spanned by y_n, y_n' with X. Notice that we have $F_n^1 \cap P = [a_n, a_n']$ and, by construction, $F_m^1 \cap P = [a_m, a_m']$ is contained in the cone with vertex x bounded by the lines $[x, a_n]$ and $[x, a_n']$. The intersection of the line spanned by $[a_m, a_m']$ with such cone is a larger segment $[b_m, b_m']$. By standard properties of the the cross ratio, we get $[b_m, y_m, y_m', b_m'] = [a_n, y_n, y_n', a_n']$. By monotonicity under inclusion (see Definition 3.13), we also get $[b_m, y_m, y_m', b_m'] \leq [a_m, y_m, y_m', a_m']$. This concludes the monotonicity part of the statement.

The upper bound follows from the semicontinuity of the extension of the Hilbert distance to the boundary (see Definition 3.13). $\qquad\square$

Let $\lambda > 1$ be the constant from Lemma 5.25. In order to get a contradiction, it is enough to prove the following:

Claim 2. For any n there exists $m \geq n$ such that $d_\Omega(y_m, y_m') \geq \lambda d_\Omega(y_n, y_n')$.

Proof of the claim. Pick $n \in \mathbb{N}$. Since $D(\xi)$ does not intersect X, there exists $k \geq 2$ such that F_{n+1}^k is empty. Take it minimal, meaning that $F_{n+1}^{k-1} \neq \emptyset$.

If $k = 2$, then $F_n^3 = \emptyset$. Set $m = n$.

If $k \geq 3$, then $F_{n+1}^{k-1} = F_{n+2}^{k-2} = \cdots = F_{n+k-2}^2$ (since all strata have codimension 2), and $F_{n+k-2}^3 = F_{n+1}^k = \emptyset$. Set $m = n + k - 2$.

Let $\mathcal{V}'$ be the component of $\mathcal{V} \smallsetminus \{D_m, D_{m+3}\}$ containing D_{m+1}. By assumption, there exists a properly convex compact set $K \subset \bigcup \mathcal{V}'$ containing $D_{m+1} \cup D_{m+2}$ such that

$$(D_m \cap D_{m+1}, K, D_{m+2} \cap D_{m+3}) \in \mathrm{GL}_{d+1}(\mathbb{R}) \cdot \mathcal{K}.$$

By Lemma 5.25, we get

$$d_\Omega(y_{m+2}, y'_{m+2}) \geq \lambda d_\Omega(y_m, y'_m) \geq \lambda d_\Omega(y_n, y'_n).$$

$\square$

Claim 3. The point $D(\xi)$ is extremal.

Proof of the claim. This is an immediate consequence of the fact that $\bar{X} \smallsetminus D(\xi)$ is convex. $\square$

Saying that p is $\mathcal{C}^1$ is equivalent (see e.g. [18, Prop. 5.4.8]) to saying that for any two rays r, r' of Ω ending at p, the distance $\inf\{d_\Omega(x, x') : (x, x') \in r \times r'\}$ is null, which can be proved using almost the same proof as above. $\square$

6. Convex-cocompactness and relative hyperbolicity

The goal of this section 3-fold: We are going to establish three geometric properties of the convex projective manifolds (potentially with totally geodesic boundary) that we construct in this paper, namely:

(a) Convex-cocompactness.
(b) Relative hyperbolicity.
(c) Extended geometric finiteness.

We now state the results. Later we will give precise definitions of their terms.

The first property we establish is convex-cocompactness. The following result is probably well-known to experts, as the proof follows a classical strategy. Recall that for us the terminology b-manifolds denotes projective manifolds with totally geodesic boundary.

Proposition 6.1. *Let M be a compact properly convex b-manifold with universal cover $\widetilde{M} \subset \mathbb{S}^d$. Suppose that every proper face of $\widetilde{M}$ touching the closure of a wall belongs to it. Then $\pi_1(M)$ acts convex-cocompactly on any invariant properly convex open set containing $\widetilde{M}$.*

Remark 6.2. In the setting of Proposition 6.1, there does not always exists an invariant properly convex open set containing $\widetilde{M}$. For instance, if M is the manifold Ω_2/Γ_2 obtained in Theorem 1.5, then $\pi_1(M) = \Gamma_2$ does not

act naively convex-cocompactly on any properly convex open set Ω (see the definition of naive convex-cocompactness in Section 6.1).

The second property we consider is relative hyperbolicity. There are two cases, depending on whether we consider a manifold with or without boundary.

Theorem 6.3. *Let M be a compact properly convex b-manifold with universal cover $\tilde{M} \subset \mathbb{S}^d$. Suppose that:*

- *All walls of $\tilde{M}$ have pairwise disjoint closures.*
- *Any point of $\partial\tilde{M}$ outside closed walls is extremal.*

Then $\pi_1(M)$ is hyperbolic relatively to *the fundamental groups of its walls, and the* Bowditch boundary *is equivariantly homeomorphic to the quotient $Y := \partial\tilde{M}/_\sim$ obtained by contracting closed walls into points.*

Relative hyperbolicity is a group theoretic generalization of the fundamental groups of geometrically finite hyperbolic manifolds, in which case the Bowditch boundary corresponds to the usual limit set (see Section 6.2 for more details).

The above theorem can be seen as a consequence of very general work of Weisman [67, Th. 1.16] or work of Islam and Zimmer [35, Th. 1.3-6], in the special case where $\pi_1(M)$ is convex-cocompact. However, notice that we will apply this theorem in a case where convex-cocompactness does not hold (see the above Remark 6.2). In fact we will exploit a different property: Namely that our subgroups with respect to which we want to prove relative hyperbolicity have codimension 1. The strategy of proof of Theorem 6.3 follows the same line as in Weisman and Islam–Zimmer's works, which themself are inspired by older works.

In the case where M is closed, relative hyperbolicity comes from the following:

Fact 6.4 (Weisman [67, Th. 1.16]). *Let $M = \Omega/\Gamma$ be a closed properly convex projective manifold. Consider closed, totally geodesic, embedded, pairwise disjoint hypersurfaces $W_j = C_j/\Gamma_j \subset M$ where $C_j = \Omega \cap H_j$ with H_j projective hyperplanes, $\Gamma_j = \mathrm{Stab}_\Gamma(H_j)$, and $j = 1\ldots n$. Suppose that:*

- *The Γ-translates of the C_j's in Ω have pairwise disjoint closures.*
- *Any point of $\partial\Omega$ outside the closures of these translates is extremal.*

Then Γ is hyperbolic relatively to $\Gamma_1, \ldots, \Gamma_n$, and the Bowditch boundary is equivariantly homeomorphic to the quotient $Y := \partial\Omega/_\sim$ obtained by contracting $\gamma\partial C_j$ for every $\gamma \in \Gamma$ and $j = 1\ldots n$.

The last property we consider is extended geometrical finiteness, a recent notion due to Weisman [68, Def. 1.3]. Extended geometrically finite discrete subgroups of semisimple Lie groups generalize the class of geometrically finite discrete subgroups of rank-one semisimple Lie groups such as $\mathrm{SO}(n, 1)$.

In particular, all extended geometrically finite groups are by definition relatively hyperbolic with respect to some proper subgroups.

Let $\mathcal{F}$ be the partial flag variety of pairs of points and hyperplanes (x, H) of $\mathbb{RP}^d$ such that $x \in H$.

Proposition 6.5. *In the settings of Theorem 6.3 and Fact 6.4, the action of Γ on $\mathcal{F}$ is* extended geometrically finite *in the sense of [68, Def. 1.3].*

Extended geometrical finiteness in the setting of Fact 6.4 and in the setting of Theorem 6.3 *with convex-cocompactness* was already known in [68, Th. 1.12].

6.1. **Convex-cocompactness.** Consider a properly convex open set $\Omega \subset \mathbb{S}^d$ and a discrete group $\Gamma \subset \mathrm{SL}_{d+1}^{\pm}(\mathbb{R})$ preserving it.

The action of Γ on Ω is said to be *naively convex-cocompact* if there exists a Γ-invariant convex subset of Ω on which Γ acts cocompactly. In such a case, and if Ω is not strictly convex, there is not always a smallest such convex set, contrarily to convex-cocompactness in hyperbolic geometry. One reason is that the set of accumulation of points of an orbit $\Gamma \cdot x$ may depend on the choice of $x \in \Omega$ (e.g. if Ω is a triangle in $\mathbb{S}^2$, and Γ is generated by an infinite-order non-proximal element). For this reason, and others, explained in [30], it appears that this notion of convex-cocompactness is not the best one; this is why another one has been introduced, as follows.

The *full orbital limit set* of Γ is the union over all $x \in \Omega$ of the set of accumulation points of the orbit $\Gamma \cdot x$, i.e. $\cup_{x \in \Omega} \overline{\Gamma x} \cap \partial \Omega$.

Definition 6.6 (Danciger–Guéritaud–Kassel [30, Def. 1.11])**.** The action of Γ on Ω is said to be *convex-cocompact* if the convex hull in Ω of the full orbital limit set is non-empty and has compact quotient by Γ.

We refer to [30, §1.4–1.6–1.7–4.1–10.7] and [31] for more details and examples on convex-cocompactness.

Proof of Proposition 6.1. Let Ω be an invariant properly convex open set containing $\widetilde{M}$ and $\Gamma = \pi_1(M)$. Let Λ be the full orbital limit set.

The convex set $\widetilde{M}$ is the union of its interior with the relative interior of countably many of its codimension 1 faces (its totally geodesic boundary). Hence $\partial_i \widetilde{M} = (\partial \widetilde{M}) \smallsetminus \widetilde{M}$ is compact, and the closure of $\widetilde{M}$ is the convex hull of $\partial_i \widetilde{M}$. Moreover, one easily checks by compactness of M that $\partial_i \widetilde{M} \subset \Lambda \subset \partial \Omega$.

To prove the proposition it is now enough to prove that any $\xi \in \Lambda$ lies in $\partial_i \widetilde{M}$. Assume by contradiction that $\xi \notin \partial_i \widetilde{M}$. There exist $x \in \Omega$ and a sequence $\gamma_n \in \subset \Gamma$ such that $\gamma_n x$ converges to ξ.

Pick y in the interior of $\widetilde{M}$. Consider $a, b \in \partial \Omega$ and z in a wall of $\widetilde{M}$ such that a, y, z, x, b are aligned in this order. Up to extracting a subsequence,

let us assume that

$$\gamma_n(a, y, z, x, b) \xrightarrow[n \to \infty]{} (\alpha, \eta, \zeta, \xi, \beta) \in \partial\Omega \times \partial_i \tilde{M} \times \partial_i \tilde{M} \times \partial\Omega \times \partial\Omega.$$

Since $\xi \notin \partial_i \tilde{M}$, one then checks that $\alpha, \eta, \zeta, \xi, \beta$ are pairwise distinct.

Let K be the closure of $\tilde{M}$ in $\mathbb{S}^d$ and $f : K \to \partial\tilde{M}$ map each p to the entry point of $[\xi, p]$ in K. By convexity, this map is continuous. Moreover, $f(p)$ belongs to a wall of $\tilde{M}$ for every p in the interior of $\tilde{M}$; by continuity and connectivity they all belong to the same wall W. By continuity again $f(\eta) \in \partial W \cap [\zeta, \xi]$.

The segment $[\eta, f(\eta)]$ is non-trivial, contained in $\partial\tilde{M}$ and touches $\bar{W}$, hence it must be contained in $\bar{W}$. Thus $\xi \in \operatorname{Span} W$.

This is absurd: $\operatorname{Span}(\xi, y) \cap \operatorname{Span} W = \{\xi\}$ whence $f(y) \notin W$. $\square$

6.2. **Yaman's characterization of relative hyperbolicity.** We now give our working definition of relative hyperbolicity, following Yaman. This is not the original definition, which is due to Bowditch [23]. There are many definitions in the literature; the following one is equivalent to Bowditch's definition.

Definition 6.7 (Relative Hyperbolicity [69, Th. 0.1]). Let Y be a compact metrisable space and Γ a discrete group acting by homeomorphisms on Y.

- The action is called a *convergence action* if Γ acts properly on the set of distinct triples of Y.
- A point $p \in Y$ is called *conical* if there exists a sequence $\gamma_n \in \Gamma$ and $a \neq b \in Y$ such that $\gamma_n x \to a$ and $\gamma_n y \to b$ for any $y \neq x$.
- A point $p \in Y$ is called *parabolic* if $\operatorname{Stab}_\Gamma(p)$ is infinite and acts properly on $Y \smallsetminus \{p\}$.
- A parabolic point $p \in Y$ is called *bounded* if $\operatorname{Stab}_\Gamma(p)$ acts cocompactly on $Y \smallsetminus \{p\}$.
- The action is *geometrically finite* if it is a convergence action and all points are conical or bounded parabolic.

Suppose the action is geometrically finite, and all stabilizers of parabolic points are finitely generated, then

(1) Γ is *relatively hyperbolic* with respect to the stabilizers of parabolic points.
(2) Y is the *Bowditch boundary.*

Remark 6.8. In the setting of Definition 6.7, the action is *uniform* if all points of Y are conical. Then Γ is hyperbolic and Y is its Gromov boundary.

Let us now consider the convex projective setting.

Fix a properly convex open set $\Omega \subset \mathbb{S}^d$, a closed subset $F \subset \Omega$, and a discrete subgroup $\Gamma \subset \operatorname{SL}_{d+1}^\pm(\mathbb{R})$ preserving Ω and F. (In our case F

will always be $\partial\Omega$, except in Proposition 6.10 and Section 6.6.) Fix a Γ-invariant equivalence relation $\sim$ on F which is closed (in the sense that $\{(x,y) \in F^2 : x \sim y\} \subset F^2$ is closed) and such that $Y := F/_\sim$ contains at least 3 points.

We say that

- F satisfies the *convexity condition* if it contains all segment $[x,y] \subset \partial\Omega$ with $x,y \in F$.
- F satisfies the *visibility condition* it contains all segments of $\partial\Omega$ touching it.
- $\sim$ satisfies the *visibility condition* if $[x,y] \subset \partial\Omega$ implies $x \sim y$ for all $x,y \in F^2$.

These assumptions imply that the convex hull of F intersects Ω.

Fact 6.9 (Islam–Zimmer [35, Prop. 8.1-2]). *If the convexity condition on F and the visibility condition on $\sim$ are satisfied, then Y is a compact metrisable space on which the action of Γ is a convergence action.*

By the fact, in order to get relative hyperbolicity, it remains to understand conical and bounded parabolic points.

6.3. **Conicality.** Consider Ω, F, Γ, $\sim$ and $Y := F/_\sim$ as in the previous section.

We first give a criterion for conicality. It is similar to a classical criterion in real hyperbolic geometry (see for instance the discussion [61, p. 75]), and it also has a similar flavor to a result of Islam and Zimmer [35, Prop. 8.4] and the argument behind a proof of Weisman [68, Prop. 8.17].

For any $p \in Y$, denote by $\mathrm{Sing}_p \subset T^1\Omega$ the set of unit tangent vectors v such that $v^+, v^- \in p$. Moreover, set $\mathrm{Sing} := \bigcup_{p \in Y} \mathrm{Sing}_p$, which is a closed subset of $T^1\Omega$.

Proposition 6.10. *Suppose the convexity condition on F and the visibility condition on $\sim$. Consider $p \in Y$, $x \in p \subset F$, and $v \in T^1\Omega$ pointing forward at x and backward at $y \in F$ with $y \not\sim x$. Then p is conical if and only if the projection in $T^1\Omega/\Gamma$ of the forward geodesic orbit of v intersects infinitely often a compact subset of $(T^1\Omega \smallsetminus \mathrm{Sing})/\Gamma$.*

Proof. Suppose that the projection of the forward geodesic orbit of v intersects infinitely often a compact subset of $(T^1\Omega \smallsetminus \mathrm{Sing})/\Gamma$, i.e. there exists $\gamma_n \in \Gamma$ and t_n going to infinity such that $\gamma_n \phi_{t_n} v$ converges to $w \notin \mathrm{Sing}$. This implies that $\gamma_n x$ and $\gamma_n y$ converge respectively to $x' := w^+$ and $y' := w^- \in F$ with $x' \not\sim y'$. It suffices to prove that $\gamma_n z$ accumulates on $[y']$ for any $z \in F \smallsetminus p$. Up to extracting a subsequence we may assume that $\gamma_n z$ converges to z'. It is enough to check that $[y', z'] \subset \partial\Omega$. We can assume that $[y,z]$ intersects Ω. It is then enough to check that the Hilbert distance from the base point of $\gamma_n \phi_{t_n} v$ to $\gamma_n[y,z]$ tends to infinity, in other

words that the distance from $\phi_{t_n} v$ to $[y, z]$ tends to infinity. This is a consequence of the fact that $z \not\sim x$ (which implies that z and x do not have the same face).

Suppose that p is conical: there exists $\gamma_n \in \Gamma$ diverging and $a \neq b \in Y$ such that $\gamma_n p$ tends to a while $\gamma_n q$ tends to b for any $q \in Y \smallsetminus \{p\}$. Up to extracting a subsequence we may assume that $\gamma_n x$ and $\gamma_n y$ converge to respectively $x' \in a$ and $y' \in b$. The segment $[x', y']$ intersects Ω since $x' \not\sim y'$. Therefore there exists $t_n \in \mathbb{R}$ such that $\gamma_n \phi_{t_n} v$ converges to some unit tangent vector w. Since γ_n diverges, the sequence t_n accumulates on $\{\pm\infty\}$. Up to extracting again a subsequence we may assume that t_n converges. If by contradiction the limit was $-\infty$, then by the first step of the present proof, for any $c \in Y \smallsetminus \{y\}$ the sequence $\gamma_n c$ would converge to a, which is absurd. $\qquad\square$

6.4. **Technical lemmas for parabolicity.**

Fact 6.11. *Consider a properly convex open set Ω and a sequence of automorphisms $g_n \in \mathrm{GL}_{d+1}(\mathbb{R})$ that converges to a non-invertible non-zero matrix g. Then the kernel and image of g intersects $\overline{\Omega}$ but not Ω.*

Lemma 6.12. *Consider a properly convex open set $\Omega \subset \mathbb{S}^d$, a closed subset $F \subset \partial\Omega$ with the visibility condition (it contains all segments of $\partial\Omega$ touching it), and a closed subgroup $\Gamma \subset \mathrm{Aut}(\Omega)$ preserving F.*

Let $X \subset \mathbb{S}^d$ be the union of supporting hyperplanes of Ω at points of F (it is closed). Let $\mathcal{O}$ be the connected component of $\mathbb{S}^d \smallsetminus X$ that contains Ω. It is open, convex, Γ-invariant, and contains $\overline{\Omega} \smallsetminus F$. Then:

- *Either Γ acts properly on $\mathcal{O}$, and every Γ-orbit of a compact subset of $\mathcal{O}$ accumulates on F,*
- *or F is reduced to a point and Γ is virtually generated by a* rank-one *automorphism in the sense of Islam [33, Def. 6.2].*

Proof. Consider a diverging sequence $g_n \in \Gamma$ such that $\|g_n^{\pm 1}\|^{-1} g_n^{\pm 1}$ converges to a non-zero non-invertible matrix $g_\pm$. By Fact 6.11, the following four compact convex sets are non-empty and contained in $\partial\Omega$.

$$\emptyset \neq I_- := \mathrm{Im}(g_-) \cap \overline{\Omega} \ \subset \ K_+ := \mathrm{Ker}(g_+) \cap \overline{\Omega} \ \subset \partial\Omega$$

$$\emptyset \neq I_+ := \mathrm{Im}(g_+) \cap \overline{\Omega} \ \subset \ K_- := \mathrm{Ker}(g_-) \cap \overline{\Omega} \ \subset \partial\Omega$$

Let us prove that if Γ is not virtually generated by a rank-one element, then K_- and K_+ are contained in F. Suppose for instance that K_+ is not contained in F. Then those two compact convex subsets of $\partial\Omega$ are disjoint by the visibility condition on F. By definition of K_+ this means that $g_n F$ tends to $g_+(F) \subset I_+$ as $t \to \infty$. But $g_n F = F$ for all n so $F \subset I_+$, and hence $I_+ = K_- = F$ by the visibility condition on F. This implies that I_- and K_- are disjoint, and hence I_- admits a compact convex neighborhood U disjoint from $K_- = F$ such that $g_n^{-1} U \subset U$ for n large. By the Brouwer

fixed point theorem, g_n has a fixed point $x \in U$, and also one $y \in F$. By [17, Cor. 6.9], g_n is a rank-one automorphism and $F = \{y\}$, and hence Γ is a rank-one group that fixes y. This implies that Γ is virtually generated by g_n by [19, Prop. 2.4].

Suppose now that K_- and K_+ are contained in F. Let us consider a converging sequence $x_n \in \mathcal{O}$ with limit $x \in \mathcal{O}$ and prove that $g_n x_n$ converge to a point of F. Consider $y \in K_+$. The kernel $\mathrm{Ker}(g_+)$ is contained in a supporting hyperplane of Ω at K_+ so it does not intersects $\mathcal{O}$ (by definition), hence $g_n x_n$ tends to $g_+ x$ as $n \to \infty$. By definition of $\mathcal{O}$, the ray $[x, y) \subset \mathcal{O}$ enters $\overline{\Omega} \smallsetminus F$ at some point x'. Since $y \in K_+$, we have $g_+ x = g_+ x' = \lim_n g_n x' \in \overline{\Omega} \cap \mathrm{Im}(g_+) = K_+$. $\square$

Lemma 6.12 has the following corollary, where is exploited the assumption in Theorem 6.3 that the subgroups "with respect to which we want to prove relative hyperbolicity" have codimension 1; assumption that, we recall, compensate for the absence of convex-cocompactness.

Corollary 6.13. *Consider a properly convex open set Ω, a closed subset $F \subset \partial\Omega$ with the visibility condition, a discrete group $\Gamma \subset \mathrm{Aut}(\Omega)$ preserving F, a Γ-invariant closed equivalence relation $\sim$ on F with at least three equivalence classes and the visibility condition (see Section 6.2), and $Y := \partial\Omega/_\sim$.*

Pick a point $y \in Y$, seen as a closed subset of $\partial\Omega$, which is not fixed by a rank-one element of Γ (for instance if y contains at least three points). Then y is a parabolic point.

Suppose moreover that:

- *$F = \partial\Omega$.*
- *$\partial\Omega \smallsetminus y$ is homeomorphic to a finite disjoint union of copies of $\mathbb{R}^{d-1}$ (e.g. if y is a face of $\partial\Omega$).*
- *The virtual cohomological dimension of $\mathrm{Stab}_\Gamma(y)$ is equal to $d - 1$ (e.g. if y is a face on which $\mathrm{Stab}_\Gamma(y)$ acts properly cocompactly).*

Then y is bounded parabolic.

Proof. Lemma 6.12 tells us $\mathrm{Stab}_\Gamma(y)$ acts properly discontinuously on $F \smallsetminus y$.

Suppose by contradiction that two points $x, x' \in Y \smallsetminus \{y\}$ are dynamically related by $x_n \in Y \smallsetminus \{y\}$ converging to x (for Y's topology), $\gamma_n \in \mathrm{Stab}_\Gamma(y)$ diverging and $\gamma_n x_n$ converging to x'.

Pick any sequence $(p_n)_n \in \prod_n x_n$. By definition of the quotient topology on Y, up to extracting we can assume that it converges to $p \in x \subset \partial\Omega \smallsetminus y$, and that $\gamma_n p_n$ converges to $p' \in x' \subset \partial\Omega \smallsetminus y$. This means $\mathrm{Stab}_\Gamma(y)$ does not act properly on $\partial\Omega \smallsetminus y$, which is a contradiction.

The last conclusion of the corollary follows immediatly from the well-known fact that a discrete group of cohomological dimension $d - 1$ acting properly on $\mathbb{R}^{d-1}$ must act cocompactly. See for instance [57]. $\square$

6.5. **Proof of Theorem 6.3 and Fact 6.4.** We are going to prove both results at the same time. Let us call "walls" the hypersurfaces $W_1, \ldots, W_n \subset M$ in the setting of Fact 6.4, as well as their lifts in $\tilde{M}$.

A point of Y corresponds either to an extremal point of $\partial \tilde{M}$ or to a closed subset of $\partial \tilde{M}$ of the form $\bar{W} \cap \partial \tilde{M}$ where W is a wall of $\tilde{M}$.

By Fact 6.9, Y is compact metrisable and the action of $\pi_1(M)$ is a convergence action.

By Corollary 6.13, every point of Y corresponding to a wall of $\tilde{M}$ is bounded parabolic.

Thus, by the working definition of relative hyperbolicity, it is enough to check that every point $x \in \partial \tilde{M}$ outside of closed walls projects to a conical point of Y, using Proposition 6.10.

Fix compact pairwise disjoint neighborhoods for the walls of M. Pick also a point p in the interior of $\tilde{M}$.

By Proposition 6.10 it is enough to check that the projection in M of the ray $[p, x)$ passes infinitely often in the complementary of our fixed set of neighborhoods. Let us assume the contrary: that there exists $q \in [p, x)$ such that $[q, x)$ is contained in a lift U of a neighborhood of a wall; denote by $W \subset U$ the lift of the wall.

Let $q_n \in [q, x)$ converge to x. Since the stabilizer $\mathrm{Stab}_\Gamma(W)$ of W acts cocompactly on it and on U, there exists a diverging sequence $g_n \in \mathrm{Stab}_\Gamma(W)$ such that $g_n q_n$ stays in a compact set K of U.

Up to extracting a subsequence we may assume that $\|g_n\|^{-1} g_n$ converges to a matrix g. By Lemma 6.12, if Ω is the interior of $\tilde{M}$ then $\mathrm{Ker}(g) \cap \overline{\Omega}$ and $\mathrm{Im}(g) \cap \overline{\Omega}$ are contained in $\overline{W}$, which does not contain x. Thus $g_n q_n$ tends to gx, which is also the limit of $g_n x$ and hence belongs to the relative boundary of W, which does not meet K: contradiction.

6.6. **Extended geometrical finiteness.** Consider a properly convex open set $\Omega \subset \mathbb{S}^d$, a closed subset $F \subset \overline{\Omega}$ with the visibility condition (it contains all segments of $\partial \Omega$ touching it), and a discrete subgroup $\Gamma \subset \mathrm{Aut}(\Omega)$ preserving F. Fix a Γ-invariant closed equivalence relation $\sim$ on F with at least three equivalence classes and the visibility condition (see Section 6.2), and $Y := F/_\sim$.

Denote by $\mathbb{RP}^{d*}$ the set of hyperplanes of $\mathbb{RP}^d$. Let $F^* \subset \partial \Omega^*$ be the set of supporting hyperplanes intersecting F. Notice that F^* is a closed subset with the visibility condition, and that it carries a natural Γ-invariant closed equivalence relation with the visibility condition ($H \sim H'$ if there are $x \in H \cap F$ and $x' \in H' \cap F$ with $x \sim x'$), such that the map $F^*/_\sim \to Y$, mapping $[H]$ to $[x]$ when $x \in H$, is well-defined and is a homeomorphism.

Let $\mathcal{F} \subset \mathbb{RP}^d \times \mathbb{RP}^{d*}$ be the partial flag variety of pairs (x, H) such that $x \in H$. Set $\Lambda := (F \times F^*) \cap \mathcal{F}$. We have a natural Γ-equivariant surjective

continuous map $\phi : \Lambda \to Y$, sending (x, H) to the equivalence class of x. This map extends to a map $\phi_1 \times \phi_2 : F \times F^* \to Y$.

Our assumptions imply that ϕ is *antipodal*: any two points $(x, H), (x', H') \in \Lambda$ with different images by ϕ are *antipodal*, in the sense that $x \notin H'$ and $x' \notin H$.

Definition 6.14 (Weisman [68, Def. 1.2]). A map $\psi : \Lambda \to Y$ *extends the convergence action of Γ on Y* if for every $z \in \Lambda$ there exists an open subset $C_z \subset \mathcal{F}$ containing $\Lambda \smallsetminus \psi^{-1}(z)$ such that every diverging sequence $\gamma_n \in \Gamma$ admits a subsequence γ_{n_k} and $z_\pm \in Y$ such that $\gamma_{n_k} K$ accumulates on $\psi^{-1}(z_+)$ for every compact subset $K \subset C_{z_-}$.

Lemma 6.15. *The map ϕ extends the convergence action of Γ on Y.*

Proof. For any $z \in Y$, let $A_z \subset \mathbb{RP}^d$ (resp. $A_z^* \subset \mathbb{RP}^{d*}$) be the connected component containing Ω (resp. Ω^*) of the (open and convex) set of x (resp. H) such that $x \notin H'$ (resp. $x' \notin H$) for any $(x, H) \in \phi^{-1}(z)$.

Consider a diverging sequence $g_n \in \Gamma$.

Up to extracting a subsequence, we may assume that $\|g_n^{\pm 1}\|^{-1} g_n^{\pm 1}$ converges to a matrix $g_\pm$. Set $K_\pm = \operatorname{Ker}(g_\pm) \cap \partial\Omega$ and $I_\pm = \operatorname{Im}(g_\pm) \cap \partial\Omega$ (note that $I_\pm \subset K_\mp$).

By Fact 6.11, for every $x \in \Omega$ the sequence $g_n^{\pm 1} x$ tends to $g_\pm x \in I_\pm$. This applies in particular to any x in the convex hull of F, and the Γ-invariance of F implies that it intersects $K_\pm$, and hence contains it by the visibility condition on F.

The visibility condition on $\sim$ implies that ϕ sends all pairs $(x, H) \in \Lambda$ with $x \in K_\pm$ on the same point of $z_\mp \in Y$.

Let us check that $g_+ A_{z_-} \subset I_+$, with the consequence that $g_n K \to \phi_1^{-1}(z_+)$ for any compact subset $K \subset A_{z_-}$. Pick $x \in A_{z_-}$ and $y \in K_+$. By definition of A_{z_-}, the segment $[x, y]$ intersects Ω at some point p. Now any lifts of x and p in $\mathbb{R}^{d+1}$ differ by a vector of $\operatorname{Ker} g_+$, so $gx = xp \in I_+$.

A duality argument (involving g_n^{-1}) ensures that for any compact subset $K \subset A_{z_-}^*$, the orbit $g_n K$ accumulates on $\phi_2^{-1}(z_+)$, which is the set of supporting hyperplanes of Ω at $\phi_1^{-1}(z_+)$.

In conclusion, for any compact subset $K \subset (A_{z_-} \times A_{z_-}^*) \cap \mathcal{F}$, the orbit $g_n K$ accumulates on $\phi^{-1}(z_+)$, i.e. ϕ extends the convergence dynamics of Γ on Y. $\qquad\square$

Corollary 6.16. *If the action of Γ on Y is geometrically finite, as in the settings of Theorem 6.3 and Fact 6.4, then the action of Γ on $\mathcal{F}$ is (by definition) extended geometrically finite in the sense of [68, Def. 1.3].*

7. Projective gluings

In this section we describe a gluing construction whose input consists of projective manifolds with totally geodesic boundary and corners and whose

output is a convex projective cone-manifold. By translating the conditions of the previous sections in the language of gluings:

- We establish when the singularities of the glued projective cone-manifold can be blown up to totally geodesic boundary components.
- We describe the shape of the resulting convex domain, in particular, we characterize the flat regions of the boundary and the extremal and $\mathcal{C}^1$ points.

7.1. **Gluings.** We start with some topological preliminaries about gluings.

Definition 7.1 (Gluing). Given a collection of b-manifolds $\mathcal{M}$, a *gluing* is a smooth involution $f : \partial\mathcal{M} \to \partial\mathcal{M}$ that does not preserve any boundary component. We denote by $\mathcal{M}_f$ the quotient of $\mathcal{M}$ under the equivalence relation $x \sim f(x)$ for $x \in \partial\mathcal{M}$.

Every component $M \subset \mathcal{M}$ can be seen as an *immersed* codimension 0 submanifold of $\mathcal{M}_f$ called a *cell*. Similarly, every boundary component $W \subset \partial\mathcal{M}$ projects to an embedded codimension 1 submanifold of $\mathcal{M}_f$ called a *wall*. The *graph of cells* is the graph whose vertices are cells and whose edges are walls (a wall links the two adjacent cells).

7.1.1. *Universal cover.* Let $\mathcal{M}$ be a collection of b-manifolds and f a gluing of the boundary. Assume that the gluing $\mathcal{M}_f$ is connected. We now describe its universal cover

$$\pi : \widetilde{\mathcal{M}}_f \to \mathcal{M}_f.$$

The preimages of the interior of the cells and the preimages of the walls give a natural decomposition of $\widetilde{\mathcal{M}}_f$

$$\widetilde{\mathcal{M}}_f = \bigsqcup_{M \subset \mathcal{M}} \pi^{-1}(\text{int}(M)) \sqcup \bigsqcup_{M \subset \mathcal{M}} \pi^{-1}(\partial M).$$

into cells which are manifolds with boundary that cover the cells of $\mathcal{M}_f$ (the components of $\mathcal{M}$).

It is convenient to realize this decomposition as a gluing of a covering $\mathcal{N}$ of $\mathcal{M}$. We now explain more formally how to do that.

The covering $\mathcal{N}$ is a pullback

$$
\begin{array}{ccc}
\mathcal{N} & \longrightarrow & \widetilde{\mathcal{M}}_f \\
\downarrow & & \downarrow{\scriptstyle \pi} \\
\mathcal{M} & \xrightarrow{\ \pi_f\ } & \mathcal{M}_f
\end{array}
$$

More precisely, $\mathcal{N} \subset \widetilde{\mathcal{M}}_f \times \mathcal{M}$ is the set of pairs $(x, y) \in \widetilde{\mathcal{M}}_f \times \mathcal{M}$ such that $\pi(x) = \pi_f(y)$. It is elementary to check that the second coordinate projection $\mathcal{N} \to \mathcal{M}$ is a covering.

Then $\widetilde{\mathcal{M}}_f$ is naturally identified with $\mathcal{N}_g$, where g is the involution of $\partial \mathcal{N} = \mathcal{N} \cap (\widetilde{\mathcal{M}}_f \times \partial \mathcal{M})$ such that $g(x, y) = (x, f(y))$ for any (x, y).

In general, the cells and walls of $\mathcal{N}_g = \widetilde{\mathcal{M}}_f$ (i.e. the components of $\mathcal{N}$ and $\partial \mathcal{N}$) are not copies of the universal covers of the corresponding cells and walls of $\mathcal{M}_f$. We now show that this is the case when the gluing satisfies a π_1-injectivity property.

Conveniently, this can be explained using the theory of graph of spaces of Scott and Wall [54]: Let $\mathcal{G}$ be the graph of cells of $\mathcal{N}_g = \widetilde{\mathcal{M}}_f$ (vertices are components of $\mathcal{N}$ and edges are images in $\mathcal{N}_g$ of components of $\partial \mathcal{N}$).

Fact 7.2. *If $\mathcal{M}$ has π_1-injective boundary, then $\mathcal{G}$ is a tree, and each cell and wall of $\widetilde{\mathcal{M}}_f$ (i.e. each component of $\mathcal{N}$ and $\partial \mathcal{N}$) is simply connected.*

Proof. Each wall of $\mathcal{M}$ admits a tubular neighborhood, and hence $\mathcal{M}_f$ is a realization of the *graph of space* associated to $\mathcal{M}$ and f and their graph of cells, in the sense of Scott and Wall [54, p. 155]. According to Scott and Wall, this means that the fundamental group of $\mathcal{M}_f$ is isomorphic to the *fundamental group of the associated graph of groups*, in which Scott and Wall [54, Prop. 3.6] prove that the fundamental groups of each cell and wall injects (see also [56, §5 Cor. 1]). This proves that each component of $\mathcal{N}$ and $\partial \mathcal{N}$ is simply connected.

The fact that $\mathcal{G}$ is a tree comes from that $\widetilde{\mathcal{M}}_f$ is also the realization of a graph of spaces with underlying graph $\mathcal{G}$. Indeed since each cell of $\widetilde{\mathcal{M}}_f$ is simply connected, the fundamental group of $\widetilde{\mathcal{M}}_f$ (which is trivial) coincides with the fundamental group of $\mathcal{G}$ by [56, p. 61]. $\square$

The gluings that we will consider from now on will always satisfy the π_1-injectivity condition of Fact 7.2 thanks to the following classical fact.

Fact 7.3. *Any wall of a properly convex b-manifold is π_1-injective.*

Proof. Let M be a properly convex b-manifold with universal cover $\widetilde{M}$. Each lift in $\widetilde{M}$ of a wall of M is mapped homeomorphically by the developing map to a convex subset of a hyperplane. In particular, it is simply connected. $\square$

7.1.2. *Projective gluing.* Next, we introduce geometric structures on gluings:

Definition 7.4 (Projective Gluing)**.** Let $\mathcal{M}$ be a collection of b-manifolds. A *projective gluing* is the data of a gluing f together with a projective structure on $\mathcal{M}_f$ that restricts to the projective structure of each component of $\mathcal{M}$.

Not all gluings admit a compatible projective structure and, if it exists, it is not necessarily unique. Note for instance that if there is compatible projective structure, then $f : \partial \mathcal{M} \to \mathcal{M}$ is an isomorphism of projective $(d-1)$-manifolds (but this is not a sufficient condition in general).

7.1.3. *Admissible gluing.* We now come to gluings of projective manifolds with totally geodesic boundary and corners. They are defined as follows:

Definition 7.5 (Admissible Gluing). Let $\mathcal{MC} = \{\mathbb{M}_j\}_{j \in J}$ be a collection of bc-manifolds with smooth locus $\mathcal{M} = \{M_j \subset \mathbb{M}_j\}_{j \in J}$. A projective gluing f for $\mathcal{M}$ is said to be *admissible* if for each corner $\mathfrak{C}$ of $\mathcal{MC}$, we can find cells $\mathbb{M}_1, \cdots, \mathbb{M}_m \subset \mathcal{MC}$ such that each $\mathbb{M}_i$ has a corner $\mathfrak{C}_i$ (with $\mathfrak{C}_0 = \mathfrak{C}$) and two adjacent walls W_i^- and W_i^+ such that

- $fW_i^+ = W_{i+1}^-$ for each i.
- $f_{|W_i^+}$ extends to a homeomorphism $f_{|\mathfrak{C}_i} : \mathfrak{C}_i \to \mathfrak{C}_{i+1}$.
- $f_{|\mathfrak{C}_{m-1}} \circ \cdots \circ f_{|\mathfrak{C}_1} \circ f_{|\mathfrak{C}_0}$ is the identity map of $\mathfrak{C}_0$.

(All the indices are thought modulo m). We denote by $\mathcal{MC}_f$ the quotient of $\mathcal{MC}$ by the closure of the relation $x \sim f(x)$ for $x \in \partial \mathcal{M}$.

A *corner* or *singularity* of $\mathcal{MC}_f$ is the projection of a corner of $\mathcal{MC}$.

A choice of $\{(\mathbb{M}_i, W_i^\pm, \mathfrak{C}_i)\}_{i \leq m}$ as in the previous paragraph is called a *cell ordering* of the corner in $\mathcal{MC}_f$.

Admissible gluings provide natural examples of projective cone-manifolds:

Lemma 7.6. *Let $\mathcal{MC}$ be a collection of bc-manifolds with smooth locus $\mathcal{M}$. Let f be an admissible gluing. Then $\mathcal{MC}_f$ is a cone-manifold (Definition 4.6).*

Proof. Fix a corner $\mathfrak{C}$. Let $\{(\mathbb{M}_i, \mathrm{Wall}_i^\pm, \mathfrak{C}_i)\}_{i \leq n}$ be a cell ordering for $\mathfrak{C}$.

One can find an open neighborhood V of $\mathfrak{C}$ that satisfies the following:

- V does not intersect any other singularity.
- $V = V_1 \cup \cdots \cup V_n$ where each V_j is a tubular neighborhood of the lift $\mathfrak{C}_j \subset \mathbb{M}_j$ of $\mathfrak{C}$.
- $W_j^\pm := V_j \cap \mathrm{Wall}_j^\pm$ is a tubular neighborhood of $\mathfrak{C}_j$ in $\mathrm{Wall}_j^\pm$.
- The universal cover $\widetilde{V}$ of V can decomposed as a gluing of universal covers $\widetilde{V}_1, \cdots, \widetilde{V}_n$ via $f_i : \widetilde{W}_i^+ \to \widetilde{W}_{i+1}^-$ such that $f_n \circ \cdots \circ f_1(x) = x$ for any $x \in \widetilde{\mathfrak{C}}_1$.
- $\pi_1(\mathfrak{C})$ naturally acts on $\widetilde{\mathfrak{C}}$ and $\widetilde{V}$ and all the $\widetilde{\mathfrak{C}}_i$'s and $\widetilde{V}_i$'s and $\widetilde{W}_i^\pm$'s, so that $\alpha(f_i(x)) = f_i(\alpha(x))$ for any i and $x \in \widetilde{W}_i^+$.

Our goal is to find a tube T and a local homeomorphism $\widetilde{V} \to T$ which is projective on the smooth locus and equivariant with respect to a morphism $\pi_1(\mathfrak{C}) \to \mathrm{Aut}(T)$.

Fix a developing map $\mathrm{dev}^1 : \widetilde{V}_1 \to \mathbb{S}^d$ that sends $\widetilde{\mathfrak{C}}_1$ to $\mathbb{S}^{d-2}$. By induction, the projective structure on $\mathcal{M}_f$ determines for each $i \leq n$ a developing map $\mathrm{dev}^i : \widetilde{V}_i \to \mathbb{S}^d$ such that $\mathrm{dev}^{i-1} = \mathrm{dev}^i \circ f_{i-1}$ on $\widetilde{W}_{i-1}^+$. It also determines a new developing map of $\widetilde{V}_1$ of the form $\mathrm{dev}^{n+1} = g \circ \mathrm{dev}^1$ where $g \in \mathrm{SL}_{d+1}(\mathbb{R})$, such that $\mathrm{dev}^n = \mathrm{dev}^{n+1} \circ f_n$ on $\widetilde{W}_n^+$.

Claim 1. g fixes every point of $\mathbb{S}^{d-2}$.

Proof of the claim. For every $x \in \mathfrak{C}_1$ we have

$$
\begin{aligned}
g \, \mathrm{dev}^1(x) &= \mathrm{dev}^{n+1}(f_n \circ \cdots \circ f_2 \circ f_1(x)) \\
&= \mathrm{dev}^n(f_{n-1} \circ \cdots \circ f_2 \circ f_1(x)) \\
&= \mathrm{dev}^1(x). \qquad\qquad\qquad \square
\end{aligned}
$$

Up to making V smaller, we may assume that for each $1 \leq i \leq n+1$ the image of dev^i is contained in the sector S_i of $\mathbb{S}^d$ between the half-hyperplanes spanned by $\mathrm{dev}^i(\widetilde{W}_i^-)$ and $\mathrm{dev}^i(\widetilde{W}_i^+)$. We choose a lift $\widetilde{S}_1 \subset \mathbf{B} \sqcup \mathbb{S}^{d-2}$ of the first sector, which determines by induction lifts of $S_2, \ldots, S_{n+1}$ to sectors $\widetilde{S}_2, \ldots, \widetilde{S}_{n+1} \subset \mathbf{B} \sqcup \mathbb{S}^{d-2}$. Then dev^i lifts in a unique way to $\widetilde{\mathrm{dev}}^i : \widetilde{V}_i \to \widetilde{S}_i$. Similarly g lifts to a unique $\widetilde{g} \in \mathrm{Aut}(\mathbf{B})$ that maps $\widetilde{S}_1$ to $\widetilde{S}_{n+1}$ (and fixes $\mathbb{S}^{d-2}$).

Let T be the tube

$$
T := \widetilde{S}_1 \cup \cdots \cup \widetilde{S}_n / \sim
$$

where $\sim$ identifies via $\widetilde{g}$ the wall of $\widetilde{S}_1$ spanned by $\widetilde{\mathrm{dev}}^1(\widetilde{W}_1^-)$ with the wall of $\widetilde{S}_n$ spanned by $\widetilde{\mathrm{dev}}^n(\widetilde{W}_n^+)$. Denote by $\pi_T : \widetilde{S}_1 \cup \cdots \cup \widetilde{S}_n \to T$ the quotient projection. We have the following:

Claim 2. The map $F : \widetilde{V} \to T$ such that $F(x) = \pi_T(\widetilde{\mathrm{dev}}^i(\alpha x))$ for all i and $x \in \widetilde{V}_i$ is well-defined and is a local embedding that respects the projective structures.

Let $\mathrm{hol} : \pi_1(\mathfrak{C}) \to \mathrm{SL}_{d+1}\mathbb{R}$ be the holonomy of dev^1, which lifts to a holonomy $\widetilde{\mathrm{hol}} : \pi_1(\mathfrak{C}) \to \mathrm{Aut}(\mathbf{B})$ of $\widetilde{\mathrm{dev}}^1$. For any $\alpha \in \pi_1(\mathfrak{C})$, the new developing map $\widetilde{\mathrm{dev}}^1 \circ \alpha$ determines by induction as above new developing maps of $\widetilde{V}_2, \ldots, \widetilde{V}_n, \widetilde{V}_1$, which can be written $\widetilde{\mathrm{dev}}^i \circ \alpha$. On the other hand, the developing map $\widetilde{\mathrm{hol}}(\alpha) \circ \widetilde{\mathrm{dev}}^1$ clearly determines the developing maps $\widetilde{\mathrm{hol}}(\alpha) \circ \widetilde{\mathrm{dev}}^i$. We check that:

Claim 3. $\widetilde{\mathrm{hol}}(\alpha)$ is an automorphism of T.

Proof of the claim. Since $\widetilde{\mathrm{dev}}^1 \circ \alpha = \widetilde{\mathrm{hol}}(\alpha) \circ \widetilde{\mathrm{dev}}^1$, we have $\widetilde{\mathrm{dev}}^i \circ \alpha = \widetilde{\mathrm{hol}}(\alpha) \circ \widetilde{\mathrm{dev}}^i$ for all $2 \leq i \leq n+1$. As a consequence, $\widetilde{\mathrm{hol}}(\alpha)$ preserves each $\widetilde{S}_i$. Moreover, it commutes with $\widetilde{g}$ since

$$
\begin{aligned}
\widetilde{\mathrm{hol}}(\alpha) \circ \widetilde{g} \circ \widetilde{\mathrm{dev}}^1 &= \widetilde{\mathrm{hol}}(\alpha) \circ \widetilde{\mathrm{dev}}^{n+1} = \widetilde{\mathrm{dev}}^{n+1} \circ \alpha \\
&= \widetilde{g} \circ \widetilde{\mathrm{dev}}^1 \circ \alpha = \widetilde{g} \circ \widetilde{\mathrm{hol}}(\alpha) \circ \widetilde{\mathrm{dev}}^1.
\end{aligned}
$$

Thus $\widetilde{\mathrm{hol}}(\alpha)$ yields an automorphism $\phi(\alpha)$ of T. $\qquad\qquad \square$

It is clear that ϕ is a morphism. To conclude, let us check that:

Claim 4. $F \circ \alpha = \phi(\alpha) \circ F$.

Proof of the claim. Let $x \in \widetilde{V}$. We have $x \in \widetilde{V}_i$ for some i. Then $\alpha(x)$ is also in $\widetilde{V}_i$, and by definition

$$F(\alpha(x)) = \pi_T(\widetilde{\mathrm{dev}}^i(\alpha x)) = \pi_T(\widetilde{\mathrm{hol}}(\alpha)\widetilde{\mathrm{dev}}^i(x)) = \phi(\alpha)\pi_T(\widetilde{\mathrm{dev}}^i(x)). \quad \square$$

This concludes the proof of the lemma. $\qquad\qquad\qquad\qquad\qquad\square$

7.2. Polars and bulging.

We describe the notion of polars and gluings adapted to polars. They come in one-parameter families obtained via a bulging procedure. The goal of this section is to explain this picture.

Let $\mathcal{MC}$ be a collection of bc-manifolds. Let f be a gluing for $\mathcal{M}$.

Definition 7.7 (Polars)**.** A *polar point* of a wall $W \subset \mathcal{M}$ is the data, for each projective chart of W in $\mathbb{S}^d$, of a point of $\mathbb{RP}^d$ transverse to the image of the chart, which behaves well with respect to transition maps.

Each developing map $\widetilde{W} \to \mathbb{S}^d$ gives rise to a holonomy invariant polar.

A projective gluing $\mathcal{M}_f$ is said to be *adapted to a given choice of polars* if for any developing map $\widetilde{\mathcal{M}}_f \to \mathbb{S}^d$ and any wall $W \subset \partial\mathcal{M}$, the associated polars of W and fW coincide.

Remark 7.8. If all walls of $\mathcal{M}$ admit a polar, then there exists a projective gluing adapted to the polars if and only if f is a projective isomorphism of $(d-1)$-dimensional projective manifolds and for any wall $W \subset \partial\mathcal{M}$, denoting by ρ (resp. ρ') the *dilation character* of W (resp. fW), we have $\rho = \rho' \circ f_*$.

The dilation character is defined as follows: For each wall $W \subset \partial\mathcal{M}$ and developing map dev $: \widetilde{W} \to \mathbb{S}^d$, whose image spans $V \subset \mathbb{R}^{d+1}$, the *dilation character* of W is the determinant of the representation of $\pi_1(W)$ to $\mathrm{GL}(\mathbb{R}^{d+1}/V)$ induced by the holonomy representation. It is a real character that does not depend on the choice of dev.

If $\mathcal{M}$ is hyperbolic, then the dilation characters are trivial, the polars are the orthogonal subspaces for the Lorentzian bilinear form, and the unique hyperbolic gluing is adapted to this choice of polars.

If $\mathcal{M}_f$ is the double of a manifold M, and all boundary components of M admit polars, then there is a natural projective gluing adapted to the polars.

Let us finally recall the definition of bulging.

Definition 7.9 (Bulging)**.** Assume that each wall of $\mathcal{M}$ admits a polar, and that $\mathcal{M}_f$ admits a projective gluing adapted to the polars with atlas $\mathcal{A}$. Let $(\mu_W)_{W \subset \partial\mathcal{M}}$ be a family of positive real parameters. The *bulging of $\mathcal{M}_f$ along the polars with parameters* $(\mu_W)_{W \subset \partial\mathcal{M}}$ is the projective gluing which includes the following charts.

For any point $p \in \mathcal{M}_f$ which is the image of $\bar{p} \in W \subset M \subset \mathcal{M}$ and $f(\bar{p}) \in fW \subset M' \subset \mathcal{M}$, for any chart ϕ of $\mathcal{A}$ defined on a small enough

neighborhood $U \sqcup U' \subset M \sqcup M'$, the map $B \circ \phi_{|U} \cup B' \circ \phi_{|U'}$ is a chart of the bulging, where B (resp. B') is the linear transformation that fixes $\phi(U \cap W)$ (resp. $\phi(U' \cap fW)$) with eigenvalue μ_W^{-1} (resp. μ_{fW}^{-1}) and also fixes the polar, with eigenvalue μ_W^d (resp. μ_{fW}^d).

7.3. **Convexity of gluings.** Having described how an admissible gluing of bc-manifold gives rise to a projective cone-manifold, we now turn to the problem of determining when (the smooth locus of) such manifold is convex and when its decomposition into gluing pieces corresponds to a periodic tessellation of a convex domain in $\mathbb{S}^d$.

In order to do so, we revisit the results of the previous sections using the terminology of admissible projective gluings and Section 7.1.

First of all, we consider a single tile $\mathbb{M} \in \mathcal{MC}$, a properly convex bc-manifold with universal cover $\widetilde{\mathbb{M}} \subset \mathbb{S}^d$ and study its geometry. We describe under which conditions it satisfies the hypothesis of Proposition 5.1.

7.3.1. *Codimensions 1 and 2 strata.* Let $\mathbb{M}$ be a properly convex bc-manifold, with universal cover $\widetilde{\mathbb{M}} \subset \mathbb{S}^d$. First, we have to make sure that the walls and corners of $\mathbb{M}$ correspond to faces of $\widetilde{\mathbb{M}}$.

Definition 7.10 (Complete Walls and Corners)**.** A wall (resp. corner) of $\mathbb{M}$ is said to be *complete* if any lift in $\widetilde{\mathbb{M}}$ consists of a full codimension 1 (resp. codimension 2) face.

Completeness of walls and corners is not difficult to check:

Fact 7.11. *We have the following:*

- *Any convex hyperbolic bc-manifold which is complete for the hyperbolic metric also has complete walls and corners, in the sense of the previous definition.*
- *Any* compact *properly convex bc-manifold has complete walls and corners.*

Proof. The proof is subdivided into claims about $(d-2)$ and $(d-1)$-manifolds. The first one is classical and not proved here. It implies that complete convex hyperbolic bc-manifolds have complete corners.

Claim 1. Any hyperbolic $(d-2)$-manifold which is complete in the metric sense is also complete in the sense of (G, X)-structures: The developing map is a bijection onto $\mathbb{H}^{d-2}$.

Next we have another classical result about hyperbolic manifold. It implies that complete convex hyperbolic bc-manifolds have complete walls.

Claim 2. Let N be a complete hyperbolic $(d-1)$-manifold with totally geodesic boundary, with universal cover $\widetilde{N} \subset \mathbb{H}^{d-1}$. Then $\widetilde{N}$ is the biggest convex subset of $\mathbb{H}^{d-1}$ delimited by the walls of $\widetilde{N}$.

We now turn to the second point of the fact. The following classical fact implies that compact properly convex bc-manifolds have complete corners. We recall the proof for the reader's convenience.

Claim 3. Let Ω/Γ be a closed convex projective $(d-2)$-manifold. Then Ω is maximal for inclusion among Γ-invariant properly convex open subsets of $\mathbb{S}^{d-2}$.

Proof. Let Ω' be Γ-invariant and contain Ω. It suffices to show Ω is closed in Ω'. Let $x \in \Omega'$ be the limit of $x_n \in \Omega$. By compactness there is $\gamma_n \in \Gamma$ such that $\gamma_n x_n$ converges to $y \in \Omega$. Since Γ acts properly on Ω', the sequence γ_n converges in Γ and $x \in \Omega$. $\qquad\square$

To conclude, the last claim implies that compact properly convex bc-manifolds have complete walls.

Claim 4. Let N be a compact properly convex projective $(d-1)$-manifold with totally geodesic boundary, with universal cover $\tilde{N} \subset \mathbb{S}^{d-1}$ and holonomy $\Gamma < \mathrm{Aut}(\mathrm{int}(\tilde{N}))$. Then $\mathrm{int}(\tilde{N})$ is maximal among Γ-invariant properly convex open subsets of $\mathbb{S}^{d-1}$ delimited by the span of the walls of $\tilde{N}$.

Proof. Let $\Omega = \mathrm{int}(\tilde{N})$ and let $\Omega' \subset \mathbb{S}^{d-1}$ be Γ-invariant, open, properly convex, and delimited by the span of the walls of $\tilde{N}$. It suffices to show Ω is closed in Ω'. Let $x \in \Omega'$ be the limit of $x_n \in \Omega$. By compactness there is $\gamma_n \in \Gamma$ such that $\gamma_n x_n$ converges to some $y \in \tilde{N}$. Since Γ acts properly on Ω', if $y \in \Omega$ then γ_n converges in Γ and $x \in \Omega$. Assume that y lies in a wall W of $\tilde{N}$ (i.e. a component of $\tilde{N} \cap \partial\tilde{N}$).

Consider a small closed half-ball B centered at y and contained in $\tilde{N}$, whose boundary is decomposed into $\partial B = (B \cap W) \sqcup \Sigma$ where $\Sigma \subset \Omega$. One may check that $\gamma_n x_n \in \mathrm{int}(B)$ for n large and that the Hilbert distance $d_{\Omega'}(\gamma_n x_n, \Sigma)$ is bounded below by some constant ϵ independent of n. Note that $d_{\Omega'}(\gamma_n x_n, \gamma_n x) = d_{\Omega'}(x_n, x)$ tends to zero, and thus is less than ϵ for n large enough. Then $\gamma_n x \in \mathrm{int}(B)$, otherwise $[\gamma_n x_n, \gamma_n x]$ would cross Σ (it cannot cross $B \cap W$) and have length at least ϵ. Therefore $x \in \Omega$. $\qquad\square$

This concludes the proof of the fact. $\qquad\square$

7.3.2. *Ghost strata.* Let $\mathbb{M}$ be a properly convex bc-manifold which we now assume to have complete walls and corners which will lift to codimension 1 and 2 faces of the universal cover $\tilde{\mathbb{M}} \subset \mathbb{S}^d$.

The next issue that we want to address is the potential presence of codimension 2 faces that do not come from corners.

Definition 7.12 (Ghost Corner)**.** A *ghost stratum* is a face of $\tilde{\mathbb{M}}$ which is the intersection of at least two closed walls but which is *not* a closed corner. It is called a *ghost corner* if it has codimension 2.

Remark 7.13. Here is an example of ghost strata: Let $C \subset \mathbb{R}^3$ be the positive cone consisting of vectors $v = (v_1, v_2, v_3) \in \mathbb{R}^3$ such that $v_1, v_2, v_3 > 0$ or $v_1, v_2 > 0 = v_3$ or $v_1, v_3 > 0 = v_2$, and let $T := \mathbb{S}(C)$. Then $\{[e_1]\}$ is a ghost corner of T and its compact quotient M under the action of any diagonal matrix with positive decreasing diagonal entries. Moreover, $T \cup \{[e_1]\}$ is a properly convex bc-manifold with complete walls and corners and without ghost corner, but, since it is compact, M cannot be described as the complement of corners in a properly convex bc-manifold with complete walls and corners and without ghost corner.

Again, there is a simple criterion to check that there are no ghost strata in the hyperbolic setting:

Fact 7.14. *Let $\mathbb{M}$ be a complete convex hyperbolic bc-manifold. Then:*

- *Any ghost stratum has dimension 0.*
- *If $\mathbb{M}$ is compact then there is no ghost stratum.*

Proof. The universal cover $\widetilde{\mathbb{M}}$ is a closed convex subset of the hyperbolic space $\mathbb{H}^d$ with non-empty interior. Let S be a ghost stratum.

It is a proper face of $\widetilde{\mathbb{M}}$ and lies in $\partial \widetilde{\mathbb{M}} \smallsetminus \widetilde{\mathbb{M}} \subset \partial \mathbb{H}^d$, and hence is a point, since $\mathbb{H}^d$ is strictly convex.

Suppose by contradiction that $\mathbb{M}$ is compact. Then there is $r > 0$ such that for any ball B of $\mathbb{H}^d$ of radius r, if B intersects two walls of $\widetilde{\mathbb{M}}$ then these walls share a corner and B intersects no other wall. Fix $p \in \widetilde{\mathbb{M}}$.

If three walls were adjacent to S, then any ball of radius r with center in $[p, S)$ close enough to S would intersect all three walls, contradicting the definition of r.

Thus S is the intersection $\overline{W}_1 \cap \overline{W}_2$ where W_1, W_2 are two walls of $\widetilde{\mathbb{M}}$. As before any ball of radius r with center in $[p, S)$ close enough to S intersects both W_1 and W_2, making S an actual corner of $\widetilde{\mathbb{M}}$, which is absurd. $\square$

7.3.3. *Local convexity.* We are now ready to translate the local convexity conditions of Proposition 5.1.

Definition 7.15 (Containment Condition)**.** Let M be a properly convex b-manifold. We say that the polar point associated to a wall W of M satisfies the *Containment Condition* if for any lift $\widetilde{W}$ of W in $\widetilde{M}$, and any developing map dev : $\widetilde{M} \to \mathbb{S}^d$, the image of dev is contained in the convex hull of $\text{dev}(\widetilde{W})$ and the polar point.

Fact 7.16. *Let $\mathcal{M}$ be a collection of properly convex b-manifolds with polar points satisfying the Containment Condition, and consider a compatible projective gluing. Then the union of any two adjacent cells of the universal cover is convex.*

Proof. This is a consequence of Remark 5.6. $\square$

In the case of convex hyperbolic manifolds, the Containment Condition follows from angle assumptions on the corners.

Fact 7.17. *Let $\mathbb{M}$ be a complete hyperbolic bc-manifold and W a wall such that all adjacent corners have angle at most $\pi/2$. Then the hyperbolic polar point of W satisfies the Containment Condition.*

Proof. Suppose $\widetilde{\mathbb{M}} \subset \mathbb{H}^d$ and $\widetilde{W} \subset \mathbb{H}^{d-1}$. Let $p \in \mathbb{S}^d$ be one of the two hyperbolic polars, such that $\widetilde{\mathbb{M}}$ intersects the cone $C = \mathcal{CH}(p, \mathbb{H}^{d-1})$. Let $X \subset \mathbb{H}^d$ be the half-space with boundary $\mathbb{H}^{d-1}$ that contains $\widetilde{\mathbb{M}}$.

Fact 7.17 is a consequence of the classical facts that:

- X satisfies the Containment Condition with respect to $\mathbb{H}^{d-1}$ and p, i.e. $X \subset C = \mathcal{CH}(\mathbb{H}^{d-1}, p)$.
- For any $x \in \mathbb{H}^{d-1}$, the projective line through x and p intersects $\mathbb{H}^d$ in exactly the geodesic perpendicular to $\mathbb{H}^{d-1}$ at x.

Indeed, let $\{\widetilde{W}_i\}_i$ be the collection of walls of $\widetilde{\mathbb{M}}$ adjacent to $\widetilde{W}$. For each i let $H_i = \mathrm{Span}(\widetilde{W}_i) \cap \mathbb{H}^d$, let $H'_i \subset \mathbb{H}^d$ be the hyperplane orthogonal to $\mathbb{H}^{d-1}$ at $\widetilde{W} \cap \widetilde{W}_i$, and let X_i and X'_i be the half-spaces with boundary H_i and H'_i containing $\widetilde{\mathbb{M}}$.

Then $Y' := X \cap \bigcap_i X'_i$ is the preimage of $\widetilde{W}$ by the orthogonal projection on $\mathbb{H}^{d-1}$, and it satisfies the Containment Condition with respect to $Y' \cap \mathbb{H}^{d-1} = \widetilde{W}$, i.e. $X' \subset \mathcal{CH}(\widetilde{W}, p)$.

Moreover, Y' contains $Y := X \cap \bigcap_i X_i$ which contains $\widetilde{\mathbb{M}}$ since all corners adjacent to $\widetilde{W}$ have angle at most $\pi/2$. Thus $\widetilde{\mathbb{M}}$ must also satisfy the Containment Condition. $\qquad\square$

7.3.4. *Global convexity.* We now rephrase Proposition 5.1 using the terminology that was introduced in the previous sections.

Theorem 7.18. *Let $\mathcal{MC}$ be a collection of bc-manifolds. Let f be a projective gluing on $\mathcal{M}_f$. Assume that:*

(1) *$\mathcal{M}_f$ is connected.*
(2) *All cells are properly convex, with complete walls and corners (Definition 7.10), and without ghost corners (Definition 7.12).*
(3) *The union of any two adjacent cells in the universal cover is convex (satisfied for instance if there are polar points satisfying the Containment Condition).*
(4) *Each singularity is uniformisable in $\mathbb{S}^d$ (see Proposition 4.7).*

Then $\mathcal{M}_f$ is convex.

Proof. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be the graph of cells associated to the gluing of $\widetilde{\mathcal{M}}_f$: each $v \in \mathcal{V}$ corresponds to a cell of $\widetilde{\mathcal{M}}_f$ and each $e \in \mathcal{E}$ corresponds to a wall of $\mathcal{M}_f$ (see Definition 7.1 and Section 7.1.1). Fix a developing map

$\widetilde{\mathcal{M}}_f \to \mathbb{S}^d$, and for each $v \in \mathcal{V}$ let D_v be the closure in $\mathbb{S}^d$ of the image of v by the developing map.

It is enough to check that $(\mathcal{G}, \{D_v\}_{v \in \mathcal{V}})$ satisfies the assumptions of Proposition 5.1, i.e. that it is a gluing kit (Definition 5.3). Note that connectedness of $\mathcal{G}$ is an immediate consequence of Property (1).

Assumption (1) of Proposition 5.1. This follows from the fact that walls of $\mathcal{M}$ are complete (Property (2)) and Property (3).

Assumption (2) of Proposition 5.1. Let $p = (v_1, v_2, \ldots, v_n) \subset \mathcal{G}$ be a path such that $F_p := D_{v_1} \cap \cdots \cap D_{v_n}$ is a codimension 2 face of D_{v_i} for every i. Since $\mathcal{M}$ has no ghost corner (Property (2)), $\text{int}(F_p)$ is a lift of a corner of $\mathcal{MC}_f$. As a consequence, p may be extended to a bi-infinite path $q = (v_k)_{k \in \mathbb{Z}}$ such that $\bigcap_{k \in \mathbb{Z}} D_{v_k} = F_p$. Then Property (4) implies by Proposition 4.7 that the sequence $(D_{v_k})_{k \in \mathbb{Z}}$ satisfies the assumptions of Proposition 5.20, whose conclusion is that F_p is contained in the boundary of a half-space of $\mathbb{S}^d$ containing $\bigcup_{n \in \mathbb{Z}} D_{v_k}$, as required by Proposition 5.1. $\quad\square$

7.4. **Totally geodesic blowups of gluings.** As we have seen in Lemma 7.6, admissible projective gluings give rise to projective cone-manifolds. Hence, as proved in Section 4, under suitable conditions on the holonomy of the meridians of the corners, we can blow up each singularity to a totally geodesic boundary component and we can also give an explicit description of such components in terms of the geometry of the corners. This is the content of the next theorem.

Theorem 7.19. *In the setting of Theorem 7.18, assume further that*

- *$\mathcal{M}_f$ is properly convex (for instance if one of the cells has an irreducible holonomy).*
- *Each singularity of $\mathcal{MC}_f$ is special (Definition 4.10).*

Then the following holds:

(1) *The totally geodesic blowup N of the singularities of $\mathcal{MC}_f$ (Definition 4.13) has complete walls and no ghost corners. If each cell of $\mathcal{M}$ has no ghost stratum then so does N.*

(2) *For each wall W of N blowing up a corner $\mathfrak{C}$ there is a developing map of N sending $\widetilde{W}$ onto the interior of $\text{Conv}(\widetilde{\mathfrak{C}}, [e_d])$ such that the holonomy of $\pi_1(W)$ is generated by that of the corner*

$$\gamma \in \pi_1(\mathfrak{C}) \longmapsto \begin{pmatrix} \phi(\gamma)^{-1}\rho'(\gamma) & \\ & \phi(\gamma)^{\frac{d-1}{2}}\mathbb{I}_2 \end{pmatrix} \in \text{SL}_{d+1}(\mathbb{R})$$

and of the meridian

$$\begin{pmatrix} \mu^{-1}\mathbb{I}_{d-1} & & \\ & \mu^{\frac{d-1}{2}}\lambda & \epsilon \\ & & \mu^{\frac{d-1}{2}}\lambda^{-1} \end{pmatrix} \in \text{SL}_{d+1}(\mathbb{R}),$$

where ρ' is a morphism, ϕ a positive character, $\lambda \geq 1$, $\epsilon = 1$ if $\lambda = 1$ and 0 otherwise, and $\mu > \lambda^{\frac{d+1}{2}}$.

(3) If $\lambda > 1$ then $[e_{d+1}]$ yields a polar of W which satisfies the Containment Condition.

Proof. We start with the description of the developing map of N sending $\widetilde{W}$ onto $\mathrm{Conv}(\widetilde{\mathfrak{C}}, [e_d])$.

By Lemma 7.6, $\mathcal{MC}_f$ is a cone-manifold: Every point of $\mathfrak{C}$ admits a neighborhood that embeds into a tube $\mathbb{T} = T \sqcup \mathbb{S}^{d-2}$. Thus we can find a tubular neighborhood $\mathbb{U} \subset \mathcal{MC}_f$ of $\mathfrak{C}$ and a developing map $\mathrm{dev}_1 : \widetilde{\mathbb{U}} \to \mathbb{T}$ with holonomy in the automorphism group of $\mathbb{T}$. Since each cell of $\mathcal{M}$ is properly convex, $\mathfrak{C}$ is uniformisable in $\mathbb{S}^{d-2}$ so dev_1 is injective on $\widetilde{\mathfrak{C}}$. Up to making $\mathbb{U}$ smaller we can assume that dev_1 is injective, which identifies $\widetilde{\mathbb{U}}$ with a subset of $\mathrm{dev}_1(\widetilde{\mathbb{U}}) := \mathbb{V} \subset \mathbb{T}$ (note that $\mathbb{V}$ is the universal cover of $\mathbb{U}$ but the smooth locus V is *not* the universal cover of the smooth locus U).

Let $\pi : T^b \to \mathbb{T} = T \cup \mathbb{S}^{d-2}$ be the totally geodesic blowup of $\mathbb{T}$. By definition, the totally geodesic blowup V^b of $\mathbb{V}$ is $\pi^{-1}(\mathbb{V})$. , and there is a developing map of the universal cover of the totally geodesic blowup U^b of $\mathbb{U}$ denoted by $\mathrm{dev}_2 : \widetilde{U}^b \to V^b$ whose holonomy consists of automorphisms of $\mathbb{T}$.

There is a developing map $\mathrm{dev}_3 : \widetilde{T}^b \to \mathbb{S}^d$ that sends the boundary onto $\mathrm{Conv}(\mathbb{S}^{d-2}, [e_d])$ such that the holonomy of the meridian has the form

$$g = \begin{pmatrix} \mu^{-1}\mathbb{I}_{d-1} & & \\ & \mu^{\frac{d-1}{2}}\lambda & \epsilon \\ & & \mu^{\frac{d-1}{2}}\lambda^{-1} \end{pmatrix} \in \mathrm{SL}_{d+1}(\mathbb{R}),$$

with $\lambda \geq 1$, $\epsilon = 1$ if $\lambda = 1$ and 0 otherwise, and $\mu > \lambda^{\frac{d+1}{2}}$, since by assumption each singularity of $\mathcal{MC}_f$ is special.

The map dev_2 lifts to a developing map $\mathrm{dev}_4 : \widetilde{U}^b \to \mathbb{S}^d$.

When restricted to a cell X of $\widetilde{\mathcal{M}}_f$ (smooth locus of $\mathbb{X}$, which is a universal cover of a cell of $\mathcal{MC}$), dev_4 extends to a developing map dev_5 of $\mathbb{X}$ that sends $\widetilde{\mathfrak{C}}$ onto $\widetilde{C} \subset \mathbb{S}^{d-2}$, which in fine does not depend on X.

The map dev_4 sends the boundary of $\widetilde{U}^b$ onto $\mathrm{Conv}(\widetilde{C}, [e_d])$, and the holonomy of the meridian is g, and the holonomy of $\mathfrak{C}$ is in $\mathrm{Aut}(\mathbf{B})$ and commutes with g, so it preserves $\mathbb{S}^{d-2}$ and $[e_d]$ and $\mathrm{Span}([e_d], [e_{d+1}])$. Since this holonomy must also preserve the image by dev_5 of the walls of $\mathbb{X}$ adjacent to $\widetilde{\mathfrak{C}}$, it has to fix every point of $\mathrm{Span}([e_d], [e_{d+1}])$ and hence have the form stated in Theorem 7.19.

By Property 1 of Proposition 5.22, $\mathrm{Conv}(\widetilde{C}, [e_d])$ corresponds to a cell at infinity of the associated gluing kit $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Property 2 of Proposition 5.22 says moreover that such a cell at infinity is a full closed face of $\partial\widetilde{N}$, which implies that N has complete walls.

By Property 1 of Proposition 5.19 and the above, a ghost stratum of $\widetilde{N}$ (the intersection of at least two closed walls of $\widetilde{N}$) is in fact the intersection of at least two corners of $\widetilde{\mathcal{MC}}_f$. This implies that N has no ghost corners, and moreover no ghost stratum at all if $\mathcal{M}$ has no ghost stratum.

That each wall of N above a singularity with hyperbolic SL_2-angle admits a polar point satisfying the Containment Condition is an immediate consequence of Property 3 of Proposition 5.22.   □

Remark 7.20. In the setting of Theorem 7.19, if every singularity has hyperbolic SL_2-angle then the double P of N admits a convex projective structure.

7.5. **Stratification of the boundary for a gluing.** Lastly, after blowing up the singularity of the convex projective cone-manifold given by a suitable admissible gluing, we want to describe the geometry of its universal cover. In order to do so, we have to understand the stratification of the boundary. We do this exploiting the work we developed in Section 5 as stated in the next result:

Theorem 7.21. *In the setting of Theorem 7.19, suppose that:*

- *For each cell $M \subset \mathcal{M}$ with universal cover $\widetilde{M}$, each non-trivial face of $\widetilde{M}$ touching a wall is contained in a wall.*
- *For each cell $M \subset \mathcal{M}$ with universal cover $\widetilde{M}$, each non-trivial face of $\widetilde{M}$ touching a corner is a wall adjacent to the corner or is contained in the corner.*
- *There is no ghost stratum.*
- *$\mathcal{MC}_f$ is compact.*

Then the associated gluing kit satisfies the assumptions of Proposition 5.28, so any cell at infinity which is not a wall of the totally geodesic blowup $\widetilde{N}$ consists of an extremal $\mathcal{C}^1$ point.

Proof. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be the gluing kit associated to $(\mathcal{M}, f)$, so that $\widetilde{M}_f = \mathring{X}$ where $X := \bigcup_{D \in \mathcal{V}} D$. Let us see how to derive the above theorem from Lemmas 5.26 and 5.27 and from our compactness assumption.

Fix a smooth structure on $\mathcal{MC}_f$ that extends that of $\mathcal{M}_f$ by using a nice parametrization. Fix also a Riemannian metric.

The group $\Gamma := \pi_1(\mathcal{M}_f)$ acts on the set of pairs (D, W) where D is a cell of $\mathcal{G}$ and W is an adjacent wall, and this action admits finitely many representatives $(D_1, W_1), \ldots, (D_n, W_n)$ since $\mathcal{MC}_f$ is compact.

Fix i. Choose for each wall W of D_i not adjacent to W_i, in a $\mathrm{Stab}_\Gamma(W_i)$-equivariant way, a segment from W_i to W with smallest length for the Riemannian metric; denote by v_W the starting vector (transverse to W_i) and by $y_W \in W$ the ending point of the segment. By compactness of $\mathcal{MC}_f$, we can find a compact set K of transverse vectors to W_i and a set $\mathcal{W}_i$ of $\mathrm{Stab}_\Gamma(W_i)$-representatives of walls W of D_i non-adjacent to W_i such that

$v_W \in K$ for any $W \in \mathcal{W}_i$. Then there exists an open neighborhood U_i of W_i such that $y_W \notin U_i$ for any $W \in \mathcal{W}_i$. By Lemma 5.27, we can find an open neighborhood U_i' of W_i such that $D_i \smallsetminus U_i'$ is convex and contains walls in $\mathcal{W}_i$. By Lemma 5.26, $(W_i, \bar{X}, D_i \smallsetminus U_i')$ is contracting.

By compactness, there is a finite number of $\mathrm{Stab}_\Gamma(W_i)$-representatives $\{W_{i,j}\}_j$ of walls of D_i adjacent to W_i. Fix j. Let $D_{i,j}$ be the cell other than D_i which is adjacent to $W_{i,j}$ and $\mathfrak{C}_{i,j} := W_i \cap W_{i,j}$. Choose for each corner $\mathfrak{C}$ of $W_{i,j}$ other than $\mathfrak{C}_{i,j}$, in a $\mathrm{Stab}_\Gamma(W_i, \mathfrak{C}_{i,j})$-equivariant way, a segment from $\mathfrak{C}_{i,j}$ to $\mathfrak{C}$ with smallest length for the Riemannian metric; denote by $v_{\mathfrak{C}}$ the starting vector (transverse to $\mathfrak{C}_{i,j}$) and by $y_{\mathfrak{C}} \in \mathfrak{C}$ the ending point of the segment.

By compactness, we can find a compact set K of vectors tangent to $W_{i,j}$ and transverse to $\mathfrak{C}_{i,j}$ and a set $\mathcal{C}_{i,j}$ of $\mathrm{Stab}_\Gamma(W_i, \mathfrak{C}_{i,j})$-representatives of corners $\mathfrak{C}$ of $W_{i,j}$ other than $\mathfrak{C}_{i,j}$ such that $v_{\mathfrak{C}} \in K$ for any $\mathfrak{C} \in \mathcal{C}_{i,j}$. Then there exists an open neighborhood $U_{i,j}$ of W_i such that $y_{\mathfrak{C}} \notin U_i$ for any $\mathfrak{C} \in \mathcal{C}_{i,j}$. By Lemma 5.27, we can find an open neighborhood $U_{i,j}'$ of W_i such that $D_i \cup D_{i,j} \smallsetminus U_{i,j}'$ is convex and contains walls adjacent to a corner of $\mathcal{C}_{i,j}$, other than $W_{i,j}$. By Lemma 5.26, $(W_i, \bar{X}, (D_i \cup D_{i,j}) \smallsetminus U_{i,j}')$ is a visible triple.

Let us now stop fixing i, j, and let $\mathcal{K}$ be the set of visible triples of the form $(W_i, \bar{X}, M)$ where $M \subset D_i \smallsetminus U_i'$ or $(D_i \cup D_{i,j}) \smallsetminus U_{i,j}'$ for some i, j.

Let A, B, C, D be a path of cells of $\mathcal{G}$ such that $A \cap B \cap C \cap D$ is empty.

- if $A \cap B \cap C$ is empty, then do the following. Find i and $g \in \Gamma$ such that $(B, A \cap B) = g(D_i, W_i)$. Find $W \in \mathcal{W}_i$ and $h \in \mathrm{Stab}_\Gamma(W_i)$ such that $B \cap C = ghW$. Then $(A \cap B, \bar{X}, C \cap D) \in gh\mathcal{K}$;
- if $B \cap C \cap D$ is empty, then apply the previous point to the path D, C, B, A, so that $(C \cap D, \bar{X}, A \cap B) \in \mathrm{GL}_{d+1}(\mathbb{R}) \cdot \mathcal{K}$;
- if $A \cap B \cap C$ and $B \cap C \cap D$ are non-empty, then do the following. Find i and $g \in \Gamma$ such that $(B, A \cap B) = g(D_i, W_i)$. Find j and $h \in \mathrm{Stab}_\Gamma(W_i)$ such that $B \cap C = hgW_{i,j}$ and $A \cap B \cap C = gh\mathfrak{C}_{i,j}$. Find $\mathfrak{C} \in \mathcal{C}_{i,j}$ and $k \in \mathrm{Stab}_\Gamma(W_i, \mathfrak{C}_{i,j})$ such that $B \cap C \cap D = ghk\mathfrak{C}$. Then $(A \cap B, \bar{X}, C \cap D) \in ghk\mathcal{K}$. $\qquad\square$

8. Hyperbolic building blocks

In this section we construct the hyperbolic building blocks that serve as input for the construction in the proof of Theorems 1.1 and 1.2. In particular we describe the results of Bonahon and Otal [21] and we prove Theorem 1.8.

8.1. **Dimension** $d = 3$. In dimension $d = 3$ there is a lot of flexibility in the choice of building blocks.

8.1.1. *3-manifold topology.* The existence of a convex hyperbolic metric with totally geodesic boundary bent along the some corners on a compact 3-manifold $(M, \partial M)$ such that the pleating locus coincides with a multicurve $\alpha = \alpha_1 \sqcup \cdots \sqcup \alpha_n \subset \partial M$ can be reduced to an explicit purely topological problem by work of Bonahon and Otal [21].

Before stating their results, we briefly review the necessary terminology and notion from 3-dimensional topology. For a more in-depth discussion, we refer to [47, Ch 9].

Definition 8.1 (Irreducible). An orientable 3-manifold M is *irreducible* if every embedded 2-sphere $\mathbb{S}^2 \subset M$ bounds a 3-ball $\mathbb{B}^3 \subset M$.

Definition 8.2 (Atoroidal). An orientable irreducible 3-manifold M is *atoroidal* if every embedded π_1-injective 2-torus $\mathbb{T}^2 \subset M$ is isotopic to a component of ∂M.

Definition 8.3 (Doubly Incompressible). Let M be a compact 3-manifold with boundary $\partial M \neq \emptyset$. An essential multicurve $\alpha = \alpha_1 \sqcup \cdots \sqcup \alpha_n \subset \partial M$ is *doubly incompressible* if:

(1) We have
$$\sum_{j \leq n} i(\alpha_j, \partial A) > 0$$
for every properly embedded essential annulus or Möbius band $(A, \partial A) \subset (M, \partial M)$.

(2) We have
$$\sum_{j \leq n} i(\alpha_j, \partial D) > 2$$
for every properly embedded essential disk $(D, \partial D) \subset (M, \partial M)$.

Here $i(\bullet, \bullet)$ denotes the *geometric intersection number* between closed curves on ∂M (see [47, Ch. 8]).

The notion of doubly incompressible multicurve was introduced by Thurston in [60]. It is well-known that doubly incompressible multicurves are abundant, in fact, there is an open set $\mathcal{O} \subset \mathcal{PML}$ of the space of projective measured laminations on ∂M (see Lecuire [42]), such that every multicurve $\alpha \in \mathcal{O}$ satisfies such property. We refer to [47, Ch. 8] for an introduction to (projective) measured laminations.

8.1.2. *Convex hyperbolic 3-manifolds with prescribed corners.* The existence of hyperbolic structures on $(M, \partial M)$ is a consequence of Thurston's Hyperbolization Theorem:

Fact 8.4 (Thurston, see [38, Th. 1.43]). *Let M be a compact orientable irreducible atoroidal 3-manifold with non-empty boundary $\partial M \neq \emptyset$. Let T be the union of all toroidal components of ∂M. Then $M - T$ admits a complete convex hyperbolic structure of finite volume, that is $M - T$ is*

homeomorphic to a quotient $\mathcal{C}/\Gamma$ where $\mathcal{C} \subset \mathbb{H}^3$ is a Γ-invariant convex set and $\Gamma < \mathrm{Isom}^+(\mathbb{H}^3)$ is a discrete and torsion free subgroup.

If M has a convex hyperbolic metric of finite volume as in Fact 8.4, then it has a large deformation space of such metrics.

In particular, Bonahon and Otal proved:

Fact 8.5 (Bonahon–Otal [21, Th. 2])**.** *Let M be a compact orientable 3-manifold with non-empty boundary $\partial M \neq \emptyset$ such that M is irreducible and atoroidal. Let $\alpha_1 \cup \cdots \cup \alpha_n \subset \partial M$ be a multicurve and let $\theta_1, \cdots, \theta_n \in [0, \pi]$ be a set of angles. Suppose that the following properties hold:*

(1) We have

$$\sum_{j \leq n} \theta_j i(\alpha_j, \partial A) > 0$$

for every properly embedded essential annulus or Möbius band $(A, \partial A) \subset (M, \partial M)$.

(2) We have

$$\sum_{j \leq n} \theta_j i(\alpha_j, \partial D) > 2\pi$$

for every properly embedded essential disk $(D, \partial D) \subset (M, \partial M)$.

Let $\mu \subset \alpha$ be the subset of curves with angle π. Then, there exists a convex hyperbolic metric on $M - \mu$, unique up to isotopy, such that

- *Each curve $\alpha_j \subset \alpha - \mu$ is a geodesic.*
- *The closure of each component of $\partial M - \alpha$ is totally geodesic.*
- *The pleating angle between the components adjacent to α_j of $\alpha - \mu$ is $\pi - \theta_j$.*
- *Every curve $\alpha_j \subset \mu$ is a rank one cusp.*

The statement of Fact 8.5 is not phrased in the same language used in Theorem 2 of [21], but is equivalent to it. We formulated it in a language adapted to this paper.

Endowed with such hyperbolic metric M is a hyperbolic 3-manifold with totally geodesic boundary and prescribed corners $\alpha_j \subset \partial M$ and corner angles θ_j.

It is immediate to check that if a multicurve $\alpha = \alpha_1 \cup \ldots \cup \alpha_n$ is doubly incompressible, then it is enough to choose angles θ_j very close to π to be sure that the requirements of Fact 8.5 are also fullfilled.

Note that Fact 8.5 guarantees some freedom in the choice of the angles at the corners. This is an important feature and we will use it to obtain a certain degree of flexibility in the choice of the projective structure and holonomy of the totally geodesic boundary tori T_j in Theorem 1.2.

8.1.3. *Deformation space.* The space of isotopy classes of hyperbolic metrics on M with totally geodesic boundary and corners or rank one cusps at the multicurve $\alpha = \alpha_1 \cup \cdots \cup \alpha_n$ has a natural topology induced by the topology of algebraic convergence of the holonomy representations $\rho : \pi_1(M, \star) \to \mathrm{Isom}^+(\mathbb{H}^3)$ where $\star \in \mathrm{int}(M)$ is a fixed basepoint. The next result of Bonahon and Otal gives a simple description of convergence of metrics in terms of convergence of angles:

Fact 8.6 (Bonahon–Otal [21, Th. 24]). *Let $\alpha = \alpha_1 \cup \cdots \cup \alpha_n \subset \partial M$ be a doubly incompressible multicurve. The map that associates to each hyperbolic metric on M with totally geodesic boundary and corners or rank one cusps at the multicurve α the pleating angles $(\theta_1, \cdots, \theta_n) \in (0, \pi]^n$ is a homeomorphism onto an open polytope $P_\alpha \subset (0, \pi]^n$. The image of those metrics that have rank one cusps at the curves $\alpha_{j_1}, \cdots, \alpha_{j_k}$ is contained in the subset with $\theta_{j_1}, \cdots, \theta_{j_k} = \pi$.*

Again, the statement of Fact 8.6 is not literally the one of [21, Th. 24], but is equivalent to it.

Using a different set of parameters, Choi and Series [28] showed that the lengths of the pleating locus of the boundary give a local parametrization of the same space of metrics considered by Bonahon and Otal [21].

Fact 8.7 (Choi–Series [28, Th. A]). *Let $\alpha = \alpha_1 \cup \cdots \cup \alpha_n \subset \partial M$ be a doubly incompressible multicurve. The map that associates to each hyperbolic metric on M with totally geodesic boundary and corners or rank one cusps at the multicurve α the pleating lengths $(\ell_1, \cdots, \ell_n) \in [0, \infty)^n$ is an injective local homeomorphism. The image of those metrics that have rank one cusps at the curves $\alpha_{j_1}, \cdots, \alpha_{j_k}$ is contained in the subset with $\ell_{j_1}, \cdots, \ell_{j_k} = 0$.*

Fact 2.1 from the introduction is a combination of Facts 8.6 and 8.7 (note that the pleating angle is π minus the corner angle).

8.2. **Dimension $d \geq 3$.** In this section we prove Theorem 1.8.

For convenience of the reader, we briefly recall our goal: We want to find a closed hyperbolic d-manifold M which contains k totally geodesic connected embedded hypersurfaces $N_1, \cdots, N_k$ whose pairwise intersection $C = N_i \cap N_j$ is a (fixed) codimension 2 connected totally geodesic submanifold and such that the angles of intersection are $\angle N_j N_{j+1} = \pi/k$ for every $j \leq k$ (indices modulo k). Furthermore, we want that the hypersurfaces N_j are pairwise isometric and related by a cyclic isometry ρ of M that fixes C pointwise and maps $\rho(N_j) = N_{j+1}$.

When the dimension is $d \geq 4$, hyperbolic d-manifolds with patterns of totally geodesic hypersurfaces can be obtained by using arithmetic techniques and separability properties of arithmetic lattices as in Gromov-Thurston [32] and Kapovich [39] and, in fact, our construction is inspired by those. Let

us point out that, while we have little control on the topology and geometry of these examples, we can, crucially, make sure that two properties are satisfied: The pattern and the angles at the intersections are controlled.

Our strategy is as follows: For every k we consider an explicit cocompact arithmetic lattice $G < \mathrm{SO}_0(d,1)$ defined in terms of a suitable quadratic form q. We first construct explicitly a convex-cocompact subgroup $Q < G$ such that $\mathbb{H}^d/Q$ contains, by construction, the desired configuration of k totally geodesic embedded hypersurfaces $N_1, \cdots, N_k$ intersecting exactly along a codimension 2 totally geodesic corner $C = N_1 \cap \cdots \cap N_k$ with angles of $\angle N_j N_{j+1} = \pi/k$. Then we use strong separability properties of G due to Bergeron, Haglund, and Wise [15] to embed $N_1 \cup \cdots \cup N_k$ in a finite cover of $\mathbb{H}^d/G$.

We begin with the description of some standard arithmetic manifolds which is the starting point of the construction.

8.2.1. *Arithmetic manifolds from quadratic forms.* From now on let $k \geq 2$ be a fixed integer.

Consider the number field $F := \mathbb{Q}(\cos(\pi/k), \sin(\pi/k))$. It is a *totally real* extension of $\mathbb{Q}$, that is, every field embedding $F \to \mathbb{C}$ has image contained in $\mathbb{R}$. We denote by $\sigma_1, \cdots, \sigma_r$ the $r = \deg(F/\mathbb{Q})$ real embeddings $\sigma_j : F \to \mathbb{R}$ where σ_1 is the trivial one $\sigma_1(x) = x$ for every $x \in F$.

Let $\mathcal{O} \subset F$ be the ring of integers of the field. It is a standard fact that every $x \in F$ can be written as $x = \frac{a}{b}$ with $a \in \mathcal{O}$ and $b \in \mathbb{N}$.

Lemma 8.8. *There exists $\tau \in \mathcal{O}$ such that*

(1) $F = \mathbb{Q}(\tau)$.
(2) $\tau = \sigma_1(\tau) > 0$.
(3) $\sigma_j(\tau) < 0$ for every $j \geq 2$.

Proof. Choose $\tau_1, \cdots, \tau_r \in F$ with $\sigma_j(\tau_j)$ pairwise distinct and with $\sigma_1(\tau_1) > 0$ and $\sigma_j(\tau_j) < 0$ for every $j > 1$. Let $\epsilon > 0$ be smaller than $\min_{j \leq r}\{|\sigma_j(\tau_j)|\}$ and $\min_{i<j}\{|\sigma_i(\tau_i) - \sigma_j(\tau_j)|\}$. By the Weak Approximation Theorem (see [48, Th. 7.20]) applied to the r absolute values $|\bullet|_j := |\sigma_j(\bullet)|$, there exists $\tau \in F$ such that $|\sigma_j(\tau) - \sigma_j(\tau_j)| < \epsilon$. By our choice of ϵ, we have that $\sigma_1(\tau), \cdots, \sigma_r(\tau)$ are pairwise distinct, in particular, τ has degree $r = \deg(F/\mathbb{Q})$ over $\mathbb{Q}$ which implies $F = \mathbb{Q}(\tau)$. Furthermore, by our choice of ϵ, we also have $\sigma_1(\tau) > 0$ and $\sigma_j(\tau) < 0$ for every $j > 1$. $\square$

Consider the quadratic form of signature $(n,1)$
$$q(x) := x_1^2 + \ldots + x_d^2 - \tau x_{d+1}^2.$$
Let $\mathrm{SO}(q)_\mathcal{O}$ be the group of orientation preserving isometries of q with coefficients in $\mathcal{O}$ that preserve the hyperboloid
$$\mathbb{H}^d := \{x \in \mathbb{R}^{d,1} \,|\, q(x) = -1, x_{d+1} > 0\}.$$

Lemma 8.9. $\mathrm{SO}(q)_\mathcal{O} < \mathrm{SO}(q)$ *is discrete and acts cocompactly on $\mathbb{H}^d$.*

Proof. This comes from the so-called restriction of scalars construction (see [12, Ex. 5 §2.1]) which requires that the quadratic forms

$$q^{\sigma_j} := x_1^2 + \ldots + x_d^2 - \sigma_j(\tau) x_{d+1}^2.$$

are all positive definite for $j > 1$ while q^{σ_1} has signature $(d, 1)$ (as ensured by Lemma 8.8). $\qquad\square$

We will work with closed oriented hyperbolic d-manifolds that are quotients of $\mathbb{H}^d$ by torsion free finite index subgroups of $\mathrm{SO}(q)_{\mathcal{O}}$. Such groups are abundant by Selberg's Lemma and Malcev's Theorem (see [50]). The group $\mathrm{SO}(q)_{\mathcal{O}}$ has also much stronger subgroup separability properties and we will need them later on:

Fact 8.10 (Bergeron–Haglund–Wise [15, Th. 1.4])**.** *There exists a torsion free finite index subgroup $G < \mathrm{SO}(q)_{\mathcal{O}}$ such that every* convex-cocompact *subgroup of G is* separable.

Let us fix such a $G < \mathrm{SO}(q)_{\mathcal{O}}$.

We briefly recall the definition of separability:

Definition 8.11 (Separable)**.** A subset $S \subset A$ of a group A is *separable* if for every $g \in A - S$ there exists a finite index subgroup $A' < A$ such that $S \subset A'$ and $g \notin A'$.

Consider the hyperplane

$$H := \{x \in \mathbb{R}^{d,1} \,|\, x_1 = 0\}$$

and the codimension 2 plane

$$V := \{x \in \mathbb{R}^{d,1} \,|\, x_1 = x_2 = 0\}.$$

Let ρ be the π/k-rotation around V

$$\rho := \begin{pmatrix} \cos(\pi/k) & -\sin(\pi/k) & \\ \sin(\pi/k) & \cos(\pi/k) & \\ & & \mathbb{I}_{d-1} \end{pmatrix}.$$

Note that $\rho \in \mathrm{SO}(q)_F$. This implies that:

Lemma 8.12. *The subgroup*

$$G \cap \rho G \rho^{-1} \cap \cdots \cap \rho^{2k-1} G \rho^{-(2k-1)}$$

has finite index in G and is normalized by ρ.

Proof. Recall that every $x \in F$ can be written as $x = \frac{a}{b}$ where $a \in \mathcal{O}$, the ring of integers of F, and $b \in \mathbb{N}$. For every $1 \leq j \leq 2k$, write every entry of ρ^j as a fraction of an element in $\mathcal{O}$ and an element in $\mathbb{N}$ and let $D \subset \mathbb{N}$ be the finite set of the denominators (note that $\rho^{-j} = \rho^{2k-j}$ for every $0 \leq j \leq 2k$). Let $\delta \in \mathbb{N}$ be the product of all the elements in D.

Let $I < \mathcal{O}$ be the ideal generated by δ^2.

In particular, as $\mathcal{O}$ is finitely generated as a $\mathbb{Z}$-module (see [48, Cor. 2.30]), $\mathcal{O}/I$ is a finite extension of the finite ring $\mathbb{Z}/\delta^2\mathbb{Z}$ and, hence, it is itself a finite ring.

Consider the congruence subgroup of G

$$G_\delta := \ker(G < \mathrm{SO}(q)_\mathcal{O} \to \mathrm{SL}_{n+1}(\mathcal{O}/I)).$$

G_δ has finite index in G. Every matrix $A \in G_\delta$ can be written as $A = I + B$ with every entry of B of the form $\delta^2 u$ for some $u \in \mathcal{O}$. Therefore, the matrix $\rho^j A \rho^{-j} = I + \rho^j B \rho^{-j}$ has coefficients in $\mathcal{O}$ and preserves q. In other words, the conjugation by ρ^j induces an injective homomorphism $G_\delta \to \mathrm{SO}(q)_\mathcal{O}$.

Note that $\rho^j G_\delta \rho^{-j} < \mathrm{SO}(q)_\mathcal{O}$, being cocompact in $\mathrm{SO}(q)$, has finite index in $\mathrm{SO}(q)_\mathcal{O}$.

As the intersection $G \cap \rho G \rho^{-1} \cap \cdots \cap \rho^{2k-1} G \rho^{-(2k-1)}$ contains the intersection of finite index subgroups $G_\delta \cap \rho G_\delta \rho^{-1} \cap \cdots \cap \rho^{2k-1} G_\delta \rho^{-(2k-1)}$, it has finite index in G. $\qquad\square$

Recall that separability passes to subgroups, meaning that, if S is separable in A and $B < A$, then $S \cap B$ is separable in B. Hence, up to replacing G with the intersection given by Lemma 8.12, we can assume that G is normalized by ρ. From now on we assume this is the case.

8.2.2. *A ping-pong argument.* We now construct a convex-cocompact subgroup $Q < G$ such that $\mathbb{H}^d/Q$ contains the desired configuration of hypersurfaces.

Recall that $H \subset \mathbb{H}^d$ is the hyperplane $\{x \in \mathbb{R}^{d,1} \,|\, x_1 = 0\}$.

Let K denote the isometry group $K := G \cap \mathrm{SO}(q|_H)_\mathcal{O}$ where we identify $\mathrm{SO}(q|_H)_\mathcal{O}$ with its natural inclusion in $\mathrm{SO}(q)_\mathcal{O}$. Observe that K acts cocompactly on H as $\mathrm{SO}(q|_H)_\mathcal{O}$ act cocompactly on H (by Lemma 8.9) and K is a finite index subgroup of it.

We have:

Proposition 8.13. *There exists a finite set $A \subset K$ such that for every finite index subgroup of $T < K$ with $T \cap A = \emptyset$ we have the following: The group*

$$Q = \langle T_1, \cdots, T_k \rangle < G,$$

where $T_j := \rho^j T \rho^{-j}$ leaves invariant the hyperplane $H_j := \rho^j H$, has the following properties:

(1) *It is convex-cocompact.*
(2) *The projection $\pi : H_j \to \mathbb{H}^d/Q$ induces an embedding $H_j/T_j \to \mathbb{H}^d/Q$.*
(3) *For every $1 \le i < j \le k$ we have $H_i/T_i \cap H_j/T_j = V/P$ where $P = \mathrm{Stab}_T(V)$.*

Proof. The idea of the proof is to construct a suitable ping-pong system adapted to the groups $T_1, \cdots, T_k$ and hyperplanes $H_1, \cdots, H_k$.

The group Q is a subgroup of G because G is normalized by ρ and $T < G$.

Let us consider a very large $R > 0$ which we will adjust for our needs later on.

First, we select a finite index subgroup $T < K$ so that

$$d_{\mathbb{H}^d}(V, \alpha V) \geq R.$$

for every $\alpha \in T - P$ where $P = \mathrm{Stab}_T(V)$.

We can find such subgroup by using hyperplane separability properties: Let $W \subset V$ be a compact fundamental domain for the action of P on V. Consider the finite set

$$A_R = \{\gamma \in K - P \mid d_{\mathbb{H}^d}(W, \gamma W) < R\}.$$

By [15, Lem. 1.7] the subgroup $P < K$ is separable, thus there exists a finite index subgroup $T < K$ containing P and such that $T \cap A_R = \emptyset$. Such subgroup satisfies the assumption: If we had $d_{\mathbb{H}^d}(V, \gamma V) \leq R$ for some $\gamma \in T - P$, then we would have $d_{\mathbb{H}^d}(\alpha W, \gamma \beta W) \leq R$ for some $\alpha, \beta \in P$ and, hence, $d_{\mathbb{H}^d}(W, \alpha^{-1}\gamma\beta W) \leq R$. As a consequence, by our choice of A_R, we would get $\alpha^{-1}\gamma\beta \in P$ and, hence, $\gamma \in P$ contradicting the initial assumption.

Consider the two hyperplanes H_j', H_j'' containing V that form with H_j angles of $\pi/2k$. The hyperplanes H_j', H_j'' divide $\mathbb{H}^d$ into four quadrants. Let us consider the union U_j of two opposite closed quadrants containing H_j. Notice that, by the angle assumption, we have $U_i \cap U_j = V$ for every $i < j$. In particular, the interiors of the sectors $U_1, \cdots, U_k$ are pairwise disjoint.
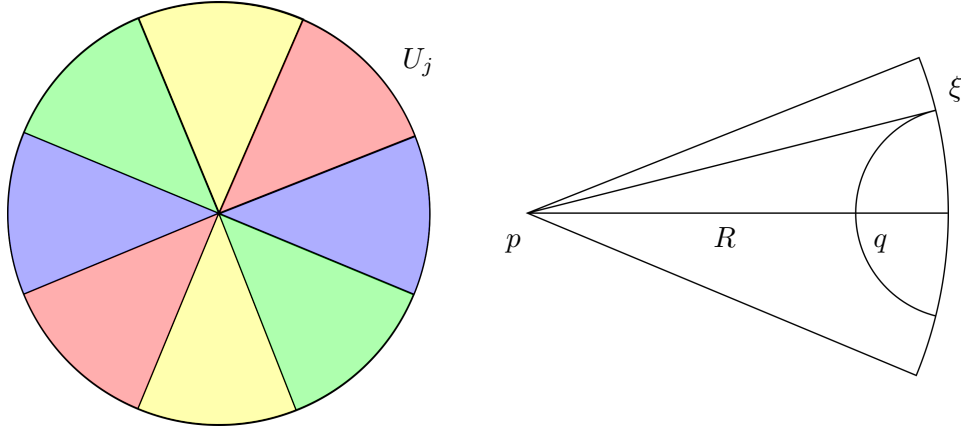

FIGURE 9. Ping-pong system.

We use these sectors as a ping-pong system.

Claim 1. We have:

(i) If $\beta \in P$, then $\beta(U_j) = U_j$.
(ii) Let R be big enough. If $\alpha_i \in T_i - P$ and $i \neq j$, then $\alpha_i(U_j) \subset \mathrm{int}(U_i)$.

Proof of the claim. The first part is clear.

For the second part, consider $H_i, H_j, \alpha_i H_j$. We need the following:

Lemma 8.14. *There is a 2-plane $Z \subset \mathbb{H}^d$ that intersects orthogonally all of them.*

Proof. We work in the projective model $\mathbb{H}^d \subset \mathbb{RP}^d$.

With a slight abuse of notation, unless it is not necessary, we will not make a distinction between a totally geodesic subspace of $\mathbb{H}^d$ and the corresponding linear subspace of $\mathbb{RP}^d$. Also recall that every linear subspace $L \subset \mathbb{RP}^d$ has a dual $L^* \subset \mathbb{RP}^d$ which corresponds to the orthogonal subspace $L^\perp$ with respect to the quadratic form q.

A 2-plane $Z \subset \mathbb{H}^d$ is orthogonal to a hyperplane $H \subset \mathbb{H}^d$ if and only if Z passes through the dual point H^*.

Let $x_i, x_j, \alpha_j x_i$ be the dual points of $H_i, H_j, \alpha_j H_i$. Let

$$Z := \mathrm{Span}\{x_i, x_j, \alpha_j x_i\}$$

be the linear subspace of $\mathbb{RP}^d$ they generate. Notice that Z is dual to the intersection (as linear subspaces) $H_i \cap H_j \cap \alpha_j H_i$. The triple intersection of the hyperplanes (as linear subspaces) avoids the closure of $\mathbb{H}^d$ as

$$(H_i \cap H_j \cap \alpha_j H_i) \cap \bar{\mathbb{H}}^d = (V \cap \alpha_j V) \cap \bar{\mathbb{H}}^d = \emptyset.$$

In particular, the restriction of the quadratic form q to the triple intersection is positive definite. As a consequence, the restriction of the quadratic form q on the dual space Z has signature $(2, 1)$ which implies that Z intersects $\mathbb{H}^d$ in a totally geodesic $\mathbb{H}^2$. Furthermore, by construction, Z passes through the dual points $x_i, x_j, \alpha_j x_i$, hence, it is orthogonal to the subspaces $H_i, H_j, \alpha_j H_i$. □

Let Z be the 2-plane provided by Lemma 8.14.

Consider the intersection points $p = Z \cap V, q := Z \cap \alpha_i V$.

By the assumption on T, we have that $d_{\mathbb{H}^d}(p, q) \geq d_{\mathbb{H}^d}(V, \alpha_i V) \geq R$.

Consider also the intersections at infinity $\{\xi, \zeta\} := \partial Z \cap \partial \alpha_i H_j$. The angles $\angle pq\xi, \angle pq\zeta$ are respectively $|j - i|\pi/k$ and $\pi - |j - i|\pi/k$. Consider the triangles $\Delta(p, q, \xi)$ and $\Delta(p, q, \zeta)$. By hyperbolic trigonometry (see []), the angles at p are very small: By the hyperbolic law of cosines and a small manipulation we have

$$\sin\left(\angle qp\xi\right), \sin\left(\angle qp\zeta\right) \leq \frac{2}{\cosh\left(d_{\mathbb{H}^n}(p, q)\right)\sin\left(|j - i|\frac{\pi}{k}\right)}$$

$$\leq \frac{2}{\cosh(R)\sin\left(|j - i|\frac{\pi}{k}\right)}.$$

If R is chosen so that the angles $\angle qp\xi, \angle qp\zeta$ are much smaller that $\pi/2k$, then $\alpha_i U_j$ is contained in U_i. □

Using the ping-pong system, we deduce the properties that we need.

Properties (2) and (3). Let us pick an element $\gamma \in Q$. We can always write it as $\gamma = \alpha_{j_1} \cdots \alpha_{j_n}$ where $\alpha_{j_k} \in T_{j_k} - P$ and $\alpha_{j_n} \in T_{j_n}$ and such that $j_k \neq j_{k+1}$ for every k.

Let us observe that, by the first claim, we have $\gamma H_i \subset U_{j_1}$ for every H_i and, furthermore, $\gamma H_i \subset \operatorname{int}(U_{j_1})$ if $n > 1$: In fact, consider first $\alpha_{j_n} H_i$ there are two cases: If $j_n = i$, then $\alpha_{j_n} H_i = H_i$ and, hence, it is contained in $U_i = U_{j_n}$. Otherwise, by the Claim, $\alpha_{j_n} H_i \subset \alpha_{j_n} U_i \subset U_{j_n}$. The conclusion follows by inductively applying the Claim using the fact that $j_k \neq j_{k+1}$:

$$
\begin{aligned}
\gamma H_i &= \alpha_{j_1} \cdots \alpha_{j_n} H_i \\
&\subset \alpha_{j_1} \cdots \alpha_{j_{n-1}} U_{j_n} \\
&\subset \alpha_{j_1} \cdots \alpha_{j_{n-2}} \operatorname{int}(U_{j_{n-1}}) \\
&\subset \cdots \\
&\subset \alpha_{j_1} U_{j_2} \subset \operatorname{int}(U_{j_1}).
\end{aligned}
$$

We can now prove the following

Claim 2. If $\gamma H_i \cap H_j \neq \emptyset$, then $\gamma = \alpha_j \alpha_i$ with $\alpha_i \in T_i$ and $\alpha_j \in T_j$.

Proof of the claim. We proceed by induction on the length of the shortest representative of γ of the above form $\gamma = \alpha_{j_1} \cdots \alpha_{j_n}$. The base case $n = 1$ is clear. If $n > 1$ then, by the above computation, $\gamma H_i \subset \operatorname{int}(U_{j_1})$. As $\gamma H_i \cap H_j \neq \emptyset$, we must have $j_1 = j$. Consider $\gamma' = \alpha_{j_1}^{-1} \gamma$. The element γ' has by construction a shorter representative than γ and satisfies $\gamma' H_i \cap H_j = \alpha_{j_1}^{-1}(\gamma H_i \cap H_j) \neq \emptyset$. Therefore, by the inductive hypothesis, we have $\gamma' = \alpha_j' \alpha_i'$ with $\alpha_j' \in T_j$ and $\alpha_i' \in T_i$. Finally, we conclude $\gamma = (\alpha_{j_1} \alpha_j') \alpha_i'$ which concludes the inductive step as $\alpha_{j_1} \alpha_j' \in T_j$. □

Claim 3. For every $j \leq k$, the natural map $H_j / T_j \to \mathbb{H}^d / Q$ is an embedding.

Proof of the claim. Suppose that $\pi(x) = \pi(y)$ for $x, y \in H_j$. Then, there exists $\gamma \in Q$ such that $y = \gamma x$ and, in particular, $\gamma H_j \cap H_j \neq \emptyset$. By the previous claim, we deduce that $\gamma \in T_j$ and, hence $[x] = [y]$ in H_j / T_j. □

Claim 4. For every $i < j$ we have $H_i / T_i \cap H_j / T_j = V / P$.

Proof of the claim. Suppose that $\pi(x) = \pi(y)$ for $x \in H_i, y \in H_j$. Then, there exists $\gamma \in Q$ such that $y = \gamma x$ and, in particular, $\gamma H_i \cap H_j \neq \emptyset$. By the previous claim, we deduce that $\gamma = \alpha_j \alpha_i$ with $\alpha_j \in T_j$ and $\alpha_i \in T_i$. In order to coclude we just observe that

$$
\alpha_j^{-1}(\gamma H_i \cap H_j) = \alpha_j^{-1}(\alpha_j \alpha_i H_i \cap H_j) = H_i \cap H_j = V
$$

so that $y \in \gamma H_i \cap H_j = \alpha_j V$ and $x = \gamma^{-1} y \in \alpha_i V$. Therefore $\pi(x) = \pi(y) \in V / P$. □

This concludes the proof of the properties (2) and (3).

Property (1). By Fact 3.20, it is enough to prove that the orbit of a point $p \in V$ is quasi-convex provided that R is large enough. As the orbit is coarsely dense in H_j for each $j \leq k$, it is enough to show that

$$C := \bigcup_{\gamma \in Q, \, j \leq k} \gamma H_j$$

is quasi-convex.

Notice that C has an intrinsic path metric. Two distinct points of C can be connected by either a geodesic segment of $\mathbb{H}^d$, if they lie on the same flat piece γH_i, or by a concatenation $\kappa = \kappa_1 \star \ldots \star \kappa_m \subset C$ of geodesic segments of $\mathbb{H}^d$ which is geodesic for C and where each κ_j lies on some translate $\gamma_j H_{i_j}$ and has endpoints on two different translates of V in $\gamma_j H_{i_j}$. We show that such intrinsic C-geodesics κ are a uniform quasi-geodesics and, therefore, by stability of quasi-geodesics, they lie at a uniform Hausdorff distance from the geodesic in $\mathbb{H}^d$ joining the same endpoints. This suffices to prove that C is quasi-convex.

In order to prove that κ is a uniform quasi-geodesic we need a couple of observations: First, the length of each κ_j is at least R as the geodesic segment joins distinct translates of V. Second, as the concatenation is a geodesic for the intrinsic path metric of C, the angle between two consecutive segments κ_j, κ_{j+1} is at least the angle between $\gamma_j H_{i_j}, \gamma_{j+1} H_{i_{j+1}}$ which is at least π/k. The conclusion follows from the following elementary fact

Fact 8.15. *For every $\theta > 0$ there exist $R > 0$ and $c > 0$ such that each concatenation $\kappa \subset \mathbb{H}^d$ of geodesic segments of length at least R forming angles of at least θ is a c-quasi-geodesic.*

This concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We are now ready to prove Theorem 1.8.

8.2.3. *The proof of Theorem 1.8.* We prove part (a).

Let $Q < G$ be the subgroup provided by Proposition 8.13. By Fact 8.10, Q is separable in G. A standard argument shows that:

Claim. There is a finite index subgroup $G' < G$ containing $Q < G'$ and such that the covering projection $\mathbb{H}^d/Q \to \mathbb{H}^d/G'$ restricts to an embedding on $N_1 \cup \cdots \cup N_k \subset \mathbb{H}^d/Q$ where $N_j := H_j/T_j$.

Proof of the claim. By Proposition 8.13, Q acts cocompactly on a convex set $\mathcal{CH} \subset \mathbb{H}^d$. The quotient $\mathcal{CH}/Q$ embeds in $\mathbb{H}^d/Q$ and contains H_j/T_j. Consider the covering projections $\pi : \mathbb{H}^d/Q \to \mathbb{H}^d/G$ and the restriction of π to $\mathcal{CH}/Q$. Observe that $\pi(p) = \pi(q)$ for some $p, q \in \mathcal{C}/Q$ if and only if there exists lifts $x, y \in F$ of p, q to some compact fundamental domain $F \subset \mathcal{CH}$ and an element $\gamma \in G$ such that $\gamma x = y$.

We use separability of Q to eliminate this configuration in an intermediate finite covering $\mathbb{H}^d/Q \to \mathbb{H}^d/G' \to \mathbb{H}^d/G$: Consider the finite set

$$A := \{\gamma \in G - Q \,|\, \gamma F \cap F \neq \emptyset\}.$$

By separability of Q in G, there exists a finite index subgroup $G' < G$ containing Q and such that $G' \cap A = \emptyset$. As G' contains Q, we have a covering projection $\pi' : \mathbb{H}^d/Q \to \mathbb{H}^d/G'$. By our choice of A and the fact that $G' \cap A = \emptyset$, the restriction of π' to $\mathcal{CH}/Q$ is now injective. $\qquad\square$

By Proposition 8.13 and the fact that the projection $\mathbb{H}^d/Q \to \mathbb{H}^d/G'$ restricts to an embedding on $N_1 \cup \cdots \cup N_k \subset \mathcal{C}/Q$, the abstract completion M of any of the complementary components of $\mathbb{H}^d/G' - N_1 \cup \cdots \cup N_k$ is a compact hyperbolic d-manifold with totally geodesic boundary and corners, all the corner angles are equal to π/k, and, if k is even, then the graph of the boundary is bipartite.

This finishes the proof. $\qquad\square$

9. Hyperbolic doubles

In this section we prove Theorems 1.1, 1.2, and 1.5.

We will deduce them from the following special case of the glue-and-bend construction, namely, the double of a compact convex hyperbolic manifold with totally geodesic boundary bent along some corners.

Theorem 9.1. *Let $\mathbb{M}$ be a compact convex hyperbolic bc-manifold with non-empty boundary of dimension $d \geq 3$. Fix a parameter for each wall. Suppose that*

  (i) *All the corner angles are smaller than $\pi/2$.*
 (ii) *For every corner $\mathfrak{C}$ with angle θ and adjacent walls W and W' with parameters μ and μ', denoting $\nu = \mu^{\frac{d+1}{2}}$ and $\nu' = \mu'^{\frac{d+1}{2}}$,*

$$t := \cos(\theta)^2 \left(\frac{\nu}{\nu'} + \frac{\nu'}{\nu} \right) - \sin(\theta)^2 \left(\nu'\nu + \frac{1}{\nu'\nu} \right) \geq 2.$$

  Set $\mu_{\mathfrak{C}} := \max \left\{ \frac{\mu}{\mu'}, \frac{\mu'}{\mu} \right\}$ and $\lambda_{\mathfrak{C}} := \frac{t}{2} + \sqrt{\frac{t^2}{4} - 1}$.

Then the following holds:

  (1) *$N := D\mathbb{M} - U$ obtained from the double $D\mathbb{M}$ of $\mathbb{M}$ by removing a tubular neighborhood U of the corners admits a convex projective structure with totally geodesic boundary.*
  (2) *For each wall W of N there is a developing map of N sending a copy of $\widetilde{W}$ onto $\mathrm{Conv}(\mathbb{H}^{d-2}, [e_d])$ such that the holonomy of $\pi_1(W)$ is the*

group generated by

$$\left(\begin{array}{cc} \Gamma_{\mathfrak{C}} & \\ & \mathbb{I}_2 \end{array} \right), \left(\begin{array}{ccc} \mu_{\mathfrak{C}}^{-1}\mathbb{I}_{d-1} & & \\ & \mu_{\mathfrak{C}}^{\frac{d-1}{2}}\lambda_{\mathfrak{C}} & \epsilon \\ & & \mu_{\mathfrak{C}}^{\frac{d-1}{2}}\lambda_{\mathfrak{C}}^{-1} \end{array} \right) \in \mathrm{SL}_{d+1}(\mathbb{R}),$$

with $\epsilon = 1$ if $\lambda = 1$ and 0 otherwise and where $\mathfrak{C}$ is the corner W blows up, with holonomy $\Gamma_{\mathfrak{C}} < \mathrm{SO}(d-2,1)$.

(3) *If $\lambda_{\mathfrak{C}} > 1$ then N satisfies the Containment Condition with respect to W and the polar $[e_{d+1}]$.*

(4) *Every pair of distinct walls of $\tilde{N} \subset \mathbb{S}^d$ have disjoint closures (i.e. N has no ghost stratum) and every point of $\partial \tilde{N}$ outside closures of walls is extremal and $\mathcal{C}^1$.*

Before proving the main result, let us fix some notations.

9.1. Some notations.

For every $\theta \in [0, 2\pi)$, we will denote by r_θ, R_θ the rotations

$$(9.1) \quad r_\theta := \left(\begin{array}{cc} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{array} \right) \in \mathrm{SL}_2(\mathbb{R}), \quad R_\theta := \left(\begin{array}{cc} \mathbb{I}_{d-1} & \\ & r_\theta \end{array} \right) \in \mathrm{SL}_{d+1}(\mathbb{R}).$$

For every $\mu > 0$, we will also denote by b_μ, b'_μ, B_μ the bulging matrices

$$(9.2) \quad b_\mu := \left(\begin{array}{cc} \frac{1}{\mu} & 0 \\ 0 & \mu \end{array} \right) \in \mathrm{SL}_2(\mathbb{R}), \quad b'_\mu := \left(\begin{array}{cc} \frac{1}{\mu} & 0 \\ 0 & \mu^d \end{array} \right) \in \mathrm{GL}_2^+(\mathbb{R}),$$

and

$$(9.3) \quad B_\mu := \left(\begin{array}{cc} \frac{1}{\mu}\mathbb{I}_d & \\ & \mu^d \end{array} \right) = \left(\begin{array}{cc} \frac{1}{\mu}\mathbb{I}_{d-1} & \\ & \mu^{\frac{d-1}{2}}b_{\mu^{(d+1)/2}} \end{array} \right) \in \mathrm{SL}_{d+1}(\mathbb{R}).$$

9.2. Projective structures on hyperbolic doubles.

The basic computation behind the proof of Theorem 9.1 is that of the holonomy of a meridian around each corner, and the tube-type of the corner. Let us use the notation from Theorem 9.1, and further denote by $\mathbb{M}'$ the copy of $\mathbb{M}$ that we glue to $\mathbb{M}$, and by M and M' the smooth loci of these manifolds.

We bulge the gluing DM of M with its double M' (and hence obtain a projective structure on DM) using the parameters of Theorem 9.1 for the walls of M, and using trivial parameters equal to 1 for the walls of M'.

9.2.1. *Holonomy of meridians.*

Consider a corner $\mathfrak{C}$ of $\mathbb{M}$ with angle θ and adjacent walls W and W' with bulging parameters μ and μ'.

Set $\nu = \mu^{(d+1)/2}$ and $\nu' = \mu'^{(d+1)/2}$.

Let $\mathbb{U} \subset D\mathbb{M}$ be an open neighborhood of a point of $\mathfrak{C}$ such that $\mathbb{U} \cap \mathbb{M}$ and $\mathbb{U} \cap \mathbb{M}'$ are both small hyperbolic balls (hence convex).

Let $\tilde{U} \subset \tilde{T}$ be the lift of U, which identifies with any lift of U in $D\tilde{M}$. It is moreover the gluing of sequences of lifts $(V_n)_{n \in \mathbb{Z}}$ and $(V'_n)_{n \in \mathbb{Z}}$ of respectively

$U \cap M$ and $U \cap M'$, along lifts $(W_n)_{n \in \mathbb{Z}}$ and $(W'_n)_{n \in \mathbb{Z}}$ of $U \cap W$ and $U \cap W'$, such that $V_n \cap V'_n = W_n$ and $V'_n \cap V_{n+1} = W'_{n+1}$.

Consider a hyperbolic developing chart ϕ of $\mathbb{U} \cap \mathbb{M}$ that sends $\mathbb{M} \cap \mathfrak{C}$ in $\mathbb{H}^{d-2}$, sends $\mathbb{U} \cap W'$ in $\text{Conv}(\mathbb{H}^{d-2}, [e_d])$, and sends $\mathbb{U} \cap \mathbb{M}$ in $\text{Conv}(\mathbb{H}^{d-1}, [e_{d+1}])$.

This determines a developing map dev for the (bulged) projective structure on $\widetilde{U}$. An elementary computation shows the following:

Lemma 9.2. *Let γ be a meridian around $\mathfrak{C}$ that starts on $U \cap W'$, goes through $U \cap M$, intersects $U \cap W$, then goes through $U \cap M'$ back to its starting point.*

- *The $\text{SL}_{d+1}(\mathbb{R})$-holonomy of γ is $\rho(\gamma) = R_\theta B_\mu R_\theta B_{\mu'}^{-1}$.*
- *The $\text{SL}_2^{\pm}$-angle of $\rho(\gamma)$ (Definition 4.4) is*

$$g(\gamma) = r_\theta b_\nu r_\theta b_{\nu'}^{-1}.$$

- *The projection of $\text{dev}(V_0 \cup V'_0)$ in $\mathbb{S}^1 = \mathbb{S}(\mathbb{R}^{d+1}/\mathbb{R}^{d-1})$ is the properly convex segment between $[e_d]$ and $r_\theta b_\nu r_\theta b_{\nu'}^{-1}[e_d]$.*

9.2.2. *The proof of Theorem 9.1.* The plan is the following.

- We check that the conditions of Theorem 7.18 are satisfied.
- We apply Theorem 7.18 to obtain that DM is convex.
- We check that the conditions of Theorem 7.19 are satisfied, in particular that DM is properly convex.
- We apply Theorem 7.19 to obtain the totally geodesic blowup N of $D\mathbb{M}$, which has no ghost stratum and satisfy the Containment Condition at walls blowing up singularities with hyperbolic SL_2-angle.
- We prove that every point of $\partial \widetilde{N}$ outside closures of walls is extremal and $\mathcal{C}^1$ using Theorem 7.21, Corollary 5.15 and Observation 5.16.

Proof of Theorem 9.1. Let us check that the assumptions of Theorem 7.18 are verified (this is where we use Lemma 9.2):

Claim 1. DM is connected.

Proof of the claim. This is clear. □

Claim 2. $\mathbb{M}$ is properly convex, with complete walls and corners, and without ghost strata.

Proof of the claim. This follows from our assumption and Fact 7.14. □

Claim 3. Every wall of $\mathbb{M}$ satisfies the Containment Condition relatively to its hyperbolic polar.

Proof of the claim. This is given by Fact 7.17. □

Claim 4. Each singularity of $D\hat{M}$ is uniformisable in $\mathbb{S}^d$ (Proposition 4.7).

Proof of the claim. Fix a corner $\mathfrak{C}$ of $D\mathbb{M}$ and take notation from Section 9.2.1 and Lemma 9.2.

By Lemma 4.9 and the last point of Lemma 9.2, we only need to check that $g(\gamma) = r_\theta b_\nu r_\theta b_{\nu'}^{-1}$ is non-trivial (this is an elementary computation) and has trace at least 2, which is exactly the assumption of Theorem 9.1 since this trace is

$$t = \cos(\theta)^2 \left(\frac{\nu}{\nu'} + \frac{\nu'}{\nu} \right) - \sin(\theta)^2 \left(\nu'\nu + \frac{1}{\nu'\nu} \right). \qquad \square$$

At this point we can use Theorem 7.18 to say that DM is convex. To be able to use Theorem 7.19, we need to check that:

Claim 5. Each singularity of $D\mathbb{M}$ is special (Definition 4.10).

and

Claim 6. DM is *properly* convex.

Proof of Claim 5. Take notation from the above proof of (4). We have

$$\rho(\gamma) = \begin{pmatrix} \left(\frac{\nu}{\nu'}\right)^{-\frac{2}{d+1}} & \\ & \left(\frac{\nu}{\nu'}\right)^{\frac{d-1}{d+1}} g(\gamma) \end{pmatrix}.$$

According to Definition 4.10 (more precisely (4.10)), we must show that $\left(\frac{\nu}{\nu'}\right)^2 - \left(\frac{\nu}{\nu'}\right) t + 1$ is positive. We compute:

$$\left(\frac{\nu}{\nu'}\right)^2 - \left(\frac{\nu}{\nu'}\right) t + 1 = \sin(\theta)^2 \left(\left(\frac{\nu}{\nu'}\right)^2 + 1 + \nu'^2 + \nu^{-2} \right) > 0. \qquad \square$$

To prove Claim 6 we will need the following.

Fact 9.3. *Any compact convex hyperbolic bc-manifold* $\mathbb{X} = (\mathcal{C} \subset \mathbb{H}^d)/\Gamma$ *of dimension at least* 3 *has irreducible holonomy (no invariant proper subspace).*

Proof of the fact. It is classical that Γ is irreducible if and only if the convex hull in $\mathbb{H}^d$ of its limit set has non-empty interior (see for instance [19, Prop. 2.5]).

As Γ acts cocompactly on the closed convex subset $\mathcal{C} \subset \mathbb{H}^d$, its limit set is $\partial\mathcal{C} \cap \partial\mathbb{H}^d$. Moreover, the closure of $\mathcal{C}$ (in $\mathbb{S}^d$) is the convex hull of its extremal points. None of these extremal points is in $\partial\mathcal{C} \cap \mathbb{H}^d$ since this set is a union of walls and corner (the corners have positive dimension since $d > 2$), so they are all in $\partial\mathcal{C} \cap \partial\mathbb{H}^d$, the limit set of Γ, whose convex hull is therefore $\mathcal{C}$, which has non-empty interior. $\qquad \square$

Proof of Claim 6. Suppose $d > 2$, then by Fact 9.3 $\pi_1(M)$ acts irreducibly on $\mathbb{S}^d$, which immediately implies, by standard arguments, that DM, which is $\pi_1(M)$-invariant, is properly convex (see Definition 3.1). $\qquad \square$

At this point we can apply Theorem 7.19.

Finally, Theorem 7.21, Corollary 5.15 and Fact 5.16 ensure that every point of $\partial \tilde{N}$ outside closures of walls is extremal and $\mathcal{C}^1$, provided that:

Claim 7. Every point of $\partial \tilde{M} \cap \partial \mathbb{H}^d$ outside the closures of the walls is an extremal $\mathcal{C}^1$ point of $\partial \tilde{M}$.

Claim 8. For every wall $\tilde{W} \subset \tilde{M}$, every point of $\partial \tilde{W} \cap \partial \mathbb{H}^d$ outside closures of a corners is an extremal $\mathcal{C}^1$ point of the boundary of the union of $\tilde{M}$ with any bulged reflection of $\tilde{M}$ across $\tilde{W}$.

The proofs of these two points are essentially the same.

Proof of Claim 7. Let $x \in \partial \tilde{M} \cap \partial \mathbb{H}^d$ be outside the closures of the walls. This point is extremal for $\tilde{M}$ because it is extremal for $\mathbb{H}^d$, which contains $\tilde{M}$.

Pick $p \in \tilde{M}$. Since $\mathbb{M}$ is compact, the fact that x does not lie in the closure of a wall says that the projection of the ray $[p, x)$ comes back infinitely often in a compact subset of the interior of M. Therefore there exists $r > 0$ and $p_n \in [p, x)$ tending to x such that the ball of $\mathbb{H}^d$ of radius r around all p_n is contained in $\tilde{M}$. This classically implies that $\partial \tilde{M}$ is $\mathcal{C}^1$ (otherwise there would be a ray $[q, x) \subset \mathbb{H}^d$ outside of $\tilde{M}$, but then by basic hyperbolic geometry this ray would eventually get arbitrarily close to $[p, x)$). $\qquad \square$

Proof of Claim 8. Consider a bulged reflection $\tilde{M}'$ of $\tilde{M}$ across $\tilde{W}$. One can stretch $\mathbb{H}^d$ to obtain a $\pi_1(W)$-invariant ellipsoid $\mathcal{E}$ that contains $\tilde{M} \cup \tilde{M}'$, and such that $\mathcal{E} \cap \mathbb{S}^{d-1} = \mathbb{H}^{d-1}$.

Let $x \in \partial \tilde{W} \cap \partial \mathbb{H}^{d-1}$ be outside the closures of corners. This point is extremal for $\tilde{M}$ because it is extremal for $\mathcal{E}$, which contains $\tilde{M} \cup \tilde{M}'$.

Pick $p \in \tilde{W}$. Since W is compact, the fact that x does not lie in the closure of a corner says that the projection of the ray $[p, x)$ comes back infinitely often in a compact subset of the interior of W. Therefore there exists $r > 0$ and $p_n \in [p, x)$ tending to x such that the ball of $\mathcal{E}$ of radius r around all p_n is contained in $\tilde{M} \cup \tilde{M}'$. This classically implies that $\partial \tilde{M} \cup \tilde{M}'$ is $\mathcal{C}^1$. $\qquad \square$

This concludes the proof of Theorem 9.1. $\qquad \square$

Remark 9.4 (The case $d = 2$). Note that the proof given above holds also in dimension $d = 2$ except in Claim 6. However, the conclusion of the claim still holds (by an elementary argument).

9.3. **The proofs of the main theorems.**

Proof of Theorems 1.3 and 1.5. Fix $k = 4$. Let M be the convex compact hyperbolic d-manifold with totally geodesic boundary and corners with angles $\pi/4$ provided by Theorem 1.8, whose graph of the boundary is bipartite.

Let us consider a coloring of the walls of M in yellow and purple such that any two adjacent walls have different colors.

For $j = 1, 2$, consider ν_j such that

$$\frac{1}{\sqrt{2}} \left(\nu_j^2 + \nu_j^{-2} \right) - \frac{1}{\sqrt{2}} \left(1 + 1 \right) = 2 + (2 - j).$$

By Theorem 9.1, by giving parameters $\nu_j^{(d+1)/2}$ and $\nu_j^{-(d+1)/2}$ to yellow and purple walls, we get a compact convex projective manifold with totally geodesic boundary N_j without corners, with complete walls (Definition 7.10), and without ghost strata (Definition 7.12). (Note that N_1 and N_2 are homeomorphic.)

Let $\tilde{N}_j \subset \mathbb{S}^d$ be the universal cover of N_j. Then:

- The description stated in Theorem 1.5 of the walls of $\tilde{N}_j$ (the lifts of the walls of N_j) as cones with stabilizer of the form $\pi_1(\mathfrak{C}) \times \mathbb{Z}$ with $\mathfrak{C} \subset M$ a corner, is given in Theorem 9.1.
- Using again Theorem 9.1, every point of ∂N_j which is not in the closure of a wall is extremal and $\mathcal{C}^1$.

Let Γ_j be the fundamental group of N_j.

Let Λ_j be the full orbital limit set of Γ_j in $\partial \Omega_j$.

- By Theorem 6.3, Γ_j is relatively hyperbolic with respect to the fundamental groups of the walls of N_j, i.e. the stabilizers of the walls of $\tilde{N}_j$.

Case $j = 1$. If $j = 1$, then:

- Each wall has a polar that satisfies the Containment Condition by Theorem 9.1.
- By Theorem 7.18, the projective structure on N_1 extends to a convex projective structure on the double $N = DN_1$, which is actually *properly convex* as $d \geq 3$ by Fact 9.3.
- Let $\Omega_1 \subset \mathbb{S}^d$ be the universal cover of N, which is divided by $\Gamma := \pi_1(N)$ since N is a closed manifold. By Proposition 6.1, Γ_1 acts convex-cocompactly on Ω_1.
- Let us call "wall" the image in N of the walls of N_1, and the lifts of these in Ω_1. By the description of the walls of $\tilde{N}_1$ and their stabilizers, the walls of Ω_1 form a Γ-invariant collection of properly embedded cones with pairwise disjoint closures, whose stabilizers have the form $\pi_1(\mathfrak{C}) \times \mathbb{Z}$ where $\mathfrak{C} \subset M$ is a corner.
- By Theorem 7.21, Corollary 5.15 and Fact 5.16, every point of $\partial \Omega_1$ which is not in the closure of a wall is extremal and $\mathcal{C}^1$.
- By Fact 6.4, Γ is relatively hyperbolic with respect to the stabilizers of the walls, i.e. the properly embedded cones. This concludes the proof of Theorem 1.3.

- By convex-cocompactness and [30, Lem. 4.1], $\Lambda_1 = \partial \widetilde{M}_1 \smallsetminus \widetilde{M}_1$, so that $\mathcal{CH}(\Lambda_1) \smallsetminus \Lambda_1 = \widetilde{M}_1$.

This finishes the proof of Theorem 1.5.

Case $j = 2$. Suppose $j = 2$. Let us check that $\Lambda_2 = \partial \widetilde{M}_2 \smallsetminus \widetilde{M}_2$.

The inclusion $\Lambda_2 \subset \partial \widetilde{M}_2 \smallsetminus \widetilde{M}_2$ is clear, let us prove the other one.

Fix $p \in \Omega_2$. Let $x \in \partial \widetilde{M}_2$ which is not in the closure of any wall. As shown in the proof of Theorem 6.3, the image of $[p, x)$ in M_2 comes back infinitely often in a compact subset of the complement of the walls, in other words there exists $x_n \in [p, x)$ going to x and $g_n \in \Gamma_2$ such that $g_n x_n$ stays in a compact subset of Ω_2. Consider $R > 0$ large enough so that the closed Hilbert ball $B_n := B_R(g_n x_n)$ contains p for every n. Since x is extremal, $g_n^{-1} B_n = B_R(x_n)$ converge to x (see for instance [18, Fact 2.1.10]). In particular, $x \in \overline{\Gamma_2 \cdot p}$.

As points in the relative boundary of walls can be approached by points outside closed walls, $\partial \widetilde{M}_2 \smallsetminus \widetilde{M}_2$ is contained in $\overline{\Gamma_2 p}$, and hence in the full orbital limit set. $\square$

Theorem 1.1 is a particular case of Theorem 1.3.

We prove Theorem 1.2.

Proof of Theorem 1.7. Take ϵ from Fact 2.1. Consider a_i, b_i, c_i, $i = 1, \ldots, n$, as in Theorem 1.7, i.e. such that $1 < a_i < e^{\epsilon/2}$, $b_i > 3$ and $1 < c_i < -2 + b_i^2/2$.

Set $\ell_i := 2 \ln(a_i) < \epsilon$, and consider the hyperbolic metric on M with totally geodesic and corners, such that the corners are the α_i's, with length ℓ_i and angle $\theta_i < \pi/4$.

The fact that $c_i < -2 + b_i^2/2$ and $\theta_i < \pi/4$ guarantees that

$$c_i + c_i^{-1} < \frac{1}{2}(b_i^2 + b_i^{-2}) - 1 < \cos^2 \theta_i (b_i^2 + b_i^{-2}) - 2 \sin^2 \theta_i,$$

and, hence, that

$$2 < \frac{\cos^2 \theta_i (b_i^2 + b_i^{-2}) - (c_i + c_i^{-1})}{\sin^2 \theta_i}.$$

Thus there exists a unique $d_i > 1$ such that

$$d_i^2 + d_i^{-2} < \frac{\cos^2 \theta_i (b_i^2 + b_i^{-2}) - (c_i + c_i^{-1})}{\sin^2 \theta_i},$$

and

$$t_i := \cos^2 \theta_i (b_i^2 + b_i^{-2}) - \sin^2 \theta_i (d_i^2 + d_i^{-2}) = c_i + c_i^{-1} > 2.$$

Set $\mu_i := \sqrt{b_i d_i}$ and $\mu_i' := \sqrt{d_i/b_i}$.

Note that $\mu_i/\mu_i' = b_i$ and $\mu_i \mu_i' = d_i$ and $c_i = t_i/2 + \sqrt{t_i^2/4 - 1}$.

We get the desired projective structure on $DM \smallsetminus U$ by applying the bulging procedure from Theorem 9.1 with parameters μ_i and μ_i' (recall that $d = 3$). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

9.4. **Commensurability.** Lastly, we show that our construction is flexible enough to produce an abundance of examples of inequivalent non-symmetric non-strictly divisible convex domains.

To begin with let us recall the definition of commensurability.

Definition 9.5 (Commensurable). Two groups G, G' are *commensurable* if they contain isomorphic finite index subgroups $H < G, H' < G'$.

We prove

Theorem 9.6. *Suppose $d \geq 5$. Let $\theta = \pi/n, \theta' = \pi/n'$ be such that the number fields $F = \mathbb{Q}(\cos(\theta), \sin(\theta)), F' = \mathbb{Q}(\cos(\theta'), \sin(\theta'))$ are different. Let $\tau \in F, \tau' \in F'$ be elements satisfying the assumptions of Lemma 8.8. Let $\Gamma, \Gamma' < \mathrm{PSL}_{d+1}(\mathbb{R})$ be the groups as in Theorem 1.3 dividing the properly convex sets $\Omega, \Omega' \subset \mathbb{RP}^d$ which are relatively hyperbolic with respect to the collections of subgroups $\{\Theta_j \times \mathbb{Z}\}_{j \in J}, \{\Theta_j' \times \mathbb{Z}\}_{j \in J'}$ with Θ_j, Θ_j' finite index subgroups of $\mathrm{SO}(q)_{\mathcal{O}}, \mathrm{SO}(q')_{\mathcal{O}'}$ where $q = x_1 + \cdots + x_{d-2}^2 - \tau y^2$ and $q' = x_1 + \cdots + x_{d-2}^2 - \tau' y^2$. Then Γ, Γ' are not commensurable.*

Proof. Suppose that Γ, Γ' are commensurable. This means that there exists a group G that embeds in both Γ, Γ' as a subgroup of finite index. Let us say $G < \Gamma$ and $\iota : G \hookrightarrow \Gamma'$ is an embedding.

As Γ, Γ' are relatively hyperbolic with respect to the collections of subgroups $\{\Theta_j \times \mathbb{Z}\}_{j \in J}, \{\Theta_j' \times \mathbb{Z}\}_{j \in J'}$ it follows that G is relatively hyperbolic with respect to $\{S_{j,\alpha} := \alpha(\Theta_j \times \mathbb{Z})\alpha^{-1} \cap G\}_{j \in J, \alpha \in \Gamma}$ and also with respect to $\{S_{j,\alpha}' := \iota^{-1}(\alpha(\Theta_j' \times \mathbb{Z})\alpha^{-1})\}_{j \in J', \alpha \in \Gamma'}$. Notice that each $S_{j,\alpha}, \iota(S_{j,\alpha}')$ is a finite index subgroup of $\alpha(\Theta_j \times \mathbb{Z})\alpha^{-1}, \alpha(\Theta_j' \times \mathbb{Z})\alpha^{-1}$.

By work of Islam and Zimmer [35, Th. 1.6], we have the following: Let $L_{j,\alpha}'$ be the (full orbital) limit set of the group $S_{j,\alpha}'$ in Ω. Then:

- $L_{j,\alpha}' \subset \partial\Omega$ is closed and $\mathcal{CH}(L_{j,\alpha}') \smallsetminus L_{j,\alpha}'$ is a convex subset of Ω
- The group $S_{j,\alpha}'$ acts cocompactly on $\mathcal{CH}(L_{j,\alpha}') \smallsetminus L_{j,\alpha}' \subset \Omega$.
- There is an equivariant homeomorphism between the Bowditch boundary of $(G, \{S_{j,\alpha}'\}_{j \in J', \alpha \in \Gamma'})$ and the space obtained from $\partial\Omega$ by collapsing each $L_{j,\alpha}'$ to a point. In particular, the sets $L_{j,\alpha}' \subset \partial\Omega$ are pairwise disjoint.
- If $[a, b] \subset \partial\Omega$ is a non-trivial line segment, then $[a, b] \subset L_{j,\alpha}'$ for some $j \in J'$ and $\alpha \in \Gamma'$.

Consider one of the properly embedded cones $\mathcal{CH}(H, p) \cap \Omega \in \mathcal{C}$ on which acts cocompactly $S_{i,\beta} = \beta(\Theta_k \times \mathbb{Z})\beta^{-1} \cap G$ for some $k \in J$ and $\beta \in \Gamma$; let $L_{i,\beta}$ be the full orbital limit set of $S_{i,\beta}$, i.e. the relative boundary of the cone.

For every $q \in \partial H$ the line segment $[q, p] \subset \partial\Omega$ is contained in some $L'_{j,\alpha}$. As all segments intersect in p we conclude that they are all contained in the same $L'_{j,\alpha}$. Therefore $L_{i,\beta} \subset L'_{j,\alpha}$ and $\mathcal{CH}(H, p) \subset \mathcal{CH}(L'_{j,\alpha})$.

Note that $S_{i,\beta}$ preserves $L'_{j,\alpha}$. Indeed if $\gamma \in S_{i,\beta}$ then $L'_{j,\iota(\gamma)\alpha} = \gamma L'_{j,\alpha}$ contains $L_{i,\beta}$ hence intersects $L'_{j,\alpha}$ and hence is equal to it.

Since $S'_{j,\alpha}$ is a finite index subgroup of $\mathrm{Stab}_G(L'_{j,\alpha})$, this means that $S_{i,\beta} \cap S'_{j,\alpha}$ is a finite index subgroup of $S_{i,\alpha}$. In particular its cohomological dimension is the same as that of $S_{i,\beta}$, which is $d-1$, which is also the cohomological dimension of $S'_{j,\alpha}$. By a standard argument (see [57]), this implies that $S_{i,\beta} \cap S'_{j,\alpha}$ is also a finite index subgroup of $S'_{j,\alpha}$.

In particular, $\Theta'_j \times \mathbb{Z}$ and $\Theta_i \times \mathbb{Z}$ are commensurable as they contain the isomorphic finite index subgroups $S_{i,\beta} \cap S'_{j,\alpha}$ and $\iota(S_{i,\beta} \cap S'_{j,\alpha})$.

Let K be a group such that every finite index subgroup of K has trivial center. For example, all Θ_i, Θ'_j have such property. Consider a finite index subgroup $S < K \times \mathbb{Z}$. Then the center of S is $Z(S) = S \cap (\{1\} \times \mathbb{Z})$ and we have

$$S/Z(S) = S/(S \cap (\{1\} \times \mathbb{Z})) \simeq \pi(S) < K$$

where $\pi : K \times \mathbb{Z} \to K$ is the projection onto the first factor.

Let us prove this claim: Consider $\pi(S) < K$. It is a finite index subgroup and, hence, it has trivial center by assumption. If $(\alpha, t) \in S$ is in the center of S, then α is in the center of $\pi(S)$ which is trivial. Vice versa it is immediate to see that $\{1\} \times \mathbb{Z}$ is in the center of $K \times \mathbb{Z}$.

We apply this fact to $S := S_{i,\beta} \cap S'_{j,\alpha} < \Theta_i \times \mathbb{Z}$ and $\iota(S) < \Theta'_j \times \mathbb{Z}$: Let π_i, π_j be the projections of $\Theta_i \times \mathbb{Z}, \Theta'_j \times \mathbb{Z}$ onto the first factors. We have

$$\pi_j(S) \simeq S/Z(S) \simeq \pi_i(\iota(S)).$$

Thus Θ'_j is commensurable with Θ_i as they contain the isomorphic finite index subgroups $\pi_j(S)$ and $\pi_i(\iota(S))$.

In order to conclude we will use a commensurability invariant of hyperbolic manifolds, namely the invariant trace field:

Definition 9.7 (Trace Field). Let $\Gamma < \mathrm{SO}_0(d-2, 1)$ be a lattice. The *invariant trace field* of Γ is

$$k(\Gamma) := \mathbb{Q}(\{\mathrm{tr}(\mathrm{Ad}(\gamma)) \mid \gamma \in \Gamma\})$$

where $\mathrm{Ad} : \mathrm{SO}_0(d-2, 1) \to \mathfrak{so}(d-2, 1)$ is the adjoint representation.

By Mostow Rigidity (see [49, Ch. 15]), if $d \geq 5$, the invariant trace field $k(\Gamma)$ is an invariant of the abstract group Γ and does not depend on the particular embedding $\Gamma < \mathrm{SO}_0(d-2, 1)$ as a lattice (which is unique up to conjugation).

Vinberg [66] shows that the trace field $k(\Gamma)$ is an invariant of the commensurability class of Γ:

Fact 9.8 (Vinberg [66])**.** *Let* $\Theta < \mathrm{SO}_0(d-2,1)$ *be a lattice. If* $\Theta_0 < \Theta$ *is a finite index subgroup, then* $k(\Theta_0) = k(\Theta)$.

The computation of the trace field in our case is not difficult:

Fact 9.9 (Prasad–Rapinchuck [52, Lem. 2.6])**.** *Let* F *be a number field with ring of integers* $\mathcal{O}$. *Let*

$$q = x_1^2 + \cdots + x_{d-2}^2 - \tau y^2$$

be a quadratic form where $\tau \in \mathcal{O}$ *satisfies the assumptions of Lemma 8.8. Then*

$$k(\mathrm{SO}(q)_{\mathcal{O}}) = F.$$

Since Θ_j', Θ_i are commensurable we must have $k(\Theta_j') = k(\Theta_i)$. However, Θ_j', Θ_i are finite index subgroups of $\mathrm{SO}(q)_{\mathcal{O}}, \mathrm{SO}(q')_{\mathcal{O}'}$ respectively where $q = x_1 + \cdots + x_{d-2}^2 - \tau y^2$ and $q' = x_1 + \cdots + x_{d-2}^2 - \tau' y^2$ and $\mathcal{O}, \mathcal{O}'$ are the ring of integers of F, F'. By the above results, we get $k(\Theta_j') = F'$ and $k(\Theta_i) = F$. As $F \neq F'$, this provides a contradiction and finishes the proof. $\square$

REFERENCES

[1] S. A. Ballas, D. Cooper, and A. Leitner, *Generalized cusps in real projective manifolds: classification*, J. Topol. **13**(2020), 1455-1496.

[2] S. A. Ballas, J. Danciger, and G.-S. Lee, *Convex projective structures on nonhyperbolic three-manifolds*, Geom. Topol. **22**(2018), 1593-1646.

[3] S. A. Ballas, L. Marquis, *Properly convex bending of hyperbolic manifolds*, Groups Geom. Dyn. **14**(2020), 653-688.

[4] W. Ballmann, *Nonpositively curved manifolds of higher rank*, Ann. Math **122**(1985), 597-609.

[5] T. Barbot, F. Bonsante, and J.-M. Schlenker, *Collisions of particles in locally AdS spacetimes I. Local description and global examples*, Comm. Math. Phys. **308**(2011), 147-200.

[6] Y. Benoist, *Automorphismes des cônes convexes*, Invent. Math. **141**(2000), 149-193.

[7] Y. Benoist, *Convex divisible I*, in Algebraic groups and arithmetic, Tata Inst. Fund. Res. Stud. Math. **17**(2004), 339-374.

[8] Y. Benoist, *Convex divisible II*, Duke Math. J. **120**(2003), 97-120.

[9] Y. Benoist, *Convex divisible III*, Ann. Sci. ENS **38**(2005), 793-832.

[10] Y. Benoist, *Convex divisible IV*, Invent. Math. **164**(2006), 249-278.

[11] Y. Benoist, *Exercises on divisible convex sets*, 2012, https://www.imo.universite-paris-saclay.fr/ benoist/prepubli/12GearJuniorRetreat.pdf

[12] Y. Benoist, *Five lectures on lattices in semisimple Lie groups*, in Géométries à courbure négative ou nulle, groupes discrets et rigidités, Sémin. Congr., vol. 18, Soc. Math. France, Paris, 2009, 117-176.

[13] Y. Benoist, *A survey on divisible convex sets*, in Geometry, analysis and topology of discrete groups, Adv. Lect. Math. (ALM), vol. 6, Int. Press, Somerville, MA, 2008, 1-18.

[14] J.-P. Benzécri, *Sur les variétés localement affines et localement projectives*, Bull. Soc. Math. Fr. **88**(1960), 229-332.

[15] N. Bergeron, F. Haglund, and D. T. Wise, *Hyperplane sections in arithmetic hyperbolic manifolds*, J. Lond. Math. Soc. **83**(2011), 431-448.

[16] G. Birkhoff, *Extensions of Jentzsch's theorem*, Trans. Amer. Math. Soc., **85**(1957), 219-227.

[17] P.-L. Blayac, *Patterson-Sullivan densities in convex projective geometry*, preprint, arXiv:2106.08089.

[18] P.-L. Blayac, *Dynamical aspects of convex projective structures*, PhD thesis, Laboratoire Alexander Grothendieck, Université Paris-Saclay, 2021.

[19] P.-L. Blayac and F. Zhu, *Ergodicity an equidistribution in Hilbert geometry*, preprint, arXiv:2106.08079.

[20] M. D. Bobb, *Convex projective manifolds with a cusp of any non-diagonalizable type*, J. Lond. Math. Soc. (2) **100**(2019), 183-202.

[21] F. Bonahon and J.-P. Otal, *Laminations mesurées de plissage des variétés hyperboliques de dimension 3*, Ann. Math. **160**(2004), 1013-1055.

[22] A. Borel, *Compact Clifford-Klein forms of symmetric spaces*, Topology **2**(1963), 111-122.

[23] B. Bowditch, *Relatively hyperbolic groups*, Internat. J. Algebra Comput., **22**(2012), 1-66,

[24] K. Burns and R. Spatzier, *Manifolds of nonpositive curvature and their buildings*, Publ. Math. IHES **65**(1987), 35-59.

[25] R. Canary, *Anosov representations: Informal lecture notes*, preprint, available at http://www.math.lsa.umich.edu/ canary/lecnotespublic.pdf.

[26] S. Choi and W. M. Goldman, *Convex real projective structures on closed surfaces are closed*, Proc. Amer. Math. Soc. **118**(1993), 657-661.

[27] S. Choi, G.-S. Lee, and L. Marquis, *Convex projective generalized Dehn filling*, Ann. Sci. École Norm. Sup. **53**(2020), 217-266.

[28] Y.-E. Choi and C. Series, *Lengths are coordinates for convex structures*, J. Diff. Geom. **73**(2006), 75-116.

[29] J. Danciger, *A geometric transition from hyperbolic to anti-de Sitter geometry*, Geom. Topol. **17**(2013), 3077-3134.

[30] J. Danciger, F. Guéritaud, and F. Kassel, *Convex-cocompact actions in real projective geometry*, preprint, arXiv:1704.08711.

[31] J. Danciger, F. Guéritaud, F. Kassel, G.-S. Lee, and L. Marquis, *Convex compactness for Coxeter groups*, preprint, arXiv:2102.02757.

[32] M. Gromov and W. Thurston, *Pinching constants for hyperbolic manifolds*, Invent. Math. **89**(1987), 1-12.

[33] M. Islam, *Rank-one Hilbert geometries*, preprint arXiv:1912.13013

[34] M. Islam and A. Zimmer, *Convex co-compact actions of relatively hyperbolic groups*, preprint, arXiv:1910.08885.

[35] M. Islam and A. Zimmer, *Relative hyperbolic groups in convex real projective geometry*, preprint, arXiv:2203.16596.

[36] D. Johnson and J. Millson, *Deformation spaces associated to compact hyperbolic manifolds*, Progress in Mathematics, Birkhäuser.

[37] V. Kac and E.Vinberg, *Quasihomogeneous cones*, Math. Notes **1**(1967), 231-235.

[38] M. Kapovich, *Hyperbolic manifolds and discrete groups*, Modern Birkhäuser Classics, Birkhäuser Boston, Inc., Boston, MA, 2009.

[39] M. Kapovich, *Convex projective structures on Gromov-Thurston manifolds*, Geom. Topol. **11**(2007), 1777-1830.

[40] M. Koecher, *The Minnesota notes on Jordan algebras and their applications*, volume 1710 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1999. Edited, annotated and with a preface by Aloys Krieg and Sebastian Walcher.

[41] J.-L. Koszul, *Déformation des connexions localement plates*, Ann. Inst. Fourier **18**(1968), 103-114.

[42] C. Lecuire, *An extension of the Masur domain*, in Y. Minsky, M. Sakuma, and C. Series (Eds.), Spaces of Kleinian Groups, London Mathematical Society Lecture Note Series, Cambridge University Press, 49-74.

[43] G.-S. Lee, L. Marquis, and S. Riolo, *A small closed convex projective 4-manifold via Dehn filling*, Publ. Mat. **66**(2022), 369-403.

[44] L. Marquis, *Espace des modules de certains polyèdres projectifs miroirs*, Geom. Dedicata **147**(2010), 47-86.

[45] L. Marquis, *Exemples de variétés projectives strictement convexes de volume fini en dimension quelconque* Enseign. Math. **58**(2012), 3-47.

[46] L. Marquis, *Around groups in Hilbert geometry*, in "Handbook of Hilbert geometry" edited by G. Besson, M. Troyanov, and A. Papadopoulos.

[47] B. Martelli, *An introduction to geometric topology*, CreateSpace Independent Publishing Platform (2016).

[48] J. S. Milne, *Algebraic Number Theory*, available at https://jmilne.org/math/CourseNotes/ANT.pdf

[49] D. W. Morris, *An introduction to arithmetic groups*, Deductive Press (2015).

[50] B. Nica, *Linear groups - Malcev's theorem and Selberg's lemma*, preprint arXiv:1306.2385.

[51] D. Osin, *Acylindrically hyperbolic groups*, Trans. Amer. Math. Soc. **368**(2016), 851-888.

[52] G. Prasad and R. Rapinchuck, *Weakly commensurable arithmetic groups and isospectral locally symmetric spaces*, Publ. Math. IHES **109**(2009), 113-184.

[53] S. Riolo and A. Seppi, *Geometric transition from hyperbolic to Anti-de Sitter structures in dimension four*, Ann. Sc. Norm. Super. Pisa Cl. Sci. **23**(2022), 115-176.

[54] P. Scott and T. Wall, *Topological methods in group theory*, in Homological group theory (Proc. Sympos., Durham, 1977), London Mathematical Society Lecture Note Series, Cambridge University Press **36**(1979), 137-203.

[55] C. Siegel, *Indefinite quadratische formen und funktionentheorie*, Math. Ann. **124**(1951), 17-54 and 364-387.

[56] J.-P. Serre, *Arbres, amalgames,* SL_2, Rédigé avec la collaboration de Hyman Bass, Astérisque **46**(1977), Société Mathématique de France, Paris.

[57] J.-P. Serre, *Cohomologie des groupes discrets*, in Prospects in Mathematics, (AM-70), vol. 70, Princeton University Press, 2016, 77-170.

[58] W. Thurston, *Geometry and topology of three-manifolds*, Princeton lecture notes, 1979.

[59] W. Thurston, *Three-dimensional manifolds, Kleinian groups and hyperbolic geometry*, Bull. Amer. Math. Soc. (N.S.), **6**(1982), 357-381.

[60] W. Thurston, *Hyperbolic structures on 3-manifolds, III: Deformations of 3-manifolds with incompressible boundary*, preprint, arXiv:math.GT/9801058.

[61] P. Tukia, *Conical limit points and uniform convergence groups*, J. Reine Angew. Math. **501**(1998), 71-98.

[62] J. Vey, *Sur les automorphismes affines des ouverts convexes saillants*, Ann. Sc. Norm. Super. Pisa Cl. Sci. **24**(1970), 641-665.

[63] E. Vinberg, *The theory of homogeneous convex cones*, Trudy Moskov. Mat. Obsc. **12**(1963), 303-358.

[64] E. Vinberg, *Structure of the group of automorphisms of a homogeneous convex cone*, Trudy Moskov. Mat. Obsc. **13**(1965), 56-83.

[65] E. Vinberg, *Discrete linear groups that are generated by reflections*, Izv. Akad. Nauk SSSR Ser. Mat. **35**(1971), 1072-1112.

[66] E. Vinberg, *Rings of definition of dense subgroups of semisimple linear groups*, Math. USSR Izv. **5**(1971), 45-55.

[67] T. Weisman, *Dynamical properties of convex-cocompact actions in projective space*, preprint arXiv:2009.10994v2.

[68] T. Weisman, *An extended definition of Anosov representation for relatively hyperbolic groups*, preprint arXiv:2205.07183.

[69] A. Yaman, *A topological characterization of relative hyperbolic groups*, J. Reine Angew. Math. **566**(2004), 41-89.

[70] A. Zimmer, *A higher rank rigidity theorem for convex real projective manifolds*, to appear in Geom. Topol.

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