PATTERSON-SULLIVAN THEORY FOR COARSE COCYCLES

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Abstract. In this paper we develop a theory of Patterson–Sullivan measures associated to coarse cocycles of convergence groups. This framework includes Patterson–Sullivan measures associated to the Busemann cocycle on the geodesic boundary of a Gromov hyperbolic metric spaces and Patterson–Sullivan measures on flag manifolds associated to Anosov (or more general transverse) subgroups of semisimple Lie groups, as well as more examples. Under some natural geometric assumptions on the coarse cocycle, we prove existence, uniqueness, and ergodicity results.

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1. Introduction

Patterson–Sullivan measures were first constructed by Patterson [Pat76] in the setting of Fuchsian groups and by Sullivan [Sul79] for Kleinian groups. They have been used to study the dynamics of the action of the recurrent part of the geodesic flow of the quotient manifold, the geometry of the limit set of the group and to obtain counting estimates for both orbit points of the group and closed geodesics in the quotient manifold. They have been generalized to many settings, including proper isometric actions on Gromov hyperbolic spaces and discrete subgroups of semi-simple Lie groups.

In this paper we develop a theory of Patterson–Sullivan measures for coarse-cocycles of convergence group actions, which encompasses many of the previous situations. When the coarse cocycle has an expanding property and a finite critical exponent, we show that Patterson–Sullivan measures exist in the critical dimension. Moreover, we establish a Shadow Lemma in the spirit of Sullivan and show that the action of the convergence group is ergodic with respect to the measure when the associated Poincaré series diverges at its critical exponent.

We also develop the notion of a coarse Gromov–Patterson–Sullivan system, which is a pair of coarse-cocycles with an associated coarse Gromov product, and establish a version of the Hopf–Tsuji–Sullivan ergodic dichotomy in this setting. In a companion paper, we will use this framework to establish mixing, equidistribution and counting results for relatively Anosov groups (and more generally for divergent GPS systems for geometrically finite convergence groups).

1.1. Main results.

Suppose $\Gamma \subset \mathsf{Homeo}(M)$ is a convergence group. A function $\sigma\colon \Gamma \times M \to \mathbb{R}$ is called a *κ-coarse-cocycle* if:

(1) For every $\gamma \in \Gamma$, the function $\sigma(\gamma, \cdot)$ is *κ-coarsely continuous*: if $x_0 \in M$, then
$$\limsup_{x \to x_0} |\sigma(\gamma, x_0) - \sigma(\gamma, x)| \leq \kappa.$$

(2) σ satisfies a coarse version of the cocycle identity: if $\gamma_1, \gamma_2 \in \Gamma$ and $x \in M$, then
$$\Big|\sigma(\gamma_1\gamma_2, x) - \Big(\sigma(\gamma_1, \gamma_2 x) + \sigma(\gamma_2, x)\Big)\Big| \leq \kappa.$$

Notice that a 0-coarse-cocycle is simply a continuous cocycle. In the classical hyperbolic setting, one usually considers the Busemann cocycle.

Given a coarse-cocycle $\sigma\colon \Gamma \times M \to \mathbb{R}$, we define the *$\sigma$-magnitude* of an element $\gamma \in \Gamma$ to be
$$\|\gamma\|_\sigma := \sup_{x \in M} \sigma(\gamma, x) \in \mathbb{R}\,.$$
Then the *σ-Poincaré series* is
$$Q_\sigma(s) = \sum_{\gamma \in \Gamma} e^{-s\|\gamma\|_\sigma} \in [0, +\infty]$$
and the *σ-critical exponent* is
$$\delta_\sigma(\Gamma) = \inf\{s > 0 : Q_\sigma(s) < +\infty\} \in [0, +\infty].$$

In Section 2, we will show that the set $\Gamma \sqcup M$ has a unique topology which makes it a compact metrizable space and where the natural action of Γ on $\Gamma \sqcup M$ is a convergence group action. We call a metric on $\Gamma \sqcup M$ which generates this topology a *compatible metric.*

We will often require that our cocycles satisfy the following weak expansion property.

Definition 1.1. Suppose $\Gamma \subset \mathsf{Homeo}(M)$ is a convergence group and d is a compatible metric on $\Gamma \sqcup M$. A coarse-cocycle $\sigma\colon \Gamma \times M \to \mathbb{R}$ is *expanding* if

- σ is *proper*: $\|\gamma_n\|_\sigma \to +\infty$ for any escaping sequence $\{\gamma_n\} \subset \Gamma$, and
- for every $\epsilon > 0$ there exists $C > 0$ such that: whenever $x \in M$, $\gamma \in \Gamma$ and $\mathrm{d}(x, \gamma^{-1}) > \epsilon$, then

$$\sigma(\gamma, x) \geq \|\gamma\|_\sigma - C.$$

We show that if a coarse-cocycle is expanding and has finite critical exponent $\delta_\sigma(\Gamma)$, then it admits a coarse Patterson–Sullivan measure of dimension $\delta_\sigma(\Gamma)$ which is supported on the limit set. Moreover, any Patterson–Sullivan measure has dimension at least $\delta_\sigma(\Gamma)$.

Definition 1.2. Suppose $\Gamma \subset \mathsf{Homeo}(M)$ is a convergence group and $\sigma\colon \Gamma \times M \to \mathbb{R}$ is a coarse-cocycle, then a probability measure μ on M is a *C-coarse σ-Patterson–Sullivan measure* of dimension δ if, for every $\gamma \in \Gamma$, the measures $\mu, \gamma_*\mu$ are absolutely continuous and

$$e^{-C-\delta\sigma(\gamma^{-1},\cdot)} \leq \frac{d\gamma_*\mu}{d\mu} \leq e^{C-\delta\sigma(\gamma^{-1},\cdot)}$$

μ-almost everywhere.

We establish a Shadow Lemma for coarse Patterson–Sullivan measures and use it to study the associated Patterson–Sullivan measures. In particular, we establish ergodicity of the action when the Poincaré series diverges at its critical exponent.

Theorem 1.3 (see Theorem 8.1 below)**.** *Suppose $\Gamma \subset \mathsf{Homeo}(M)$ is a convergence group and $\sigma\colon \Gamma \times M \to \mathbb{R}$ is an expanding coarse-cocycle with $\delta := \delta_\sigma(\Gamma) < +\infty$. If μ is a C-coarse σ-Patterson–Sullivan measure of dimension δ and*

$$\sum_{\gamma \in \Gamma} e^{-\delta\|\gamma\|_\sigma} = +\infty,$$

then:

1. *Γ acts ergodically on (M, μ).*
2. *μ is coarsely unique in the following sense: if λ is a C-coarse σ-Patterson–Sullivan measure of dimension δ, then $e^{-4C}\mu \leq \lambda \leq e^{4C}\mu$.*
3. *The conical limit set of Γ has full μ-measure.*

As an application of ergodicity in the divergent case, we prove the following rigidity result for Patterson–Sullivan measures.

Proposition 1.4 (see Propositions 14.1 and 14.2)**.** *Suppose $\Gamma \subset \mathsf{Homeo}(M)$ is a convergence group and $\sigma_1, \sigma_2\colon \Gamma \times M \to \mathbb{R}$ are expanding coarse-cocycles with finite critical exponents $\delta_1 := \delta_{\sigma_1}(\Gamma)$, $\delta_2 := \delta_{\sigma_2}(\Gamma)$. For $i = 1, 2$, let μ_i be a coarse σ_i-Patterson–Sullivan measure of dimension δ_i.*

If $\sum_{\gamma\in\Gamma} e^{-\delta_1\|\gamma\|_{\sigma_1}} = +\infty$, then either:

1. *$\mu_1 \perp \mu_2$.*
2. *$\mu_1 \ll \mu_2$ and $\mu_2 \ll \mu_1$. Moreover, in this case*

$$\sup_{\gamma\in\Gamma} \left|\delta_1 \|\gamma\|_{\sigma_1} - \delta_2 \|\gamma\|_{\sigma_2}\right| < \infty.$$

Remark 1.5. Dongryul Kim [Kim24] has informed us that in forthcoming work, which studies higher rank analogues of conformal measure rigidity theorems, they establish similar results in the special case of coarse-cocycles associated to transverse Zariski dense discrete subgroups of semisimple Lie groups.

Using this rigidity result we establish a strict convexity result for the critical exponent.

Theorem 1.6 (see Theorem 15.1 below)**.** *Suppose* $\Gamma \subset \mathsf{Homeo}(M)$ *is a convergence group and* $\sigma_1, \sigma_2 \colon \Gamma \times M \to \mathbb{R}$ *are expanding coarse-cocycles with finite critical exponents* $\delta_{\sigma_1}(\Gamma) = \delta_{\sigma_2}(\Gamma) = 1$. *For* $0 < \lambda < 1$, *let* $\sigma_\lambda = \lambda\sigma_0 + (1-\lambda)\sigma_1$. *Then*

$$\delta_{\sigma_\lambda}(\Gamma) \leq 1.$$

Moreover, if $\sum_{\gamma\in\Gamma} e^{-\delta_{\sigma_\lambda}(\Gamma)\|\gamma\|_{\sigma_\lambda}} = +\infty$, *then the following are equivalent:*

(1) $\delta_{\sigma_\lambda}(\Gamma) = 1$.

(2) $\sup_{\gamma\in\Gamma} \left| \|\gamma\|_{\sigma_0} - \|\gamma\|_{\sigma_1} \right| < +\infty$.

In the context of Theorem 1.6, if σ_1 and σ_2 do not have coarsely equivalent magnitudes, then one obtains a drop in critical exponent when taking a convex combination of σ_0 and σ_1. These types of strict convexity results can be used to prove entropy rigidity results, see for instance [PS17].

We further study coarse-cocycles which have a well-behaved "dual cocycle" and coarse Gromov product.

Definition 1.7. Suppose $\Gamma \subset \mathsf{Homeo}(M)$ is a convergence group and let $M^{(2)} = \{(x,y) \in M^2 : x \neq y\}$. We say that $(\sigma, \bar\sigma, G)$ is a κ-coarse *Gromov–Patterson–Sullivan system (or GPS system)* if $\sigma, \bar\sigma : \Gamma \times M \to \mathbb{R}$ are κ-coarse-cocycles, $G \colon M^{(2)} \to [0,\infty)$ is a locally bounded function, and

$$\left| \Big(\bar\sigma(\gamma, x) + \sigma(\gamma, y)\Big) - \Big(G(\gamma x, \gamma y) - G(x,y)\Big) \right| \leq \kappa$$

for all $\gamma \in \Gamma$ and $x, y \in M$ distinct.

We construct a measurable flow space associated to a GPS system and use the Patterson–Sullivan measures of σ and $\bar\sigma$ and the Gromov product to give it a Bowen–Margulis–Sulivan measure. We will show that the dynamics of this flow space are controlled by the behavior of the Poincaré series at the critical exponent and use this to establish the following version of the Hopf–Tsuji–Sullivan dichotomy.

Theorem 1.8 (see Section 12)**.** *Suppose* $(\sigma, \bar\sigma, G)$ *is a coarse GPS system and* $\delta_\sigma(\Gamma) < +\infty$. *Let* μ, $\bar\mu$ *be Patterson–Sullivan measures of dimension* δ *for* σ, $\bar\sigma$ *respectively. Then there exists a measurable nonnegative function* $\tilde G$ *on* $M^{(2)}$ *such that*

$$\nu := e^{\delta \tilde G} \bar\mu \otimes \mu$$

is Γ*-invariant. Moreover we have the following dichotomy:*

(1) If $\sum_{\gamma\in\Gamma} e^{-\delta\|\gamma\|_\sigma} = +\infty$, *then:*
   - *(a)* $\delta = \delta_\sigma(\Gamma)$.
   - *(b)* $\mu(\Lambda^{\rm con}(\Gamma)) = 1 = \bar\mu(\Lambda^{\rm con}(\Gamma))$.
   - *(c) The* Γ *action on* $(M^{(2)}, \nu)$ *is ergodic and conservative.*

(2) If $\sum_{\gamma\in\Gamma} e^{-\delta\|\gamma\|_\sigma} < +\infty$, *then:*
   - *(a)* $\delta \geq \delta_\sigma(\Gamma)$.
   - *(b)* $\mu(\Lambda^{\rm con}(\Gamma)) = 0 = \bar\mu(\Lambda^{\rm con}(\Gamma))$.

(c) The Γ action on $(M^{(2)}, \nu)$ is non-ergodic and dissipative.

In the theorem above, $\Lambda^{\mathrm{con}}(\Gamma)$ denotes the set of conical limit points. We provide the definitions of conservative and dissipative actions, and state their basic properties, in Appendix A.

1.2. **Motivating examples.** We now discuss a range of examples which our approach to Patterson–Sullivan theory treats in a unified way.

1.2.1. *Transverse subgroups of semi-simple Lie groups.* In the sequel to this paper [BCZZ24] we apply the framework developed here to study Patterson–Sullivan measures for certain class of discrete subgroups of semi-simple Lie groups. We show, among other things, that the ergodic dichotomy for transverse groups established in [CZZ23, KOW23] is a particular case of the dichotomy established in this paper. For more details, see [BCZZ24, Section 11].

1.2.2. *Proper actions on Gromov hyperbolic spaces.* If X is a proper geodesic Gromov hyperbolic metric space and $\Gamma \subset \mathsf{Isom}(X)$ is discrete, then Γ acts on the Gromov boundary $\partial_\infty X$ as a convergence group (see [Tuk94, Th. 3A] or [Fre95]). If we fix a base point $o \in X$, we can define, and for each $x \in \partial_\infty X$, a *Busemann function*

$$b_x \colon X \to \mathbb{R} \quad \text{by setting} \quad b_x(q) = \limsup_{p \to x} \mathrm{d}(p,q) - \mathrm{d}(p,o).$$

The *Busemann coarse-cocycle* $\beta \colon \Gamma \times \partial_\infty X \to \mathbb{R}$ is defined by

$$\beta(\gamma, x) = b_x(\gamma^{-1}(o)).$$

When X is CAT(-1) (e.g. $X = \mathbb{H}^n$) this is a continuous cocycle, but in general it will only be a coarse-cocycle.

The Gromov product $G \colon \partial_\infty X^{(2)} \to \mathbb{R}$ is classically defined by

$$G(x,y) = \limsup_{p \to x, q \to y} d(o,p) + d(o,q) - d(p,q).$$

Then (β, β, G) is a coarse GPS system, which is not always continuous. One can show that

$$\sup_{\gamma \in \Gamma} \Big| \|\gamma\|_\beta - d(o, \gamma(o)) \Big| < +\infty.$$

Hence, $\delta_\beta(\Gamma)$ is also the critical exponent of the series

$$Q(s) = \sum_{\gamma \in \Gamma} e^{-s \, \mathrm{d}(o, \gamma(o))}.$$

When X is CAT(-1), Roblin [Rob03] proved the Hopf–Tsuji–Sullivan dichotomy for the GPS system (β, β, G), see also work of Burger–Mozes [BM96]. Building upon work of Bader–Furman [BF17], Coulon–Dougall–Schapira–Tapie [CDST18] extended this to the case of general proper geodesic Gromov hyperbolic metric spaces.

1.2.3. *Coarsely additive potentials.* We continue to assume that X is a proper geodesic Gromov hyperbolic metric space and $\Gamma \subset \mathsf{Isom}(X)$ is discrete.

Adapting a definition of Cantrell–Tanaka [CT22, Defn. 2.2], we make the following definition.

Definition 1.9. A function $\psi\colon X \times X \to \mathbb{R}$ is a *coarsely additive potential* if

(1) $\lim_{r\to\infty} \inf_{\mathrm{d}_X(p,q)\geq r} \psi(p,q) = +\infty$,
(2) for any $r > 0$,
$$\sup_{\mathrm{d}_X(p,q)\leq r} |\psi(p,q)| < +\infty,$$
(3) for every $r > 0$ there exists $\kappa = \kappa(r) > 0$ such that: if u is contained in the r-neighborhood of a geodesic in d joining p to q, then
$$\left|\psi(p,q) - \big(\psi(p,u) + \psi(u,q)\big)\right| \leq \kappa.$$

Remark 1.10. Cantrell–Tanaka consider the case when Γ is word hyperbolic and $X = \Gamma$ with a word metric. In this case they introduce *tempered potentials* which are functions $\psi : \Gamma \times \Gamma \to \mathbb{R}$ which satisfy (3) and another property they call (QE). In their results they consider the case when ψ is Γ-invariant (which implies (2)) and has finite "exponent" (which implies (1)). In Lemma 17.7, we show that a version of their property (QE) holds for any coarsely additive potential.

We will show that any Γ-invariant potential gives rise to any expanding coarse-cocycle on $\partial_\infty X$ and when Γ acts co-compactly on X, then every expanding coarse-cocycle arises in this way.

Theorem 1.11 (see Theorem 17.1 below)**.** *Suppose ψ is a Γ-invariant coarsely additive potential. Define functions $\sigma_\psi, \bar{\sigma}_\psi\colon \Gamma \times \partial_\infty X \to \mathbb{R}$ and $G_\psi\colon \partial_\infty X^{(2)} \to [0,\infty)$ by*
$$\sigma_\psi(\gamma, x) = \limsup_{p\to x} \psi(\gamma^{-1}o, p) - \psi(o,p),$$
$$\bar{\sigma}_\psi(\gamma, x) = \limsup_{p\to x} \psi(p, \gamma^{-1}o) - \psi(p,o),$$
$$G_\psi(x,y) = \limsup_{p\to x, q\to y} \psi(p,o) + \psi(o,q) - \psi(p,q).$$
Then there exists $\kappa_1 > 0$ such that $(\bar{\sigma}_\psi, \sigma_\psi, G_\psi + \kappa_1)$ is a coarse GPS-system and
$$\sup_{\gamma\in\Gamma} \left| \|\gamma\|_{\sigma_\psi} - \psi(o, \gamma o)\right| < +\infty.$$

Theorem 1.12 (see Theorem 17.2)**.** *Suppose Γ acts co-compactly on X and $\sigma : \Gamma \times \partial_\infty X \to \mathbb{R}$ is an expanding coarse-cocycle. Then there exists a Γ-invariant coarsely additive potential where*
$$\sup_{\gamma\in\Gamma, x\in\partial_\infty X} |\sigma_\psi(\gamma, x) - \sigma(\gamma, x)| < +\infty.$$
In particular, σ is contained in a GPS-system.

One can also interpret coarsely additive potentials as Γ-invariant coarsely-geodesic quasimetrics on X, see Section 17.1 below.

The next two subsections highlight two previously studied examples that can be interpreted in terms of GPS systems associated to coarsely additive potentials.

1.2.4. *Hölder potentials and cocycles.* Next we describe the setting studied in work of Paulin–Pollicott–Schapira [PPS15, Section 3], see also earlier work of Ledrappier [Led95].

Let X be a simply connected complete Riemannian manifold with pinched negative curvature and suppose $\Gamma \subset \mathsf{Isom}(X)$ is discrete. Then let $F\colon T^1X \to \mathbb{R}$ be a Γ-invariant Hölder function with

$$0 < \inf_{v \in T^1X} F(v) \leq \sup_{v \in T^1X} F(v) < +\infty.$$

Then for $p, q \in X$ define

$$\int_p^q F := \int_0^T F(\ell'(t))dt$$

where $\ell\colon [0,T] \to X$ is the unit speed geodesic joining p to q. Among (many) other things, [PPS15] consider counting for the "magnitudes"

$$\int_o^{\gamma o} F.$$

To accomplish this they develop a theory of Patterson–Sullivan measures, Busemann cocycles, and Gromov products in this setting.

By [PPS15, Lemma 3.2], the function $(p,q) \mapsto \int_p^q F$ is a coarsely additive potential. Further the Busemann cocycle and Gromov product introduced in [PPS15] (essentially) correspond to the definitions in Theorem 1.11. Hence this setting fits into the general GPS systems framework.

1.2.5. *Hitting measures of random walks.* Next let Γ be a word hyperbolic group and let λ be a finitely-supported probability measure on Γ with $\langle \operatorname{supp} \lambda \rangle = \Gamma$. If $g_1, g_2, \dots \subset \Gamma$ are random group elements following the distribution μ, then the location of the random walk $X_n = g_1 \cdots g_n$ follows the distribution λ^{*n}. The Green metric, introduced in [BB07], is the left-invariant function d_λ on $\Gamma \times \Gamma$ defined by $\mathrm{d}_\lambda(x,y) = -\log F(x,y)$, where $F(x,y)$ is the probability that the random walk started at x ever hits y.

We claim that d_λ is a coarsely additive potential. Property (2) follows from the fact that

$$\mathrm{d}_\lambda(\alpha, \beta) \leq \inf_{n \geq 1} -\log \lambda^{*n}(\alpha^{-1}\beta).$$

Property (1) follows from [BHM08, Prop. 3.1]. Property (3) follows from a result of Ancona (see [Woe00, Thm. 27.11]): for any $r \geq 0$, there exists a positive constant $C(r)$ such that

$$F(\mathrm{id}, \gamma) \leq C(r) F(\mathrm{id}, \gamma') F(\gamma', \gamma)$$

whenever $\gamma, \gamma' \in \Gamma$ and γ' at (word) distance at most r from a geodesic segment between id and γ in a Cayley graph. Hence d_λ is a coarsely additive potential.

Thus Theorem 1.11 can be applied to conclude: $(\sigma, \bar\sigma, G) := (\sigma_\lambda, \sigma_{\bar\lambda}, G_\lambda)$ is a GPS system for $\Gamma \subset \mathsf{Homeo}(\partial_\infty \Gamma)$, where

- $\sigma_\lambda(\gamma, x) = \limsup_{\alpha \to x} \mathrm{d}_\lambda(\mathrm{id}, \gamma\alpha) - \mathrm{d}_\lambda(\mathrm{id}, \alpha)$;
- $\bar\lambda$ is the probability measure on Γ defined by $\bar\lambda(\gamma) := \lambda(\gamma^{-1})$;
- $G_\lambda(x,y) := \limsup_{\alpha \to x, \beta \to y} \mathrm{d}_\lambda(\alpha, \mathrm{id}) + \mathrm{d}_\lambda(\mathrm{id}, \beta) - \mathrm{d}_\lambda(\alpha, \beta)$.

The cocycle σ_λ also satisfies

$$\sigma_\lambda(\gamma, \xi) = -\log \frac{d\gamma_*^{-1}\nu}{d\nu}(\xi)$$

where ν is the unique λ-stationary measure on $\partial\Gamma$, i.e. harmonic measure or hitting measure associated to λ [GMM18, Prop. 2.5]. The σ-Patterson–Sullivan measures are absolutely continuous with respect to the hitting measure ν.

1.2.6. *Proper actions on CAT(0) visibility spaces.* Finally, we briefly discuss another set of examples our results encompass, involving spaces which need not be uniformly hyperbolic. Let X be a proper CAT(0) space with base point $o \in X$ and visual boundary ∂X. Suppose X is *visible*, i.e. all $\xi \neq \eta \in \partial X$ can be connected by a bi-infinite geodesic in X (this notion was introduced by Eberlein–O'Neill [EO73], see also [BH99, Def III.9.28]). Let Γ be a discrete group of isometries of X.

Then Γ acts on ∂X as a convergence group [Kar05, Th. 1]. The Busemann functions $(x, y, z) \in X^3 \mapsto b_z(x, y) = d(x, z) - d(y, z)$ extend continuously to $(x, y, z) \in X^2 \cup (X \cup \partial X)$ (see [BH99, p. 267]), and $\sigma(\gamma, \xi) = b_\xi(\gamma^{-1}o, o)$ defines a continuous cocycle $\Gamma \times \partial X \to \mathbb{R}$. Finally, setting $G(\xi, \eta) = -\inf_{x\in X}(b_\xi + b_\eta)(x, o)$, we obtain a continuous GPS system (σ, σ, G) by work of Ricks [Ric17, p. 948]. (Because of the visibility assumption, every geodesic in X is rank-one in the sense Ricks uses.)

If Γ acts cocompactly on X, then X is Gromov hyperbolic ([EO73], see also [BH99, III.H.1.4]). Otherwise, X may not be Gromov hyperbolic. For example, given geodesics in the hyperbolic plane at distance at least 1 from one another, the surface obtained by grafting flat strips (of any widths) along these geodesics is always CAT(0) and visible.

1.3. **Outline of the paper.** In many theories of Patterson–Sullivan measures, the measures live on the boundary of a metric space and this metric space is used in an essential way in the study these measures. The first part of this paper (Sections 2 to 6) is devoted to developing a perspective for studying these measures without the presence of a metric space.

We first observe, in Section 2, that the set $\Gamma \sqcup M$ has a topology which makes it a compact metrizable space (the existence of this topology is implicit in work of Bowditch, see [Bow99]). In Section 3, we study properties of cocycles and prove that the coarse-cocycles in a coarse GPS system are expanding.

Among other things, we establish the following property, which allows us to regard our cocycles as the "Busemann cocycle" on the "Busemann boundary" associated to the metric-like function $\rho(\alpha, \beta) = \left\|\alpha^{-1}\beta\right\|_\sigma$ on Γ.

Proposition 1.13 (see Proposition 3.2 for more properties)**.** *Suppose $\Gamma \subset \mathsf{Homeo}(M)$ is a convergence group and $\sigma\colon \Gamma \times M \to \mathbb{R}$ is an expanding κ-coarse-cocycle. If $x \in \Lambda(\Gamma)$ and $\gamma \in \Gamma$, then*

$$\limsup_{\alpha\to x} \left|\sigma(\gamma, x) - (\|\gamma\alpha\|_\sigma - \|\alpha\|_\sigma)\right| \leq 2\kappa.$$

We use this result and Patterson's original argument to show that Patterson–Sullivan measures exist in the critical dimension.

Theorem 4.1. *If σ is an κ-coarse expanding cocycle for a convergence group $\Gamma \subset \mathsf{Homeo}(M)$ and $\delta := \delta_\sigma(\Gamma) < +\infty$, then there exists a $2\kappa\delta$-coarse σ-Patterson–Sullivan measure of dimension δ on M, which is supported on the limit set $\Lambda(\Gamma)$.*

One nearly immediate consequence of the existence of a Patterson–Sullivan measure is a result guaranteeing decrease of critical exponent in the spirit of Dal'bo–Otal–Peigné [DOP00, Prop. 2].

Theorem 4.2. *Suppose* $\Gamma \subset \mathsf{Homeo}(M)$ *is a convergence group,* σ *is an expanding coarse-cocycle, and* $\delta_\sigma(\Gamma) < +\infty$*. If* $G \subset \Gamma$ *is a subgroup where* $\Lambda(G)$ *is a strict subset of* $\Lambda(\Gamma)$ *and*

$$\sum_{g \in G} e^{-\delta_\sigma(G)\|g\|_\sigma} = +\infty,$$

then $\delta_\sigma(G) < \delta_\sigma(\Gamma)$*.*

The next key step in the paper is to define shadows in our setting and prove a version of shadow lemma. To define shadows we borrow an idea from the theory of Patterson–Sullivan measures associated to Zariski-dense discrete subgroups in semisimple Lie groups (compare the shadows below to the sets $\gamma B^{\epsilon}_{\theta,\gamma}$ in [Qui02, Lem. 8.2]).

Definition 1.14. Suppose $\Gamma \subset \mathsf{Homeo}(M)$ is a convergence group and d is a compatible metric on $\Gamma \sqcup M$. Given $\epsilon > 0$ and $\gamma \in \Gamma$, the associated *shadow* is

$$\mathcal{S}_\epsilon(\gamma) := \gamma\left(M - B_\epsilon(\gamma^{-1})\right).$$

where $B_\epsilon(\gamma^{-1})$ is the open ball centered at γ^{-1} of radius ϵ with respect to the metric d.

In Section 5, we establish some basic properties of shadows, relate shadows to a notion of uniformly conical limit points, and compare these shadows to the classically defined shadows in the Gromov hyperbolic setting. In Section 6 we prove our version of the Shadow Lemma:

The Shadow Lemma (see Theorem 6.1) *Suppose* $\Gamma \subset \mathsf{Homeo}(M)$ *is a convergence group,* $\sigma\colon \Gamma \times M \to \mathbb{R}$ *is an expanding coarse-cocycle, and* μ *is a coarse* σ*-Patterson–Sullivan measure on* M *of dimension* δ*. For any sufficiently small* $\epsilon > 0$ *there exists* $C = C(\epsilon) > 1$ *such that*

$$\frac{1}{C} e^{-\delta\|\gamma\|_\sigma} \leq \mu\left(\mathcal{S}_\epsilon(\gamma)\right) \leq C e^{-\delta\|\gamma\|_\sigma}$$

for all $\gamma \in \Gamma$*.*

We then establish some standard consequences of the Shadow Lemma in our setting.

Proposition 6.3. *Suppose* $\Gamma \subset \mathsf{Homeo}(M)$ *is a convergence group,* $\sigma\colon \Gamma \times M \to \mathbb{R}$ *is an expanding coarse-cocycle, and* μ *is a coarse* σ*-Patterson–Sullivan measure on* M *of dimension* β*. Then:*

1. *If* $y \in M$ *is a conical limit point, then* $\mu(\{y\}) = 0$*.*
2. *If*
$$\sum_{\gamma \in \Gamma} e^{-\beta\|\gamma\|_\sigma} < +\infty,$$
then $\mu(\Lambda^{\mathrm{con}}(\Gamma)) = 0$*.*
3. $\beta \geq \delta_\sigma(\Gamma)$*.*
4. *There exists* $C > 0$ *such that*
$$\#\{\gamma \in \Gamma : \|\gamma\|_\sigma \leq R\} \leq C e^{\delta_\sigma(\Gamma) R}$$
for any $R > 0$*.*

In the second part of the paper, we use the framework developed in the first part to study the ergodicity properties of Patterson–Sullivan measures. In Sections 7 and 8 we prove Theorem 1.3. In Section 9 we study the action of Γ on $M^{(2)}$. In Section 10 we introduce a flow space which admits a measurable action of Γ. In Section 11 we use this flow space to establish ergodicity of the action of Γ on $(M^{(2)}, \nu)$ in Theorem 1.8. Finally, in Section 12 we complete the proof of Theorem 1.8.

The constructions and arguments in Sections 10 and 11 uses ideas from the work of Bader–Furman [BF17].

For continuous GPS systems (i.e. when $\kappa = 0$ in Definition 1.7), there is a well-defined continuous flow space $\psi^t : U_\Gamma \to U_\Gamma$ and when the Poincaré series diverges at its critical exponent, there is a unique Bowen–Margulis–Sullivan measure (see Section 10.4 for details). The arguments establishing Theorem 1.8 show that the flow is conservative and ergodic in this case.

Theorem 11.2. *If $(\sigma, \bar{\sigma}, G)$ is a continuous GPS system with $\delta := \delta_\sigma(\Gamma) < +\infty$ and*

$$\sum_{\gamma \in \Gamma} e^{-\delta \|\gamma\|_\sigma} = +\infty,$$

then the flow ψ^t on (U_Γ, m_Γ) is conservative and ergodic.

In the third part of the paper, we consider applications of our ergodicity results and examine more deeply relations between expanding cocycles and GPS systems.

In Section 13 we observe that two expanding coarse-cocycles with coarsely the same magnitudes have coarsely the same periods, and vice versa. In Section 14 we establish Proposition 1.4 and in Section 15 we establish Theorem 1.6.

The results in the next two sections partly answer the question of whether every expanding (coarse-)cocycle is part of a (coarse) GPS system, in addition to describing a systematic way to find expanding cocycles. In Section 16 we define what it means for a coarse-cocycle to be coarsely-symmetric and prove that any expanding coarsely-symmetric coarse-cocycle is part of a GPS system. In Section 17 we study the coarsely additive potentials introduced in Definition 1.9 above.

Finally, in Appendix A we define the notions of conservativity, dissipativity and Hopf decompositions for a general group action. We also prove that quotient measures exist when the action is dissipative, which is an essential point in our construction of a measurable flow space.

1.4. **Other approaches and related results.** In recent work Cantrell–Tanaka [CT21, CT22] study general cocycles on the Gromov boundary $\partial_\infty \Gamma$ of a word hyperbolic group. They show that if two cocycles have a corresponding Gromov product, then it is possible to use Patterson–Sullivan measures to build a Γ-invariant measure on $\partial^{(2)}\Gamma$ [CT22, Prop. 2.8] and prove ergodicity of the Γ action [CT22, Thm. 3.1]. They also consider a slightly more restrictive notion of the coarsely additive potentials introduced above (see Remark 1.10 above) and show that they give rise to coarse-cocycles satisfying these hypotheses. Our definition of GPS systems can be viewed as an extension of some of their ideas to general convergence groups.

A number of recent papers study Patterson–Sullivan theory for metric spaces where the group need not act as a convergence group on the boundary of the metric space. The most general of these investigations are perhaps independent works of

Coulon [Cou22, Cou23] and Yang [Yan23] which consider the case when X is a proper geodesic metric space and Γ is a group acting properly on X by isometries with a contracting element. In this case the boundary is the horoboundary of X and the cocycle is Busemann cocycle. The group action on this boundary may not be a convergence group action, but satisfies certain contracting properties.

In many ways our approach is orthogonal to Coulon and Yang's. In our approach, we start with a convergence group action and find large classes of cocycles that are amenable to Patterson–Sullivan theory. In Coulon and Yang's approach, one studies large classes of metric spaces where the Busemann cocycle is amenable to Patterson–Sullivan theory. It would also be interesting to develop a uniform framework which contains both theories.

Part 1. **Foundations**

2. Convergence groups

When M is a compact metrizable space, a subgroup $\Gamma \subset \mathsf{Homeo}(M)$ is called a (discrete) *convergence group* if for every sequence $\{\gamma_n\}$ of distinct elements in Γ, there exist points $x, y \in M$ and a subsequence $\{\gamma_{n_j}\}$ such that $\gamma_{n_j}|_{M \smallsetminus \{y\}}$ converges locally uniformly to x. This notion was first introduced in [GM87].

Bowditch proved that this is equivalent to asking that Γ acts properly discontinuously on the set of distinct triples of M [Bow99, Prop. 1.1].

Given a convergence group, we define the following:

(1) The *limit set* $\Lambda(\Gamma)$ is the set of points $x \in M$ where there exist $y \in M$ and a sequence $\{\gamma_n\}$ in Γ so that $\gamma_n|_{M \smallsetminus \{y\}}$ converges locally uniformly to x.

(2) A point $x \in \Lambda(\Gamma)$ is a *conical limit point* if there exist distinct points $a, b \in M$ and a sequence of elements $\{\gamma_n\}$ in Γ where $\lim_{n\to\infty} \gamma_n(x) = a$ and $\lim_{n\to\infty} \gamma_n(y) = b$ for all $y \in M \smallsetminus \{x\}$.

We say that a convergence group Γ is *non-elementary* if $\Lambda(\Gamma)$ contains at least 3 points. In this case $\Lambda(\Gamma)$ is the smallest Γ-invariant closed subset of M.

The elements in a convergence group can be characterized as follows.

Fact 2.1 ([Tuk94, Th. 2B])**.** *Suppose* $\Gamma \subset \mathsf{Homeo}(M)$ *is a convergence group, then every element* $\gamma \in \Gamma$ *is either*

- loxodromic*: it has two fixed points* γ^+ *and* γ^- *in the limit set* $\Lambda(\Gamma) \subset M$ *such that* $\gamma^{\pm n}|_{M \smallsetminus \{\gamma^{\mp}\}}$ *converges locally uniformly to* $\gamma^{\pm}$*,*
- parabolic*: it has one fixed point* $p \in \Lambda(\Gamma)$ *such that* $\gamma^{\pm n}|_{M \smallsetminus \{p\}}$ *converges locally uniformly to* p*, or*
- elliptic*: it has finite order.*

We next observe that $\Gamma \sqcup M$ admits a metrizable compact topology. This topology plays a similar role in our work as the topology on the union of a transverse group and its limit set did in [CZZ23]. Our argument is similar to a construction of Bowditch [Bow99, pg. 4 & Prop. 1.8] which produces a natural compact topology on $M^{(3)} \sqcup M$ by seeing it as a quotient of M^3, where $M^{(3)}$ is the space of ordered pairwise distinct elements of M^3.

Definition 2.2. Given a convergence group $\Gamma \subset \mathsf{Homeo}(M)$, a *compactifying topology* on $\Gamma \sqcup M$ is a topology such that:

- $\Gamma \sqcup M$ is a compact metrizable space.

- The inclusions $\Gamma \hookrightarrow \Gamma \sqcup M$ and $M \hookrightarrow \Gamma \sqcup M$ are embeddings (where in the first embedding Γ has the discrete topology).
- Γ acts as a convergence group on $\Gamma \sqcup M$.

A metric d on $\Gamma \sqcup M$ is called *compatible* if it induces a compactifying topology.

Proposition 2.3. *If $\Gamma \subset \mathsf{Homeo}(M)$ is a convergence group, then there exists a unique compactifying topology. Moreover, with respect to this topology the following hold:*

(1) *If $\{\gamma_n\} \subset \Gamma$ is a sequence where $\gamma_n \to a \in M$ and $\gamma_n^{-1} \to b \in M$, then $\gamma_n|_{M \smallsetminus \{b\}}$ converges locally uniformly to a.*
(2) *A sequence $\{\gamma_n\} \subset \Gamma$ converges to $a \in M$ if and only if for every subsequence $\{\gamma_{n_j}\}$ there exist $b \in M$ and a further subsequence $\{\gamma_{n_{j_k}}\}$ such that $\gamma_{n_{j_k}}|_{M \smallsetminus \{b\}}$ converges locally uniformly to a.*
(3) *For any compatible metric d and any $\epsilon > 0$ there exists a finite set $F \subset \Gamma$ such that*
$$\gamma\left(M \smallsetminus B_\epsilon(\gamma^{-1})\right) \subset B_\epsilon(\gamma)$$
for every $\gamma \in \Gamma \smallsetminus F$ (where $B_r(x)$ is the open ball of radius r centered at x with respect to d).
(4) *Γ is open in $\Gamma \sqcup M$ and its closure is $\Gamma \sqcup \Lambda(\Gamma)$.*

Proof. We first show that $\Gamma \sqcup M$ has a compactifying topology.

Fix three distinct points $x_1, x_2, x_3 \in M$. For any open set $U \subset M$, let $\Gamma_U \subset \Gamma$ be the set of γ such that $\#(\{\gamma x_1, \gamma x_2, \gamma x_3\} \cap U) \geq 2$. Fix a countable basis $\mathcal{B}$ of open sets of M. Let $\mathcal{B}'$ be the set of singletons of Γ and subsets of $\Gamma \sqcup M$ of the form $\Gamma_U \cup U$ for some $U \in \mathcal{B}$. It is straightforward to check that the topology generated by $\mathcal{B}'$ is compact Hausdorff and second-countable. Hence, it is metrizable by Urysohn's metrization theorem. It remains to show that Γ acts on $\Gamma \sqcup M$ as a convergence group.

Suppose $\{\gamma_n\} \subset \Gamma$ is a sequence of distinct elements. Then there exist points $a, b \in M$ and a subsequence $\{\gamma_{n_j}\}$ such that $\gamma_{n_j}|_{M \smallsetminus \{b\}}$ converges locally uniformly to a. We claim that $\gamma_{n_j}|_{\Gamma \sqcup M \smallsetminus \{b\}}$ converges locally uniformly to a. Suppose not. Then after passing to a subsequence we can find $\{z_j\} \subset \Gamma \sqcup M$ where $z_j \to z \neq b$ and $\gamma_{n_j}(z_j) \to c \neq a$. Passing to a further subsequence we can consider two cases:

Case 1: Assume $\{z_j\} \subset M$. Then by the choice of $\{\gamma_{n_j}\}$ we have $\gamma_{n_j}(z_j) \to a$, which contradicts our assumptions.

Case 2: Assume $\{z_j\} \subset \Gamma$. First suppose that $z \in \Gamma$, then passing to a subsequence we can suppose that $z_j = z$ for all j. At least two zx_1, zx_2, zx_3 do not equal b. So after relabelling we can suppose that $zx_1 \neq b$ and $zx_2 \neq b$. Then $(\gamma_{n_j} z_j)(x_1) \to a$ and $(\gamma_{n_j} z_j)(x_2) \to a$. So by the definition of the topology $\gamma_{n_j} z_j \to a$. So we have a contradiction.

Next suppose that $z \in M$. Then by definition of the topology and passing to a subsequence we can assume that $z_j|_{M \setminus \{b'\}}$ converges locally uniformly to z. Since $z \neq b$, then $(\gamma_{n_j} z_j)|_{M \setminus \{b'\}}$ converges locally uniformly to a, which implies that $\gamma_{n_j} z_j \to a$. So we have a contradiction.

Thus Γ acts on $\Gamma \sqcup M$ as a convergence group and hence $\Gamma \sqcup M$ has a compactifying topology.

Next we consider $\Gamma \sqcup M$ with some compactifying topology and prove the assertions in the "moreover" part of the proposition. Notice that part (2) will imply that there is a unique compactifying topology. Let d be a metric which induces this topology.

(1) Assume $\gamma_n \to a$ and $\gamma_n^{-1} \to b$. Suppose for a contradiction that $\gamma_n|_{M \smallsetminus \{b\}}$ does not converge locally uniformly to a. Then after passing to a subsequence there exist $\epsilon > 0$ and $\{c_n\} \subset M \smallsetminus B(b,\epsilon)$ such that $\{\gamma_n(c_n)\} \subset M \smallsetminus B(a,\epsilon)$. Since Γ acts as a convergence group on $\Gamma \sqcup M$, passing to a further subsequence we can suppose that $\gamma_n|_{\Gamma \sqcup M \smallsetminus \{b'\}}$ converges locally uniformly to a' for some $a', b' \in \Gamma \sqcup M$. Since Γ acts by homeomorphisms on M, we must have $b' \in M$ (otherwise when n is large $\gamma_n|_M$ would not map onto M). So

$$a = \lim_{n\to\infty} \gamma_n = \lim_{n\to\infty} \gamma_n(\mathrm{id}) = a'.$$

Also notice that $\gamma_n(\gamma_n^{-1}) = \mathrm{id}$ for all n and so we must have $b = b'$. Then $\gamma_n(c_n) \to a$ and we have a contradiction.

(2) ($\Rightarrow$): Suppose $\gamma_n \to a$ and fix a subsequence $\{\gamma_{n_j}\}$. Since $\Gamma \sqcup M$ is compact, there exists a subsequence with $\gamma_{n_{j_k}}^{-1} \to b$. Then by (1), $\gamma_{n_{j_k}}|_{M \smallsetminus \{b\}}$ converges locally uniformly to a.

($\Leftarrow$): Suppose $a \in M$ and $\{\gamma_n\} \subset \Gamma$ has the property that for every subsequence $\{\gamma_{n_j}\}$ there exist $b \in M$ and a further subsequence $\{\gamma_{n_{j_k}}\}$ such that $\gamma_{n_{j_k}}|_{M \smallsetminus \{b\}}$ converges locally uniformly to a. Since $\Gamma \sqcup M$ is compact, to show that γ_n converges to a it suffices to show that every convergent subsequence converges to a. So suppose that $\gamma_{n_j} \to a'$. Passing to a subsequence we can suppose that $\gamma_{n_j}^{-1} \to b$. Then by (1), $\gamma_{n_j}|_{M \smallsetminus \{b\}}$ converges locally uniformly to a'. So by hypothesis, $a = a'$.

(3) Fix $\epsilon > 0$ and suppose not. Then there exist a sequence $\{\gamma_n\}$ of distinct elements and a sequence $\{x_n\} \subset \Gamma \sqcup M$ such that

$$\mathrm{d}(\gamma_n(x_n), \gamma_n) \geq \epsilon \quad \text{and} \quad \mathrm{d}(x_n, \gamma_n^{-1}) \geq \epsilon.$$

Passing to a subsequence, we can suppose that $\gamma_n \to a \in M$ and $\gamma_n^{-1} \to b \in M$. Then by (1), $\gamma_n|_{M \smallsetminus \{b\}}$ converges locally uniformly to a. Since

$$\lim_{n\to\infty} \mathrm{d}(x_n, b) = \lim_{n\to\infty} \mathrm{d}(x_n, \gamma_n^{-1}) \geq \epsilon,$$

then $\gamma_n(x_n) \to a$. So

$$\epsilon \leq \lim_{n\to\infty} \mathrm{d}(\gamma_n(x_n), \gamma_n) = \mathrm{d}(a,a) = 0$$

and we have a contradiction.

(4) Since M is compact, it must be closed in $\Gamma \sqcup M$. Hence Γ must be open. Part (2) implies that the closure of Γ in $\Gamma \sqcup M$ is $\Gamma \sqcup \Lambda(\Gamma)$. □

3. Cocycles and GPS systems

In this subsection, we record basic properties of coarse-cocycles and GPS systems. We begin with a few simple properties shared by all coarse-cocycles.

Observation 3.1. Suppose $\Gamma \subset \mathsf{Homeo}(M)$ is a convergence group and σ is a κ-coarse-cocycle. Then:

1. If $\delta_\sigma(\Gamma) < +\infty$, then σ is proper.
2. $|\sigma(\mathrm{id}, x)| \leq \kappa$ for any $x \in M$.
3. If $\gamma \in \Gamma$ and $x \in M$, then

$$\left|\sigma(\gamma, \gamma^{-1}x) + \sigma(\gamma^{-1}, x)\right| \leq 2\kappa.$$

(4) If $\gamma_1, \gamma_2 \in \Gamma$, then

$$\max\left\{\|\gamma_1\|_\sigma - \left\|\gamma_2^{-1}\right\|_\sigma, \|\gamma_2\|_\sigma - \left\|\gamma_1^{-1}\right\|_\sigma\right\} - \kappa \leq \|\gamma_1\gamma_2\|_\sigma \leq \|\gamma_1\|_\sigma + \|\gamma_2\|_\sigma + \kappa.$$

Proof. Part (1) follows immediately from the definitions. For part (2), notice that

$$|\sigma(\mathrm{id}, x)| = \left|\sigma(\mathrm{id}^2, x) - (\sigma(\mathrm{id}, \mathrm{id}(x)) + \sigma(\mathrm{id}, x))\right| \leq \kappa.$$

Part (3) follows from part (2) and the fact that

$$\left|\sigma(\gamma, \gamma^{-1}x) + \sigma(\gamma^{-1}, x) - \sigma(\mathrm{id}, x)\right| \leq \kappa.$$

For part (4), notice that

$$\begin{aligned}\|\gamma_1\gamma_2\|_\sigma &= \sup_{x\in M} \sigma(\gamma_1\gamma_2, x) \leq \kappa + \sup_{x \in M}\Big(\sigma(\gamma_1, \gamma_2 x) + \sigma(\gamma_2, x)\Big)\\ &\leq \kappa + \|\gamma_1\|_\sigma + \|\gamma_2\|_\sigma\end{aligned}$$

for all $\gamma_1, \gamma_2 \in \Gamma$, giving us the upper bound. For the lower bound, note

$$\begin{aligned}\|\gamma_1\|_\sigma - \left\|\gamma_2^{-1}\right\|_\sigma &= \left\|\gamma_1\gamma_2\gamma_2^{-1}\right\|_\sigma - \left\|\gamma_2^{-1}\right\|_\sigma\\ &\leq \|\gamma_1\gamma_2\|_\sigma + \left\|\gamma_2^{-1}\right\|_\sigma - \left\|\gamma_2^{-1}\right\|_\sigma + \kappa\\ &= \|\gamma_1\gamma_2\|_\sigma + \kappa\end{aligned}$$

and

$$\begin{aligned}\|\gamma_2\|_\sigma - \left\|\gamma_1^{-1}\right\|_\sigma &= \left\|\gamma_1^{-1}\gamma_1\gamma_2\right\|_\sigma - \left\|\gamma_1^{-1}\right\|_\sigma\\ &\leq \left\|\gamma_1^{-1}\right\|_\sigma + \|\gamma_1\gamma_2\|_\sigma - \left\|\gamma_1^{-1}\right\|_\sigma + \kappa\\ &= \|\gamma_1\gamma_2\|_\sigma + \kappa\end{aligned}$$

for all $\gamma_1, \gamma_2 \in \Gamma$. □

In the majority of our work we will further require that our coarse-cocycles are expanding, see Definition 1.1. The next result establishes a number of useful properties for such cocycles.

Proposition 3.2. *Suppose $\Gamma \subset \mathsf{Homeo}(M)$ is a convergence group, d is a compatible metric on $\Gamma \sqcup M$, and σ is an expanding κ-coarse-cocycle, then:*

(1) If $\gamma \in \Gamma$ is loxodromic, then

$$-\kappa + \liminf_{n\to\infty} \frac{1}{n}\|\gamma^n\|_\sigma \leq \sigma(\gamma, \gamma^+) \leq \kappa + \limsup_{n\to\infty}\frac{1}{n}\|\gamma^n\|_\sigma$$

and

$$-\kappa < \sigma(\gamma, \gamma^+).$$

(2) If $\gamma \in \Gamma$ is parabolic with fixed point $p \in M$, then

$$-2\kappa \leq \sigma(\gamma, p) \leq 4\kappa.$$

(3) If $\{\gamma_n\} \subset \Gamma$ is an escaping sequence, $\{y_n\} \subset M$ and $\{\sigma(\gamma_n, y_n)\}$ is bounded below, then

$$\lim_{n\to\infty} \mathrm{d}(\gamma_n y_n, \gamma_n) = 0.$$

(4) For any $\alpha \in \Gamma$, the function

$$x \in \Gamma \sqcup M \longmapsto \begin{cases}\sigma(\alpha, x) & \text{if } x \in M\\ \|\alpha x\|_\sigma - \|x\|_\sigma & \text{if } x \in \Gamma\end{cases}$$

is (2κ)-coarsely-continuous.

(5) *For any* $\epsilon > 0$ *there exists* $C > 0$ *such that: if* $\alpha, \beta \in \Gamma$ *and* $\mathrm{d}(\alpha^{-1}, \beta) \geq \epsilon$, *then*

$$\|\alpha\|_\sigma + \|\beta\|_\sigma - C \leq \|\alpha\beta\|_\sigma .$$

(6) *For any* $\epsilon > 0$ *there exists a finite subset* $F \subset \Gamma$ *such that: if* $\alpha, \beta \in \Gamma$, $\|\alpha\|_\sigma \leq \|\beta\|_\sigma$ *and* $\beta^{-1}\alpha \notin F$, *then*

$$\mathrm{d}(\beta^{-1}, \beta^{-1}\alpha) \leq \epsilon.$$

Proof of (1). Suppose $\gamma \in \Gamma$ is loxodromic. Since $\gamma^{-n} \to \gamma^-$ when $n \to \infty$ and $\gamma^+ \neq \gamma^-$, the expanding property implies that there exists $C > 0$ such that

$$\|\gamma^n\|_\sigma - C \leq \sigma(\gamma^n, \gamma^+) \leq \|\gamma^n\|_\sigma$$

for all $n \geq 1$. By the coarse cocycle property,

$$n\sigma(\gamma, \gamma^+) - (n-1)\kappa \leq \sigma(\gamma^n, \gamma^+) \leq n\sigma(\gamma, \gamma^+) + (n-1)\kappa$$

Combining the two estimates and sending n to infinity yields the first set of inequalities.

By the properness assumption there exists $N \geq 1$ such that $\left\|\gamma^N\right\|_\sigma > C$. Then

$$\sigma(\gamma, \gamma^+) \geq \frac{1}{N}\Big(\sigma(\gamma^N, \gamma^+) - (N-1)\kappa\Big) \geq \frac{1}{N}\left(\left\|\gamma^N\right\|_\sigma - C\right) - \kappa > -\kappa. \qquad \square$$

Proof of (2). Suppose $\gamma \in \Gamma$ is parabolic with fixed point $p \in M$. Fix $y \in M \smallsetminus \{p\}$. Since $\gamma^{\pm n} \to p$ when $n \to \infty$, by the expanding property there exists $C > 0$ such that

$$\sigma(\gamma^{\pm n}, y) \geq \left\|\gamma^{\pm n}\right\|_\sigma - C$$

for all $n \geq 1$. So by the properness assumption, both $\sigma(\gamma^n, y)$ and $\sigma(\gamma^{-n}, y)$ are nonnegative for n large. Therefore

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n}\left(\sigma(\gamma^{\pm1}, \gamma^{\pm(n-1)}y) + \sigma(\gamma^{\pm1}, \gamma^{\pm(n-2)}y) + \cdots + \sigma(\gamma^{\pm1}, y)\right)\\ \geq \limsup_{n\to\infty} \frac{1}{n}\left(\sigma(\gamma^{\pm n}, y) - (n-1)\kappa\right) \geq -\kappa.\end{aligned}$$

Since σ is κ-coarsely continuous and $\gamma^{\pm n}y \to p$, we see that

$$\limsup_{n\to\infty} \frac{1}{n}\left(\sigma(\gamma^{\pm1}, \gamma^{\pm(n-1)}y) + \sigma(\gamma^{\pm1}, \gamma^{\pm(n-2)}y) + \cdots + \sigma(\gamma^{\pm1}, y)\right) \leq \sigma(\gamma^{\pm1}, p) + \kappa.$$

Thus, $\sigma(\gamma^{\pm1}, p) \geq -2\kappa$.

Finally, by the coarse cocycle identity, see Observation 3.1(3),

$$\sigma(\gamma, p) + \sigma(\gamma^{-1}, p) \leq 2\kappa.$$

Hence, $\sigma(\gamma^{\pm1}, p) \leq 4\kappa$. $\square$

Proof of (3). We prove the contrapositive: if $\{\mathrm{d}(\gamma_n y_n, \gamma_n)\}$ does not converge to 0, then

$$\liminf_{n\to\infty} \sigma(\gamma_n, y_n) = -\infty.$$

Passing to a subsequence, we may suppose that there exists $\epsilon > 0$ such that

$$\mathrm{d}(\gamma_n, \gamma_n y_n) \geq \epsilon$$

for all $n \geq 1$. Then by the expanding property, there exists $C > 0$ such that

$$\sigma(\gamma_n^{-1}, \gamma_n y_n) \geq \left\|\gamma_n^{-1}\right\|_\sigma - C$$

for all $n \geq 1$. In particular, since σ is proper, we have

$$\liminf_{n\to\infty} \sigma(\gamma_n^{-1}, \gamma_n y_n) = +\infty$$

Since

$$\left|\sigma(\gamma_n, y_n) - \sigma(\gamma_n^{-1}, \gamma_n y_n)\right| \leq 2\kappa$$

for all n (see Observation 3.1), this implies that $\liminf_{n\to\infty} \sigma(\gamma_n, y_n) = -\infty$. □

Proof of (4). It suffices to fix a sequence $\{\gamma_n\}$ in Γ converging to $x \in M$ and show that

$$\limsup_{n\to\infty} \left|\sigma(\alpha, x) - \Big(\left\|\alpha\gamma_n\right\|_\sigma - \left\|\gamma_n\right\|_\sigma \Big)\right| \leq 2\kappa.$$

For each n, fix $y_n \in M$ such that

$$\left\|\alpha\gamma_n\right\|_\sigma - \frac{1}{2^n} \leq \sigma(\alpha\gamma_n, y_n).$$

Notice that $\alpha\gamma_n \to \alpha x$ and so part (3) implies that $(\alpha\gamma_n)y_n \to \alpha x$. Hence $\gamma_n y_n \to x$. Then

$$\left\|\alpha\gamma_n\right\|_\sigma - \left\|\gamma_n\right\|_\sigma \leq \sigma(\alpha\gamma_n, y_n) - \sigma(\gamma_n, y_n) + \frac{1}{2^n} \leq \kappa + \sigma(\alpha, \gamma_n y_n) + \frac{1}{2^n}$$

and so

$$\limsup_{n\to\infty} \left\|\alpha\gamma_n\right\|_\sigma - \left\|\gamma_n\right\|_\sigma \leq 2\kappa + \sigma(\alpha, x).$$

Next for each n, fix $z_n \in M$ such that

$$\left\|\gamma_n\right\|_\sigma - \frac{1}{2^n} \leq \sigma(\gamma_n, z_n).$$

Then part (3) implies that $\gamma_n z_n \to x$. Then

$$\left\|\alpha\gamma_n\right\|_\sigma - \left\|\gamma_n\right\|_\sigma \geq \sigma(\alpha\gamma_n, z_n) - \sigma(\gamma_n, z_n) - \frac{1}{2^n} \geq \sigma(\alpha, \gamma_n z_n) - \kappa - \frac{1}{2^n}$$

and so

$$\liminf_{n\to\infty} \left\|\alpha\gamma_n\right\|_\sigma - \left\|\gamma_n\right\|_\sigma \geq \sigma(\alpha, x) - 2\kappa.$$

□

Proof of (5). Suppose not. Then there exist sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in Γ where $\mathrm{d}(\alpha_n^{-1}, \beta_n) \geq \epsilon$ and

$$\lim_{n\to\infty} \left\|\alpha_n\right\|_\sigma + \left\|\beta_n\right\|_\sigma - \left\|\alpha_n\beta_n\right\|_\sigma = +\infty.$$

By Observation 3.1,

$$\left\|\alpha_n\beta_n\right\|_\sigma \geq \left\|\alpha_n\right\|_\sigma + \left\|\beta_n\right\|_\sigma - (\left\|\beta_n\right\|_\sigma + \left\|\beta_n^{-1}\right\|_\sigma) - \kappa$$

and hence $\{\beta_n\}$ is an escaping sequence. For similar reasons, $\{\alpha_n\}$ is an escaping sequence. Then passing to a subsequence we can suppose that $\beta_n \to x \in M$ and $\alpha_n^{-1} \to y \in M$. Notice that, by assumption, $x \neq y$.

For each n, fix $x_n \in M$ such that

$$\sigma(\beta_n, x_n) \geq \left\|\beta_n\right\|_\sigma - 1.$$

Then part (3) implies that $\beta_n x_n \to x$. So

$$\liminf_{n\to\infty} \mathrm{d}(\alpha_n^{-1}, \beta_n x_n) \geq \epsilon.$$

So by the expanding property, there exists $C' > 0$ such that

$$\sigma(\alpha_n, \beta_n x_n) \geq \left\|\alpha_n\right\|_\sigma - C'$$

for all $n \geq 1$. But then

$$\begin{aligned}\|\alpha_n\beta_n\|_\sigma &\geq \sigma(\alpha_n\beta_n, x_n) \geq \sigma(\alpha_n, \beta_n x_n) + \sigma(\beta_n, x_n) - \kappa \\ &\geq \|\alpha_n\|_\sigma + \|\beta_n\|_\sigma - C' - 1 - \kappa\end{aligned}$$

and we have a contradiction. □

Proof of (6). This is essentially the contrapositive of Part (5). By that part there exists $C > 0$ such that: if $\alpha, \beta \in \Gamma$ and $\mathrm{d}(\alpha^{-1}, \beta) \geq \epsilon$, then

$$\|\alpha\|_\sigma + \|\beta\|_\sigma - C \leq \|\alpha\beta\|_\sigma .$$

Then let

$$F := \{\gamma \in \Gamma : \|\gamma\|_\sigma \leq C\}.$$

Notice that F is finite since σ is proper.

Now if $\alpha, \beta \in \Gamma$, $\|\alpha\|_\sigma \leq \|\beta\|_\sigma$, and $\beta^{-1}\alpha \notin F$, then

$$\|\beta\|_\sigma + \left\|\beta^{-1}\alpha\right\|_\sigma - C > \|\alpha\|_\sigma = \left\|\beta\beta^{-1}\alpha\right\|_\sigma .$$

So by our choice of C we must have $\mathrm{d}(\beta^{-1}, \beta^{-1}\alpha) < \epsilon$. □

Finally, we observe that the coarse-cocycles in a coarse GPS system are expanding.

Proposition 3.3. *If $\Gamma \subset \mathsf{Homeo}(M)$ is a convergence group and $(\sigma, \bar\sigma, G)$ is a coarse GPS system where $\delta_\sigma(\Gamma) < +\infty$, then:*

(1) There exists $C > 0$ such that

$$\left\|\gamma^{-1}\right\|_{\bar\sigma} - C \leq \|\gamma\|_\sigma \leq \left\|\gamma^{-1}\right\|_{\bar\sigma} + C$$

for all $\gamma \in \Gamma$.

(2) σ and $\bar\sigma$ are expanding.

(3) **If G is κ-coarsely continuous***, then there exists $C' > 0$ such that for any $(a, b) \in \Lambda(\Gamma)^{(2)}$,*

$$-C' + \limsup_{\alpha\to a, \beta\to b} G(\alpha, \beta) \leq G(a, b) \leq \liminf_{\alpha\to a, \beta\to b} G(\alpha, \beta) + C',$$

where, given $\alpha, \beta \in \Gamma$ we write

$$G(\alpha, \beta) := \left\|\alpha^{-1}\right\|_\sigma + \|\beta\|_\sigma - \left\|\alpha^{-1}\beta\right\|_\sigma .$$

Proof of (1). Fix $\epsilon > 0$ such that for every $x \in M$ there exists $y \in M$ with $\mathrm{d}(x, y) \geq \epsilon$. Then let

$$C := \sup\{G(x, y) : \mathrm{d}(x, y) \geq \epsilon\}.$$

Notice that $C < +\infty$ since G is locally bounded.

Fix $\gamma \in \Gamma$. Then fix $y \in M$ such that $\|\gamma\|_\sigma - 1 \leq \sigma(\gamma, y)$. Pick $x' \in M$ such that $\mathrm{d}(x', \gamma y) \geq \epsilon$ and let $x := \gamma^{-1}(x')$. Then

$$\begin{aligned}\|\gamma\|_\sigma &\leq \sigma(\gamma, y) + 1 \leq G(\gamma x, \gamma y) - G(x, y) - \bar\sigma(\gamma, x) + \kappa + 1 \\ &\leq G(\gamma x, \gamma y) - G(x, y) + \bar\sigma(\gamma^{-1}, \gamma x) + 3\kappa + 1 \\ &\leq C - 0 + \left\|\gamma^{-1}\right\|_{\bar\sigma} + 3\kappa + 1 = \left\|\gamma^{-1}\right\|_{\bar\sigma} + C + 3\kappa + 1.\end{aligned}$$

The same reasoning can be used to show that

$$\left\|\gamma^{-1}\right\|_{\bar\sigma} \leq \|\gamma\|_\sigma + C + 3\kappa + 1.$$

□

Proof of (2). Fix $\epsilon > 0$. We wish to find $C > 0$ such that $\sigma(\gamma, x) > \|\gamma\|_\sigma - C$ whenever $\mathrm{d}(\gamma^{-1}, x) > \epsilon$.

To this end: fix $\epsilon' \in (0, \epsilon)$ such that $M \not\subset B_{\epsilon'}(p) \cup B_{\epsilon'}(q)$ for all $p, q \in \Gamma \sqcup M$. Let

$$C_1 := \sup\{G(p,q) : \mathrm{d}(p,q) \geq \epsilon'/2\}.$$

Notice that $C_1 < +\infty$ since G is locally bounded. By Proposition 2.3 there exists a finite subset $F \subset \Gamma$ such that

$$\gamma^{-1}\left(M - B_{\epsilon'/2}(\gamma)\right) \subset B_{\epsilon'/2}(\gamma^{-1}) \tag{1}$$

for any $\gamma \in \Gamma - F$. Then let

$$C_2 := \sup\left\{\|g\|_\sigma - \sigma(g, x) : g \in F \text{ and } x \in M\right\}.$$

We claim that $C := \max\{C_2, 2C_1 + 2\kappa + 1\}$ suffices.

Fix $\gamma \in \Gamma$ and $x \in M - B_\epsilon(\gamma^{-1})$. If $\gamma \in F$, then

$$\sigma(\gamma, x) \geq \|\gamma\|_\sigma - C_2.$$

Otherwise fix $y \in M$ such that $\|\gamma\|_\sigma \leq 1 + \sigma(\gamma, y)$. By the definition of ϵ' there exists $z' \in M - (B_{\epsilon'}(\gamma y) \cup B_{\epsilon'}(\gamma))$. Then let $z := \gamma^{-1}(z')$. By Equation (1),

$$z \in B_{\epsilon'/2}(\gamma^{-1})$$

and hence

$$\mathrm{d}(x, z) \geq \mathrm{d}(x, \gamma^{-1}) - \mathrm{d}(z, \gamma^{-1}) > \epsilon - \epsilon'/2 > \epsilon'/2.$$

Then

$$\begin{aligned}
\sigma(\gamma, x) - \|\gamma\|_\sigma &\geq \sigma(\gamma, x) + \bar\sigma(\gamma, z) - (\sigma(\gamma, y) + \bar\sigma(\gamma, z)) - 1 \\
&\geq G(\gamma z, \gamma x) - G(z, x) - G(\gamma z, \gamma y) + G(z, y) - 2\kappa - 1 \\
&\geq -G(z, x) - G(\gamma z, \gamma y) - 2\kappa - 1 \geq -2C_1 - 2\kappa - 1.
\end{aligned}$$

□

Proof of (3). Fix $\epsilon > 0$ such that $M \not\subset B_\epsilon(x) \cup B_\epsilon(y)$ for all $x, y \in \Gamma \sqcup M$.

Fix $a \neq b \in \Lambda(\Gamma)$ and sequences $\{\alpha_n\}, \{\beta_n\} \subset \Gamma$ converging to a, b respectively. Passing to a subsequence we can assume that $\alpha_n^{-1} \to a_-$ and $\beta_n^{-1} \to b_-$. Note that $a \neq b$ implies that $\alpha_n^{-1}\beta_n \to a_-$ and $\beta_n^{-1}\alpha_n \to b_-$.

Fix $x, y, z \in M$ such that

$$\mathrm{d}(x, a),\ \mathrm{d}(y, b_-),\ \mathrm{d}(z, b),\ \mathrm{d}(z, a_-) > \epsilon.$$

Passing to a further subsequence and using the facts that $\alpha_n^{-1}x \to a_-$, $\alpha_n z \to a$ and $\beta_n^{-1}\alpha_n z \to b_-$, we can assume that

$$\mathrm{d}(\alpha_n, a),\ \mathrm{d}(\beta_n^{-1}, b_-),\ \mathrm{d}(\alpha_n^{-1}x, a_-),\ \mathrm{d}(\alpha_n^{-1}\beta_n, a_-),\ \mathrm{d}(\alpha_n z, a),\ \mathrm{d}(\beta_n^{-1}\alpha_n z, b_-) < \frac{\epsilon}{2}.$$

This implies that

$$\mathrm{d}(\alpha_n, x),\ \mathrm{d}(\beta_n^{-1}, y),\ \mathrm{d}(z, \alpha_n^{-1}x),\ \mathrm{d}(z, \alpha_n^{-1}\beta_n),\ \mathrm{d}(\alpha_n z, x),\ \mathrm{d}(\beta_n^{-1}\alpha_n z, y) \geq \frac{\epsilon}{2}.$$

Then using the constant C from Part (1) we have

$$G(\alpha_n, \beta_n) \leq \left\|\alpha_n^{-1}\right\|_\sigma + \|\beta_n\|_\sigma - \left\|\beta_n^{-1}\alpha_n\right\|_{\bar\sigma} + C.$$

Since $\sigma, \bar\sigma$ are expanding, there exists C_ϵ such that

$$\sigma(\gamma, p) \geq \|\gamma\|_\sigma - C_\epsilon \quad \text{and} \quad \bar\sigma(\gamma, p) \geq \|\gamma\|_{\bar\sigma} - C_\epsilon$$

whenever $\mathrm{d}(p, \gamma^{-1}) \geq \frac{\epsilon}{2}$. Since $\mathrm{d}(\alpha_n, x) \geq \frac{\epsilon}{2}$ and $\mathrm{d}(\beta_n^{-1}, y) \geq \frac{\epsilon}{2}$, we get

$$G(\alpha_n, \beta_n) \leq \sigma(\alpha_n^{-1}, x) + \sigma(\beta_n, y) - \bar\sigma(\beta_n^{-1}\alpha_n, z) + C + 2C_\epsilon.$$

Using the fact that $\bar{\sigma}$ is a coarse-cocycle and Observation 3.1(3), it follows that

$$\begin{aligned} G(\alpha_n,\beta_n) &\leq \sigma(\alpha_n^{-1},x)+\sigma(\beta_n,y)+\bar{\sigma}(\alpha_n^{-1}\beta_n,\beta^{-1}\alpha_n z)+C+2C_\epsilon+2\kappa \\ &\leq \sigma(\alpha_n^{-1},x)+\sigma(\beta_n,y)+\bar{\sigma}(\alpha_n^{-1},\alpha_n z)+\bar{\sigma}(\beta_n,\beta_n^{-1}\alpha_n z)+C+2C_\epsilon+3\kappa. \end{aligned}$$

Next we use the GPS system property (Definition 1.7), which implies

$$G(\alpha_n,\beta_n) \leq G(z,\alpha_n^{-1}x)-G(\alpha_n z,x)+G(\alpha_n z,\beta_n y)-G(\beta_n^{-1}\alpha_n z,y)+C+2C_\epsilon+5\kappa.$$

Finally since G is locally finite there is $C'_\epsilon > 0$ such that $G(p,q) \leq C'_\epsilon$ whenever $\mathrm{d}(p,q) \geq \frac{\epsilon}{2}$. Since G is nonnegative and $\mathrm{d}(z,\alpha_n^{-1}x) \geq \frac{\epsilon}{2}$, this implies

$$G(\alpha_n,\beta_n) \leq G(\alpha_n z,\beta_n y)+C+2C_\epsilon+5\kappa+C'_\epsilon.$$

We get a lower bound for $G(\alpha_n,\beta_n)$ in a similar way:

$$\begin{aligned} G(\alpha_n,\beta_n) &\geq \left\|\alpha_n^{-1}\right\|_\sigma+\|\beta_n\|_\sigma-\left\|\beta_n^{-1}\alpha_n\right\|_{\bar{\sigma}}-C \\ &\geq \sigma(\alpha_n^{-1},x)+\sigma(\beta_n,y)-\bar{\sigma}(\beta_n^{-1}\alpha_n,z)-C-C_\epsilon \\ &\geq \sigma(\alpha_n^{-1},x)+\sigma(\beta_n,y)+\bar{\sigma}(\alpha_n^{-1}\beta_n,\beta^{-1}\alpha_n z)-C-C_\epsilon-2\kappa \\ &\geq \sigma(\alpha_n^{-1},x)+\sigma(\beta_n,y)+\bar{\sigma}(\alpha_n^{-1},\alpha_n z)+\bar{\sigma}(\beta_n,\beta_n^{-1}\alpha_n z)-C-C_\epsilon-3\kappa \\ &\geq G(z,\alpha_n^{-1}x)-G(\alpha_n z,x)+G(\alpha_n z,\beta_n y)-G(\beta_n^{-1}\alpha_n z,y)-C-C_\epsilon-5\kappa \\ &\geq G(\alpha_n z,\beta_n y)-C-C_\epsilon-5\kappa-2C'_\epsilon. \end{aligned}$$

As $\beta_n y \to b$ and $\alpha_n z \to a$, we can conclude using the κ-coarse continuity of G. □

4. Patterson–Sullivan measures

Using the results established in Proposition 3.2, we can carry out the standard construction of a Patterson–Sullivan measure due to Patterson [Pat76] in the presence of an expanding coarse-cocyle.

Theorem 4.1. *If σ is an expanding κ-coarse-cocycle for a convergence group $\Gamma \subset \mathsf{Homeo}(M)$ and $\delta := \delta_\sigma(\Gamma) < +\infty$, then there exists a $2\kappa\delta$-coarse σ-Patterson–Sullivan measure of dimension δ on M, which is supported on the limit set $\Lambda(\Gamma)$.*

Proof. By [Pat76, Lemma 3.1], there exists a non-decreasing function $\chi\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 1}$ such that

(a) For every $\epsilon > 0$ there exists $R > 0$ such that $\chi(r+t) \leq e^{\epsilon t}\chi(r)$ for any $r \geq R$ and $t \geq 0$,
(b) $\sum_{g\in\Gamma} \chi(\|g\|_\sigma)e^{-\delta\|g\|_\sigma} = +\infty$

(when $\sum_{g\in\Gamma} e^{-\delta\|g\|_\sigma} = +\infty$, we can take $\chi \equiv 1$).

Endow $\Gamma \sqcup M$ with the compactifying topology (see Proposition 2.3). For $x \in \Gamma \sqcup M$, let $\mathcal{D}_x$ denote the Dirac measure centered at x.

For $s > \delta$, define a Borel probability measure on $\Gamma \sqcup M$ by

$$\mu_s := \frac{1}{Q_\sigma^\chi(s)} \sum_{g\in\Gamma} \chi(\|g\|_\sigma)e^{-s\|g\|_\sigma}\mathcal{D}_g,$$

where $Q_\sigma^\chi(s) := \sum_{g\in\Gamma} \chi(\|g\|_\sigma)e^{-s\|g\|_\sigma}$. Then fix $s_n \searrow \delta$ such that $\mu_{s_n} \to \mu$ in the weak-$*$ topology. We claim that μ is a $2\kappa\delta$-coarse σ-Patterson–Sullivan measure of dimension δ on M.

By property (b) of χ,

$$\lim_{s \searrow \delta_\sigma} Q_\sigma^\chi(s) = +\infty.$$

Hence μ is supported on $\Lambda(\Gamma)$ by Proposition 2.3.

To verify the Radon-Nikodym derivative condition, fix $\gamma \in \Gamma$. Then define $\chi_\gamma \colon \Gamma \sqcup M \to \mathbb{R}$ by

$$\chi_\gamma(x) = \begin{cases} \chi(\left\|\gamma^{-1}x\right\|_\sigma)/\chi(\|x\|_\sigma) & x \in \Gamma \\ 1 & x \in M. \end{cases}$$

Property (a) of χ and Observation 3.1 imply that χ_γ is continuous. Next define $f_\gamma \colon M \sqcup \Gamma \to \mathbb{R}$ by

$$f_\gamma(x) = \begin{cases} \left\|\gamma^{-1}x\right\|_\sigma - \|x\|_\sigma & x \in \Gamma \\ \limsup\limits_{x_n \to x,\ \{x_n\} \subset \Gamma} \left\|\gamma^{-1}x_n\right\|_\sigma - \|x_n\|_\sigma & x \in \Lambda(\Gamma) \\ \left\|\gamma^{-1}\right\|_\sigma & x \in M \smallsetminus \Lambda(\Gamma). \end{cases}$$

Then by definition f_γ is upper semicontinuous and hence Borel measurable. Further Proposition 3.2(4) implies that

$$\left|f_\gamma(x) - \sigma(\gamma^{-1}, x)\right| \leq 2\kappa$$

when $x \in \Lambda(\Gamma)$.

Then

$$\begin{aligned} \gamma_*\mu_s &= \frac{1}{Q_\sigma^\chi(s)} \sum_{g \in \Gamma} \chi(\|g\|_\sigma) e^{-s\|g\|_\sigma} \mathcal{D}_{\gamma g} = \frac{1}{Q_\sigma^\chi(s)} \sum_{g' \in \Gamma} \chi(\left\|\gamma^{-1}g'\right\|_\sigma) e^{-s\left\|\gamma^{-1}g'\right\|_\sigma} \mathcal{D}_{g'} \\ &= \frac{1}{Q_\sigma^\chi(s)} \sum_{g' \in \Gamma} \chi_\gamma(g') e^{-sf_\gamma(g')} \chi(\|g'\|_\sigma) e^{-s\|g'\|_\sigma} \mathcal{D}_{g'} = \chi_\gamma e^{-sf_\gamma} \mu_s. \end{aligned}$$

Since χ_γ is continuous and μ is supported on $\Lambda(\Gamma)$, taking the limit $s_n \searrow \delta$ we obtain that μ and $\gamma_*\mu$ are absolutely continuous and that

$$e^{-2\kappa\delta - \delta\sigma(\gamma^{-1}, \cdot)} \leq \frac{d\gamma_*\mu}{d\mu} \leq e^{2\kappa\delta - \delta\sigma(\gamma^{-1}, \cdot)},$$

so μ is a $2\kappa\delta$-coarse σ-Patterson–Sullivan measure of dimension δ. □

One can use the above Patterson–Sullivan measure to obtain the following classical entropy gap result (see [DOP00, Prop. 2]).

Theorem 4.2. *Suppose $\Gamma \subset \mathsf{Homeo}(M)$ is a convergence group, σ is an expanding coarse-cocycle, and $\delta_\sigma(\Gamma) < +\infty$. If $G \subset \Gamma$ is a subgroup where $\Lambda(G)$ is a strict subset of $\Lambda(\Gamma)$ and*

$$\sum_{g \in G} e^{-\delta_\sigma(G)\|g\|_\sigma} = +\infty,$$

then $\delta_\sigma(G) < \delta_\sigma(\Gamma)$.

Proof. Fix an open set $U \subset M$ such that $U \cap \Lambda(\Gamma) \neq \varnothing$ and $\overline{U} \cap \Lambda(G) = \varnothing$. By the definition of $\Lambda(G)$, G acts properly discontinuously on $M \smallsetminus \Lambda(G)$. Hence there exists $N > 0$ such that every point in M is contained in at most N different G-translates of U.

Now fix a C_2-coarse σ-Patterson–Sullivan μ for Γ of dimension $\delta_\sigma(\Gamma)$ supported on $\Lambda(\Gamma)$, so

$$e^{-C_2-\delta_\sigma(\Gamma)\sigma(\gamma^{-1},\cdot)} \leq \frac{d\gamma_*\mu}{d\mu} \leq e^{C_2-\delta_\sigma(\Gamma)\sigma(\gamma^{-1},\cdot)}$$

for all $\gamma \in \Gamma$, for some constant $C_2 > 0$.

Suppose for a contradiction that $\delta_\sigma(G) = \delta_\sigma(\Gamma)$. Since Γ acts minimally on $\Lambda(\Gamma)$ we must have $\mu(U) > 0$. Then

$$\begin{aligned} N &\geq \sum_{g\in G} \mu(gU) = \sum_{g\in G} (g_*^{-1})\mu(U) \geq e^{-C_2} \sum_{g\in G} \int_U e^{-\delta_\sigma(\Gamma)\sigma(g,x)} d\mu(x) \\ &\geq \frac{\mu(U)}{e^{C_2}} \sum_{g\in G} e^{-\delta_\sigma(G)\|g\|_\sigma} = +\infty. \end{aligned}$$

So we have a contradiction. ☐

5. Shadows and their properties

In this section we define our shadows, establish some of their basic properties, relate them to a notion of uniformly conical limit points, and compare these shadows to the classically defined shadows in the Gromov hyperbolic setting.

5.1. **Basic properties.** Suppose for the rest of the section that $\Gamma \subset \mathsf{Homeo}(M)$ is a convergence group. Fix a compatible metric d on $\Gamma \sqcup M$ and let $B_r(x) \subset \Gamma \sqcup M$ denote the open ball of radius $r > 0$ centered at x. Given $\epsilon > 0$ and $\gamma \in \Gamma$, the associated *shadow* is

$$\mathcal{S}_\epsilon(\gamma) := \gamma\left(M - B_\epsilon(\gamma^{-1})\right).$$

Proposition 5.1. *If $\epsilon > 0$ and $\sigma\colon \Gamma \times M \to \mathbb{R}$ is an expanding coarse-cocycle, then:*

1. *There exists $C_1 > 0$ such that: if $x \in \mathcal{S}_\epsilon(\gamma)$, then*
$$\sigma(\gamma, \gamma^{-1}(x)) \geq \|\gamma\|_\sigma - C_1.$$
2. *If $\{\gamma_n\} \subset \Gamma$ is an escaping sequence, then*
$$\lim_{n\to\infty} \operatorname{diam} \mathcal{S}_\epsilon(\gamma_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} \mathrm{d}(\gamma_n, \mathcal{S}_\epsilon(\gamma_n)) = 0,$$
where the diameter is with respect to d. *In particular, the Hausdorff distance with respect to* d *between the sets $\{\gamma_n\}$ and $\mathcal{S}_\epsilon(\gamma_n)$ converges to zero.*
3. *There exists $C_2 > 0$ such that: if $\alpha, \beta \in \Gamma$, $\|\alpha\|_\sigma \leq \|\beta\|_\sigma$, and*
$$\mathcal{S}_\epsilon(\alpha) \cap \mathcal{S}_\epsilon(\beta) \neq \varnothing,$$
then
$$\|\beta\|_\sigma \geq \left\|\alpha^{-1}\beta\right\|_\sigma + \|\alpha\|_\sigma - C_2.$$
4. *There exists $0 < \epsilon' < \epsilon$ such that: if $\alpha, \beta \in \Gamma$, $\|\alpha\|_\sigma \leq \|\beta\|_\sigma$, and $\mathcal{S}_\epsilon(\alpha) \cap \mathcal{S}_\epsilon(\beta) \neq \varnothing$, then*
$$\mathcal{S}_\epsilon(\beta) \subset \mathcal{S}_{\epsilon'}(\alpha).$$
5. *There exists $0 < \epsilon' < \epsilon$ such that: if $I \subset \Gamma$, then there exists $J \subset I$ such that the shadows $\{\mathcal{S}_\epsilon(\gamma) : \gamma \in J\}$ are disjoint and*
$$\bigcup_{\gamma\in I} \mathcal{S}_\epsilon(\gamma) \subset \bigcup_{\gamma \in J} \mathcal{S}_{\epsilon'}(\gamma).$$

Proof. Part (1) follows from the definition of expanding coarse-cocycles and part (2) is a consequence of Proposition 2.3(3).

Part (3): suppose for a contradiction that the claim is false. Then for each $n \geq 1$ there exist $\alpha_n, \beta_n \in \Gamma$ such that $\|\alpha_n\|_\sigma \leq \|\beta_n\|_\sigma$,

$$\mathcal{S}_\epsilon(\alpha_n) \cap \mathcal{S}_\epsilon(\beta_n) \neq \varnothing \quad \text{and} \quad \|\beta_n\|_\sigma \leq \left\|\alpha_n^{-1}\beta_n\right\|_\sigma + \|\alpha_n\|_\sigma - n.$$

In particular

$$\left\|\alpha_n^{-1}\beta_n\right\|_\sigma \geq \|\beta_n\|_\sigma - \|\alpha_n\|_\sigma + n \geq n,$$

so $\{\alpha_n^{-1}\beta_n\}$ is escaping. Since $\|\beta_n\|_\sigma \geq \|\alpha_n\|_\sigma$, this implies that $\{\beta_n\}$ is also escaping. Then by Proposition 3.2(6) we have

$$\lim_{n\to\infty} \mathrm{d}(\beta_n^{-1}\alpha_n, \beta_n^{-1}) = 0.$$

For each n, fix $x_n \in \mathcal{S}_\epsilon(\alpha_n) \cap \mathcal{S}_\epsilon(\beta_n)$. By definition, $\mathrm{d}(\beta_n^{-1}x_n, \beta_n^{-1}) \geq \epsilon$ and so

$$\mathrm{d}(\beta_n^{-1}x_n, \beta_n^{-1}\alpha_n) \geq \mathrm{d}(\beta_n^{-1}x_n, \beta_n^{-1}) - \mathrm{d}(\beta_n^{-1}\alpha_n, \beta_n^{-1}) \geq \epsilon/2$$

for n large enough. Also, $\mathrm{d}(\alpha_n^{-1}x_n, \alpha_n^{-1}) \geq \epsilon$ for any n.

Since σ is expanding, there exists $C > 0$ such that

$$\sigma(\gamma, x) \geq \|\gamma\|_\sigma - C$$

for all $\gamma \in \Gamma$ and $x \in M - B_{\epsilon/2}(\gamma^{-1})$. Thus

$$\begin{aligned}
\|\beta_n\|_\sigma &\geq \sigma(\beta_n, \beta_n^{-1}x_n) = \sigma(\alpha_n\alpha_n^{-1}\beta_n, \beta_n^{-1}x_n) \\
&\geq \sigma(\alpha_n, \alpha_n^{-1}x_n) + \sigma(\alpha_n^{-1}\beta_n, \beta_n^{-1}x_n) - \kappa \\
&\geq \|\alpha_n\|_\sigma - C + \left\|\alpha_n^{-1}\beta_n\right\|_\sigma - C - \kappa
\end{aligned}$$

and we have a contradiction. Thus part (3) is true.

Part (4): suppose for a contradiction that there exist $\{\alpha_n\}, \{\beta_n\} \subset \Gamma$ and $\epsilon_n \to 0$ such that $\|\alpha_n\|_\sigma \leq \|\beta_n\|_\sigma$, $\mathcal{S}_\epsilon(\alpha_n) \cap \mathcal{S}_\epsilon(\beta_n) \neq \varnothing$, and $\mathcal{S}_\epsilon(\beta_n) \not\subset \mathcal{S}_{\epsilon_n}(\alpha_n)$. Then

$$\alpha_n^{-1}\beta_n(M - B_\epsilon(\beta_n^{-1})) = \alpha_n^{-1}\mathcal{S}_\epsilon(\beta_n) \not\subset \alpha_n^{-1}\mathcal{S}_{\epsilon_n}(\alpha_n) = M - B_{\epsilon_n}(\alpha_n^{-1})$$

for all $n \geq 1$. Since $\epsilon_n \to 0$, by continuity of the action of Γ on $\Gamma \sqcup M$, the sequence $\{\alpha_n^{-1}\beta_n\}$ must be escaping.

Then $\{\beta_n^{-1}\alpha_n\}$ is also escaping and so by Proposition 3.2(6) we have

$$\lim_{n\to\infty} \mathrm{d}(\beta_n^{-1}, \beta_n^{-1}\alpha_n) = 0.$$

Thus for n large enough

$$\alpha_n^{-1}\mathcal{S}_\epsilon(\beta_n) = \alpha_n^{-1}\beta_n\left(M - B_\epsilon(\beta_n^{-1})\right) \subset \alpha_n^{-1}\beta_n\left(M - B_{\epsilon/2}(\beta_n^{-1}\alpha_n)\right) = \mathcal{S}_{\epsilon/2}(\alpha_n^{-1}\beta_n).$$

Then, applying part (2) to the escaping sequence $\{\alpha_n^{-1}\beta_n\}$, we obtain that the diameter of

$$\alpha_n^{-1}\mathcal{S}_\epsilon(\beta_n) \subset \mathcal{S}_{\epsilon/2}(\alpha_n^{-1}\beta_n)$$

tends to zero, and hence is less than $\epsilon/2$ for n large enough. Further, by assumption, $\alpha_n^{-1}\mathcal{S}_\epsilon(\beta_n)$ intersects $\alpha_n^{-1}\mathcal{S}_\epsilon(\alpha_n) = M - B_\epsilon(\alpha_n^{-1})$ for all n. Hence

$$\alpha_n^{-1}\mathcal{S}_\epsilon(\beta_n) \subset M - B_{\epsilon/2}(\alpha_n^{-1})$$

for n large enough, which implies that $\mathcal{S}_\epsilon(\beta_n) \subset \mathcal{S}_{\epsilon_n}(\alpha_n)$ for n large enough. Thus we have a contradiction.

Part (5): using part (4), the proof of the proposition is standard, see e.g. [Fol99, Lemma 3.15].

Let ϵ' be as in part (4). Enumerate $I = \{\gamma_1, \gamma_2, \dots\}$ such that

$$\|\gamma_1\|_\sigma \leq \|\gamma_2\|_\sigma \leq \dots$$

Inductively define $j_1 < j_2 < \dots$ as follows: let $j_1 = 1$, then supposing $j_1, \dots, j_k$ have been selected pick j_{k+1} to be the smallest index greater than j_k such that

$$\mathcal{S}_\epsilon(\gamma_{j_{k+1}}) \cap \bigcup_{i=1}^{k} \mathcal{S}_\epsilon(\gamma_{j_i}) = \varnothing.$$

We claim that $J = \{\gamma_{j_1}, \gamma_{j_2}, \dots\}$ suffices (it is possible for J to be finite). By definition the shadows $\{\mathcal{S}_\epsilon(\gamma) : \gamma \in J\}$ are disjoint. Further if $\gamma_n \notin J$, then there exists some index $j_k < n$ such that

$$\mathcal{S}_\epsilon(\gamma_n) \cap \mathcal{S}_\epsilon(\gamma_{j_k}) \neq \varnothing$$

(otherwise we would have $\gamma_n \in J$). Then part (4) implies that

$$\mathcal{S}_\epsilon(\gamma_n) \subset \mathcal{S}_{\epsilon'}(\gamma_{j_k}).$$

So

$$\bigcup_{\gamma \in I} \mathcal{S}_\epsilon(\gamma) \subset \bigcup_{\gamma \in J} \mathcal{S}_{\epsilon'}(\gamma). \qquad \square$$

5.2. **Uniformly conical limit points.** Next we introduce a notion of uniformly conical limit points and relate them to the shadows defined above.

Definition 5.2. Given $\epsilon > 0$, the *ϵ-uniform conical limit set*, denoted $\Lambda^{\rm con}_\epsilon(\Gamma)$, is the set of points $y \in M$ such that there exist $a, b \in M$ and a sequence of elements $\{\gamma_n\}$ in Γ where $\mathrm{d}(a,b) \geq \epsilon$, $\lim_{n\to\infty} \gamma_n(y) = b$, and $\lim_{n\to\infty} \gamma_n(x) = a$ for all $x \in M \smallsetminus \{y\}$.

Notice that by definition

$$\Lambda^{\rm con}(\Gamma) = \bigcup_{\epsilon > 0} \Lambda^{\rm con}_\epsilon(\Gamma) = \bigcup_{n=1}^{\infty} \Lambda^{\rm con}_{\frac{1}{n}}(\Gamma). \tag{2}$$

We also observe that these limit sets are invariant.

Observation 5.3. If $\epsilon > 0$, then $\Lambda^{\rm con}_\epsilon(\Gamma)$ is Γ-invariant.

Proof. Fix $y \in \Lambda^{\rm con}_\epsilon(\Gamma)$ and $\gamma \in \Gamma$. Then there exist $a, b \in M$ and a sequence of elements $\{\gamma_n\}$ in Γ where $\mathrm{d}(a,b) \geq \epsilon$, $\lim_{n\to\infty} \gamma_n(y) = b$, and $\lim_{n\to\infty} \gamma_n(x) = a$ for all $x \in M \smallsetminus \{y\}$. Then $\lim_{n\to\infty} \gamma_n\gamma^{-1}(\gamma y) = b$ and $\lim_{n\to\infty} \gamma_n\gamma^{-1}(x) = a$ for all $x \in M \smallsetminus \{\gamma y\}$. So $\gamma y \in \Lambda^{\rm con}_\epsilon(\Gamma)$. $\square$

Next we relate the shadows to this notion of uniformly conical limit set.

Lemma 5.4.

1. *If $x \in \Lambda^{\rm con}_\epsilon(\Gamma)$ and $0 < \epsilon' < \epsilon$, then there exists an escaping sequence $\{\gamma_n\} \subset \Gamma$ such that $x \in \bigcap_n \mathcal{S}_{\epsilon'}(\gamma_n)$.*
2. *If there exists an escaping sequence $\{\gamma_n\} \subset \Gamma$ such that $x \in \bigcap_n \mathcal{S}_\epsilon(\gamma_n)$, then $x \in \Lambda^{\rm con}_\epsilon(\Gamma)$.*

Proof. First suppose that $x \in \Lambda^{\rm con}_\epsilon(\Gamma)$. Then by definition there exist $a, b \in M$ so that $\mathrm{d}(a,b) \geq \epsilon$ and $\{\gamma_n\} \subset \Gamma$ such that $\gamma_n^{-1} x \to a$ and $\gamma_n^{-1} y \to b$ for any $y \in M - \{x\}$. Thus $\gamma_n^{-1} \to b \neq a$. So if $\epsilon' < \epsilon$, then $\mathrm{d}(\gamma_n^{-1} x, \gamma_n^{-1}) > \epsilon'$ for n sufficiently large. Thus

$$x = \gamma_n \gamma_n^{-1}(x) \in \gamma_n(M - B_{\epsilon'}(\gamma_n^{-1})) = \mathcal{S}_{\epsilon'}(\gamma_n)$$

for n sufficiently large.

Next suppose that $x \in \bigcap_n \mathcal{S}_\epsilon(\gamma_n)$ for some $\epsilon > 0$ and some escaping $\{\gamma_n\} \subset \Gamma$. Passing to a subsequence we can assume that $\gamma_n^{-1} x \to a$, $\gamma_n^{-1} \to b$ and $\gamma_n \to c$. In particular $\gamma_n^{-1} y \to b$ for any $y \in M \smallsetminus \{c\}$. Since $x \in \mathcal{S}_\epsilon(\gamma_n)$ for every n, we have by definition $\mathrm{d}(\gamma_n^{-1} x, \gamma_n^{-1}) \geq \epsilon$. Passing to the limit we get $\mathrm{d}(a,b) \geq \epsilon$, so $a \neq b$. Moreover $x = c$, as otherwise $\{\gamma_n^{-1} x\}$ would have to converge to b. Hence $x \in \Lambda^{\rm con}_\epsilon(\Gamma)$. □

5.3. **Comparison to classical shadows.** Suppose that X is a proper geodesic Gromov hyperbolic space. Let $\Gamma \subset \mathsf{Isom}(X)$ be a discrete group. Then Γ acts as a convergence group on the Gromov boundary $\partial_\infty X$ of X.

Given $b, p \in X$ and $r > 0$ the associated shadow $\mathcal{O}_r(b,p) \subset \partial_\infty X$ is the set of all $x \in \partial_\infty X$ where there is some geodesic ray $\ell \colon [0,\infty) \to X$ where $\ell(0) = b$, $\lim_{t\to\infty} \ell(t) = x$, and ℓ intersects the open ball of radius $r > 0$ centered at p.

Now fix a compatible metric d on $\Gamma \sqcup \partial_\infty X$, and for $\epsilon > 0$ and $\gamma \in \Gamma$ let $\mathcal{S}_\epsilon(\gamma) \subset \partial_\infty X$ denote the shadow defined above.

Proposition 5.5.

(1) *For any $b \in X$ and $r > 0$ there exists $\epsilon > 0$ such that*

$$\mathcal{O}_r(b, \gamma(b)) \subset \mathcal{S}_\epsilon(\gamma)$$

for all $\gamma \in \Gamma$.

(2) *For any $b \in X$ and $\epsilon > 0$ there exists $r > 0$ such that*

$$\mathcal{S}_\epsilon(\gamma) \subset \mathcal{O}_r(b, \gamma(b))$$

for all $\gamma \in \Gamma$.

Proof. (1): Suppose that no such $\epsilon > 0$ exists. Then there exist $\{\gamma_n\} \subset \Gamma$ and $\{\epsilon_n\}$ such that $\epsilon_n \to 0$ and $\mathcal{O}_r(b, \gamma_n(b)) \not\subset \mathcal{S}_{\epsilon_n}(\gamma_n)$ for all n. Equivalently, for each n there exists

$$x_n \in \mathcal{O}_r(\gamma_n^{-1}(b), b) \smallsetminus \Big(\partial_\infty X - B_{\epsilon_n}(\gamma_n^{-1})\Big) = \mathcal{O}_r(\gamma_n^{-1}(b), b) \cap B_{\epsilon_n}(\gamma_n^{-1}).$$

Passing to a subsequence we can suppose that $x_n \to x$ and $\gamma_n^{-1} \to a$. Then by definition there exists a geodesic line $\ell \colon \mathbb{R} \to X$ where $\lim_{t\to\infty} \ell(t) = a$, $\lim_{t\to-\infty} \ell(t) = x$, and ℓ intersects the closed ball of radius r centered at b. In particular, $a \neq x$ and hence $x_n \notin B_{\epsilon_n}(\gamma_n^{-1})$ for n sufficiently large. So we have a contradiction.

(2): This is very similar to the proof of (1). □

6. The Shadow Lemma and its consequences

In this section we establish a version of the classical Shadow Lemma. We then derive some of its immediate consequences.

Theorem 6.1 (The Shadow Lemma)**.** *Suppose $\Gamma \subset \mathsf{Homeo}(M)$ is a convergence group, $\sigma \colon \Gamma \times M \to \mathbb{R}$ is an expanding coarse-cocycle, and μ is a coarse σ-Patterson–Sullivan measure on M of dimension δ. For any sufficiently small $\epsilon > 0$ there exists $C = C(\epsilon) > 1$ such that*

$$\frac{1}{C} e^{-\delta \|\gamma\|_\sigma} \leq \mu\left(\mathcal{S}_\epsilon(\gamma)\right) \leq C e^{-\delta\|\gamma\|_\sigma}$$

for all $\gamma \in \Gamma$.

Using the results established in Sections 3 and 5, the proof of the shadow lemma is essentially the same as Sullivan's original argument [Sul79, Proposition 3].

Lemma 6.2. *For every $\eta > \sup_{x \in M} \mu(\{x\})$ there exists $\epsilon > 0$ such that*

$$\mu\left(\gamma^{-1}\mathcal{S}_\epsilon(\gamma)\right) = \mu\left(M - B_\epsilon(\gamma)\right) \geq 1 - \eta$$

for all $\gamma \in \Gamma$.

Proof. Otherwise there would exist $\{\gamma_n\} \subset \Gamma$ and $\{\epsilon_n\}$ such that $\epsilon_n \to 0$ and $\mu\left(M \cap B_{\epsilon_n}(\gamma_n)\right) \geq \eta$ for all n. Passing to a subsequence we can suppose that $\gamma_n \to x \in \Gamma \sqcup M$. Let $\delta_n := \mathrm{d}(\gamma_n, x)$ and pass to a further subsequence so that $\{\epsilon_n + \delta_n\}$ is decreasing. Then

$$\mu(\{x\}) = \lim_{n \to \infty} \mu\left(M \cap B_{\epsilon_n + \delta_n}(x)\right) \geq \eta,$$

which contradicts our choice of η. □

Proof of Theorem 6.1. Notice that

$$\sup_{x \in M} \mu(\{x\}) < 1.$$

Otherwise, μ would be supported on a single point, which is impossible since Γ is non-elementary. Hence by Lemma 6.2, there exists $\epsilon_0 > 0$ such that

$$\delta_0 := \inf_{\gamma \in \Gamma} \mu\Big(\gamma^{-1} S_{\epsilon_0}(\gamma)\Big)$$

is positive.

Fix $\epsilon < \epsilon_0$. By Proposition 5.1(1) there exists $C_1 > 1$ such that: if $\gamma \in \Gamma$, then

$$\frac{1}{C_1} e^{-\delta\|\gamma\|_{\sigma_2}} \leq \frac{d\left(\gamma^{-1}\right)_* \mu}{d\mu} \leq C_1 e^{-\delta\|\gamma\|_{\sigma_2}}$$

almost everywhere on $\gamma^{-1}\mathcal{S}_\epsilon(\gamma)$.

Fix $\gamma \in \Gamma$. Then

$$\mu\big(\mathcal{S}_\epsilon(\gamma)\big) = \left(\gamma^{-1}\right)_* \mu\big(\gamma^{-1}\mathcal{S}_\epsilon(\gamma)\big) = \int_{\gamma^{-1}\mathcal{S}_\epsilon(\gamma)} \frac{d\left(\gamma^{-1}\right)_* \mu}{d\mu} d\mu.$$

Hence

$$\frac{\delta_0}{C_1} e^{-\delta\|\gamma\|_{\sigma_2}} \leq \mu\big(\mathcal{S}_\epsilon(\gamma)\big) \leq C_1 e^{-\delta\|\gamma\|_{\sigma_2}}.$$ □

The following results now also follow from the standard arguments from the classical case.

Proposition 6.3. *Suppose $\Gamma \subset \mathsf{Homeo}(M)$ is a convergence group, $\sigma \colon \Gamma \times M \to \mathbb{R}$ is an expanding coarse-cocycle, and μ is a coarse σ-Patterson–Sullivan measure on M of dimension β. Then:*

(1) *If $y \in M$ is a conical limit point, then $\mu(\{y\}) = 0$.*
(2) *If*
$$\sum_{\gamma \in \Gamma} e^{-\beta \|\gamma\|_\sigma} < +\infty,$$
then $\mu(\Lambda^{\rm con}(\Gamma)) = 0$.
(3) *$\beta \geq \delta_\sigma(\Gamma)$.*
(4) *There exists $C > 0$ such that*
$$\#\{\gamma \in \Gamma : \|\gamma\|_\sigma \leq R\} \leq C e^{\delta_\sigma(\Gamma) R}$$
for any $R > 0$.

Proof of (1). Suppose y is a conical limit point. By Equation (2) and Lemma 5.4, there exist $\epsilon > 0$ and an escaping sequence $\{\gamma_n\} \subset \Gamma$ such that $y \in \mathcal{S}_\epsilon(\gamma_n)$ for all n. Hence, by the Shadow Lemma (Theorem 6.1), there exists $C > 0$ such that
$$\mu(\{y\}) \leq \mu(\mathcal{S}_\epsilon(\gamma_n)) \leq C e^{-\beta \|\gamma_n\|_\sigma}$$
for all n. Since σ is expanding, it is proper (by definition) and so $\|\gamma_n\|_\sigma \to +\infty$. Hence $\mu(\{y\}) = 0$. ☐

Proof of (2). By Lemma 5.4 for every $m_0 > 0$ we have
$$\Lambda^{\rm con}(\Gamma) \subset \bigcup_{m \geq m_0} \bigcap_{n \geq 1} \bigcup_{\|\gamma\|_\sigma \geq n} \mathcal{S}_{1/m}(\gamma).$$
By the Shadow Lemma (Theorem 6.1), for all m sufficiently large there exists $C_m > 0$ such that
$$\mu(\mathcal{S}_{1/m}(\gamma)) \leq C_m e^{-\beta \|\gamma\|_\sigma}$$
for all $\gamma \in \Gamma$. Hence for all m sufficiently large,
$$\mu\left(\bigcap_{n \geq 1} \bigcup_{\|\gamma\|_\sigma \geq n} \mathcal{S}_{1/m}(\gamma) \right) \leq \lim_{n \to \infty} \sum_{\|\gamma\|_\sigma \geq n} C_m e^{-\beta \|\gamma\|_\sigma}$$
which equals zero by assumption. Thus $\mu(\Lambda^{\rm con}(\Gamma)) = 0$. ☐

The final two parts of the proposition require a lemma.

Lemma 6.4. *Then there exists $C > 0$ such that*
$$\#\{\gamma \in \Gamma : \|\gamma\|_\sigma \leq R\} \leq C e^{\beta R}$$
for any $R > 0$.

Proof. By the Shadow Lemma (Theorem 6.1) there exist $\epsilon > 0$ and $C_1 > 1$ such that
$$\mu(\mathcal{S}_\epsilon(\gamma)) \geq C_1^{-1} e^{-\beta \|\gamma\|_\sigma} \tag{3}$$
for all $\gamma \in \Gamma$. By Proposition 5.1(3), there exists C_2 such that: if $\gamma, \gamma' \in \Gamma$,
$$\big| \|\gamma\|_\sigma - \|\gamma'\|_\sigma \big| \leq 1,$$
and $\mathcal{S}_\epsilon(\gamma) \cap \mathcal{S}_\epsilon(\gamma') \neq \varnothing$, then $\left\| \gamma^{-1} \gamma' \right\|_\sigma \leq C_2$. Let
$$C_3 := \#\{\gamma \in \Gamma : \|\gamma\|_\sigma \leq C_2\}$$
(which is finite since σ is proper). Then, for all $x \in M$ and $R > 0$,
$$\#\{\gamma \in \Gamma : R - 1 \leq \|\gamma\|_\sigma \leq R \text{ and } x \in \mathcal{S}_\epsilon(\gamma)\} \leq C_3. \tag{4}$$

Then

$$\#\{\gamma \in \Gamma : R-1 \leq \|\gamma\|_\sigma \leq R\} = \sum_{\substack{\gamma\in\Gamma \\ R-1\leq\|\gamma\|_\sigma\leq R}} 1 \leq C_1 e^{\beta R} \sum_{\substack{\gamma\in\Gamma \\ R-1\leq\|\gamma\|_\sigma\leq R}} \mu(\mathcal{S}_\epsilon(\gamma))$$
$$\leq C_1 C_3 \mu(M) e^{\beta R} = C_1 C_3 e^{\beta R}.$$

We complete the proof by summing this inequality over $\mathbb{N}$. ☐

Proof of (3). This follows immediately from Lemma 6.4 and the definition of the critical exponent $\delta_\sigma(\Gamma)$. ☐

Proof of (4). By Theorem 4.1 there exists a Patterson–Sullivan measure with dimension $\delta_\sigma(\Gamma)$. Then part (4) follows immediately from applying Lemma 6.4 to this measure. ☐

Part 2. Dynamics of Patterson–Sullivan measures

7. Conical limit points have full measure in the divergent case

In this section we show that any Patterson–Sullivan measure with dimension equal to the critical exponent is supported on the conical limit set in case when the associated Poincaré series diverges at its critical exponent. The proof is similar to Roblin's argument for the analogous result for Busemann cocycles in CAT(-1) spaces [Rob03], in that we use a variant of the Borel–Cantelli Lemma. However, we use a different variant of the lemma and apply it to a different collection of sets. This approach seems slightly simpler and was also used in [CZZ23].

Proposition 7.1. *Suppose* $\Gamma \subset \mathsf{Homeo}(M)$ *is a convergence group and* $\sigma\colon \Gamma\times M \to \mathbb{R}$ *is an expanding coarse-cocycle with* $\delta := \delta_\sigma(\Gamma) < +\infty$. *If* μ *is a coarse* σ-*Patterson–Sullivan measures of dimension* δ *and*

$$\sum_{\gamma\in\Gamma} e^{-\delta\|\gamma\|_\sigma} = +\infty,$$

then $\mu\left(\Lambda^{\rm con}(\Gamma)\right) = 1$.

We first show that $\mu(\Lambda^{\rm con}(\Gamma)) > 0$. To accomplish this we use the following variant of the Borel–Cantelli Lemma.

Lemma 7.2 (Kochen–Stone Borel–Cantelli Lemma)**.** *Let* (X,ν) *be a finite measure space. If* $\{A_n\}$ *is a sequence of measurable sets where*

$$\sum_{n=1}^\infty \nu(A_n) = +\infty \quad \textit{and} \quad \liminf_{N\to\infty} \frac{\sum_{1\leq m,n\leq N}\nu(A_n\cap A_m)}{\left(\sum_{n=1}^N \nu(A_n)\right)^2} < +\infty,$$

then

$$\nu\left(\{x\in M : x \textit{ is contained in infinitely many of } A_1, A_2, \dots\}\right) > 0.$$

Fix a compatible metric d on $\Gamma \sqcup M$, and for $\epsilon > 0$ and $\gamma\in\Gamma$ let $\mathcal{S}_\epsilon(\gamma) \subset M$ denote the shadow defined in Section 5. Using the Shadow Lemma (Theorem 6.1), fix $\epsilon > 0$ and a constant $C_1 > 1$ such that

$$\frac{1}{C_1} e^{-\delta\|\gamma\|_\sigma} \leq \mu\Big(\mathcal{S}_\epsilon(\gamma)\Big) \leq C_1 e^{-\delta\|\gamma\|_\sigma}$$

for all $\gamma \in \Gamma$. Next fix an enumeration $\Gamma = \{\gamma_n\}$ such that

$$\|\gamma_1\|_\sigma \leq \|\gamma_2\|_\sigma \leq \dots$$

and let

$$A_n := \mathcal{S}_\epsilon(\gamma_n).$$

We will show that the sets $\{A_n\}$ satisfy the hypothesis of the Kochen–Stone lemma.

One part is easy: By assumption

$$\sum_{n=1}^{\infty} \mu(A_n) \geq \frac{1}{C_1} \sum_{\gamma \in \Gamma} e^{-\delta \|\gamma\|_\sigma} = +\infty.$$

The other part is only slightly more involved. Using Proposition 5.1(3) there exists $C_2 > 0$ such that: if $1 \leq n \leq m$ and $A_n \cap A_m \neq \varnothing$, then

$$\|\gamma_n\|_\sigma + \left\|\gamma_n^{-1}\gamma_m\right\|_\sigma \leq \|\gamma_m\|_\sigma + C_2.$$

Hence, in this case, $\left\|\gamma_n^{-1}\gamma_m\right\|_\sigma \leq \|\gamma_m\|_\sigma + C_2$ and

$$\mu(A_n \cap A_m) \leq \mu(A_m) \leq C_1 e^{-\delta\|\gamma_m\|_\sigma} \leq C_3 e^{-\delta\|\gamma_n\|_\sigma} e^{-\delta\left\|\gamma_n^{-1}\gamma_m\right\|_\sigma}$$

where $C_3 := C_1 e^{\delta C_2}$. So, if $f(N) := \max\{n : \|\gamma_n\| \leq \|\gamma_N\| + C_2\}$, then

$$\begin{aligned}\sum_{m,n=1}^{N} \mu(A_n \cap A_m) &\leq 2 \sum_{1 \leq n \leq m \leq N} \mu(A_n \cap A_m) \leq 2C_3 \sum_{1 \leq n \leq m \leq N} e^{-\delta\|\gamma_n\|_\sigma} e^{-\delta\left\|\gamma_n^{-1}\gamma_m\right\|_\sigma} \\ &\leq 2C_3 \sum_{n=1}^{N} e^{-\delta\|\gamma_n\|_\sigma} \sum_{n=1}^{f(N)} e^{-\delta\|\gamma_n\|_\sigma}.\end{aligned}$$

Thus to apply the Kochen–Stone lemma, it suffices to observe the following.

Lemma 7.3. *There exists $C_4 > 0$ such that:*

$$\sum_{n=1}^{f(N)} e^{-\delta\|\gamma_n\|_\sigma} \leq C_4 \sum_{n=1}^{N} e^{-\delta\|\gamma_n\|_\sigma}$$

for all $N \geq 1$.

Proof. Notice if $N < n \leq m \leq f(N)$ and $A_n \cap A_m \neq \varnothing$, then

$$\left\|\gamma_n^{-1}\gamma_m\right\|_\sigma \leq \|\gamma_m\| - \|\gamma_n\|_\sigma + C_2 \leq 2C_2.$$

So if $C_4 := \#\{\gamma \in \Gamma : \|\gamma\|_\sigma \leq 2C_2\}$, then

$$\sum_{n=N+1}^{f(N)} e^{-\delta\|\gamma_n\|_\sigma} \leq C_1 \sum_{n=N+1}^{f(N)} \mu(A_n) \leq C_1 C_4 \mu\left(\bigcup_{n=N+1}^{f(N)} A_n\right) \leq C_1 C_4.$$

Hence

$$\sum_{n=1}^{f(N)} e^{-\delta\|\gamma_n\|_\sigma} \leq \left(1 + C_1 C_4 e^{\delta\|\gamma_1\|_\sigma}\right) \sum_{n=1}^{N} e^{-\delta\|\gamma_n\|_\sigma}.$$

□

So by the Kochen–Stone lemma the set

$$X := \{x \in M : x \text{ is contained in infinitely many of } A_1, A_2, \dots\}$$

has positive μ measure. By Lemma 5.4, $X \subset \Lambda^{\mathrm{con}}(\Gamma)$. Hence $\mu(\Lambda^{\mathrm{con}}(\Gamma)) > 0$.

Now suppose for a contradiction that $\mu(\Lambda^{\rm con}(\Gamma)) < 1$. Let $Y := M - \Lambda^{\rm con}(\Gamma)$ and define a measure $\hat{\mu}$ on M by

$$\hat{\mu}(\cdot) = \frac{1}{\mu(Y)}\mu(Y \cap \cdot).$$

This is also a coarse σ-Patterson–Sullivan measure of dimension δ and so the argument above implies that

$$0 < \hat{\mu}(\Lambda^{\rm con}(\Gamma)) = \mu(Y \cap \Lambda^{\rm con}(\Gamma)) = 0.$$

So we have a contradiction.

8. Ergodicity and uniqueness of Patterson–Sullivan measures

In this section we establish uniqueness and ergodicity of Patterson–Sullivan measures in the divergent case. Our argument is similar to the proof of statement (g) in [Rob03, pg. 22], see also [DK22, Sublemma 8.7].

For the rest of the section, suppose $\Gamma \subset \mathsf{Homeo}(M)$ is a convergence group and $\sigma\colon \Gamma \times M \to \mathbb{R}$ is an expanding coarse-cocycle with $\delta := \delta_\sigma(\Gamma) < +\infty$.

Theorem 8.1. *If μ is a C-coarse σ-Patterson–Sullivan measure of dimension δ and*

$$\sum_{\gamma \in \Gamma} e^{-\delta \|\gamma\|_\sigma} = +\infty,$$

then:

1. *Γ acts ergodically on (M, μ).*
2. *μ is coarsely unique in the following sense: if λ is a C-coarse σ-Patterson–Sullivan measure of dimension δ, then $e^{-4C}\mu \le \lambda \le e^{4C}\mu$.*
3. *$\mu(\Lambda^{\rm con}_\epsilon(\Gamma)) = 1$ when $\epsilon > 0$ is sufficiently small (recall that $\Lambda^{\rm con}_\epsilon(\Gamma)$ was defined in Definition 5.2).*

The rest of the section is devoted to the proof of Theorem 8.1. We will prove that Γ acts ergodically on (M, μ) and then use ergodicity to deduce the other claims. To prove ergodicity we will first establish a version of the Lebesgue differentiation theorem (as in [Rob03, Lemme 2]).

Fix a compatible metric d on $\Gamma \sqcup M$.

Lemma 8.2. *Suppose $\epsilon_0 > 0$ satisfies the Shadow Lemma (Theorem 6.1). If $f \in L^1(M, \mu)$, then for μ-almost every $x \in M$ we have*

$$f(x) = \lim_{n \to \infty} \frac{1}{\mu(\mathcal{S}_\epsilon(\gamma_n))} \int_{\mathcal{S}_\epsilon(\gamma_n)} f(y) d\mu(y)$$

for every $0 < \epsilon \le \epsilon_0$ and escaping sequence $\{\gamma_n\} \subset \Gamma$ with

$$x \in \bigcap_{n \ge 1} \mathcal{S}_\epsilon(\gamma_n).$$

Proof. Using Proposition 5.1(5), the proof is very similar to the proof of the Lebesgue differentiation theorem, see e.g. [Fol99, Theorem 3.18].

Let $\epsilon_j := \epsilon_0/j$. For $f \in L^1(M, \mu)$ and $j \ge 1$, define $A_j f, B_j f\colon M \to [0, \infty)$ by setting

$$A_j f(x) = \lim_{R \to \infty} \sup_{\substack{\|\gamma\|_\sigma \ge R \\ x \in \mathcal{S}_{\epsilon_j}(\gamma)}} \frac{1}{\mu(\mathcal{S}_{\epsilon_j}(\gamma))} \int_{\mathcal{S}_{\epsilon_j}(\gamma)} |f(y) - f(x)|\, d\mu(y)$$

and

$$B_j f(x) = \lim_{R\to\infty} \sup_{\substack{\|\gamma\|_\sigma \geq R \\ x\in \mathcal{S}_{\epsilon_j}(\gamma)}} \frac{1}{\mu(\mathcal{S}_{\epsilon_j}(\gamma))} \int_{\mathcal{S}_{\epsilon_j}(\gamma)} |f(y)|\, d\mu(y)$$

if $x \in \Lambda^{\rm con}_{2\epsilon_0}(\Gamma)$ and $Af(x) = Bf(x) = 0$ otherwise.

Now fix $f \in L^1(M,\mu)$. We claim that $A_j f(x) = 0$ for μ-almost every $x \in M$. To show this it suffices to fix $\alpha > 0$ and show that

$$\mu(\{x \in M : A_j f(x) > \alpha\}) = 0.$$

Fix $\eta > 0$ and let g be a continuous function with

$$\int_M |f-g|\, d\mu < \eta.$$

Then

$$0 \leq A_j f(x) \leq B_j(f-g)(x) + |f(x)-g(x)| + A_j g(x).$$

Since g is continuous, Proposition 5.1(2) implies that $A_j g(x) = 0$. Hence

$$\{x \in M : A_j f(x) > \alpha\} \subset N_j \cup \{x\in M : |f(x)-g(x)| > \alpha/2\}$$

where

$$N_j := \{x \in M : B_j(f-g)(x) > \alpha/2\}.$$

The measure of the second set is easy to bound:

$$\mu(\{x \in M : |f(x)-g(x)| > \alpha/2\}) \leq \frac{2}{\alpha}\int_M |f-g|\, d\mu < \frac{2\eta}{\alpha}. \tag{5}$$

To bound the measure of the first set, notice that for every $x \in N_j$, there exists $\gamma_x \in \Gamma$ such that $x \in \mathcal{S}_{\epsilon_j}(\gamma_x)$ and

$$\mu(\mathcal{S}_{\epsilon_j}(\gamma_x)) \leq \frac{2}{\alpha}\int_{\mathcal{S}_{\epsilon_j}(\gamma_x)} |f(y)-g(y)|\, d\mu(y).$$

Using Proposition 5.1(5), we can find $N_j' \subset N_j$ and $\epsilon_j' < \epsilon_j$ such that the shadows $\{\mathcal{S}_{\epsilon_j}(\gamma_x) : x \in N_j'\}$ are disjoint and

$$N_j \subset \bigcup_{x\in N_j} \mathcal{S}_{\epsilon_j}(\gamma_x) \subset \bigcup_{x\in N_j'} \mathcal{S}_{\epsilon_j'}(\gamma_x).$$

Applying the Shadow Lemma (Theorem 6.1), there exists a constant $C_j > 1$ such that

$$\mu(\mathcal{S}_{\epsilon_j'}(\gamma)) \leq C_j \mu(\mathcal{S}_{\epsilon_j}(\gamma))$$

for all $\gamma \in \Gamma$. Then

$$\begin{aligned}
\mu(N_j) &\leq \sum_{x\in N_j'} \mu(\mathcal{S}_{\epsilon_j'}(\gamma_x)) \leq C_j \sum_{x\in N_j'} \mu(\mathcal{S}_{\epsilon_j}(\gamma_x)) \\
&\leq \frac{2C_j}{\alpha} \sum_{x\in N_j'} \int_{\mathcal{S}_{\epsilon_j}(\gamma_x)} |f(y)-g(y)|\, d\mu(y) \\
&= \frac{2C_j}{\alpha} \int_{\bigcup_{x\in N_j'} \mathcal{S}_{\epsilon_j}(\gamma_x)} |f(y)-g(y)|\, d\mu(y) \leq \frac{2C_j\eta}{\alpha}.
\end{aligned}$$

Then using Equation (5),

$$\mu(\{x \in M : A_j f(x) > \alpha\}) \leq \frac{2(1+C_j)\eta}{\alpha}.$$

Since $\eta > 0$ was arbitrary, we have $\mu(\{x \in M : A_j f(x) > \alpha\}) = 0$. Since $\alpha > 0$ was arbitrary, we see that $A_j f(x) = 0$ for μ-almost every $x \in M$.

Next we show that the full μ-measure set $\cap_{j \geq 1}\{x : Af(x) = 0\}$ satisfies the lemma. To that end, fix x with $A_j f(x) = 0$ for all $j \geq 1$, $\epsilon \in (0, \epsilon_0]$ and an escaping sequence $\{\gamma_n\}$ where

$$x \in \bigcap_{n \geq 1} \mathcal{S}_\epsilon(\gamma_n).$$

Fix $j \geq 1$ such that $\epsilon_j < \epsilon$. Then $\mathcal{S}_{\epsilon_j}(\gamma) \supset \mathcal{S}_\epsilon(\gamma)$. By the Shadow Lemma (Theorem 6.1), there exists a constant $c > 0$ such that

$$\mu(\mathcal{S}_\epsilon(\gamma)) \geq c\mu(\mathcal{S}_{\epsilon_j}(\gamma))$$

for all $\gamma \in \Gamma$. Then

$$\begin{aligned}
\limsup_{n\to\infty} & \left| f(x) - \frac{1}{\mu(\mathcal{S}_\epsilon(\gamma_n))} \int_{\mathcal{S}_\epsilon(\gamma_n)} f(y) d\mu(y) \right| \\
& \leq \limsup_{n\to\infty} \frac{1}{\mu(\mathcal{S}_\epsilon(\gamma_n))} \int_{\mathcal{S}_\epsilon(\gamma_n)} |f(x) - f(y)| \, d\mu(y) \\
& \leq \limsup_{n\to\infty} \frac{c^{-1}}{\mu(\mathcal{S}_{\epsilon_j}(\gamma_n))} \int_{\mathcal{S}_{\epsilon_j}(\gamma_n)} |f(x) - f(y)| \, d\mu(y) \\
& \leq c^{-1} A_j f(x) = 0.
\end{aligned}$$

□

Next we use Lemma 8.2 to prove the following.

Lemma 8.3. *Suppose $\epsilon_0 > 0$ satisfies the Shadow Lemma (Theorem 6.1). If $E \subset M$ is measurable, then for μ-almost every $x \in E$ we have*

$$1 = \lim_{n\to\infty} \mu(\gamma_n^{-1} E)$$

for every $0 < \epsilon \leq \epsilon_0$ and escaping sequence $\{\gamma_n\} \subset \Gamma$ with

$$x \in \bigcap_{n \geq 1} \mathcal{S}_\epsilon(\gamma_n).$$

Proof. Applying Lemma 8.2 to 1_E, there is a full μ-measure set $N \subset M$ such that

$$1 = \lim_{n\to\infty} \frac{1}{\mu(\mathcal{S}_\epsilon(\gamma_n))} \int_{\mathcal{S}_\epsilon(\gamma_n)} 1_E(y) d\mu(y) = \lim_{n\to\infty} \frac{\mu(E \cap \mathcal{S}_\epsilon(\gamma_n))}{\mu(\mathcal{S}_\epsilon(\gamma_n))}$$

whenever $x \in N \cap E$, $0 < \epsilon \leq \epsilon_0$ and $\{\gamma_n\} \subset \Gamma$ is an escaping sequence with

$$x \in \bigcap_{n \geq 1} \mathcal{S}_\epsilon(\gamma_n).$$

We claim that the full μ-measure set N satisfies the lemma. To that end, fix $x \in E \cap N$, $0 < \epsilon \leq \epsilon_0$ and escaping sequence $\{\gamma_n\} \subset \Gamma$ with

$$x \in \bigcap_{n \geq 1} \mathcal{S}_\epsilon(\gamma_n).$$

Notice that

$$x \in \bigcap_{n \geq 1} \mathcal{S}_{\epsilon/j}(\gamma_n).$$

for all $j \geq 1$, since $\mathcal{S}_\epsilon(\gamma) \subset \mathcal{S}_{\epsilon/j}(\gamma)$. So we have

$$1 = \lim_{n \to \infty} \frac{\mu(E \cap \mathcal{S}_{\epsilon/j}(\gamma_n))}{\mu(\mathcal{S}_{\epsilon/j}(\gamma_n))}$$

for every $j \geq 1$. Now

$$\begin{aligned} \mu(E \cap \mathcal{S}_{\epsilon/j}(\gamma_n)) &= \mu(\mathcal{S}_{\epsilon/j}(\gamma_n)) - \mu(E^c \cap \mathcal{S}_{\epsilon/j}(\gamma_n)) \\ &= \mu(\mathcal{S}_{\epsilon/j}(\gamma_n)) - (\gamma_n^{-1})_*\mu(\gamma_n^{-1}E^c \cap \gamma_n^{-1}\mathcal{S}_{\epsilon/j}(\gamma_n)). \end{aligned}$$

Hence

$$0 = \lim_{n \to \infty} \frac{(\gamma_n^{-1})_*\mu(\gamma_n^{-1}E^c \cap \gamma_n^{-1}\mathcal{S}_{\epsilon/j}(\gamma_n))}{(\gamma_n^{-1})_*\mu(\gamma_n^{-1}\mathcal{S}_{\epsilon/j}(\gamma_n))}.$$

By Proposition 5.1(1) , there exists $C_j > 1$ (independent of n) such that

$$\frac{1}{C_j} e^{-\|\gamma_n\|_\sigma} \leq \frac{d(\gamma_n^{-1})_*\mu}{d\mu} \leq C_j e^{-\|\gamma_n\|_\sigma}$$

almost everywhere on $\gamma_n^{-1}\mathcal{S}_{\epsilon/j}(\gamma_n)$. Thus

$$0 = \lim_{n \to \infty} \frac{\mu(\gamma_n^{-1}E^c \cap \gamma_n^{-1}\mathcal{S}_{\epsilon/j}(\gamma_n))}{\mu(\gamma_n^{-1}\mathcal{S}_{\epsilon/j}(\gamma_n))}.$$

Recall that $\gamma_n^{-1}\mathcal{S}_{\epsilon/j}(\gamma_n)) = M - B_{\epsilon/j}(\gamma_n)$. Further, by Proposition 7.1 and Proposition 6.3, μ has no atoms. Hence

$$\lim_{j \to \infty} \inf_{n \geq 1} \mu\left(\gamma_n^{-1}\mathcal{S}_{\epsilon/j}(\gamma_n))\right) = 1.$$

Thus $\mu(\gamma_n^{-1}E^c) \to 0$, which implies that $\mu(\gamma_n^{-1}E) \to 1$. □

Now we are ready to prove the three assertions in Theorem 8.1.

Lemma 8.4. *Γ acts ergodically on (M, μ).*

Proof. Lemma 8.3 implies that any Γ-invariant set with positive μ-measure has full measure. □

Lemma 8.5. *If λ is a C-coarse σ-Patterson–Sullivan measure of dimension δ, then $e^{-4C}\mu \leq \lambda \leq e^{4C}\mu$.*

Proof. For any $t \in [0,1]$ the measure $\mu_t := (1-t)\mu + t\lambda$ is also a C-coarse σ-Patterson–Sullivan measure of dimension δ. Indeed, for any $\gamma \in \Gamma$, letting $f(x) = e^{-\delta\sigma(\gamma^{-1},x)}$ we have $C^{-1}f\mu \leq \gamma_*\mu \leq Cf\mu$ and $C^{-1}f\lambda \leq \gamma_*\lambda \leq Cf\lambda$, so

$$\gamma_*\mu_t = \gamma_*((1-t)\mu + t\lambda) = (1-t)\gamma_*\mu + t\gamma_*\lambda \leq (1-t)Cf\mu + tCf\lambda = Cf\mu_t,$$

and similarly $\gamma_*\mu_t \geq C^{-1}f\mu_t$.

Fix $s, t \in (0,1)$. Then the measures μ_s and μ_t are absolutely continuous. Since μ_t and μ_s are both coarse Patterson–Sullivan measures of the same dimension, the Radon-Nikodym derivative $\frac{d\mu_t}{d\mu_s}$ is coarsely Γ-invariant, more precisely: for any $\gamma \in \Gamma$ we have

$$e^{-2C}\frac{d\mu_t}{d\mu_s} \leq \frac{d\mu_t}{d\mu_s} \circ \gamma \leq e^{2C}\frac{d\mu_t}{d\mu_s}$$

μ_s-almost everywhere.

Next fix $\epsilon_j \searrow 0$. Then for each j there exists $r_j \in \mathbb{R}$ such that the set $A_j := \{r_j \leq \frac{d\mu_t}{d\mu_s} \leq r_j + \epsilon_j\}$ has positive μ_s-measure. Then $\Gamma \cdot A_j$ is Γ-invariant and hence, by ergodicity, must have full measure. Further,

$$\Gamma \cdot A_j \subset \{e^{-2C} r_j \leq \frac{d\mu_t}{d\mu_s} \leq e^{2C} r_j + e^{2C}\epsilon_j\}$$

and so

$$e^{-2C} r_j d\mu_s \leq d\mu_t \leq (e^{2C} r_j + e^{2C}\epsilon_j) d\mu_s.$$

Since μ_t and μ_s are both probability measures, we must have

$$e^{-2C} r_j \leq 1 \quad \text{and} \quad e^{2C} r_j + e^{2C}\epsilon_j \geq 1$$

for all j. Thus any limit point of $\{r_j\}$ is in $[e^{-2C}, e^{2C}]$, which implies that

$$e^{-4C} d\mu_s \leq d\mu_t \leq e^{4C} d\mu_s.$$

Since $s, t \in (0,1)$ were arbitrary, we then see that $e^{-4C}\mu \leq \lambda \leq e^{4C}\mu$. □

Lemma 8.6. $\mu(\Lambda^{\rm con}_\epsilon(\Gamma)) = 1$ *when* $\epsilon > 0$ *is sufficiently small.*

Proof. Proposition 7.1 implies that $\mu(\Lambda^{\rm con}(\Gamma)) = 1$. Since $\Lambda^{\rm con}(\Gamma) = \bigcup_{\epsilon>0} \Lambda^{\rm con}_\epsilon(\Gamma)$, this implies that $\mu(\Lambda^{\rm con}_\epsilon(\Gamma)) > 0$ when $\epsilon > 0$ is sufficiently small. By Observation 5.3, the set $\Lambda^{\rm con}_\epsilon(\Gamma)$ is Γ-invariant. Hence, by ergodicity, $\mu(\Lambda^{\rm con}_\epsilon(\Gamma)) = 1$ for all sufficiently small $\epsilon > 0$. □

9. BMS measures on $M^{(2)}$, conservativity and dissipativity

Suppose $\Gamma \subset \mathsf{Homeo}(M)$ is a convergence group and let

$$M^{(2)} := \{(x,y) \in M^2 : x \neq y\}.$$

In this section we study the action of Γ on $M^{(2)}$.

9.1. **BMS measures.** We first observe that a coarse GPS system can be used to produce a Γ-invariant measure on $M^{(2)}$. To that end, suppose $(\sigma, \bar\sigma, G)$ is a coarse GPS system, and $\mu, \bar\mu$ are coarse Patterson–Sullivan measures of dimension $\delta \geq 0$ for $\sigma, \bar\sigma$ respectively.

We use a lemma from [BF17] to show that $\bar\mu \otimes \mu$ can be scaled to become Γ-invariant. Note that this lemma is unnecessary in the continuous case, i.e. when $\kappa = 0$ in Definition 1.7.

Lemma 9.1. *There exists a Borel measurable function* $\tilde G \colon M^{(2)} \to [0,\infty)$ *such that* $(\sigma, \bar\sigma, \tilde G)$ *is a coarse GPS system and the measure*

$$\nu := e^{\delta \tilde G} \bar\mu \otimes \mu.$$

is locally finite and Γ*-invariant. We call* ν *a* BMS (Bowen–Margulis–Sullivan) measure of dimension δ *on* $M^{(2)}$ *associated to* $(\sigma, \bar\sigma, G, \mu, \bar\mu)$.

Proof. Define $H \colon M^{(2)} \to [0,\infty)$ by

$$H(x,y) = \limsup_{p \to x, q \to y} G(p,q).$$

Since $(\sigma, \bar\sigma, G)$ is a coarse GPS system, we see that $(\sigma, \bar\sigma, H)$ is a coarse GPS system. By construction H is upper semicontinuous and hence Borel measurable (while G may not be).

Let $\nu_0 := e^{\delta H}\bar{\mu} \otimes \mu$ and

$$\rho(\gamma, x, y) := -\frac{1}{\delta} \log \frac{d\gamma_*^{-1}\nu_0}{d\nu_0}(x, y).$$

By uniqueness of Radon–Nikodym derivatives, there is a full $(\bar{\mu} \otimes \mu)$-measure Γ-invariant Borel measurable subset $E \subset M^{(2)}$ such that $\rho(\gamma, x, y)$ is defined for all $\gamma \in \Gamma$ and $(x, y) \in E$, and $\rho(\gamma\gamma', x, y) = \rho(\gamma, \gamma' x, \gamma' y) + \rho(\gamma', x, y)$ for any additional $\gamma' \in \Gamma$. We extend ρ to a cocycle on the whole set $M^{(2)}$ by setting it to zero on the complement of E. Further, since $(\sigma, \bar{\sigma}, H)$ is a coarse GPS system and $\mu, \bar{\mu}$ are coarse Patterson–Sullivan measures for $\sigma, \bar{\sigma}$, one may check that ρ is bounded on a full measure set. So up to changing it on a null measure set we may assume that

$$\sup_{\gamma \in \Gamma, (x,y) \in M^{(2)}} |\rho(\gamma, x, y)| < +\infty.$$

By [BF17, Lem. 3.4] there exists a bounded Borel function $\phi \colon M^{(2)} \to \mathbb{R}$ such that

$$\rho(\gamma, x, y) = \phi(\gamma x, \gamma y) - \phi(x, y).$$

for all $\gamma \in \Gamma$ and $(x, y) \in M^{(2)}$. Then let

$$\tilde{G} = H + \phi - \inf_{(x,y) \in M^{(2)}} \phi(x, y)$$

(notice that the constant term is added so that $\tilde{G}$ is non-negative).

Since ϕ is bounded, $\tilde{G}$ is at bounded distance from H, which immediately implies that $(\sigma, \bar{\sigma}, \tilde{G})$ is a coarse GPS system. The fact that $\nu := e^{\delta\tilde{G}}\bar{\mu} \otimes \mu$ is locally finite comes from the fact that $\tilde{G}$ is locally finite. To see the Γ-invariance of ν, note

$$\gamma_*^{-1}\nu = Ce^{\delta\phi\circ\gamma}\gamma_*^{-1}\nu_0 = Ce^{\delta\phi\circ\gamma}e^{-\delta\rho(\gamma,\cdot)}\nu_0 = Ce^{\delta\phi}\nu_0 = \nu. \qquad \square$$

9.2. **Conservative–dissipative dichotomy for BMS measures.** In this section, we consider the conservativity/dissipativity of the Γ action on $M^{(2)}$.

We say that an orbit $\Gamma(x, y) \subset M^{(2)}$ is *escaping* if $\{\gamma \in \Gamma : \gamma(x, y) \in K\}$ is finite for any compact subset $K \subset M^{(2)}$.

Lemma 9.2. *An orbit $\Gamma(x, y) \subset M^{(2)}$ is escaping if and only both x and y are not conical limit points.*

Proof. Let d be a compatible metric on $\Gamma \sqcup M$.

Suppose one of x, y is conical, say x. Then there exists $\{\gamma_n\} \subset \Gamma$ and $a \neq b \in M$ such that $\gamma_n x \to a$ while $\gamma_n z \to b$ for any $z \in M \smallsetminus \{x\}$. In particular $\gamma_n(x, y) \to (a, b) \in M^{(2)}$, so $\Gamma(x, y)$ is not escaping.

Suppose $\Gamma(x, y)$ is not escaping, i.e. $\mathrm{d}(\gamma_n x, \gamma_n y)$ stays away from zero for some escaping sequence $\{\gamma_n\} \subset \Gamma$. Passing to a subsequence, there are $c, b \in M$ such that $\gamma_n z \to b$ for any $z \in M \smallsetminus \{c\}$. Since $\{\gamma_n x\}$ and $\{\gamma_n y\}$ cannot both converge to b, one of x, y must be equal to c, say x. Then passing to a further subsequence, $\gamma_n x \to a \in M \smallsetminus \{b\}$, so x is conical. $\square$

As a corollary we obtain the following dichotomy, which is a part of our Hopf–Tsuji–Sullivan dichotomy (Theorem 1.8).

Corollary 9.3. *Let $(\sigma, \bar{\sigma}, G)$ be a coarse GPS system and let ν be a BMS measure of dimension δ on $M^{(2)}$.*

- *If $\sum_{\gamma\in\Gamma} e^{-\delta\|\gamma\|_\sigma} = +\infty$, then the action of Γ on $(M^{(2)}, \nu)$ is conservative.*
- *If $\sum_{\gamma\in\Gamma} e^{-\delta\|\gamma\|_\sigma} < +\infty$, then the action of Γ on $(M^{(2)}, \nu)$ is dissipative.*

Proof. By definition $\nu = e^{\delta \tilde{G}} \bar{\mu} \otimes \mu$ where $\mu, \bar{\mu}$ are coarse Patterson–Sullivan measures for $\sigma, \bar{\sigma}$ and $\tilde{G} : M^{(2)} \to \mathbb{R}$. Suppose $\sum e^{-\delta \|\gamma\|_\sigma} = +\infty$ (resp. $< +\infty$). By Proposition 7.1 (resp. Proposition 6.3(2)) and Proposition 3.3(1), μ and $\bar{\mu}$ give full measure to $\Lambda^{\rm con}(\Gamma)$ (resp. $M - \Lambda^{\rm con}(\Gamma)$). Hence ν gives full measure to $\Lambda^{\rm con}(\Gamma)^{(2)}$ (resp. $(M - \Lambda^{\rm con}(\Gamma))^{(2)}$) in $M^{(2)}$, and hence gives full measure to the set of Γ-orbits in $M^{(2)}$ that do not escape (resp. that do escape) by Lemma 9.2, which is the conservative (resp. dissipative) part in the Hopf decomposition of Lemma A.9. □

10. A flow space

In this section, we use our Patterson–Sullivan measure to define a flow space which admits a measurable action by Γ. In the presence of a GPS system we construct a Γ-invariant flow-invariant measure on this flow space. The construction of the measurable action comes from work of Bader–Furman [BF17].

For the rest of the section suppose $\Gamma \subset \mathsf{Homeo}(M)$ is a convergence group and $\sigma \colon \Gamma \times M \to \mathbb{R}$ is an expanding coarse-cocycle. As in Section 9, let

$$M^{(2)} := \{(x, y) \in M^2 : x \neq y\}.$$

The space $M^{(2)} \times \mathbb{R}$ has a natural flow defined by

$$\psi^t(x, y, s) = (x, y, s + t).$$

10.1. **An action of Γ on $M^{(2)} \times \mathbb{R}$.** In this section we show that any Patterson–Sullivan induces a measurable action of Γ on $M^{(2)} \times \mathbb{R}$.

Suppose μ is a coarse σ-Patterson–Sullivan measure of dimension δ. Then let $\sigma_{\rm PS} \colon \Gamma \times M \to \mathbb{R}$ be the measurable cocycle defined by

$$\sigma_{\rm PS}(\gamma, x) = -\frac{1}{\delta} \log \frac{d\gamma_*^{-1}\mu}{d\mu}(x).$$

Observation 10.1. We can assume that $\sigma_{\rm PS}$ is everywhere defined and that $\sigma_{\rm PS}$ is a cocycle:

$$\sigma_{\rm PS}(\gamma_1\gamma_2, x) = \sigma_{\rm PS}(\gamma_1, \gamma_2 x) + \sigma_{\rm PS}(\gamma_2, x)$$

for all $\gamma_1, \gamma_2 \in \Gamma$ and $x \in M$.

Proof. By uniqueness of Radon–Nikodym derivatives, there exists a Γ-invariant set $E \subset M$ where $\mu(E) = 1$ and

$$\sigma_{\rm PS}(\gamma_1\gamma_2, x) = \sigma_{\rm PS}(\gamma_1, \gamma_2 x) + \sigma_{\rm PS}(\gamma_2, x)$$

for all $\gamma_1, \gamma_2 \in \Gamma$ and $x \in E$. Since $\mu(E^c) = 0$, we may assume that $\sigma_{\rm PS}|_{\Gamma \times E^c} \equiv 0$. Then $\sigma_{\rm PS}$ is a cocycle. □

Using Observation 10.1 we can define a Γ action on $M^{(2)} \times \mathbb{R}$ by

$$\gamma \cdot (x, y, t) = (\gamma x, \gamma y, t + \sigma_{\rm PS}(\gamma, y)).$$

Notice that this action commutes with the flow ψ^t.

10.2. **A measure on the flow space.** Now we assume that σ is part of a coarse GPS system $(\sigma, \bar{\sigma}, G)$ and $\bar{\mu}$ is a coarse $\bar{\sigma}$-Patterson–Sullivan measure of dimension δ. In this case, we will construct a flow-invariant measure on $M^{(2)} \times \mathbb{R}$.

Let $\nu = e^{\delta \tilde{G}} \bar{\mu} \otimes \mu$ be a Γ-invariant BMS measure associated to $(\sigma, \bar{\sigma}, G, \mu, \bar{\mu})$ as in Section 9.1, where $\tilde{G} : M^{(2)} \to [0, \infty)$ is measurable and $(\sigma, \bar{\sigma}, \tilde{G})$ is a coarse GPS system.

Then let $\tilde{m} := \nu \otimes dt$, which is a measure on $M^{(2)} \times \mathbb{R}$. Notice that:

(1) Since $\tilde{G}$ is locally bounded on $M^{(2)}$, the measure $\tilde{m}$ is locally finite on $M^{(2)} \times \mathbb{R}$.
(2) $\tilde{m}$ is Γ-invariant and ψ^t-invariant.

Next we show that the action of Γ on $(M^{(2)} \times \mathbb{R}, \tilde{m})$ is dissipative (see Appendix A for the definition).

Since μ is a coarse Patterson–Sullivan measure, there exists $C > 0$ such that for each $\gamma \in \Gamma$ there is some $M_\gamma \subset M$ with $\mu(M_\gamma) = 1$ and

$$\sup_{x \in M_\gamma} |\sigma_{\rm PS}(\gamma, x) - \sigma(\gamma, x)| < C.$$

Let

$$M' := \bigcap_{\alpha \in \Gamma} \alpha \left(\bigcap_{\gamma \in \Gamma} M_\gamma \right). \tag{6}$$

Then M' is Γ-invariant, $\mu(M') = 1$, and

$$\sup_{x \in M', \gamma \in \Gamma} |\sigma_{\rm PS}(\gamma, x) - \sigma(\gamma, x)| < C. \tag{7}$$

Finally let

$$Z := \{(x, y, t) \in M^{(2)} \times \mathbb{R} : y \in M'\}.$$

Then Z is Γ-invariant and ψ^t-invariant, and has full $\tilde{m}$-measure.

The next result implies that if $v \in Z$, then its Γ-orbit is escaping, i.e. $\{\gamma : \gamma v \in K\}$ is finite for any compact set K. In particular, $\tilde{m}$-almost every orbit is escaping.

Proposition 10.2. *For any compact subset $K \subset M^{(2)} \times \mathbb{R}$ the set*

$$\{\gamma \in \Gamma : (K \cap Z) \cap \gamma(K \cap Z) \neq \varnothing\}$$

is finite. In particular, the action of Γ on $(M^{(2)} \times \mathbb{R}, \tilde{m})$ is dissipative.

Proof. Suppose for a contradiction that there exist a compact set $K \subset M^{(2)} \times \mathbb{R}$ and a sequence $\{\gamma_n\}$ of distinct elements of Γ such that

$$(K \cap Z) \cap \gamma_n(K \cap Z) \neq \varnothing$$

for all n. Passing to a subsequence we can assume that $\gamma_n \to a \in M$ and $\gamma_n^{-1} \to b \in M$, i.e. $\gamma_n^{-1}|_{M - \{a\}}$ converges locally uniformly to b.

For each n fix

$$(x_n, y_n, t_n) \in (K \cap Z) \cap \gamma_n(K \cap Z).$$

Passing to a subsequence we can suppose that $x_n \to x$ and $y_n \to y \neq x$.

Since $\{t_n\}$ is bounded and

$$(\gamma_n^{-1} x_n, \gamma_n^{-1} y_n, t_n + \sigma_{\rm PS}(\gamma_n^{-1}, y_n)) = \gamma_n^{-1}(x_n, y_n, t_n) \in K \cap Z,$$

we see that $\{\sigma_{\rm PS}(\gamma_n^{-1}, y_n)\}$ is bounded. Then Equation (7) implies that $\{\sigma(\gamma_n^{-1}, y_n)\}$ is bounded. Then Proposition 3.2(3) implies that $\gamma_n^{-1} y_n \to b$. Since

$$\liminf_{n\to\infty} \mathrm{d}(\gamma_n^{-1} x_n, \gamma_n^{-1} y_n) > 0$$

and $\gamma_n^{-1}|_{M-\{a\}}$ converges locally uniformly to b, we must have $x = a$. Hence $y \neq a$. Since σ is expanding, there exists $C' > 0$ such that

$$\sigma(\gamma_n^{-1}, y_n) \geq \left\|\gamma_n^{-1}\right\|_\sigma - C'$$

for all $n \geq 1$. In particular, this quantity diverges to $+\infty$ as $n \to \infty$, which contradicts our earlier observation that $\{\sigma(\gamma_n^{-1}, y_n)\}$ must be bounded. □

10.3. **The quotient flow space and quotient measure.** In this section we show that the quotient $\Gamma\backslash M^{(2)} \times \mathbb{R}$ is a reasonable measure space, the flow descends to a measurable flow on the quotient, and the measure $\tilde{m}$ descends to a flow-invariant measure on the quotient.

Endow the quotient $\Gamma\backslash M^{(2)} \times \mathbb{R}$ with the quotient sigma-algebra (of the Borel sigma-algebra). By Proposition 10.2, the action of Γ is dissipative with respect to the measure $\tilde{m} = \nu \otimes dt = e^{\delta \tilde{G}} \bar{\mu} \otimes \mu \otimes dt$. Thus by the discussion in Section A.3 the space $\Gamma\backslash M^{(2)} \times \mathbb{R}$ admits a quotient measure m, which we also call a *BMS measure* associated to $(\sigma, \bar{\sigma}, G, \mu, \bar{\mu})$.

Recall that the flow $\psi^t(x, y, s) = (x, y, t + s)$ commutes with the Γ action. So ψ^t descends to a measurable flow on the quotient space $\Gamma\backslash M^{(2)} \times \mathbb{R}$, which we also denote by ψ^t. Since $\tilde{m}$ is ψ^t-invariant, the uniqueness of quotient measures, again see Section A.3, implies that m is ψ^t-invariant.

Finally, by the discussion in Section A.2, $\Gamma\backslash M^{(2)} \times \mathbb{R}$ has a ψ^t-invariant full m-measure subset that is standard (i.e. measurably embeds into $[0, 1]$).

10.4. **The continuous case.** The construction above involves a number of choices, for instance a different choice of Patterson–Sullivan measure could lead to a different Γ action on $M^{(2)} \times \mathbb{R}$ and hence a different quotient space.

In this section we show that in the continuous case, some of the technicalities and all of the choices made in the above construction can be avoided.

First suppose that $\sigma : \Gamma \times M \to \mathbb{R}$ is an expanding 0-coarse-cocycle. Then, in the discussion above, can assume that $\sigma_{\rm PS} = \sigma$, $M' = M$, and $Z = M^{(2)} \times \mathbb{R}$. Then (the proof of) Proposition 10.2 implies that Γ acts properly discontinuously on $M^{(2)} \times \mathbb{R}$ and hence the quotient

$$U_\Gamma := \Gamma\backslash\Lambda(\Gamma)^{(2)} \times \mathbb{R},$$

is a metrizable locally compact topological space. Further the flow ψ^t descends to a continuous flow, also called ψ^t, on U_Γ.

Next we assume that σ is part of a continuous (i.e. $\kappa = 0$ in Definition 1.7) GPS system $(\sigma, \bar{\sigma}, G)$ with $\delta := \delta_\sigma(\Gamma) < +\infty$ and $\sum_{\gamma\in\Gamma} e^{-\delta\|\gamma\|_\sigma} = +\infty$. By Theorems 4.1 and 8.1 there are unique probability measures μ, $\bar{\mu}$ on M which satisfy

$$\frac{d\gamma_*\mu}{d\mu} = e^{-\delta\sigma(\gamma^{-1},\cdot)} \quad \text{and} \quad \frac{d\gamma_*\bar{\mu}}{d\bar{\mu}} = e^{-\delta\bar{\sigma}(\gamma^{-1},\cdot)}.$$

Then, since $(\sigma, \bar{\sigma}, G)$ is a continuous GPS system, the measure $\nu := e^{\delta G} \bar{\mu} \otimes \mu$ on $M^{(2)}$ is Γ-invariant. Note ν is supported on $\Lambda(\Gamma)^{(2)}$.

Finally, the measure $\tilde{m} := e^{\delta G} d\bar{\mu} \otimes d\mu \otimes dt$ on $\Lambda(\Gamma)^{(2)} \times \mathbb{R}$ descends to a ψ^t-invariant Borel measure m_Γ on U_Γ. In this construction, no choices were made and so we call m **the** *Bowen–Margulis–Sullivan (BMS) measure associated to* $(\sigma, \bar{\sigma}, G)$ and denote it by m_Γ.

11. Ergodicity of product measures

In this section we prove ergodicity of the product action for coarse GPS systems whose Poincaré series diverges at the critical exponent.

Theorem 11.1. *Suppose* $(\sigma, \bar{\sigma}, G)$ *is a coarse GPS system with* $\delta := \delta_\sigma(\Gamma) < +\infty$ *and* μ, $\bar{\mu}$ *are coarse Patterson–Sullivan measures of dimension* δ *for* σ, $\bar{\sigma}$ *respectively. If*

$$\sum_{\gamma \in \Gamma} e^{-\delta \|\gamma\|_\sigma} = +\infty,$$

then Γ *acts ergodically on* $(M^{(2)}, \bar{\mu} \otimes \mu)$.

As described in Section 10.4, in the continuous case there is a canonical flow space and in this case our arguments will yield the following, see Section 11.5 for the proof.

Theorem 11.2. *If* $(\sigma, \bar{\sigma}, G)$ *is a continuous GPS system with* $\delta := \delta_\sigma(\Gamma) < +\infty$ *and*

$$\sum_{\gamma \in \Gamma} e^{-\delta \|\gamma\|_\sigma} = +\infty,$$

then the flow ψ^t *on* (U_Γ, m_Γ) *is conservative and ergodic, where* m_Γ *is the (unique) BMS measure associated to* $(\sigma, \bar{\sigma}, G)$ *defined in Section 10.4.*

The general strategy of the proof goes back to Sullivan's original work in real hyperbolic geometry [Sul79]. In particular, we use the Hopf ratio ergodic theorem to prove ergodicity of the flow space introduced in Section 10, which in turn will imply ergodicity of the action of Γ on $M^{(2)}$. Some of our arguments also use ideas from work of Bader–Furman [BF17].

11.1. Notations.

We will freely use the notations and objects introduced in Sections 9 and 10, in particular:

(1) the measurable cocycle σ_{PS} introduced in Section 10.1, the associated action of Γ on $M^{(2)} \times \mathbb{R}$ given by

$$\gamma \cdot (x, y, t) = (\gamma x, \gamma y, t + \sigma_{\mathrm{PS}}(\gamma, y)),$$

and the associated measurable quotient $\Gamma \backslash M^{(2)} \times \mathbb{R}$;

(2) the Γ-invariant measure $\nu = e^{\delta \tilde{G}} \bar{\mu} \otimes \mu$ on $M^{(2)}$ constructed in Section 9.1;

(3) the flow $\psi^t(x, y, s) = (x, y, t + s)$ on $M^{(2)} \times \mathbb{R}$ and the quotient flow, also denoted by ψ^t, on $\Gamma \backslash M^{(2)} \times \mathbb{R}$;

(4) the flow-invariant measure $\tilde{m} = \nu \otimes dt$ on $M^{(2)} \times \mathbb{R}$ and the associated flow-invariant quotient measure m on $\Gamma \backslash M^{(2)} \times \mathbb{R}$ described in Section 10.3;

(5) the set $M' \subset M$ defined in Equation (6), which is Γ-invariant, has full μ-measure, and where

$$C := \sup_{\gamma \in \Gamma, \ y \in M'} |\sigma(\gamma, y) - \sigma_{\mathrm{PS}}(\gamma, y)| < +\infty. \tag{8}$$

We will also use the following notation from Section A.3. For $f \in L^1(M^{(2)} \times \mathbb{R}, \tilde{m})$ let $\tilde{P}(f)$ be the $\tilde{m}$-almost everywhere defined function on $M^{(2)} \times \mathbb{R}$ given by

$$\tilde{P}(f)(v) = \sum_{\gamma \in \Gamma} f(\gamma \cdot v)$$

and let $P(f)$ be the m-almost everywhere defined function on the quotient given by $P(f)([v]) = \tilde{P}(f)(v)$. By Equation 19,

$$\int P(f) dm = \int f d\tilde{m} \tag{9}$$

for all $f \in L^1(M^{(2)} \times \mathbb{R}, \tilde{m})$ and the map

$$P \colon L^1(M^{(2)} \times \mathbb{R}, \tilde{m}) \to L^1(\Gamma \backslash M^{(2)} \times \mathbb{R}, m)$$

is continuous. We also observe that

$$\tilde{P}(f)(\psi^t(v)) = P(f)(\psi^t([v])) \tag{10}$$

whenever both sides are defined.

Finally, given $\theta \in L^1(\mathbb{R})$ and $f \in L^1(M^{(2)}, \nu)$, let $f \otimes \theta \in L^1(M^{(2)} \times \mathbb{R}, \tilde{m})$ denote the function

$$(f \otimes \theta)(x, y, t) = f(x, y)\, \theta(t).$$

Notice that with a, b fixed, the map

$$f \in L^1(M^{(2)}, \nu) \mapsto f \otimes 1_{[a,b]} \in L^1(M^{(2)} \times \mathbb{R}, \tilde{m})$$

is a continuous operator.

11.2. **Constructing a weight function for the Hopf ratio ergodic theorem.** In this section we construct a weight function to use in the Hopf ratio ergodic theorem.

We begin by relating conical limit points to recurrence properties of the flow. To that end, fix a compatible metric d on $\Gamma \sqcup M$. Then given $\epsilon > 0$, let

$$K_\epsilon := \{(x, y, 0) : \mathrm{d}(x, y) \geq \epsilon\}.$$

Proposition 11.3. *Fix $0 < \epsilon' < \epsilon$ and $y \in M'$.*

(1) If $y \in \Lambda_\epsilon^{\mathrm{con}}(\Gamma)$, then there exists a sequence of distinct elements $\{\gamma_n\} \subset \Gamma$ such that: for any $x \in M \smallsetminus \{y\}$, there is a sequence $\{t_n\} \subset \mathbb{R}$ with $t_n \to +\infty$ so that $(x, y, t_n) \in \gamma_n(K_{\epsilon'})$ for n sufficiently large.

(2) If there exist a sequence of distinct elements $\{\gamma_n\} \subset \Gamma$, $x \in M \smallsetminus \{y\}$, and a sequence $\{t_n\} \subset \mathbb{R}$ so that $\{t_n\}$ is bounded below and $(x, y, t_n) \in \gamma_n K_\epsilon$ for all n, then $y \in \Lambda_\epsilon^{\mathrm{con}}(\Gamma)$.

Proof. (1) If $y \in \Lambda_\epsilon^{\mathrm{con}}(\Gamma)$, there exist $a, b \in M$ so that $d(a, b) \geq \epsilon$ and a sequence $\{\gamma_n\} \subset \Gamma$ such that $\gamma_n^{-1} y \to a$ and $\gamma_n^{-1} x \to b$ for all $x \in M \smallsetminus \{y\}$. In particular, $\gamma_n^{-1} \to b$ and $\gamma_n \to y$.

Fix $x \in M \smallsetminus \{y\}$. Then $(\gamma_n^{-1} x, \gamma_n^{-1} y) \to (b, a)$ and so $(\gamma_n^{-1} x, \gamma_n^{-1} y, 0) \in K_{\epsilon'}$ for all sufficiently large n. Then

$$\gamma_n(\gamma_n^{-1} x, \gamma_n^{-1} y, 0) = (x, y, \sigma_{\mathrm{PS}}(\gamma_n, \gamma_n^{-1} y)) \in \gamma_n(K_{\epsilon'})$$

for sufficiently large n. Since $\gamma_n^{-1} y \to a$, $\gamma_n^{-1} \to b$, and $a \neq b$, Equation (8) and the expanding property for σ imply that

$$\lim_{n \to \infty} \sigma_{\mathrm{PS}}(\gamma_n, \gamma_n^{-1} y) \geq -C + \lim_{n \to \infty} \sigma(\gamma_n, \gamma_n^{-1} y) = +\infty.$$

Hence if $t_n := \sigma_{\rm PS}(\gamma_n, \gamma_n^{-1} y)$, then $t_n \to +\infty$ and $(x, y, t_n) \in \gamma_n K_{\epsilon'}$ for all sufficiently large n.

(2) Now suppose there exist a sequence of distinct elements $\{\gamma_n\} \subset \Gamma$, $x \in M \smallsetminus \{y\}$, and a sequence $\{t_n\} \subset \mathbb{R}$ so that $\{t_n\}$ is bounded below and $(x, y, t_n) \in \gamma_n(K_\epsilon)$ for all n. Passing to subsequence, we may assume that $\gamma_n^{-1}(y) \to a \in M$ and $\gamma_n^{\pm 1} \to b^{\pm}$. By assumption, $\mathrm{d}(\gamma_n^{-1} x, \gamma_n^{-1} y) \geq \epsilon$ for all n, so $\mathrm{d}(a, b^-) \geq \epsilon$, and hence $y \in \Lambda^{\rm con}_\epsilon(\Gamma)$. □

Using Theorem 8.1, we can fix $\epsilon_0 > 0$ sufficiently small so that

$$\mu(\Lambda^{\rm con}_{\epsilon_0}(\Gamma)) = 1.$$

Then by Proposition 11.3, there exists a compact subset $K \subset M^{(2)} \times \mathbb{R}$ such that for every $v \in M^{(2)} \times \mathbb{R}$ with $v^+ \in \Lambda^{\rm con}_{\epsilon_0}(\Gamma) \cap M'$ there exist sequences $\{\gamma_n\} \subset \Gamma$ and $\{t_n\} \subset [0, \infty)$ where $t_n \to +\infty$ and

$$\psi^{t_n}(v) \in \gamma_n(K)$$

for all $n \geq 1$. Then fix a non-negative $\rho_0 \in C_c(M^{(2)})$ and $R > 0$ such that

$$\rho_0 \otimes 1_{[-R,R]} \geq 1$$

on $\bigcup_{t \in [0,1]} \psi^t(K)$. Then let

$$\tilde{\rho} := \tilde{P}(\rho_0 \otimes 1_{[-R,R]}) \quad \text{and} \quad \rho := P(\rho_0 \otimes 1_{[-R,R]}).$$

Notice that $\rho \in L^1(\Gamma \backslash M^{(2)} \times \mathbb{R}, m)$.

Lemma 11.4. *If $v \in M^{(2)} \times \mathbb{R}$ and $v^+ \in \Lambda^{\rm con}_{\epsilon_0}(\Gamma) \cap M'$, then*

$$\lim_{T \to \infty} \int_0^T \tilde{\rho}(\psi^t(v)) dt = +\infty.$$

In particular,

$$\lim_{T \to \infty} \int_0^T \rho(\psi^t(v)) dt = +\infty$$

for m-almost every $v \in \Gamma \backslash M^{(2)} \times \mathbb{R}$ and so the quotient flow $\psi^t \colon \Gamma \backslash M^{(2)} \times \mathbb{R} \to \Gamma \backslash M^{(2)} \times \mathbb{R}$ is conservative (see Fact A.3).

The fact that ψ^t is conservative can also be deduced from Corollary 9.3 and [Bla21, Fact 2.29].

Proof. Fix $v \in M^{(2)} \times \mathbb{R}$ with $v^+ \in \Lambda^{\rm con}_{\epsilon_0}(\Gamma) \cap M'$. By our choice of K, there exist $\{\gamma_n\} \subset \Gamma$ and $\{t_n\} \subset [0, \infty)$ where $t_n + 1 < t_{n+1}$ and

$$\psi^{t_n}(v) \in \gamma_n(K)$$

for all $n \geq 1$. Then

$$\liminf_{T \to \infty} \int_0^T \tilde{\rho}(\psi^t(v)) dt \geq \sum_{n=1}^\infty \int_{t_n}^{t_n+1} \tilde{\rho}(\psi^t(v)) dt = +\infty,$$

since $\tilde{\rho}(\psi^t(v)) \geq 1$ for any $t \in [t_n, t_n + 1]$.

The "in particular" statement then follows from Equation (10). □

11.3. **Applying the Hopf ratio ergodic theorem.** Next we apply the Hopf ratio ergodic theorem to the conservative flow ψ^t on $(\Gamma\backslash M^{(2)} \times \mathbb{R}, m)$.

This theorem was first proved by Stepanoff [Ste36] and Hopf [Hop37]. For a modern reference: Krengel states the result for discrete actions of $\mathbb{Z}_{\geq 1}$ [Kre85, Th. 2.7 & 3.4] and explains how to then deduce the result for flows [Kre85, §2 p.10].

This theorem yields the following. If $f \in L^1(\Gamma\backslash M^{(2)} \times \mathbb{R}, m)$, then the limit

$$\Phi(f)(v) = \lim_{T\to\infty} \frac{\int_0^T f(\psi^t(v))dt}{\int_0^T \rho(\psi^t(v))dt}$$

exists for every v in a ψ^t-invariant set of m-full measure. Further, the m-almost everywhere defined function $\Phi(f)$ is measurable and ψ^t-invariant, and $\Phi(f)\rho$ is integrable with

$$\int_A \Phi(f)\rho \, dm = \int_A f \, dm \tag{11}$$

for any ψ^t-invariant subset $A \subset \Gamma\backslash M^{(2)} \times \mathbb{R}$. Since $|\Phi(f)| \leq \Phi(|f|)$, Equation (11) implies that

$$\Phi \colon L^1(\Gamma\backslash M^{(2)} \times \mathbb{R}, m) \to L^1(\Gamma\backslash M^{(2)} \times \mathbb{R}, \rho m)$$

is continuous.

We will also let $\tilde{\Phi}(f) \colon M^{(2)} \times \mathbb{R} \to \mathbb{R}$ denote the lift of $\Phi(f)$, which is $\tilde{m}$-almost everywhere defined, Γ-invariant and ψ^t-invariant.

Using a Hopf Lemma type argument, we will deduce the following.

Proposition 11.5. *If $f \in C_c(M^{(2)})$ and $a < b$, then $\Phi \circ P(f \otimes 1_{[a,b]})$ is constant m-almost everywhere.*

This proposition will be proved by first showing $\tilde{\Phi} \circ P(f \otimes 1_{[a,b]})$ is almost surely constant along "weak stable manifolds" of the form $M \times \{y\} \times \mathbb{R}$, which are parametrized by $y \in M$. Thus $\tilde{\Phi} \circ P(f \otimes 1_{[a,b]})$ induces a Γ-invariant function on M defined by

$$y \mapsto \tilde{\Phi} \circ P(f \otimes 1_{[a,b]})(M, y, \mathbb{R}).$$

Then Theorem 8.1, which says that Γ acts ergodically on (M, μ), implies that this function is constant.

Delaying the proof of Proposition 11.5, we deduce Theorem 11.1.

Lemma 11.6. Γ *acts ergodically on $(M^{(2)}, \nu)$ and hence also on $(M^{(2)}, \bar{\mu} \otimes \mu)$.*

Proof. Suppose for a contradiction that there exists a Γ-invariant measurable set $A \subset M^{(2)}$ where $\nu(A) > 0$ and $\nu(A^c) > 0$. By inner regularity, there exists a compact subset $K \subset A^c$ with $\nu(K) > 0$.

Let $\{f_n\}$ be a sequence of compactly supported continuous functions on $M^{(2)}$ converging to 1_K in $L^1(M^{(2)}, \nu)$. Since Φ, P and $\cdot \otimes 1_{[0,1]}$ are continuous operators, we have

$$\Phi \circ P(f_n \otimes 1_{[0,1]}) \to \Phi \circ P(1_K \otimes 1_{[0,1]}) = \Phi \circ P(1_{K\times[0,1]})$$

in $L^1(\Gamma\backslash M^{(2)} \times \mathbb{R}, \rho m)$.

By Proposition 11.5, each $\Phi \circ P(f_n \otimes 1_{[0,1]})$ is constant m-almost everywhere and hence constant ρm-almost everywhere. Hence the limit $\Phi \circ P(1_{K\times[0,1]})$ is constant ρm-almost everywhere (since the convergence is in $L^1(\Gamma\backslash M^{(2)} \times \mathbb{R}, \rho m)$).

By definition, $K \subset A^c$ and so $\Phi \circ P(1_{K\times[0,1]})$ is well defined and equal to zero on $\Gamma\backslash A \times \mathbb{R}$. This set is ψ^t-invariant and has positive m-measure (see Remark A.5), hence it also has positive ρm-measure since $\int_0^\infty \rho(\phi^t(v))dt = +\infty$ for m-almost $v \in \Gamma\backslash M^{(2)} \times \mathbb{R}$ by Lemma 11.4.

So $\Phi \circ P(1_{K\times[0,1]}) = 0$ ρm-almost everywhere. Hence

$$\int \Phi \circ P(1_{K\times[0,1]})\rho\, dm = 0.$$

However, by Equations (11) and (9),

$$\int \Phi \circ P(1_{K\times[0,1]})\rho\, dm = \int P(1_{K\times[0,1]})\, dm = \int_{M^{(2)}\times\mathbb{R}} 1_{K\times[0,1]}\, d\tilde{m} > 0.$$

So we have a contradiction. □

11.4. **Proof of Proposition 11.5.** We start with a technical lemma similar to [BF17, Lem. 2.6]. The statement of the lemma is somewhat opaque, but can be interpreted as a boundary version of the assertion that the flow $\psi^t \colon M^{(2)} \times \mathbb{R} \to M^{(2)} \times \mathbb{R}$ has "weak stable manifolds" of the form $M \times \{y\} \times \mathbb{R}$. In the case when the GPS system is continuous, this assertion about "weak stable manifolds" can be made precise, see [BCZZ24, §3].

Recall d is a compatible metric on $\Gamma \sqcup M$.

Lemma 11.7. *For any $\epsilon, r > 0$ and $b \in \mathbb{R}$ there exists a finite subset $F \subset \Gamma$ such that: if $x_1, x_2, y \in M$, $\gamma \in \Gamma \smallsetminus F$, $\sigma(\gamma, y) \leq b$ and*

$$\min\{\mathrm{d}(x_1, y), \mathrm{d}(x_2, y)\} \geq r,$$

then

$$\mathrm{d}(\gamma x_1, \gamma x_2) < \epsilon.$$

Proof. Suppose not. Then there exists a sequence $\{\gamma_n\} \subset \Gamma$ of distinct elements such that for every $n \geq 1$ there are $x_{1,n}, x_{2,n}, y_n \in M$ where

$$\min\{\mathrm{d}(x_{1,n}, y_n), \mathrm{d}(x_{2,n}, y_n)\} \geq r, \quad \sigma(\gamma_n, y_n) \leq b, \quad \text{and} \quad \mathrm{d}(\gamma_n x_{1,n}, \gamma_n x_{2,n}) \geq \epsilon.$$

Passing to a subsequence we can suppose that $\gamma_n^{\pm 1} \to a^{\pm} \in M$. Since $\{\gamma_n\}$ are distinct, the properness property of expanding coarse-cocycles implies that $\|\gamma_n\|_\sigma \to +\infty$. Then since

$$\sigma(\gamma_n, y_n) \leq b$$

and σ is expanding, we must have $y_n \to a^-$. Then since

$$\min\{\mathrm{d}(x_{1,n}, y_n), \mathrm{d}(x_{2,n}, y_n)\} \geq r,$$

we have $\gamma_n x_{1,n} \to a^+$ and $\gamma_n x_{2,n} \to a^+$. So

$$\lim_{n\to\infty} \mathrm{d}(\gamma_n x_{1,n}, \gamma_n x_{2,n}) = 0$$

and we have a contradiction. □

We now begin our investigation of functions of the form $P(f \otimes 1_{[a,b]})$.

Lemma 11.8. *Suppose $f \in C_c(M^{(2)})$, $a < b$, $g := \tilde{P}(f \otimes 1_{[a,b]})$, and $h := \tilde{P}(1_{\mathrm{supp}(f)} \otimes 1_{[a,b]})$. If $v, w \in M^{(2)} \times \mathbb{R}$ satisfy $v^+ = w^+ \in M'$ and $\epsilon > 0$, then there exists $C = C(g, v, w, \epsilon) > 0$ such that*

$$\left| \int_0^T g(\psi^t(v))dt - \int_0^T g(\psi^t(w))dt \right| \leq C + \epsilon \left(\int_0^T h(\psi^t(v))dt + \int_0^T h(\psi^t(w))dt \right)$$

for all $T \geq 0$.

Proof. Let $v = (x_1, y, s)$ and $w = (x_2, y, s')$. By Proposition 10.2, there is $N \in \mathbb{N}$ such that for any u with $u^+ \in M'$, at most N elements $\gamma \in \Gamma$ send u in $\mathrm{supp}(f) \times [a, b]$, which implies $|g(u)| \leq N \|f\|_\infty$. Then, since

$$\left| \int_0^T g(\psi^t(x_2, y, s'))dt - \int_0^T g(\psi^t(x_2, y, s))dt \right| \leq 2N \|f\|_\infty |s - s'|,$$

we can assume that $s = s'$.

Let λ be the Lebesgue measure on $\mathbb{R}$ and let

$$L_\gamma(T) := \lambda\Big([0, T] \cap [a - s - \sigma_{\mathrm{PS}}(\gamma, y), b - s - \sigma_{\mathrm{PS}}(\gamma, y)]\Big).$$

Then

$$\int_0^T g(\psi^t(v))dt - \int_0^T g(\psi^t(w))dt = \sum_{\gamma \in \Gamma} (f(\gamma x_1, \gamma y) - f(\gamma x_2, \gamma y)) L_\gamma(T).$$

So by the uniform continuity of f and Lemma 11.7, there exists a finite set $F \subset \Gamma$ such that: if $\gamma \in \Gamma \smallsetminus F$ and $L_\gamma(T) \neq 0$ (hence $\sigma_{\mathrm{PS}}(\gamma, y) \leq b - s$), then

$$|f(\gamma x_1, \gamma y) - f(\gamma x_2, \gamma y)| \leq \epsilon.$$

Then, writing $S := \mathrm{supp}(f)$,

$$\begin{aligned} &\left| \int_0^T g(\psi^t(v))dt - \int_0^T g(\psi^t(w))dt \right| \\ &\quad \leq \sum_{\gamma \in F} 2 \|f\|_\infty (b - a) + \epsilon \sum_{\gamma \in \Gamma \smallsetminus F} \big(1_S(\gamma x_1, \gamma y) + 1_S(\gamma x_2, \gamma y)\big) L_\gamma(T) \\ &\quad \leq \sum_{\gamma \in F} 2 \|f\|_\infty (b - a) + \epsilon \left(\int_0^T h(\psi^t(v))dt + \int_0^T h(\psi^t(w))dt \right). \end{aligned}$$

□

Recall that Lemma 11.4 says that $\lim_{T \to \infty} \int_0^T \tilde{\rho}(\psi^t(v))dt = +\infty$ for any v with $v^+ \in \Lambda^{\mathrm{con}}_{\epsilon_0}(\Gamma) \cap M'$. The next lemma shows that, on a full measure set, the convergence to infinity is asymptotically identical for flow lines with the same forward endpoint.

Lemma 11.9. *There is a full $\tilde{m}$-measure set $Y_\rho \subset M^{(2)} \times \mathbb{R}$ such that: If $v, w \in Y_\rho$ satisfy $v^+ = w^+ \in \Lambda^{\mathrm{con}}_{\epsilon_0}(\Gamma) \cap M'$, then*

$$\lim_{T \to \infty} \frac{\int_0^T \tilde{\rho}(\psi^t(v))dt}{\int_0^T \tilde{\rho}(\psi^t(w))dt} = 1.$$

Proof. Recall that $\tilde{\rho} = \tilde{P}(\rho_0 \otimes 1_{[-R,R]})$ where $\rho_0 \in C_c(M^{(2)})$. Let $h := P(1_{\mathrm{supp}(\rho_0)} \otimes 1_{[-R,R]})$. By the Hopf ratio ergodic theorem, there is a full measure subset Y_ρ where $\tilde{\Phi}(h)$ exists.

Fix $v, w \in Y_\rho$ with $v^+ = w^+ \in \Lambda^{\mathrm{con}}_{\epsilon_0}(\Gamma) \cap M'$ and let

$$r_T := \frac{\int_0^T \tilde{\rho}(\psi^t(v))dt}{\int_0^T \tilde{\rho}(\psi^t(w))dt}$$

By Lemma 11.4, there exists $T_0 > 0$ such that $r_T \in (0, \infty)$ for all $T \geq T_0$.

Suppose for a contradiction that

$$\lim_{T\to\infty} r_T \neq 1.$$

Then after possibly relabeling v, w there exists $T_n \to \infty$ such that

$$r_\infty := \lim_{n\to\infty} r_{T_n} \in (1, +\infty].$$

By Lemma 11.8, for any $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$\begin{aligned} r_T &= 1 - \frac{\int_0^T \tilde\rho(\psi^t(v))dt - \int_0^T \tilde\rho(\psi^t(w))dt}{\int_0^T \tilde\rho(\psi^t(w))dt} \\ &\leq 1 + \frac{C_\epsilon}{\int_0^T \tilde\rho(\psi^t(w))dt} + \epsilon\left(r_T \frac{\int_0^T h(\psi^t(v))dt}{\int_0^T \tilde\rho(\psi^t(v))dt} + \frac{\int_0^T h(\psi^t(w))dt}{\int_0^T \tilde\rho(\psi^t(w))dt}\right) \end{aligned}$$

for all $T \geq T_0$. Hence,

$$\left(1 - \epsilon\frac{\int_0^T h(\psi^t(v))dt}{\int_0^T \tilde\rho(\psi^t(v))dt}\right) r_T \leq 1 + \frac{C_\epsilon}{\int_0^T \tilde\rho(\psi^t(w))dt} + \epsilon\frac{\int_0^T h(\psi^t(w))dt}{\int_0^T \tilde\rho(\psi^t(w))dt}$$

for all $T \geq T_0$. Lemma 11.4 implies that

$$\lim_{T\to\infty} \int_0^T \tilde\rho(\psi^t(w))dt = +\infty.$$

So for any $\epsilon > 0$ we have

$$\left(1 - \epsilon\tilde\Phi(h)(v)\right) r_\infty \leq 1 + \epsilon\tilde\Phi(h)(w)$$

Since $\epsilon > 0$ is arbitrary, we have $r_\infty \leq 1$ which is a contradiction. □

We are now ready to finish the proof of Proposition 11.5. Fix $f \in C_c(M^{(2)})$ and $a < b$. Let $g := \tilde P(f \otimes 1_{[a,b]})$ and $h := \tilde P(1_{\operatorname{supp}(f)} \otimes 1_{[a,b]})$. By the Hopf ratio ergodic theorem, there is a full $\tilde m$-measure set Y where $\tilde\Phi(g)$ and $\tilde\Phi(h)$ both exist.

We claim that

$$\tilde\Phi(g)(v) = \tilde\Phi(g)(w) \tag{12}$$

when $v, w \in Y$ satisfy $v^+ = w^+ \in \Lambda^{\rm con}_{\epsilon_0}(\Gamma) \cap M'$. Indeed, for such vectors v, w, Lemma 11.4 implies that

$$\lim_{T\to\infty} \int_0^T \tilde\rho(\psi^t(v))dt = +\infty.$$

So by Lemmas 11.8 and 11.9, for any $\epsilon > 0$ we have

$$\left|\tilde\Phi(g)(v) - \tilde\Phi(g)(w)\right| \leq \epsilon\left(\tilde\Phi(h)(v) + \tilde\Phi(h)(w)\right).$$

So $\tilde\Phi(g)(v) = \tilde\Phi(g)(w)$.

Now suppose for a contradiction that $\tilde\Phi(g)$ is not constant $\tilde m$-almost everywhere. Then there exists a measurable set $A \subset \mathbb{R}$ such that the sets $\{v : \tilde\Phi(g)(v) \in A\}$ and $\{v : \tilde\Phi(g)(v) \in A^c\}$ both have positive $\tilde m$-measure. As before, let λ denote the Lebesgue measure on $\mathbb{R}$. Then let

$$A' := \{y \in M : \tilde\Phi(g)(x, y, t) \in A \text{ for } \mu \otimes \lambda\text{-a.e. } (x, t)\}.$$

Since $\tilde\Phi(g)$ is a Γ-invariant function, A' is a Γ-invariant set. Further Equation (12) implies that $\mu(A') > 0$. Since Γ acts ergodically on (M, μ), see Theorem 8.1, we then have $\mu(A')=1$. But this contradicts the assumption that $\{v : \tilde{\Phi}(g)(v) \in A^c\}$ has positive m-measure.

Thus $\tilde{\Phi}(g)$ is constant $\tilde{m}$-almost everywhere, which implies that $\Phi \circ P(f \otimes 1_{[a,b]})$ is constant m-almost everywhere. ☐

11.5. **Proof of Theorem 11.2: The continuous case.** In this section, we observe that the arguments we have just given immediately establish Theorem 11.2. We will freely use the objects introduced in Section 10.4.

First notice that the flow $\psi^t \colon (U_\Gamma, m_\Gamma) \to (U_\Gamma, m_\Gamma)$ introduced in Section 10.4 coincides with the flow $\psi^t \colon (\Gamma \backslash M^{(2)} \times \mathbb{R}, m) \to (\Gamma \backslash M^{(2)} \times \mathbb{R}, m)$ considered in the proof of Theorem 11.1. So Lemma 11.4 implies immediately that ψ^t is conservative on (U_Γ, m_Γ).

If ψ_t is not ergodic on (U_Γ, m_Γ), then there exists a flow-invariant subset A of U_Γ so that $m_\Gamma(A) > 0$ and $m_\Gamma(A^c) > 0$. Then A lifts to a flow-invariant, Γ-invariant subset $\tilde{A}$ of $M^{(2)} \times \mathbb{R}$ of the form $B \times \mathbb{R}$. Then, $(\bar{\mu} \otimes \mu)(B) > 0$ and $(\bar{\mu} \otimes \mu)(B^c) > 0$, which contradicts the ergodicity of the action of Γ on $(M^{(2)}, \bar{\mu} \otimes \mu)$. Therefore, ψ^t is ergodic on (U_Γ, m_Γ).

12. Proof of dichotomy

In this section we complete the proof of Theorem 1.8. Suppose $(\sigma, \bar{\sigma}, G)$ is a coarse GPS system and $\delta_\sigma(\Gamma) < +\infty$. Let μ, $\bar{\mu}$ be coarse Patterson–Sullivan measures of dimension δ for σ, $\bar{\sigma}$ respectively. By Lemma 9.1, there exists a measurable nonnegative function $\tilde{G}$ on $M^{(2)}$ such that $\nu := e^{\delta \tilde{G}} \bar{\mu} \otimes \mu$ is Γ-invariant.

We already have most of the proof. There is one lemma left to prove:

Lemma 12.1. *If the action of* Γ *on* $(M^{(2)}, \nu)$ *is ergodic, then* ν *has no atoms, and hence the* Γ *action on* $(M^{(2)}, \nu)$ *is also conservative.*

Proof. We argue by contradiction: suppose the Γ action on $(M^{(2)}, \nu)$ is ergodic and (ξ, η) is an atom. Then $\mathcal{O} := \Gamma \cdot (\xi, \eta) \cap M^{(2)}$ must have full ν-measure by ergodicity. Now note that we can find $\gamma \in \Gamma$ such that $(\xi, \gamma\eta) \notin \mathcal{O}$, which contradicts the fact that $\mathcal{O}$ has full measure.

Conservativity of the Γ action then follows by [Aar97, Prop. 1.6.6] (see also [Bur]). ☐

12.1. **Divergent case.** First suppose $\sum_{\gamma \in \Gamma} e^{-\delta \|\gamma\|_\sigma} = +\infty$.

(a) By the definition of the critical exponent, $\delta \leq \delta_\sigma(\Gamma)$. By Proposition 6.3(3), $\delta \geq \delta_\sigma(\Gamma)$. Hence $\delta = \delta_\sigma(\Gamma)$.

(b) By Proposition 7.1, $\mu\left(\Lambda^{\mathrm{con}}(\Gamma)\right) = 1$.

(c) By Theorem 11.1, the action of Γ on $(M^{(2)}, \nu)$ is ergodic. Conservativity of the action can be seen from Corollary 9.3, or from Lemma 12.1.

12.2. **Convergent case.** Now suppose $\sum_{\gamma \in \Gamma} e^{-\delta \|\gamma\|_\sigma} < +\infty$.

(a) By the definition of the critical exponent, $\delta \geq \delta_\sigma(\Gamma)$.

(b) By Proposition 6.3(2), $\mu\left(\Lambda^{\mathrm{con}}(\Gamma)\right) = 0$.

(c) The Γ action on $(M^{(2)}, \bar{\mu} \otimes \mu)$ is dissipative by Corollary 9.3. Non-ergodicity of the action then follows from Lemma 12.1.

Part 3. Applications, Examples, and other Remarks

13. Periods versus magnitudes

In this section we observe that two expanding coarse-cocycles have coarsely the same magnitudes if and only if they have coarsely the same periods. The key observation, Lemma 13.3, that relates these two quantities is a convergence group action version of the fact (see [AMS95]) that elements in a strongly irreducible linear group are uniformly close to compact sets of proximal elements.

Proposition 13.1. *If $\Gamma \subset \mathsf{Homeo}(M)$ is a convergence group and $\sigma_1, \sigma_2 \colon \Gamma \times M \to \mathbb{R}$ are two expanding κ-coarse-cocycles, then the following are equivalent:*

(1) $\displaystyle\sup_{\gamma \in \Gamma,\ x \in M} |\sigma_1(\gamma, x) - \sigma_2(\gamma, x)| < +\infty$,

(2) $\displaystyle\sup_{\gamma \in \Gamma} \big| \|\gamma\|_{\sigma_1} - \|\gamma\|_{\sigma_2} \big| < +\infty$,

(3) $\displaystyle\sup_{\substack{\gamma \in \Gamma \\ \gamma \text{ loxodromic}}} |\sigma_1(\gamma, \gamma^+) - \sigma_2(\gamma, \gamma^+)| \leq 2\kappa$,

(4) $\displaystyle\sup_{\substack{\gamma \in \Gamma \\ \gamma \text{ loxodromic}}} |\sigma_1(\gamma, \gamma^+) - \sigma_2(\gamma, \gamma^+)| < +\infty$.

Notice that the implication (1) $\Rightarrow$ (2) is clear, the implication (2) $\Rightarrow$ (1) follows from Proposition 3.2(4), the implication (2) $\Rightarrow$ (3) follows from Proposition 3.2(1) and the implication (3) $\Rightarrow$ (4) is clear. To show that (4) $\Rightarrow$ (2), we will use the following two lemmas to relate a general element to a loxodromic one.

Lemma 13.2 ([Tuk94, Lem. 2C])**.** *Suppose $\Gamma \subset \mathsf{Homeo}(M)$ is a convergence group. If $\{\gamma_n\}$ is a sequence of distinct elements, $\gamma_n \to a$, $\gamma_n^{-1} \to b$, and $a \neq b$, then for n sufficiently large γ_n is loxodromic and $\gamma_n^+ \to a$, $\gamma_n^- \to b$.*

Lemma 13.3. *Suppose $\Gamma \subset \mathsf{Homeo}(M)$ is a non-elementary convergence group and d is a compatible metric on $\Gamma \sqcup M$. Then there exist $\epsilon > 0$ and a finite set $F \subset \Gamma$ with the following property: for any $\gamma \in \Gamma$ there is some $f \in F$ where γf is loxodromic and*

$$\min\left\{\mathrm{d}((\gamma f)^+, (\gamma f)^-),\, \mathrm{d}(\gamma f, (\gamma f)^-),\, \mathrm{d}((\gamma f)^+, (\gamma f)^{-1})\right\} > \epsilon.$$

Proof. Fix four distinct points $x_1, x_2, x_3, x_4 \in M$ in the limit set of Γ. Let

$$\epsilon := \frac{1}{4} \min_{1 \leq i < j \leq 4} \mathrm{d}(x_i, x_j).$$

Since Γ acts minimally on its limit set, for every $i \neq j \in \{1, 2, 3, 4\}$ we can find an element $g_{i,j} \in \Gamma$ such that

$$g_{i,j}\Big(M \setminus B_\epsilon(x_j)\Big) \subset B_\epsilon(x_i) \quad \text{and} \quad g_{i,j}^{-1}\Big(M \setminus B_\epsilon(x_i)\Big) \subset B_\epsilon(x_j).$$

We claim that there exists a finite set $F_0 \subset \Gamma$ such that: if $\gamma \in \Gamma \setminus F_0$, then there exist $i \neq j \in \{1, 2, 3, 4\}$ such that $\gamma g_{i,j}$ is loxodromic and

$$\min\left\{\mathrm{d}((\gamma g_{i,j})^+, (\gamma g_{i,j})^-),\, \mathrm{d}(\gamma g_{i,j}, (\gamma g_{i,j})^-),\, \mathrm{d}((\gamma g_{i,j})^+, (\gamma g_{i,j})^{-1})\right\} > \epsilon.$$

Suppose not. Then there exists an escaping sequence $\{\gamma_n\}$ in Γ where each γ_n does not have this property. Passing to a subsequence we can suppose that there is $a, b \in M$ such that $\gamma_n \to a$ and $\gamma_n^{-1} \to b$.

Since the balls $\{B_{2\epsilon}(x_i)\}_{1\leq i\leq 4}$ are pairwise disjoint we can pick $i \neq j \in \{1,2,3,4\}$ such that $a, b \notin B_{2\epsilon}(x_i) \cup B_{2\epsilon}(x_j)$. Then $\gamma_n g_{i,j} \to a$ and $(\gamma_n g_{i,j})^{-1} \to g_{i,j}^{-1} b \in B_\epsilon(x_j)$. Then, by our choice of i and j,

$$\mathrm{d}(a, g_{i,j}^{-1}(b)) > \epsilon.$$

Thus Lemma 13.2 implies that $\gamma_n g_{i,j}$ is loxodromic for n sufficiently large. Further, $(\gamma_n g_{i,j})^+ \to a$ and $(\gamma_n g_{i,j})^- \to g_{i,j}^{-1}(b)$. So for n sufficiently large we have

$$\min\left\{\mathrm{d}((\gamma_n g_{i,j})^+, (\gamma_n g_{i,j})^-), \mathrm{d}(\gamma_n g_{i,j}, (\gamma g_{i,j})^-), \mathrm{d}((\gamma_n g_{i,j})^+, (\gamma_n g_{i,j})^{-1})\right\} > \epsilon.$$

Thus we have a contradiction. Thus there exists a finite set $F_0 \subset \Gamma$ with the desired property.

Now fix a loxodromic element h with

$$\min\left\{\mathrm{d}(h^+, h^-), \mathrm{d}(h, h^-), \mathrm{d}(h^+, h^{-1})\right\} > \epsilon.$$

Then the set

$$F := \left\{g_{i,j} : i \neq j \in \{1,2,3,4\}\right\} \cup \{f^{-1}h : f \in F_0\}$$

satisfies the lemma. □

Proof of Proposition 13.1. As discussed above, it only remains to show that (4) ⇒ (2). Suppose that

$$C_1 := \sup_{\substack{\gamma\in\Gamma \\ \gamma \text{ loxodromic}}} \left|\sigma_1(\gamma, \gamma^+) - \sigma_2(\gamma, \gamma^+)\right| < +\infty.$$

Fix a compatible metric d on $\Gamma \sqcup M$. Then fix a finite set $F \subset \Gamma$ and $\epsilon > 0$ as in Lemma 13.3. By the expanding property, there exists $C_2 > 0$ such that

$$\|x\|_{\sigma_j} - C_2 \leq \sigma_j(\gamma, x) \leq \|x\|_{\sigma_j}$$

for $j \in \{1,2\}$, $\gamma \in \Gamma$, and $x \in M$ with $\mathrm{d}(x, \gamma^{-1}) \geq \epsilon$. Let

$$C_3 := \max_{f\in F}\left(\|f\|_{\sigma_1} + \left\|f^{-1}\right\|_{\sigma_1} + \|f\|_{\sigma_2} + \left\|f^{-1}\right\|_{\sigma_2}\right).$$

Given $\gamma \in \Gamma$, choose $f \in F$ as in Lemma 13.3. Then Observation 3.1 implies that

$$\begin{aligned}\left|\|\gamma\|_{\sigma_1} - \|\gamma\|_{\sigma_2}\right| &\leq C_3 + 2\kappa + \left|\|\gamma f\|_{\sigma_1} - \|\gamma f\|_{\sigma_2}\right| \\ &\leq C_3 + 2\kappa + 2C_2 + \left|\sigma_1(\gamma f, (\gamma f)^+) - \sigma_2(\gamma f, (\gamma f)^+)\right| \\ &\leq C_3 + 2\kappa + 2C_2 + C_1.\end{aligned}$$

□

14. Rigidity of Patterson–Sullivan measures

In this section, we prove that in the divergent case Patterson–Sullivan measures are either absolutely continuous or mutually singular. Furthermore, we characterize the absolutely continuous case in terms of rough similarity between magnitudes.

For the rest of the section, suppose $\Gamma \subset \mathsf{Homeo}(M)$ is a convergence group and $\sigma_1, \sigma_2 \colon \Gamma \times M \to \mathbb{R}$ are two expanding coarse-cocycles with finite critical exponents $\delta_1 := \delta_{\sigma_1}(\Gamma)$, $\delta_2 := \delta_{\sigma_2}(\Gamma)$. For $i = 1, 2$, let μ_i be a coarse σ_i-Patterson–Sullivan measure of dimension δ_i.

Proposition 14.1. *If* $\sum_{\gamma\in\Gamma} e^{-\delta_1\|\gamma\|_{\sigma_1}} = +\infty$, *then either:*

1. $\mu_1 \ll \mu_2$ *and* $\mu_2 \ll \mu_1$, *or*
2. $\mu_1 \perp \mu_2$.

Proof. By the Lebesgue decomposition theorem, we can write

$$d\mu_1 = d\lambda + f d\mu_2$$

where $\lambda \perp \mu_2$ and f is a non-negative μ_2-measurable function.

Since $\lambda \perp \mu_2$, we can fix a decomposition $M = A \cup B$ where A has full λ-measure, B has full μ_2-measure, and $\lambda(B) = \mu_2(A) = 0$. Since μ_2 is Γ-quasi-invariant, by replacing A by $\bigcup_{\gamma \in \Gamma} \gamma A$, we can assume that A is Γ-invariant. Then by ergodicity, see Theorem 8.1, A either has zero or full μ_1-measure. If A has zero μ_1-measure, then $\mu_1 = f\mu_2$ and $\mu_1 \ll \mu_2$. If A has full μ_1-measure, then $\mu_2 = \lambda$ and $\mu_1 \perp \mu_2$.

It remains to show that $\mu_1 \ll \mu_2$ implies $\mu_2 \ll \mu_1$. Since $\sum_{\gamma \in \Gamma} e^{-\delta_1 \|\gamma\|_{\sigma_1}} = +\infty$, Theorem 1.8 implies that $\mu_1(\Lambda^{\rm con}(\Gamma)) = 1$. Then since $\mu_1 \ll \mu_2$, this implies that $\mu_2(\Lambda^{\rm con}(\Gamma)) > 0$. So by Theorem 1.8 we must have $\sum_{\gamma \in \Gamma} e^{-\delta_2 \|\gamma\|_{\sigma_2}} = +\infty$. Then we can repeat the argument in the first two paragraphs to see that $\mu_2 \ll \mu_1$. $\square$

We can also characterize when the measures are absolutely continuous.

Proposition 14.2. *If $\sum_{\gamma \in \Gamma} e^{-\delta_1 \|\gamma\|_{\sigma_1}} = +\infty$, then the following are equivalent:*

(1) $\mu_1 \ll \mu_2$.
(2) $\mu_1 \ll \mu_2$ *and* $\mu_2 \ll \mu_1$.
(3) $\sup_{\gamma \in \Gamma} \left| \delta_1 \|\gamma\|_{\sigma_1} - \delta_2 \|\gamma\|_{\sigma_2} \right| < \infty$.
(4) There exist $C > 0$ *such that* $C^{-1}\mu_1 \le \mu_2 \le C\mu_1$.

Proposition 14.1 implies that (1) ⇔ (2). By definition (4) ⇒ (2). By the Shadow Lemma (Theorem 6.1), (4) ⇒ (3).

The implication (3) ⇒ (4) is a consequence of Proposition 13.1 and Theorem 8.1. Indeed, if (3) holds, then Proposition 13.1 implies that

$$\sup_{\gamma \in \Gamma,\, x \in M} |\delta_1 \sigma_1(\gamma, x) - \delta_2 \sigma_2(\gamma, x)| < \infty.$$

This in turn implies that μ_2 is a coarse σ_1-Patterson–Sullivan measure of dimension δ_1. Then by the coarse uniqueness statement in Theorem 8.1 we see that (4) is true.

We will complete the proof by showing that (1) ⇒ (4).

14.1. **Proof of** (1) ⇒ (4)**.** By Theorem 8.1 there exists $\epsilon_0 > 0$ such that $\mu_2(\Lambda^{\rm con}_{2\epsilon_0}(\Gamma)) = 1$. By shrinking $\epsilon_0 > 0$ we may also assume that it satisfies the hypothesis of the Shadow Lemma (Theorem 6.1) for μ_1 and μ_2.

Let $f := \frac{d\mu_1}{d\mu_2}$. Notice that for every $\gamma \in \Gamma$,

$$f \circ \gamma = \frac{d\gamma_*^{-1}\mu_1}{d\gamma_*^{-1}\mu_2} = \frac{d\gamma_*^{-1}\mu_1}{d\mu_1} \frac{d\mu_2}{d\gamma_*^{-1}\mu_2} f = e^{\delta_2\sigma_2(\gamma,\cdot) - \delta_1\sigma_1(\gamma,\cdot)} f \tag{13}$$

μ_2-almost everywhere. Since μ_2 and $\mu_1 = f\mu_2$ are probability measures, there exists $D_0 > 1$ such that the set

$$E := \{D_0^{-1} \le f \le D_0\}$$

has positive μ_2-measure.

Using Lemmas 8.2 and 8.3 we can fix $x_0 \in E \cap \Lambda^{\rm con}_{2\epsilon_0}(\Gamma)$ such that: if $0 < \epsilon \le \epsilon_0$ and $\{\gamma_n\} \subset \Gamma$ is an escaping sequence with $x_0 \in \bigcap_{n \ge 1} \mathcal{S}_\epsilon(\gamma_n)$, then

$$1 = \lim_{n \to \infty} \mu_2(\gamma_n^{-1} E) \tag{14}$$

and

$$f(x_0) = \lim_{n\to\infty} \frac{1}{\mu_2(\mathcal{S}_\epsilon(\gamma_n))} \int_{\mathcal{S}_\epsilon(\gamma_n)} f(y) d\mu_2(y) = \lim_{n\to\infty} \frac{\mu_1(\mathcal{S}_\epsilon(\gamma_n))}{\mu_2(\mathcal{S}_\epsilon(\gamma_n))}. \tag{15}$$

We will construct two sequences of group elements to use in the above limits. Since $x_0 \in \Lambda^{\rm con}_{2\epsilon_0}(\Gamma)$, Lemma 5.4 implies that there exists an escaping sequence $\{\gamma_{1,n}\}$ such that

$$x_0 \in \bigcap_{n\geq 1} \mathcal{S}_{\epsilon_0}(\gamma_{1,n}).$$

Passing to a subsequence we can suppose that $\gamma_{1,n}^{-1}(x_0) \to a_1 \in M$ and $\gamma_{1,n}^{-1} \to b_1 \in M$. Then fix $\alpha \in \Gamma$ such that $\alpha^{-1} b_1 \neq b_1$. Then let $\gamma_{2,n} := \gamma_{1,n}\alpha$ and

$$b_2 := \alpha^{-1} b_1 = \lim_{n\to\infty} \gamma_{2,n}^{-1}.$$

Lemma 14.3. *After passing to a subsequence, there exists $\epsilon > 0$ such that*

$$x_0 \in \bigcap_{n\geq 1} \mathcal{S}_\epsilon(\gamma_{2,n}).$$

Proof. Notice that $\gamma_{1,n}^{-1}(x_0) \in \gamma_{1,n}^{-1}\mathcal{S}_\epsilon(\gamma_{1,n}) = M - B_\epsilon(\gamma_{1,n}^{-1})$. So $a_1 \neq b_1$. Hence $\epsilon := 2\,\mathrm{d}(\alpha^{-1}a_1, \alpha^{-1}b_1)$ is positive. Since $(\gamma_{1,n}\alpha)^{-1}x_0 \to \alpha^{-1}a_1$ and $(\gamma_{1,n}\alpha)^{-1} \to \alpha^{-1}b_1$, after passing to a subsequence we can suppose that $\mathrm{d}((\gamma_{1,n}\alpha)^{-1}x_0, (\gamma_{1,n}\alpha)^{-1}) > \epsilon$ for all n. Then $x_0 \in \mathcal{S}_\epsilon(\gamma_{1,n}\alpha) = \mathcal{S}_\epsilon(\gamma_{2,n})$ for all n. □

Shrinking $\epsilon > 0$ we can assume

$$\big(M - B_{2\epsilon}(b_1)\big) \cup \big(M - B_{2\epsilon}(b_2)\big) = M$$

and that $\epsilon \leq \epsilon_0$. Since $\gamma_{i,n}^{-1} \to b_i$, passing to a further subsequence we can also suppose that

$$M - B_{2\epsilon}(b_i) \subset M - B_\epsilon(\gamma_{i,n}^{-1})$$

for all $n \geq 1$.

Lemma 14.4. *There exists $D_1 > 1$ such that*

$$D_1^{-1} \leq f(x) \leq D_1$$

for μ_2-almost every point $x \in M$.

Proof. Since $\epsilon \leq \epsilon_0$, Equation (15) implies

$$f(x_0) = \lim_{n\to\infty} \frac{\mu_1(\mathcal{S}_\epsilon(\gamma_{i,n}))}{\mu_2(\mathcal{S}_\epsilon(\gamma_{i,n}))}.$$

Since $x_0 \in E$, we have $f(x_0) \in [D_0^{-1}, D_0]$. So by the Shadow Lemma (Theorem 6.1), there exists $C_1 > 0$ such that

$$\left| \delta_1 \left\| \gamma_{i,n} \right\|_{\sigma_1} - \delta_2 \left\| \gamma_{i,n} \right\|_{\sigma_2} \right| \leq C_1$$

for every $n \geq 1$. Then, since the cocycles are expanding, this implies that there exists $C_2 > 0$ such that

$$|\delta_1 \sigma_1(\gamma_{i,n}, x) - \delta_2 \sigma_2(\gamma_{i,n}, x)| \leq C_2$$

when $x \in M - B_{2\epsilon}(b_i) \subset M - B_\epsilon(\gamma_{i,n}^{-1})$. So Equation (13) implies that

$$D_0^{-1} e^{-C_2} \leq f(x) \leq D_0 e^{C_2}$$

μ_2-almost everywhere on $\gamma_n^{-1}E \cap (M - B_{2\epsilon}(b_i))$. Then Equation (14) implies that

$$D_0^{-1}e^{-C_2} \leq f(x) \leq D_0 e^{C_2}$$

μ_2-almost everywhere on $M - B_{2\epsilon}(b_i)$. Since $\big(M - B_{2\epsilon}(b_1)\big) \cup \big(M - B_{2\epsilon}(b_2)\big) = M$, this completes the proof. □

15. Convexity of critical exponent

In this section we prove Theorem 1.6, which we restate below. For the rest of the section fix a convergence group $\Gamma \subset \mathsf{Homeo}(M)$ and two expanding coarse-cocycles $\sigma_0, \sigma_1 \colon \Gamma \times M \to \mathbb{R}$ such that $\delta_{\sigma_0}(\Gamma) = 1 = \delta_{\sigma_1}(\Gamma)$. For $0 < \lambda < 1$, notice that

$$\sigma_\lambda = (1-\lambda)\sigma_0 + \lambda\sigma_1$$

is also a coarse-cocycle.

Theorem 15.1. *If $0 < \lambda < 1$, then*

$$\delta_{\sigma_\lambda}(\Gamma) \leq 1.$$

Moreover, if $\sum_{\gamma\in\Gamma} e^{-\delta_{\sigma_\lambda}(\Gamma)\|\gamma\|_{\sigma_\lambda}} = +\infty$, then the following are equivalent:

(1) $\delta_{\sigma_\lambda}(\Gamma) = 1$.
(2) $\sup_{\gamma\in\Gamma} \big| \|\gamma\|_{\sigma_0} - \|\gamma\|_{\sigma_1} \big| < +\infty$.

We start that by observing that the magnitudes of group elements behave nicely under convex combinations of cocycles.

Lemma 15.2. *σ_λ is expanding and there exists $D > 0$ such that:*

$$(1-\lambda)\|\gamma\|_{\sigma_0} + \lambda\|\gamma\|_{\sigma_1} - D \leq \|\gamma\|_{\sigma_\lambda} \leq (1-\lambda)\|\gamma\|_{\sigma_0} + \lambda\|\gamma\|_{\sigma_1}$$

for all $\gamma \in \Gamma$.

Proof. The upper bound on $\|\gamma\|_{\sigma_\lambda}$ is by definition. Then the expanding properties for σ_0 and σ_1 imply the lower bound on $\|\gamma\|_{\sigma_\lambda}$ and the fact that σ_λ is expanding. □

Lemma 15.3. $\delta_{\sigma_\lambda}(\Gamma) \leq 1$.

Proof. By Hölder's inequality and the previous lemma,

$$\sum_{\gamma\in\Gamma} e^{-s\|\gamma\|_{\sigma_\lambda}} \leq e^{sD}\left(\sum_{\gamma\in\Gamma} e^{-s\|\gamma\|_{\sigma_0}}\right)^{\lambda}\left(\sum_{\gamma\in\Gamma} e^{-s\|\gamma\|_{\sigma_1}}\right)^{1-\lambda}.$$

Hence $\delta_{\sigma_\lambda}(\Gamma) \leq 1$. □

We now consider the "moreover" part of the theorem. So fix $\lambda \in (0,1)$ where

$$\sum_{\gamma\in\Gamma} e^{-\|\gamma\|_{\sigma_\lambda}} = \sum_{\gamma\in\Gamma} e^{-\delta_{\sigma_\lambda}(\Gamma)\|\gamma\|_{\sigma_\lambda}} = +\infty.$$

It is clear that (2) ⇒ (1). The proof that (1) ⇒ (2) is much more complicated and will occupy the rest of the section.

To that end, suppose that $\delta_{\sigma_\lambda}(\Gamma) = 1$. For $t \in \{0, \lambda, 1\}$, let μ_t be a coarse σ_t-Patterson-Sullivan measure of dimension 1 (which exists by Theorem 4.1).

The key step in the proof is to show that μ_λ is absolutely continuous to $\mu_0 + \mu_1$.

Proposition 15.4. $\mu_\lambda \ll \mu_0 + \mu_1$.

Assuming Proposition 4.1 for a moment we finish the proof that (1) $\Rightarrow$ (2). Since $\mu_\lambda \ll \mu_0 + \mu_1$, at least one of μ_0 or μ_1 is not singular to μ_λ. So by relabelling we can assume that μ_λ is not singular to μ_0. Then Proposition 14.2 implies that

$$\sup_{\gamma\in\Gamma} \left| \|\gamma\|_{\sigma_0} - \|\gamma\|_{\sigma_\lambda} \right| < +\infty.$$

Then Lemma 15.2 implies that

$$\sup_{\gamma\in\Gamma} \left| \|\gamma\|_{\sigma_0} - \|\gamma\|_{\sigma_1} \right| < +\infty.$$

15.1. **Proof of Proposition 15.4.** The idea is to use the Shadow Lemma to relate the measures.

Fix a compatible metric on $\Gamma \sqcup M$ and let $\mathcal{S}_\epsilon(\gamma)$ denote the associated shadows. By the Shadow Lemma (Theorem 6.1), there exists $\epsilon_0 > 0$ such that for every $0 < \epsilon \leq \epsilon_0$ there is constant $C_0(\epsilon) > 1$ where

$$\frac{1}{C_0(\epsilon)} e^{-\|\gamma\|_{\sigma_t}} \leq \mu_t\Big(\mathcal{S}_\epsilon(\gamma)\Big) \leq C_0(\epsilon) e^{-\|\gamma\|_{\sigma_t}}$$

for all $\gamma \in \Gamma$ and all $t \in \{0, \lambda, 1\}$.

We first establish bounds for the measure of shadows and then extend these bounds to arbitrary sets using the covering result in Proposition 5.1(5).

Lemma 15.5. *For any $0 < \epsilon \leq \epsilon_0$ there exists $C_1 = C_1(\epsilon) > 1$ such that: if $\gamma \in \Gamma$, then*

$$\mu_\lambda\left(\mathcal{S}_\epsilon(\gamma)\right) \leq C_1 (\mu_0 + \mu_1)\left(\mathcal{S}_\epsilon(\gamma)\right).$$

Proof. By the Shadow Lemma and Lemma 15.2,

$$\begin{aligned} \mu_\lambda\left(\mathcal{S}_\epsilon(\gamma)\right) &\leq C_0(\epsilon) e^{-\|\gamma_2\|_{\sigma_\lambda}} \leq C_0(\epsilon) e^D e^{-(1-\lambda)\|\gamma_2\|_{\sigma_0} - \lambda\|\gamma_2\|_{\sigma_1}} \\ &\leq C_0(\epsilon)^2 e^D \mu_0\left(\mathcal{S}_\epsilon(\gamma)\right)^{1-\lambda} \mu_1\left(\mathcal{S}_\epsilon(\gamma)\right)^{\lambda}. \end{aligned}$$

Then the desired estimate follows from the weighted arithmetic-geometric mean inequality. □

Lemma 15.6. *There exists $C_2 > 1$ such that: if $A \subset M$ is Borel measurable, then*

$$\mu_\lambda(A) \leq C_2(\mu_0 + \mu_1)(A).$$

Hence $\mu_\lambda \ll \mu_0 + \mu_1$.

Proof. Fix $\eta > 0$. By outer regularity we can find an open set $U \subset M$ such that $A \subset U$ and

$$(\mu_0 + \mu_1)(U) < \eta + (\mu_0 + \mu_1)(A).$$

Using Theorem 8.1 and possibly shrinking $\epsilon_0 > 0$, we may assume that

$$\mu_\lambda(\Lambda^{\rm con}_{\epsilon_0}(\Gamma)) = 1. \tag{16}$$

Now fix $0 < \epsilon < \epsilon_0$ and let

$$I := \{\gamma \in \Gamma : \mathcal{S}_\epsilon(\gamma) \subset U\}.$$

By Lemma 5.4, for each $x \in \Lambda^{\rm con}_{\epsilon_0}(\Gamma)$ there exists an escaping sequence $\{\gamma_n\}$ such that

$$x \in \bigcap_{n=1}^{\infty} \mathcal{S}_\epsilon(\gamma_n).$$

Moreover, for each such sequence, Proposition 5.1(2) implies that $\operatorname{diam} \mathcal{S}_\epsilon(\gamma_n) \to 0$ as $n \to \infty$. Hence

$$U \cap \Lambda^{\rm con}_{\epsilon_0}(\Gamma) \subset \bigcup_{\gamma \in I} \mathcal{S}_\epsilon(\gamma) \subset U.$$

So by Equation (16),

$$\mu_\lambda(U) = \mu_\lambda\left(\bigcup_{\gamma \in I} \mathcal{S}_\epsilon(\gamma) \right).$$

Let $J \subset I$ and $\epsilon' < \epsilon$ satisfy Proposition 5.1(5). Then repeatedly using the Shadow Lemma,

$$\begin{aligned}
\mu_\lambda(A) \leq \mu_\lambda(U) = \mu_\lambda\left(\bigcup_{\gamma \in I} \mathcal{S}_\epsilon(\gamma) \right) &\leq \sum_{\gamma \in J} \mu_\lambda\left(\mathcal{S}_{\epsilon'}(\gamma)\right) \\
&\leq C_0(\epsilon) C_0(\epsilon') \sum_{\gamma \in J} \mu_\lambda\left(\mathcal{S}_\epsilon(\gamma)\right) \leq C_0(\epsilon) C_0(\epsilon') C_1(\epsilon) \sum_{\gamma \in J} (\mu_0 + \mu_1)\left(\mathcal{S}_\epsilon(\gamma)\right) \\
&\leq C_0(\epsilon) C_0(\epsilon') C_1(\epsilon) (\mu_0 + \mu_1)(U) \leq C_0(\epsilon) C_0(\epsilon') C_1(\epsilon) \Big((\mu_0 + \mu_1)(A) + \eta \Big).
\end{aligned}$$

Since $\eta > 0$ was arbitrary, this completes the proof. □

16. Symmetric coarse-cocycles

In this section we consider the case when an expanding coarse-cocycle is "coarsely-symmetric." For the rest of the section, fix a convergence group $\Gamma \subset \mathsf{Homeo}(M)$. A coarse-cycle $\sigma : \Gamma \times M \to \mathbb{R}$ is *coarsely-symmetric* if

$$\sup_{\substack{\gamma \in \Gamma \\ \gamma \text{ loxodromic}}} \left| \sigma(\gamma, \gamma^+) - \sigma(\gamma^{-1}, \gamma^-) \right| < +\infty.$$

The next result shows that coarsely-symmetric can also be defined using magnitudes and that expanding coarsely-symmetric coarse-cocycles are always contained in a coarse GPS system.

Proposition 16.1. *Suppose $\sigma \colon \Gamma \times M \to \mathbb{R}$ is an expanding coarse-cocycle. Then σ is coarsely-symmetric if and only if*

$$\sup_{\gamma \in \Gamma} \left| \|\gamma\|_\sigma - \left\|\gamma^{-1}\right\|_\sigma \right| < +\infty.$$

Moreover, if σ is coarsely-symmetric, then (σ, σ, G) is a GPS system for Γ acting on $\Lambda(\Gamma) \subset M$, where $G \colon \Lambda(\Gamma)^{(2)} \to [0, \infty)$ is defined by

$$G(x,y) = \kappa + \limsup_{\alpha \to x, \beta \to y} \|\alpha\|_\sigma + \left\|\beta^{-1}\right\|_\sigma - \left\|\beta^{-1}\alpha\right\|_\sigma$$

(notice that $G \geq 0$ by Observation 3.1).

Remark 16.2. One could also define a Gromov product using a limit infimum instead of a limit supremum.

As a corollary to Proposition 16.1 and Theorem 11.1, we have the following.

Corollary 16.3. *Suppose $\sigma\colon \Gamma\times M \to \mathbb{R}$ is a coarsely-symmetric expanding coarse-cocycle with $\delta := \delta_\sigma(\Gamma) < +\infty$ and μ is a coarse σ-Patterson–Sullivan measure of dimension δ. If*

$$\sum_{\gamma\in\Gamma} e^{-\delta\|\gamma\|_\sigma} = +\infty,$$

then Γ acts ergodically on $(M^{(2)}, \mu\otimes\mu)$.

We prove the proposition via a series of lemma. Fix, for the rest of the section, an expanding κ-coarse-cocycle $\sigma\colon \Gamma\times M\to\mathbb{R}$ and a compatible distance d on $\Gamma\sqcup M$.

Lemma 16.4. *σ is coarsely-symmetric if and only if*

$$\sup_{\gamma\in\Gamma}\left|\|\gamma\|_\sigma - \left\|\gamma^{-1}\right\|_\sigma\right| < +\infty.$$

Proof. ($\Leftarrow$): This follows from Proposition 3.2(1).

($\Rightarrow$): Fix $\epsilon > 0$ and a finite set $F\subset\Gamma$ satisfying Lemma 13.3. By Observation 3.1,

$$C_1 := \sup_{\gamma\in\Gamma, f\in F} \left|\|\gamma\|_\sigma - \|\gamma f\|_\sigma\right| + \left|\|f\gamma\|_\sigma - \|\gamma\|_\sigma\right|$$

is finite. Further, by the expanding property, there exists $C_2 > 0$ such that: if $\gamma\in\Gamma$ and $\mathrm{d}(x,\gamma^{-1}) > \epsilon$, then

$$\left|\sigma(\gamma,x) - \|\gamma\|_\sigma\right| \le C_2.$$

Now fix $\gamma\in\Gamma$. Then there exists $f\in F$ such that γf is loxodromic and

$$\min\left\{\mathrm{d}((\gamma f)^+,(\gamma f)^-),\ \mathrm{d}(\gamma f,(\gamma f)^-),\ \mathrm{d}((\gamma f)^+,(\gamma f)^{-1})\right\} > \epsilon.$$

Then

$$\begin{aligned}\left|\|\gamma\|_\sigma - \left\|\gamma^{-1}\right\|_\sigma\right| &\le C_1 + \left|\|\gamma f\|_\sigma - \left\|(\gamma f)^{-1}\right\|_\sigma\right|\\ &\le C_1 + 2C_2 + \left|\sigma(\gamma f,(\gamma f)^+) - \sigma((\gamma f)^{-1},(\gamma f)^-)\right|.\end{aligned}$$

Then, since σ is coarsely-symmetric,

$$\sup_{\gamma\in\Gamma}\left|\|\gamma\|_\sigma - \left\|\gamma^{-1}\right\|_\sigma\right| < +\infty.$$

□

Lemma 16.5. *If σ is coarsely-symmetric, then (σ,σ,G) is a GPS system on $\Lambda(\Gamma)$.*

Proof. Notice that G is locally bounded by Proposition 3.2(5). Also, by Proposition 3.2(4), for every $x\in\Lambda(\Gamma)$ and $\gamma\in\Gamma$ we have

$$-2\kappa + \limsup_{\alpha\to x}\|\gamma\alpha\|_\sigma - \|\alpha\|_\sigma \le \sigma(\gamma,x) \le 2\kappa + \liminf_{\alpha\to x}\|\gamma\alpha\|_\sigma - \|\alpha\|_\sigma.$$

Then by the previous lemma, there exists $C > 0$ such that

$$-C + \limsup_{\alpha\to x}\left\|\alpha^{-1}\gamma^{-1}\right\|_\sigma - \left\|\alpha^{-1}\right\|_\sigma \le \sigma(\gamma,x) \le C + \liminf_{\alpha\to x}\left\|\alpha^{-1}\gamma^{-1}\right\|_\sigma - \left\|\alpha^{-1}\right\|_\sigma.$$

Fix $(x,y)\in\Lambda(\Gamma)^{(2)}$ and $\gamma\in\Gamma$. By definition there exist $\alpha_n\to x$ and $\beta_n\to y$ such that

$$G(\gamma x,\gamma y) = \kappa + \lim_{n\to\infty}\|\gamma\alpha_n\|_\sigma + \left\|\beta_n^{-1}\gamma^{-1}\right\|_\sigma - \left\|\beta_n^{-1}\alpha_n\right\|.$$

Then

$$G(x,y) \ge \kappa + \limsup_{n\to\infty}\|\alpha_n\|_\sigma + \left\|\beta_n^{-1}\right\|_\sigma - \left\|\beta_n^{-1}\alpha_n\right\|.$$

So

$$\begin{aligned} G(\gamma x,\gamma y) - G(x,y) &\leq \liminf_{n\to\infty} \left\|\gamma\alpha_n\right\|_\sigma - \left\|\alpha_n\right\|_\sigma + \left\|\beta_n^{-1}\gamma^{-1}\right\|_\sigma - \left\|\beta_n^{-1}\right\|_\sigma \\ &\leq 2\kappa + C + \sigma(\gamma,x) + \sigma(\gamma,y). \end{aligned}$$

Using the definition of G again, there exist $\hat{\alpha}_n \to x$ and $\hat{\beta}_n \to y$ such that

$$G(x,y) = \kappa + \lim_{n\to\infty} \left\|\hat{\alpha}_n\right\|_\sigma + \left\|\hat{\beta}_n^{-1}\right\|_\sigma - \left\|\hat{\beta}_n^{-1}\hat{\alpha}_n\right\|.$$

Then

$$G(\gamma x,\gamma y) \geq \kappa + \limsup_{n\to\infty} \left\|\gamma\hat{\alpha}_n\right\|_\sigma + \left\|\hat{\beta}_n^{-1}\gamma^{-1}\right\|_\sigma - \left\|\hat{\beta}_n^{-1}\hat{\alpha}_n\right\|.$$

So

$$\begin{aligned} G(\gamma x,\gamma y) - G(x,y) &\geq \limsup_{n\to\infty} \left\|\gamma\hat{\alpha}_n\right\|_\sigma - \left\|\hat{\alpha}_n\right\|_\sigma + \left\|\hat{\beta}_n^{-1}\gamma^{-1}\right\|_\sigma - \left\|\hat{\beta}_n^{-1}\right\|_\sigma \\ &\geq -2\kappa - C + \sigma(\gamma,x) + \sigma(\gamma,y). \end{aligned}$$

Thus

$$\left|\Big(\sigma(\gamma,x) + \bar{\sigma}(\gamma,y)\Big) - \Big(G(\gamma(x),\gamma(y)) - G(x,y)\Big)\right| \leq 2\kappa + C$$

and hence (σ,σ,G) is a GPS system. □

17. Potentials on Gromov hyperbolic spaces

For the rest of the section let (X, d_X) be a proper geodesic Gromov hyperbolic metric space and fix a basepoint $o \in X$. Also, let $\Gamma \subset \mathsf{Isom}(X)$ be a discrete group. Then Γ acts on the Gromov boundary $\partial_\infty X$ as a convergence group.

In this section we consider coarsely additive potentials on X, as defined in Definition 1.9, and prove Theorems 1.11 and 1.12 (which we restate here).

Theorem 17.1. *Suppose ψ is a Γ-invariant coarsely additive potential. Define functions $\sigma_\psi, \bar{\sigma}_\psi \colon \Gamma \times \partial_\infty X \to \mathbb{R}$ and $G_\psi \colon \partial_\infty X^{(2)} \to [0,\infty)$ by*

$$\begin{aligned} \sigma_\psi(\gamma,x) &= \limsup_{p\to x}\ \psi(\gamma^{-1}o,p) - \psi(o,p), \\ \bar{\sigma}_\psi(\gamma,x) &= \limsup_{p\to x}\ \psi(p,\gamma^{-1}o) - \psi(p,o), \\ G_\psi(x,y) &= \limsup_{p\to x, q\to y}\ \psi(p,o) + \psi(o,q) - \psi(p,q). \end{aligned}$$

Then there exists $\kappa_1 > 0$ such that $(\bar{\sigma}_\psi, \sigma_\psi, G_\psi + \kappa_1)$ is a coarse GPS system and

$$\sup_{\gamma\in\Gamma} \left| \left\|\gamma\right\|_{\sigma_\psi} - \psi(o,\gamma o)\right| < +\infty.$$

Theorem 17.2. *Suppose Γ acts co-compactly on X and $\sigma : \Gamma \times \partial_\infty X \to \mathbb{R}$ is an expanding coarse-cocycle. Then there exists a Γ-invariant coarsely additive potential where*

$$\sup_{\gamma\in\Gamma, x\in\partial_\infty X} \left|\sigma_\psi(\gamma,x) - \sigma(\gamma,x)\right| < +\infty.$$

In particular, σ is contained in a GPS system.

17.1. **Metric perspective.** If $\psi : X \times X \to \mathbb{R}$ is a Γ-invariant coarsely additive potential, then by Lemma 17.3 below there exists a constant $C > 0$ such that the function

$$\mathrm{d}_\psi(p,q) = \begin{cases} \psi(p,q) + C & \text{if } p \neq q \\ 0 & \text{if } p = q \end{cases}$$

is a Γ-invariant quasimetric, that is a function that satisfies all the axioms of a metric except for the symmetry property. Using properties (1) and (3) in Definition 1.9 one can show that (X, d_ψ) is quasi-isometric to (X, d_X) and that (X, d_ψ) is coarsely-geodesic, i.e. there is some $C > 0$ such that every two points in X are joined by a $(1, C)$-quasi-geodesic with respect to the quasimetric d_ψ.

Conversely, given a Γ-invariant coarsely-geodesic quasimetric d on X which is quasi-isometric to (X, d_X), the Morse lemma implies that the function $\psi(x,y) = \mathrm{d}(x,y)$ is a Γ-invariant coarsely additive potential.

Hence Theorems 17.1 and 17.2 could be instead be stated in terms of Γ-invariant coarsely-geodesic quasimetrics which are quasi-isometric to (X, d_X).

17.2. **Proof of Theorem 17.1.** Suppose $\psi : X \times X \to \mathbb{R}$ is a Γ-invariant coarsely additive potential and $\kappa : [0,\infty) \to [0,\infty)$ is the function in property (3).

Since (X, d_X) is Gromov hyperbolic, there exists $\delta > 0$ such that every geodesic triangle in (X, d_X) is δ-slim.

We first show that ψ satisfies a coarse version of the triangle inequality.

Lemma 17.3.

(1) *For every $r > 0$ there exists $C(r) > 0$ such that:*
$$|\psi(p,q) - \psi(p',q')| \leq C(r)$$
when $\mathrm{d}_X(p,p'), \mathrm{d}_X(q,q') \leq r$.

(2) *There exists $\kappa_1 > 0$ such that:*
$$\psi(p_1,p_2) \leq \psi(p_1,q) + \psi(q,p_2) + \kappa_1$$
for all $p_1, p_2, q \in X$.

Proof. (1). Notice that p' is in the $(r+1)$-neighborhood of any geodesic joining p to q and q' is in the $(r+1)$-neighborhood of any geodesic joining p' to q. So

$$\left|\psi(p,q) - \big(\psi(p,p') + \psi(p',q') + \psi(q',q)\big)\right| \leq 2\kappa(r+1).$$

Hence

$$|\psi(p,q) - \psi(p',q')| \leq 2\kappa(r+1) + 2 \sup_{\mathrm{d}_X(u,v) \leq r} |\psi(u,v)|,$$

which is finite by property (2).

(2). Let $m := \inf_{p,q \in X} \psi(p,q)$, which is finite by property (1).

Fix $p_1, p_2, q \in X$ and a geodesic triangle $[p_1,p_2] \cup [p_2,q] \cup [q,p_1]$ in X with vertices p_1, p_2, q. Since every geodesic triangle is δ-slim, there exists $u \in [p_1,p_2]$, $p_1' \in [q,p_1]$ and $p_2' \in [p_2,q]$ such that

$$\mathrm{d}_X(p_1',u), \mathrm{d}_X(p_2',u) < \delta.$$

Then

$$\begin{aligned}\psi(p_1,p_2) &\leq \psi(p_1,u) + \psi(u,p_2) + \kappa(0) \leq \psi(p_1,p_1') + \psi(p_2',p_2) + 2C(\delta) + \kappa(0) \\ &\leq \psi(p_1,q) - \psi(p_1',q) + \psi(q,p_2) - \psi(q,p_2') + 2C(\delta) + 3\kappa(0) \\ &\leq \psi(p_1,q) + \psi(q,p_2) - 2m + 2C(\delta) + 3\kappa(0).\end{aligned}$$

□

The next lemma states that it coarsely doesn't matter what sequence we use to define σ_ψ.

Lemma 17.4. *There exists $\kappa_2 > 0$ such that: if $x \in \partial_\infty X$ and $\gamma \in \Gamma$, then*

$$\limsup_{p,q \to x} \left| \big(\psi(\gamma^{-1}o, p) - \psi(o,p)\big) - \big(\psi(\gamma^{-1}o, q) - \psi(o,q)\big) \right| \leq \kappa_2.$$

Proof. Since geodesic triangles are δ-slim, for any two geodesic rays $r_1, r_2 : [0,\infty) \to (X, \mathrm{d}_X)$ with $\lim_{t\to\infty} r_1(t) = \lim_{t\to\infty} r_2(t)$ there exists $T > 0$ such that $r_1([T,\infty))$ is contained in the 2δ-neighborhood of r_2.

Fix $x \in \partial_\infty \Gamma$ and $\gamma \in \Gamma$. Then fix sequences $\{p_n\}, \{q_n\} \subset \Gamma$ converging to x where

$$L := \lim_{n\to\infty} \left| \big(\psi(\gamma^{-1}o, p_n) - \psi(o,p_n)\big) - \big(\psi(\gamma^{-1}o, q_n) - \psi(o,q_n)\big) \right|$$

equals the limit supremum in the lemma statement. Using the fact mentioned above, after passing to a subsequence, we can find $u \in X$ such that u is contained in the $(2\delta+1)$-neighborhood of any geodesic joining o to either p_n or q_n, and u is contained in the $(2\delta+1)$-neighborhood of any geodesic joining $\gamma^{-1}o$ to either p_n or q_n. Then

$$\begin{aligned}
&\left| \big(\psi(\gamma^{-1}o, p_n) - \psi(o,p_n)\big) - \big(\psi(\gamma^{-1}o, u) - \psi(o,u)\big) \right| \\
&\quad = \left| \big(\psi(\gamma^{-1}o, p_n) - \psi(\gamma^{-1}o, u) - \psi(u,p_n)\big) - \big(\psi(o,p_n) - \psi(o,u) - \psi(u,p_n)\big) \right| \\
&\quad \leq 2\kappa(2\delta+1).
\end{aligned}$$

Likewise,

$$\left| \big(\psi(\gamma^{-1}o, q_n) - \psi(o,q_n)\big) - \big(\psi(\gamma^{-1}o, u) - \psi(o,u)\big) \right| \leq 2\kappa(2\delta+1).$$

So $L \leq 4\kappa(2\delta+1)$. □

Lemma 17.5. $\sigma_\psi : \Gamma \times \partial_\infty X \to \mathbb{R}$ *is a coarse-cocyle.*

Proof. Fix $\gamma_1, \gamma_2 \in \Gamma$ and a sequence $\{p_n\} \subset X$ converging to $x \in \partial_\infty X$. Then by Lemma 17.4,

$$\begin{aligned}
&|\sigma_\psi(\gamma_1\gamma_2, x) - \sigma_\psi(\gamma_1, \gamma_2 x) - \sigma_\psi(\gamma_2, x)| \\
&\quad \leq 3\kappa_2 + \limsup_{n\to\infty} \big| \psi(\gamma_2^{-1}\gamma_1^{-1}o, p_n) - \psi(o,p_n) - \psi(\gamma_1^{-1}o, \gamma_2 p_n) + \psi(o, \gamma_2 p_n) \\
&\qquad\qquad - \psi(\gamma_2^{-1}o, p_n) + \psi(o,p_n) \big| \\
&\quad = 3\kappa_2.
\end{aligned}$$

Next fix $\gamma \in \Gamma$ and $\{x_n\} \subset \partial_\infty X$ converging to x. Then we can fix $\{p_{n,j}\} \subset X$ such that $\lim_{j\to\infty} p_{n,j} = x_n$. Then Lemma 17.4, we can fix $\{j_n\}$ such that

$$\sup_{n\geq 1} \left| \sigma_\psi(\gamma, x_n) - \psi(\gamma^{-1}o, p_{n,j_n}) + \psi(o, p_{n,j_n}) \right| \leq \kappa_2$$

and $p_{n,j_n} \to x$. Then again using Lemma 17.4,

$$\begin{aligned}
\limsup_{n\to\infty} |\sigma_\psi(\gamma,x) - \sigma_\psi(\gamma,x_n)| &\leq \kappa_2 + \limsup_{n\to\infty} \left| \sigma_\psi(\gamma,x) - \psi(\gamma^{-1}o, p_{n,j_n}) + \psi(o, p_{n,j_n}) \right| \\
&\leq 2\kappa_2.
\end{aligned}$$

Thus σ_ψ is a $(3\kappa_2)$-coarse-cocycle. □

Lemma 17.6. $\sup_{\gamma\in\Gamma} \left| \|\gamma\|_{\sigma_\psi} - \psi(o, \gamma o) \right| < +\infty.$

Proof. Fix a geodesic line ℓ in X and let $x, y \in \partial_\infty \Gamma$ denote the endpoints of ℓ. Fix $\{p_n\}, \{q_n\} \subset X$ converging to x, y respectively along the geodesic line ℓ. Also fix some $u \in \ell$.

Fix $\gamma \in \Gamma$. Notice that a geodesic triangle with vertices $p_n, q_n, \gamma^{-1}o$ is δ-slim and so after relabelling we can assume that u is a δ-neighborhood of a geodesic segment from $\gamma^{-1}o$ to p_n. So

$$\begin{aligned}\sigma_\psi(\gamma, x) &\geq -\kappa_2 + \limsup_{n\to\infty} \psi(\gamma^{-1}o, p_n) - \psi(o, p_n)\\ &\geq -\kappa_2 - \kappa(\delta) + \limsup_{n\to\infty} \psi(\gamma^{-1}o, u) + \psi(u, p_n) - \psi(o, p_n).\end{aligned}$$

By Lemma 17.3,

$$\psi(\gamma^{-1}o, o) \leq \psi(\gamma^{-1}o, u) + \psi(u, o) + \kappa_1$$

and

$$|\psi(u, p_n) - \psi(o, p_n)| \leq C(\mathrm{d}_X(u, o)).$$

So

$$\sigma_\psi(\gamma, x) \geq -\kappa_2 - \kappa(\delta) - C(\mathrm{d}_X(u, o)) - \psi(u, o) - \kappa_1 + \psi(\gamma^{-1}o, o).$$

For the other direction, fix $z \in \partial_\infty X$ and $\{p_n\} \subset X$ converging to z. Then

$$\sigma_\psi(\gamma, z) \leq \kappa_2 + \limsup_{n\to\infty} \psi(\gamma^{-1}o, p_n) - \psi(o, p_n) \leq \kappa_2 + \psi(\gamma^{-1}o, o) + \kappa_1.$$

So $\left| \|\gamma\|_{\sigma_\psi} - \psi(o, \gamma o) \right| = \left| \|\gamma\|_{\sigma_\psi} - \psi(\gamma^{-1}o, o) \right|$ is uniformly bounded. □

The next lemma states that it coarsely doesn't matter what sequence we use to define G_ψ.

Lemma 17.7. *There exists $\kappa_3 > 0$ such that: if ℓ is a geodesic line in (X, d_X) with endpoints $x, y \in \partial_\infty X$, then*

$$\limsup_{p\to x, q\to y} \left| \psi(p, o) + \psi(o, q) - \psi(p, q) - \min_{u\in\ell} \psi(u, o) + \psi(o, u) \right| \leq \kappa_3.$$

Proof. Fix a geodesic line ℓ with endpoints $x, y \in \partial_\infty X$. Then fix sequences $\{p_n\}, \{q_n\} \subset X$ converging to x, y which realize the limit supremum in the lemma statement. Passing to a subsequence we can suppose that

$$L := \lim_{n\to\infty} \psi(p_n, o) + \psi(o, q_n) - \psi(p_n, q_n)$$

exists in $[-\infty, +\infty]$. Then Lemma 17.3 implies that $L \in [-\kappa_1, +\infty]$.

Fix a geodesic $[p_n, q_n]$ joining p_n to q_n. Passing to a subsequence we can suppose that $[p_n, q_n]$ converges to a geodesic line $\hat{\ell}$ with endpoints x, y. Since every geodesic triangle is δ-slim, ℓ must be contained in a (2δ)-neighborhood of $\hat{\ell}$ and $\hat{\ell}$ must be contained in a (2δ)-neighborhood of ℓ.

First suppose that $u \in \ell$. Then for n sufficiently large, u is in the $(2\delta + 1)$-neighborhood of $[p_n, q_n]$. Hence

$$\psi(p_n, q_n) \geq \psi(p_n, u) + \psi(u, q_n) - \kappa(2\delta + 1),$$

which implies

$$\begin{aligned}L &\leq \kappa(2\delta + 1) + \lim_{n\to\infty} \psi(p_n, o) - \psi(p_n, u) + \psi(o, q_n) - \psi(u, q_n)\\ &\leq \kappa(2\delta + 1) + 2\kappa_1 + \psi(o, u) + \psi(u, o).\end{aligned}$$

Thus

$$L \leq \kappa(2\delta+1) + 2\kappa_1 + \min_{u\in\ell} \psi(u,o) + \psi(o,u).$$

Notice that this implies that $L < +\infty$.

For each n, let $[o,p_n] \cup [p_n,q_n] \cup [q_n,o]$ be a geodesic triangle with vertices o, p_n, q_n. Since every geodesic triangle is δ-slim, exist $u_n \in [p_n,q_n]$, $p_n' \in [0,p_n]$, and $q_n' \in [0,q_n]$ such that

$$\mathrm{d}_X(p_n', u_n), \mathrm{d}_X(q_n', u_n) < \delta.$$

Then

$$\begin{aligned}
&\psi(p_n,o) + \psi(o,q_n) - \psi(p_n,q_n)\\
&\quad \geq \psi(p_n,u_n) + \psi(u_n,o) + \psi(o,u_n) + \psi(u_n,q_n) - \psi(p_n,u_n) - \psi(u_n,q_n) - 3\kappa(\delta)\\
&\quad = \psi(u_n,o) + \psi(o,u_n) - 3\kappa(\delta).
\end{aligned}$$

Hence

$$L \geq -3\kappa(\delta) + \limsup_{n\to\infty} \psi(u_n,o) + \psi(o,u_n).$$

Since $L < +\infty$, property (1) implies that $\{u_n\}$ is relatively compact in X. So for n sufficiently large, u_n is contained in a $(2\delta+1)$-neighborhood of ℓ. Thus by Lemma 17.3,

$$L \geq -3\kappa(\delta) - 2C(2\delta+1) + \min_{u\in\ell} \psi(u,o) + \psi(o,u). \qquad \square$$

Lemma 17.8. *$(\bar\sigma_\psi, \sigma_\psi, G_\psi + \kappa_1)$ is a coarse GPS system.*

Proof. Lemma 17.3 implies that $G_\psi + \kappa_1$ is non-negative. Notice that $\bar\psi(p,q) = \psi(q,p)$ defines a Γ-invariant coarsely additive potential. So Lemma 17.5 implies that $\bar\sigma_\psi = \sigma_{\bar\psi}$ is a coarse-cocycle.

Next we show that G_ψ is locally finite. Fix a compact set $K \subset \partial_\infty X^{(2)}$. Then there exists $r > 0$ such that any geodesic line in (X, d_X) joining points in K intersects the ball of radius $r > 0$ centered at o. Then Lemma 17.7 and property (2) imply that

$$\sup_{(x,y)\in K} G_\psi(x,y) < +\infty.$$

Hence G_ψ is locally finite.

Finally, arguing exactly as in the proof of Lemma 16.5 there exists a constant $C > 0$ such that

$$|G_\psi(\gamma x, \gamma y) - G_\psi(x,y) - \bar\sigma_\psi(\gamma,x) - \sigma_\psi(\gamma,y)| \leq C$$

for all $\gamma \in \Gamma$ and $(x,y) \in \partial_\infty X^{(2)}$.

Hence $(\bar\sigma_\psi, \sigma_\psi, G_\psi + \kappa_1)$ is a coarse GPS system. $\square$

17.3. **Proof of Theorem 17.2.** Suppose $\sigma\colon \Gamma \times \partial_\infty X \to \mathbb{R}$ is an expanding κ-coarse-cocycle and Γ acts co-compactly on X. Fix $r > 0$ such that $X = \Gamma \cdot B_r(o)$.

For $p \in X$ let $A_p := \{\gamma \in \Gamma : \mathrm{d}_X(p, \gamma(o)) < r\}$. Then define $\psi\colon X \times X \to \mathbb{R}$ by

$$\psi(p,q) = \frac{1}{\#A_p \#A_q} \sum_{\gamma_1\in A_p, \gamma_2 \in A_q} \left\|\gamma_1^{-1}\gamma_2\right\|_\sigma.$$

We will show that ψ is a Γ-invariant coarsely additive potential.

Since $\gamma A_p = A_{\gamma p}$, the function ψ is Γ-invariant. Since σ is proper,

$$\lim_{r\to\infty} \inf_{\mathrm{d}_X(p,q)\geq r} \psi(p,q) = +\infty.$$

Since Γ acts properly on X, for any $r > 0$ we have

$$\sup_{\mathrm{d}_X(p,q)\leq r} |\psi(p,q)| < +\infty.$$

Let $B := \{\gamma \in \Gamma : \mathrm{d}_X(o,\gamma(o)) < 2r\}$. Then Observation 3.1(4) implies that if $\alpha \in A_x$ and $\beta \in A_y$, then

$$\left| \left\| \alpha^{-1}\beta \right\|_\sigma - \psi(x,y) \right| \leq C := 2 \max_{\gamma\in B} \|\gamma\|_\sigma . \tag{17}$$

In particular,

$$\left| \|\gamma\|_\sigma - \psi(o,\gamma o) \right| \leq C \tag{18}$$

for all $\gamma \in \Gamma$.

Lemma 17.9. *For every $r > 0$ there exists $\kappa = \kappa(r) > 0$ such that: if u is contained in the r-neighborhood of a geodesic in (X, d_X) joining p to q, then*

$$\left| \psi(p,q) - \big(\psi(p,u) + \psi(u,q)\big) \right| \leq \kappa.$$

Proof. Fix $r > 0$ and suppose no such $\kappa(r) > 0$ exists. Then for each $n \geq 1$ we can find $p_n, q_n, u_n \in X$ such that u_n is contained in the r-neighborhood of a geodesic joining p_n to q_n and

$$\left| \psi(p_n,q_n) - \big(\psi(p_n,u_n) + \psi(u_n,q_n)\big) \right| \geq n.$$

Translating by Γ, we can assume that $p_n \in B_r(o)$, which implies that $\mathrm{id} \in A_{p_n}$. Fix $\alpha_n \in A_{u_n}$ and $\beta_n \in A_{q_n}$. Then Equation (17) implies that

$$\left| \|\beta_n\|_\sigma - \|\alpha_n\|_\sigma - \left\|\alpha_n^{-1}\beta_n\right\|_\sigma \right| \geq n - 3C.$$

However, Proposition 3.2(5) implies that there exists $C' > 0$ such that

$$\|\beta_n\|_\sigma \geq \|\alpha_n\|_\sigma + \left\|\alpha_n^{-1}\beta_n\right\|_\sigma - C'$$

for all $n \geq 1$ and by Observation 3.1(4)

$$\|\beta_n\|_\sigma \leq \|\alpha_n\|_\sigma + \left\|\alpha_n^{-1}\beta_n\right\|_\sigma + \kappa.$$

So we have a contradiction. □

Thus ψ is a Γ-invariant coarsely additive potential. Finally, by the definition of σ_ψ, Equation (18), and Proposition 3.2(4) we have

$$\sup_{\gamma\in\Gamma, x\in\partial_\infty X} |\sigma_\psi(\gamma,x) - \sigma(\gamma,x)| < +\infty.$$

This completes the proof of Theorem 17.2.

Appendix A. Conservativity, dissipativity and quotient measures

In this appendix we define the notions of conservativity, dissipativity and Hopf decompositions for a general group action, check that it coincides with several other definitions in the literature [Kai10, Aar97, Rob03], and also that it is consistent with the classical theory of Hopf decompositions for actions of $\mathbb{Z}$. This expands on the discussion in [Bla21]. We also prove that quotient measures exist when the action is dissipative.

We include this appendix because the references we found on this topic were not entirely suitable for this paper: some sources [Rob03, Bla21] are missing details, while others [Kai10, Aar97] only apply to free actions (while here we allow actions which are not free).

For the rest of the section fix a measurable space X, a unimodular, locally compact second-countable group G acting measurably on X, and an G-invariant sigma-finite measure m. We denote by dg a fixed choice of Haar measure on G: since G is unimodular, this measure is invariant under both left and right multiplication, and under the involution $g \mapsto g^{-1}$.

A.1. The Hopf decomposition.

There are several reasonable definitions of wandering sets, which generalize in different ways the classical notion of wandering sets for actions of $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{Z}_{\geq 1}$. We use the following:

Definition A.1. A measurable subset $W \subset X$ is called *wandering* (resp. *exactly wandering*) if $\{g \in \mathsf{G} : gx \in W\}$ is relatively compact for m-almost any (resp. for any) $x \in W$.

When G is discrete, in particular if $\mathsf{G} = \mathbb{Z}$, then a set W is sometimes called wandering if it satisfies the stronger property that $W \cap gW = \varnothing$ for any $g \in \mathsf{G}$, or $m(W \cap gW) = 0$, see [Aar97, Kai10]. We will see, in Section A.2, the link between this stronger definition and ours. Roblin defines W to be wandering if it satisfies the weaker property that $\int 1_W(gx)dg < +\infty$ for almost any $x \in W$ [Rob03, p.17]. This gives the same notions of conservativity and dissipativity, as explained below.

Definition A.2. The action of G on (X, m) is called *conservative* if every wandering set has measure zero, The action is called *dissipative* if X is a countable union of wandering sets.

A *Hopf decomposition* of X is a decomposition $X = C \sqcup D$ into disjoint G-invariant measurable sets such that the action on C is conservative and the action on D is dissipative.

Notice that if $X = C \sqcup D$ and $X = C' \sqcup D'$ are both Hopf decompositions, then $C' \cap D$ is a countable union of wandering sets. Since every wandering set in C' has measure zero, we see that $m(C' \cap D) = 0$, and similarly $m(C \cap D') = 0$. So, up to a set of measure zero, there is a unique Hopf decomposition.

There is another classical characterization of conservativity, dissipativity and Hopf decompositions in terms of integrable functions. This characterization also proves the existence of Hopf decompositions.

Fact A.3 ([Bla21, Fact 2.27])**.** *For any positive integrable function f on X, the sets $C := \{x : \int_{\mathsf{G}} f(gx)\, dg = +\infty\}$ and $D := \{x : \int_{\mathsf{G}} f(gx)\, dg < +\infty\}$ form a Hopf decomposition.*

In particular, the action of G *is conservative (resp. dissipative) if and only if for any/some positive integrable function* f *on* X*, we have* $\int_{\mathsf{G}} f(gx)\,dg = +\infty$ *(resp.* $< +\infty$*) for* m*-almost any* $x \in X$*.*

This result implies that our notion of Hopf decomposition coincides with that of Roblin [Rob03, p.17]. If not, there would be $W \subset C$ with positive measure such that $h(x) = \int 1_W(gx)\,dg$ is finite for m-almost every $x \in W$, and up to reducing W we can assume there exists R such that $\{h(x) \leq R\}$ has full m-measure in W. Then $\int_W h(x)f(x)\,dx < +\infty$, but this quantity equals $\int_{x\in W} \int f(gx)\,dg\,dm(x)$, which is infinite since $\int f(gx)\,dg = +\infty$ on C.

A consequence of Fact A.3 is that a group G acting on X has the same Hopf decomposition as any lattice of G acting on X (if G has lattices), see e.g. [Aar97, Th. 1.6.4] in the case of free actions.

A.2. **The case of discrete groups.** In this section we suppose that G is discrete and X is standard, i.e. X can be measurably embedded in $[0, 1]$. The goal is to construct a measurable fundamental domain for the action of G on the dissipative part. This will allow us to check that our definition of Hopf decomposition agree with other definitions [Aar97, Kai10] when G is torsion-free.

Lemma A.4. *Let* $A \subset X$ *be a* G*-invariant measurable subset which can be written as a countable union of exactly wandering sets. Then there exists a measurable subset* $\mathcal{F} \subset A$ *such that every orbit* $\mathsf{G} \cdot x$ *intersects* $\mathcal{F}$ *at exactly one point. In particular,* $\mathcal{F}$ *is measurably isomorphic to* $\mathsf{G}\backslash A$ *endowed with the quotient sigma-algebra, and this quotient is hence standard (can be measurably embedded in* $[0, 1]$*).*

Moreover, there exists such an $A \subset X$ *such that* $X = A \sqcup (X - A)$ *is a Hopf decomposition.*

Proof. Let $\{W_n\}$ be a sequence of exactly wandering sets with $A = \bigcup_n W_n$. Let $W_1' := W_1$, and let $W_n' := W_n - \mathsf{G}(W_1 \cup \cdots \cup W_{n-1})$ for all $n > 1$. Then $\{W_n'\}$ is a sequence of exactly wandering sets such that the orbits $\mathsf{G} \cdot W_n'$ form a partition of A. To conclude the proof, it suffices to find a fundamental domain in each W_n', i.e. to select in a measurable way one representative for each orbit which intersects W_n'.

Let $\phi\colon X \to [0, 1]$ be a measurable embedding. Now for each $n \geq 1$, we select in each orbit $\mathsf{G} \cdot x$ the point $y \in W_n'$ whose image under ϕ is the smallest, i.e. $y = \phi^{-1}(\min \phi(\mathsf{G} \cdot x \cap W_n'))$. So

$$\mathcal{F}_n := \{x \in W_n' : \phi(x) \leq \phi(gx) \quad \text{for any } g \in \mathsf{G} \text{ with } gx \in W_n'\}.$$

This set is measurable since $x \in \mathcal{F}_n$ if and only if $\phi(x) \leq \phi(gx) + 1_{X-W_n'}(gx)$ for any $g \in \mathsf{G}$, which are countably many measurable conditions. Then $\mathcal{F} := \bigcup_n \mathcal{F}_n$ is a measurable fundamental domain.

To construct A satisfying the "moreover" statement, consider a Hopf decomposition $X = C \sqcup D$, write D as a countable union of wandering sets $\{W_n\}$, and then let W_n' be the set of $x \in W_n$ such that $\{g : gx \in W_n\}$ is finite, so that W_n' is exactly wandering and has full measure in W_n. Finally set $A := \bigcup_n \mathsf{G} \cdot W_n'$. □

If G is torsion-free, then the action of G on the set $A \subset X$ constructed in the last part above is free. As a corollary, any Hopf decomposition $X = C \sqcup D$ for our definition in Section A.1 is also a Hopf decomposition in the sense of Aaronson [Aar97, §1.6] and Kaimanovich [Kai10]: every positive measure subset $B \subset C$ is *recurrent*, meaning that for almost any $x \in B$ the orbit eventually returns to B (because B is not wandering), and D admits a subset $\mathcal{F}$ such that $\{g\mathcal{F}\}_{g\in\mathsf{G}}$ are pairwise disjoint and $\mathsf{G}\cdot\mathcal{F}$ has full measure in D.

A.3. **Quotient measures.** In this section we assume the action of G on X is dissipative. Let $\pi\colon X \to \mathsf{G}\backslash X$ denote the projection map associated to the action and endow $\mathsf{G}\backslash X$ with the quotient sigma-algebra.

For any non-negative measurable function $f\colon X \to [0,+\infty]$, the function

$$\tilde{P}(f)(x) := \int_{g\in\mathsf{G}} f(gx)\,dg$$

is measurable and G-invariant, hence it descends to a measurable function on $\mathsf{G}\backslash X$ which we denote by $P(f)$.

We say that a measure m' on $\mathsf{G}\backslash X$ is a *quotient measure for* m if for any non-negative measurable function $f : X \to [0,+\infty]$ we have

$$\int_{x\in X} f(x)\,dm(x) = \int_{q\in\mathsf{G}\backslash X} P(f(q)\,dm'(q). \tag{19}$$

For instance, if X is a smooth manifold, G is discrete and acts freely and properly discontinuously on X, and m comes from a smooth G-invariant volume form α, then $\mathsf{G}\backslash X$ is a manifold and the quotient measure is induced by the volume form $\pi_*\alpha$.

We will show that quotient measures exist and are unique.

Remark A.5. First we make some observations.

(1) The quotient measure m' is automatically sigma-finite, since $P(f)$ is a positive function in $L^1(\mathsf{G}\backslash X, m')$ whenever f is a positive function in $L^1(X,m)$.

(2) A G-invariant measurable subset $A \subset X$ has zero m-measure if and only if its projection $\pi(A)$ (which is measurable) has zero m'-measure. Indeed, let f be a positive integrable function on X. Then $P(f1_A) = P(f)1_{\pi(A)}$. If $m(A) = 0$ then $\int P(f)1_{\pi(A)}dm' = \int f1_A dm = 0$ so $P(f)1_{\pi(A)} = 0$ almost everywhere, so $m'(\pi(A)) = 0$. Conversely, if $m'(\pi(A)) = 0$ then $\int f1_A dm = 0$ so $m(A) = 0$.

(3) If $f \in L^1(X,m)$, then $\tilde{P}(f)(x) = \int_{g\in\mathsf{G}} f(gx)\,dg$ is an m-almost everywhere defined measurable function and hence it descends to a measurable m'-almost everywhere defined function on $\mathsf{G}\backslash X$ which we denote by $P(f)$. Equation (19) implies that

$$\int P(f)dm' = \int f dm$$

for all $f \in L^1(X,m)$. Since $|P(f)| \le P(|f|)$, Equation (19) also implies that

$$P : L^1(X,m) \to L^1(\mathsf{G}\backslash X, m')$$

is continuous.

Fact A.6. *There is a unique quotient measure on* $\mathsf{G}\backslash X$*, and it is given by the formula*

$$m' = \frac{1}{P(f_0)}\pi_*(f_0 m),$$

where f_0 *is any integrable positive function on* X*. Moreover, for any* $\chi\colon \mathsf{G}\backslash X \to \mathbb{R}_{\ge 0}$*, if* $f = \frac{f_0}{\tilde{P}(f_0)}\chi\circ\pi$ *then* $P(f) = \chi$ m'*-almost surely.*

Proof. Since the action of G on X is dissipative, $P(f_0)$ is finite m-almost surely by Fact A.3.

Let us prove uniqueness: let m_1, m_2 be quotient measures. Fix $\chi\colon \mathsf{G}\backslash X \to [0,+\infty]$ measurable. Set $f(x) = \chi(\pi(x))\frac{f_0(x)}{P(f_0)(x)}$ and observe that $P(f)(q) = \chi(q)$ for any $q \in \mathsf{G}\backslash X$ such that $P(f_0)(q) < \infty$, which occurs m_i-almost surely since $\int P(f_0)dm_i = \int f_0 dm < \infty$, for any $i = 1,2$. Thus $\int \chi dm_1 = \int f dm = \int \chi dm_2$, which implies $m_1 = m_2$ since χ was an arbitrary non-negative measurable function.

Let us now check that $m' = P(f_0)^{-1}\pi_*(f_0 m)$ is a quotient measure. Fix $f\colon X \to [0,+\infty]$. Then by Fubini, the G-invariance of m, and the invariance of the Haar measure under $g \mapsto g^{-1}$, we have

$$\begin{aligned}\int_{q\in\mathsf{G}\backslash X} P(f)(q)\,dm'(q) &= \int_{q\in\mathsf{G}\backslash X} \frac{P(f)(q)}{P(f_0)(q)}\,d\pi_*(f_0 m)(q)\\ &= \int_{\mathsf{G}}\int_X f(gx)\frac{f_0(x)}{P(f_0)(gx)}\,dm(x)\,dg\\ &= \int_X f(y)\int_{\mathsf{G}} \frac{f_0(g^{-1}y)}{P(f_0)(y)}\,dg\,dm(y) = \int_X f(y)\,dm(y). \quad\square\end{aligned}$$

If G is discrete, then one can use the existence of a fundamental domain from Section A.2 to give a more concrete description of the quotient measure.

Fact A.7. *Suppose X is standard and G is discrete. Let $\mathcal{F} \subset X$ be a measurable subset that intersects every Γ-orbit at most once and such that $\Gamma\cdot\mathcal{F}$ has full measure (as in the "moreover" part of Lemma A.4).*

Then $\pi_*(f_0 m_{|\mathcal{F}})$ is the quotient measure, where $f_0(x) = \frac{1}{\#\{\gamma\in\Gamma:\gamma x = x\}}$.

Proof. Let f be a measurable non-negative function on X.

For any finite subgroup $K \subset \mathsf{G}$, let $\mathcal{F}_K$ be the set of $x \in \mathcal{F}$ whose stabilizer is K. Then $\mathcal{F}$ is the disjoint countable union of the $\mathcal{F}_K$'s, and we have $\sum_\gamma 1_{\mathcal{F}_K}\circ\gamma = \#K \cdot 1_{\mathsf{G}\mathcal{F}_K}$ and $f_0(x) = (\#K)^{-1}$ for any $x \in \mathcal{F}_K$. So

$$\begin{aligned}\int P(f)d\pi_*(f_0 m_{|\mathcal{F}}) &= \int_{x\in\mathcal{F}} \sum_\gamma f(\gamma x) f_0(x) dm(x)\\ &= \sum_{K\subset\mathsf{G}} \frac{1}{\#K}\int_{x\in\mathcal{F}}\sum_\gamma 1_{\mathcal{F}_K}(x) f(\gamma x) dm(x)\\ &= \sum_{K\subset\mathsf{G}}\int_{y\in\mathsf{G}\mathcal{F}_K} f(y)dm(y) = \int f dm. \quad\square\end{aligned}$$

A.4. **The case $\mathsf{G} = \mathbb{Z}$.** In this section we consider the case when $\mathsf{G} = \mathbb{Z}$. There is an abundant literature on the notions of conservativity, dissipativity and Hopf decomposition in this case, and more generally in the case of actions of the semigroup $\mathbb{Z}_{\geq 1}$. We denote by T^n the transformation of X associated to an element n.

For any reasonable choice of definitions, it is obvious that conservativity of $\mathbb{Z}_{\geq 1}$ always implies conservativity of $\mathbb{Z}$ and that dissipativity of $\mathbb{Z}$ implies dissipativity of $\mathbb{Z}_{\geq 1}$. It is well-known, although nontrivial, that the converses are also true. We shall use Krengel as a reference, and check that our definitions are consistent with the definitions there:

Fact A.8. *Consider a decomposition $X = C \sqcup D$ which is a Hopf decomposition in the sense of Krengel* [Kre85, Th. 3.2]*: D admits a measurable subset W_0 such that*

$D = T^{\mathbb{Z}}W_0$ and $W_0 \cap T^n W_0 = \varnothing$ for any $n \neq 0$, and every subset $W \subset C$ with $W \cap T^n W = \varnothing$ for any $n \neq 0$ has measure zero.

Then $X = C \sqcup D$ is a Hopf decomposition for our definition.

Proof. The action on D is clearly dissipative for our definition.

If every subset $W \subset C$ with $W \cap T^n W = \varnothing$ for any $n \neq 0$ has measure zero, then for any measurable $A \subset C$, for almost any $x \in A$ there exist infinitely many n' such that $T^n x \in A$ [Kre85, Th. 3.1]. This implies that the action on C is conservative for our definition. □

A.5. **A topological Hopf decomposition.** Suppose the sigma-algebra of X comes from a locally compact second-countable topology and the action of G is by homeomorphisms. In this case there is a natural Hopf decomposition that does not depend on m.

We say an orbit $\mathsf{G} \cdot x$ is *escaping* if for any compact set K the set $\{g : gx \in K\}$ is relatively compact, i.e. $gx \to \infty$ as $g \to \infty$. Let $D \subset X$ be the set of x such that $\mathsf{G} \cdot x$ is escaping, and $C = X - D$. Note that D is measurable because it is a countable intersection of closed sets of the form

$$\{x : (\mathsf{G} - L) \cdot x \subset X - \mathrm{int}(K)\} = \bigcap_{g \in \mathsf{G} - L} g^{-1}(X - \mathrm{int}(K))$$

for some compact sets $K \subset X$ and $L \subset \mathsf{G}$.

Lemma A.9. *$X = C \sqcup D$ is a Hopf decomposition for any G-invariant locally finite measure.*

Proof. Our assumptions imply the existence of a positive continuous integrable function f. Then $\int f(gx)\, dx = +\infty$ for any $x \in C$. Indeed let $x \in C$. Then there is a compact subset $K \subset X$ such that $\{g : gx \in K\}$ is not relatively compact. Fix $U \subset \mathsf{G}$ a compact neighborhood of the identity. By continuity there is some $\epsilon > 0$ such that $f(uy) > \epsilon$ when $u \in U$ and $y \in K$. Let $\{g_n\} \subset \mathsf{G}$ be an escaping sequence such that Ug_n are pairwise disjoint and $g_n x \in K$ for any n. Then

$$\int_{\mathsf{G}} f(gx)\, dg \geq \sum_n \int_{u \in U} f(ug_n x)\, du \geq \sum_n \epsilon \cdot \mathrm{Haar}(U) = +\infty.$$

It remains to prove that D is a countable union of wandering sets. In fact it is a countable union of exactly wandering sets W_n ($\{g : gx \in W_n\}$ is relatively compact for any $x \in W_n$). Indeed let $\{K_n\}$ be a sequence of compact sets covering X, and let $W_n = K_n \cap D$. Then W_n is exactly wandering. □

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