COUNTING, MIXING AND EQUIDISTRIBUTION FOR GPS SYSTEMS WITH APPLICATIONS TO RELATIVELY ANOSOV GROUPS

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Abstract. We establish counting, mixing and equidistribution results for finite BMS measures on flow spaces associated to geometrically finite convergence group actions. We show that, in particular, these results apply to flow spaces associated to relatively Anosov groups.

Contents

1. Introduction

In previous work [8], we introduced the theory of Patterson–Sullivan measures for Gromov–Patterson–Sullivan (GPS) systems. We recall that a GPS system consists of a pair of cocycles for a discrete convergence groups action which are related by a Gromov product. More precisely, given a GPS system we constructed Patterson–Sullivan measures, a flow space and a Bowen–Margulis–Sullivan (BMS) measure. We established a Hopf–Tusji–Sullivan dichotomy for these flows and their Bowen–Margulis–Sullivan (BMS) measures. In this paper, we obtain mixing and equidistribution results when the BMS measure is finite and the length spectrum is non-arithmetic.

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In the case when the convergence group action is geometrically finite, we are able to establish a stronger equidistribution result which will allow us to obtain counting results. We apply our abstract results to obtain equidistribution and counting results for relatively Anosov groups in semisimple Lie groups which generalize earlier work of Sambarino [42, 44] for Anosov groups. In the relatively Anosov setting such counting results were previously known only when the semisimple Lie groups has rank-one, or for special classes of Anosov groups, see the discussion after Corollary 1.7 for more details.

1.1. **Main results.** We will assume through the paper that M is a compact metrizable space and $\Gamma \subset \mathsf{Homeo}(M)$ is a non-elementary convergence group. A continuous cocycle $\sigma \colon \Gamma \times M \to \mathbb{R}$ defines a natural *magnitude*

$$\|\gamma\|_\sigma := \max_{x \in M} \sigma(\gamma, x)$$

and *period*

$$\ell_\sigma(\gamma) := \begin{cases} \sigma(\gamma, \gamma^+) & \text{if } \gamma \text{ is loxodromic} \\ 0 & \text{otherwise} \end{cases}.$$

One of the primary aims of this paper is to study counting results for these quantities. To that end, we restrict our study to the case when σ is *proper*, that is if $\{\gamma_n\}$ is a sequence of distinct elements then $\|\gamma_n\|_\sigma \to +\infty$. In this case the *critical exponent* is defined to be

$$\delta_\sigma(\Gamma) := \limsup_{R \to \infty} \frac{1}{R} \log \#\{\gamma \in \Gamma : \|\gamma\|_\sigma \leq R\}.$$

We also assume our cocycle is part of a (continuous) *Gromov–Patterson–Sullivan (GPS) system*, which is a triple $(\sigma, \bar\sigma, G)$ where $\sigma, \bar\sigma \colon \Gamma \times M \to \mathbb{R}$ are continuous proper cocycles and $G \colon M^{(2)} \to [0, \infty)$ is a continuous function such that

$$\bar\sigma(\gamma, x) + \sigma(\gamma, y) = G(\gamma x, \gamma y) - G(x, y) \tag{1}$$

for all $\gamma \in \Gamma$ and $(x, y) \in M^{(2)}$, where as usual $M^{(2)} \subset M \times M$ denotes the space of distinct pairs. Many geometrically important cocycles appear as part of a GPS system, see [8] for examples.

Given a GPS system, the *length spectrum* is

$$\mathcal{L}(\sigma, \bar\sigma, G) := \{\ell_\sigma(\gamma) + \ell_{\bar\sigma}(\gamma) : \gamma \in \Gamma \text{ is loxodromic}\}$$

and we say that $\mathcal{L}(\sigma, \bar\sigma)$ is *non-arithmetic* if it generates a dense subgroup of $\mathbb{R}$.

In the case when Γ is geometrically finite we obtain the following counting result for periods of elements in $[\Gamma_{\mathrm{lox}}]^w$, the set of weak conjugacy classes of loxodromic elements (defined in Section 10 below). If Γ is torsion-free, then $[\Gamma_{\mathrm{lox}}]^w$ is simply the set of conjugacy classes of loxodromic elements.

Theorem 1.1 (Corollary 10.2). *Suppose $(\sigma, \bar\sigma, G)$ is a continuous GPS system for a geometrically finite convergence group $\Gamma \subset \mathsf{Homeo}(M)$ where $\delta := \delta_\sigma(\Gamma) < +\infty$. If*

(1) $\mathcal{L}(\sigma, \bar\sigma, G)$ is non-arithmetic and
(2) $\delta_\sigma(P) < \delta$ for all maximal parabolic subgroups $P \subset \Gamma$,

then

$$\#\{[\gamma]^w \in [\Gamma_{\mathrm{lox}}]^w : 0 < \ell_\sigma(\gamma) \leq R\} \sim \frac{e^{\delta R}}{\delta R},$$

i.e. the ratio of the two sides goes to 1 as $R \to +\infty$.

Theorem 1.1, in combination with the main result of [14], will allow us to prove counting results for relatively Anosov representations. We will describe these results in Section 1.2 below after developing the appropriate terminology to state them.

Remark 1.2. The condition that $\mathcal{L}(\sigma,\bar{\sigma},G)$ is non-arithmetic holds whenever Γ contains a parabolic element. Moreover, it can be replaced with the weaker assumption that the cross ratio spectrum $\mathcal{CR}$, see Section 4, is non-arithmetic which holds whenever some infinite path component of the limit set $\Lambda(\Gamma)$ contains a loxodromic fixed point, see Section 5. In particular, in the geometrically finite case, the cross ratio spectrum is non-arithmetic whenever Γ is not a virtually free uniform convergence group.

Motivated by previous work concerning specific GPS systems, the proof of Theorem 10.2 is based on studying the dynamical properties of the flow space associated to the GPS system. We briefly introduce this flow space and state the dynamical results we prove. For more precise definitions, see Section 2.

Let $\Lambda(\Gamma) \subset M$ denote the limit set of Γ and let $\Lambda(\Gamma)^{(2)}$ denote the space of distinct pairs in $\Lambda(\Gamma)$. One may define a flow space

$$\tilde{U}_\Gamma := \Lambda(\Gamma)^{(2)} \times \mathbb{R}$$

with flow

$$\psi^t(x,y,s) = (x,y,s+t),$$

which corresponds via the famous Hopf parametrisation to the (nonwandering part of the) unit tangent bundle of $\mathbb{H}^n$ when $\Gamma \subset \mathrm{Isom}(\mathbb{H}^n)$, $M = \partial\mathbb{H}^n$ and σ is the Busemann cocycle from the hyperbolic metric. Using this analogy, one then defines an action of Γ on $\widetilde{U}_\Gamma$ by

$$\gamma(x,y,s) = (\gamma(x),\gamma(y),s+\sigma(\gamma,y)).$$

When σ is part of a GPS system, the action of Γ on $\tilde{U}_\Gamma$ is properly discontinuous (see [8, Prop. 10.2]) and commutes with the flow, so ψ^t descends to a flow on the quotient

$$U_\Gamma := \Gamma\backslash\tilde{U}_\Gamma$$

which we also denote ψ^t.

In the case of GPS systems associated to Anosov groups this flow space was constructed by Sambarino [42, 43, 44] and in the case of GPS systems associated to transverse groups this flow space was constructed by Kim–Oh–Wang [30]. In the general setting considered here, one can easily adapt the proof of [30, Th. 9.1] to show that Γ acts properly discontinuously on $\widetilde{U}_\Gamma$.

In our previous paper [8], we showed that if the Poincaré series associated to σ, given by

$$Q_\sigma(s) = \sum_{\gamma\in\Gamma} e^{-s\|\gamma\|_\sigma},$$

diverges at the critical exponent $\delta_\sigma(\Gamma)$, then U_Γ has a unique *(BMS) Bowen–Margulis–Sullivan measure* m_Γ and the flow $\psi^t\colon (U_\Gamma, m_\Gamma) \to (U_\Gamma, m_\Gamma)$ is ergodic and conservative (see Section 2 for the definition of m_Γ and a precise statement).

In this paper, we further show that the flow is mixing when the BMS measure is finite and the length spectrum is non-arithmetic.

Theorem 1.3 (see Theorem 4.2 below)**.** *Suppose $(\sigma, \bar{\sigma}, G)$ is a continuous GPS system for a convergence group $\Gamma \subset \mathsf{Homeo}(M)$ where $\delta := \delta_\sigma(\Gamma) < +\infty$ and $Q_\sigma(\delta) = +\infty$. If the BMS measure m_Γ is finite and the length spectrum $\mathcal{L}(\sigma, \bar{\sigma}, G)$ is non-arithmetic, then the flow $\psi^t \colon (U_\Gamma, m_\Gamma) \to (U_\Gamma, m_\Gamma)$ is mixing.*

In the geometrically finite setting, we are able to adapt arguments of Dal'bo–Otal–Peigné [20] in the setting of geometrically finite negatively curved manifolds to provide the following criterion for when the BMS measure to be finite.

Theorem 1.4 (see Theorem 9.1 below)**.** *Suppose $(\sigma, \bar{\sigma}, G)$ is a continuous GPS system for a geometrically finite convergence group $\Gamma \subset \mathrm{Homeo}(M)$ where $\delta := \delta_\sigma(\Gamma) < +\infty$. If $\delta_\sigma(P) < \delta$ for any maximal parabolic subgroup P of Γ, then $Q_\sigma(\delta) = +\infty$ and the BMS measure m_Γ is finite.*

As an application of mixing, we prove an equidistribution result for fixed points of loxodromic elements in terms of the Patterson–Sullivan measures associated to the cocycles in a GPS system. In our earlier work [8], we proved that Patterson–Sullivan measures exist in the critical dimension and are unique when the Poincaré series diverges at its critical exponent, see Section 2.2 for details. In the statement below, $\Gamma_{\rm lox}$ denotes the set of loxodromic elements of γ and $\mathcal{D}_{\gamma^\pm}$ denotes the unit Dirac mass based at the attracting/repelling fixed point of $\gamma \in \Gamma_{\rm lox}$.

Theorem 1.5 (see Theorem 6.1 below)**.** *Suppose $(\sigma, \bar{\sigma}, G)$ is a continuous GPS system for a convergence group $\Gamma \subset \mathsf{Homeo}(M)$ where $\delta := \delta_\sigma(\Gamma) < +\infty$ and $Q_\sigma(\delta) = +\infty$. Let μ be the unique σ-Patterson–Sullivan measure of dimension δ and let $\bar{\mu}$ be the unique $\bar{\sigma}$-Patterson–Sullivan measure of dimension δ.*

If the BMS measure m_Γ is finite and mixing, then

$$\lim_{T \to \infty} \delta e^{-\delta T} \sum_{\substack{\gamma \in \Gamma_{\rm lox} \\ \ell_\sigma(\gamma) \leq T}} \mathcal{D}_{\gamma^-} \otimes \mathcal{D}_{\gamma^+} = \frac{1}{\|m_\Gamma\|} e^{\delta G(x,y)} d\bar{\mu}(x) \otimes d\mu(y)$$

in the dual of compactly supported continuous functions.

The above theorem can be expressed as an equidistribution result for closed orbits of the geodesic flow on U_Γ. For every R, let $\tilde{m}_R$ be the sum of Lebesgue measures on axes of loxodromic elements of Γ with period at most R, which is a locally finite measure on $M^{(2)} \times \mathbb{R}$. Denote by m_R the quotient measure on U_Γ. Then the conclusion of Theorem 1.5 can be reformulated as

$$\lim_{R \to \infty} \delta e^{-\delta R} \int f dm_R = \frac{1}{\|m_\Gamma\|} \int f dm_\Gamma$$

for any continuous function $f \colon U_\Gamma \to \mathbb{R}$ with compact support.

In the context of geometrically finite convergence groups, we can establish the following stronger equidistribution result which is needed to obtain our counting results.

Theorem 1.6 (see Theorem 10.1 below)**.** *Suppose $(\sigma, \bar{\sigma}, G)$ is a continuous GPS system for a geometrically finite convergence group $\Gamma \subset \mathsf{Homeo}(M)$ where $\delta := \delta_\sigma(\Gamma) < +\infty$. If*

(1) $\mathcal{L}(\sigma, \bar{\sigma}, G)$ is non-arithmetic and
(2) $\delta_\sigma(P) < \delta$ for all maximal parabolic subgroups $P \subset \Gamma$,

then

$$\lim_{R\to\infty} \delta e^{-\delta R} \int f dm_R = \frac{1}{\|m_\Gamma\|} \int f dm_\Gamma$$

for any bounded continuous function $f\colon U_\Gamma \to \mathbb{R}$.

The counting result in Theorem 1.1 is obtained by applying Theorem 1.6 to the constant function with value 1.

1.2. **Applications to relatively Anosov groups.** We now develop the terminology necessary to explain how to obtain counting result for relatively Anosov subgroups of $\mathsf{SL}(d,\mathbb{R})$ from Theorem 1.1. In Section 11 we describe the same result in the more general setting of semisimple Lie groups.

Let $\mathfrak{sl}(d,\mathbb{R}) = \mathfrak{k} \oplus \mathfrak{p}$ denote the standard Cartan decomposition of the Lie algebra of $\mathsf{SL}(d,\mathbb{R})$, where $\mathfrak{k}$ is the Lie algebra of $\mathsf{SO}(d)$ and $\mathfrak{p}$ consists of symmetric matrices with trace zero. Let $\mathfrak{a} \subset \mathfrak{p}$ denote the standard Cartan subalgebra consisting of diagonal matrices with trace zero and let

$$\mathfrak{a}^+ = \{\mathrm{diag}(a_1,\dots,a_d) \in \mathfrak{a} : a_1 \geq a_2 \geq \cdots \geq a_d\}$$

denote the standard choice of positive Weyl chamber. The associated simple roots are

$$\Delta = \{\alpha_1,\dots,\alpha_{d-1}\}$$

where $\alpha_j(\mathrm{diag}(a_1,\dots,a_d)) = a_j - a_{j+1}$. The *Cartan projection* $\kappa\colon \mathsf{SL}(d,\mathbb{R}) \to \mathfrak{a}^+$ is given by

$$\kappa(\gamma) = \mathrm{diag}\big(\log\sigma_1(\gamma),\dots,\log\sigma_d(\gamma)\big)$$

where $\sigma_1(\gamma) \geq \cdots \geq \sigma_d(\gamma)$ are the singular values of γ and the *Jordan projection* $\lambda\colon \mathsf{SL}(d,\mathbb{R}) \to \mathfrak{a}^+$ is given by

$$\lambda(\gamma) = \mathrm{diag}\big(\log\lambda_1(\gamma),\dots,\log\lambda_d(\gamma)\big)$$

where $\lambda_1(\gamma) \geq \cdots \geq \lambda_d(\gamma)$ are the moduli of the generalized eigenvalues of γ.

Given $\phi \in \mathfrak{a}^*$, one can define the *ϕ-period* of $\gamma \in \mathsf{SL}(d,\mathbb{R})$ by

$$\ell^\phi(\gamma) := \phi(\lambda(\gamma))$$

and the *ϕ-magnitude* by $\phi(\kappa(\gamma))$. Also, given a discrete subgroup $\Gamma \subset \mathsf{SL}(d,\mathbb{R})$ and $\phi \in \mathfrak{a}^*$, one can define a, possibly infinite, critical exponent

$$\delta^\phi(\Gamma) := \limsup_{R\to\infty} \frac{1}{R} \log \#\{\gamma \in \Gamma : \phi(\kappa(\gamma)) \leq R\}.$$

Given a subset $\theta \subset \Delta$, let $\mathcal{F}_\theta$ denote the partial flag manifold associated to θ, i.e. $\mathcal{F}_\theta$ is the set of partial flags with subspaces of dimensions $\{j : \alpha_j \in \theta\}$. When θ is symmetric (i.e. $\alpha_j \in \theta \Leftrightarrow \alpha_{d-j} \in \theta$), a discrete subgroup $\Gamma \subset \mathsf{SL}(d,\mathbb{R})$ is P_θ*-relatively Anosov* if Γ (as an abstract group) is relatively hyperbolic with respect to a finite collection $\mathcal{P}$ of finitely generated subgroups of Γ and there exists a Γ-equivariant embedding of the Bowditch boundary $\partial(\Gamma, P)$ into $\mathcal{F}_\theta$ with good dynamical properties.

Associated to $\theta \subset \Delta$ is a natural subspace of $\mathfrak{a}$ defined by

$$\mathfrak{a}_\theta := \{a \in \mathfrak{a} : \beta(a) = 0 \text{ if } \beta \in \Delta - \theta\}.$$

Then $\mathfrak{a}_\theta^*$ is generated by $\{\omega_j|_{\mathfrak{a}_\theta} : \alpha_j \in \theta\}$ where $\omega_j \in \mathfrak{a}^*$ is the fundamental weight associated to α_j and satisfies

$$\omega_j(\mathrm{diag}(a_1,\dots,a_d)) = a_1 + \cdots + a_j.$$

Hence we can identify $\mathfrak{a}_\theta^*$ as a subspace of $\mathfrak{a}^*$.

In Section 11 we will show that if $\Gamma \subset \mathsf{SL}(d,\mathbb{R})$ is P_θ-relatively Anosov, $\phi \in \mathfrak{a}_\theta^*$ and $\delta^\phi(\Gamma) < +\infty$, then there exists a GPS system on the Bowditch boundary of Γ whose periods are exactly the ϕ-lengths defined above (in fact we will show that this is more generally true for the wider class of P_θ-transverse groups in semisimple Lie groups). We then use results in [14] to verify that the conditions of Theorem 1.1 are satisfied and obtain the following counting result.

Corollary 1.7 (see Section 11 below)**.** *Suppose $\theta \subset \Delta$ is symmetric and $\Gamma \subset \mathsf{SL}(d,\mathbb{R})$ is P_θ-relatively Anosov with respect to $\mathcal{P}$. If $\phi \in \mathfrak{a}_\theta^*$ and $\delta := \delta^\phi(\Gamma) < +\infty$, then*

$$\#\{[\gamma]^w \in [\Gamma_{\mathrm{lox}}]^w : 0 < \phi(\lambda(\gamma)) \leq R\} \sim \frac{e^{\delta R}}{\delta R}.$$

Corollary 1.7 was previously known only when G has rank-one (see Roblin [41]), when Γ is the image of a relatively Anosov representation of a finitely generated torsion-free Fuchsian group [12], and when Γ is $\{\alpha_1, \alpha_{d-1}\}$-Anosov in $\mathsf{PGL}_d(\mathbb{R})$, preserves a properly convex domain in $\mathbb{P}(\mathbb{R}^d)$ and $\phi = \omega_1 + \omega_{d-1}$ [9, 50].

1.3. **Historical remarks.** There is a long history of using dynamical methods to obtain asymptotic counting results for the number of closed orbits of a flow. The first counting result of the form of Corollary 1.6 was established by Huber [26] for cocompact Fuchsian groups, as an application of Selberg's trace formula. In his Ph.D. thesis, Margulis [34] established mixing, counting and equidistribution results for negatively curved manifolds, and his proofs provide the template for much subsequent work. More generally, Margulis' work applies to all flows ϕ^t on closed manifolds M that are *Anosov*, i.e. for which there is an invariant splitting of the tangent bundle of M into the flow direction, a "stable" sub-bundle which is exponentially contracted by the flow, and a "unstable" sub-bundle which is exponentially dilated.

Margulis' approach, combined with the theory of Patterson–Sullivan measures [36, 46], was used in Roblin's work on (not necessarily cocompact) discrete isometry groups of CAT(-1) spaces [41]. This is the approach we use here. In the context of discrete subgroups of Lie groups, which is our main application, this method was also used in [50, 9] to obtain counting results for certain subgroups of $\mathsf{PGL}_n(\mathbb{R})$ preserving convex domains of $\mathbb{P}(\mathbb{R}^n)$. The results use the geometry of these domains, and are for a certain length function called the Hilbert length. The combination of Margulis's ideas and Patterson–Sullivan theory was also used to prove counting results for groups acting on rank-one CAT(0) spaces in [40]. Note that these convex projective and rank-one CAT(0) settings are not encompassed by the present work, as the natural boundary action in these cases are not necessarily convergence actions, due to the presence of flats.

Parry and Pollicott [35, Th. 2] used symbolic dynamics and the Thermodynamic Formalism to establish counting results for Axiom A flows, which are generalizations of Anosov flows to compact spaces other than manifolds, and this was extended to an even more general class of flows on compact spaces called metric Anosov or Smale flows in [38, Th. 8]. Sambarino [42, 43] established counting, mixing and equidistribution results for Anosov groups isomorphic to the fundamental groups of negatively curved manifolds M, by showing that the length functions coming from Anosov groups corresponds to periods of a reparametrization of the geodesic flow on T^1M, and this reparametrization is a metric Anosov flow, where counting results are already known. With additional results established by Bridgeman–Canary–Labourie–Sambarino [7], Sambarino's arguments generalize to all Anosov groups. A similar idea was used earlier by Benoist [4, Cor. 5.7] in the setting of Anosov representations acting cocompactly on strictly convex projective domains. Later Chow–Fromm [16] established a general result that implies Sambarino's counting result for Anosov groups.

Lalley [32] used symbolic dynamics and a renewal theorem to establish counting and equidistribution results for convex cocompact Fuchsian groups. This approach was generalized by Dal'bo and Peigné [21] to the setting of geometrically finite negatively curved manifolds with free fundamental group. This approach was later implemented in Bray–Canary–Kao–Martone [12] for relatively Anosov representations of finitely generated torsion-free Fuchsian groups.

"Local" mixing results can be used to obtain finer counting estimates. Chow–Sarkar [15] establish stronger mixing results in the Anosov case, which were used to obtain counting results in affine symmetric spaces by Edwards–Lee–Oh [22]. Recently, Delarue–Monclair–Sanders [19] obtained exponential mixing results in the $\{\alpha_1, \alpha_{d-1}\}$-Anosov case, yielding a counting estimate with an exponential error term.

1.4. **Outline of paper.** In Section 2, we recall some results from the theory of convergence group actions and also some results from [8] about GPS systems.

The first part of the paper, Sections 3, 4 and 6, follows a classical strategy to prove equidistribution of closed geodesics (Theorem 1.5), combining Margulis's ideas and Patterson–Sullivan theory. This portion of the work relies heavily, and follows fairly quickly, from the machinery developed in [8]. In Section 5, we adapt several classical criteria for non-arithmeticity of length spectra to our setting.

In Section 3, we construct natural stable/unstable manifolds in U_Γ. In Section 4, we use these stable/unstable manifolds and Coudène's criterion for mixing [17] to establish mixing (under a non-arithmeticity assumption). Finally, in Section 6, we combine the mixing property, the explicit product structure of the BMS measure, and a closing lemma (Lemma 6.7), to establish equidistribution of closed geodesics. In contrast to Roblin's [41] work in the setting of CAT(-1) spaces, we do not prove nor use an equidistribution of double Γ-orbits.

One notable difference with previous works concerns our construction of stable and unstable manifolds. Unlike previous settings such as those of CAT(-1) spaces or Hilbert geometries (see [6, Fact 6.11 & Prop. 6.14]), we do not have a natural metric on the flow space U_Γ to use to check that our algebraic definition matches Coudène's metric definition. We address this issue by observing that in fact any metric on the one-point compactification of U_Γ does the job.

The second part of the paper specializes to geometrically finite groups, and is more involved than the first. To be able to apply the results of the first part to geometrically finite groups, we need to build a decomposition of U_Γ into a union of a compact part and finitely many disjoint cusp-like parts, which are quotients of horoballs in $\tilde{U}_\Gamma$, and to understand the behavior of the BMS measure in these cusps.

In Section 7, we define a convenient topology on $\tilde{U}_\Gamma \sqcup \Lambda(\Gamma)$ that compactifies $\tilde{U}_\Gamma$, and we define a Dirchlet domain for the action of Γ in $\tilde{U}_\Gamma$. In Section 8, we define horoballs in the space $\tilde{U}_\Gamma$. We show that it is possible to construct a Γ-equivariant choice of disjoint horoballs at bounded parabolic points and that Γ acts cocompactly on the complement in $\tilde{U}_\Gamma$ of such a collection. In Section 9, we prove Theorem 1.4 which gives a criterion guaranteeing that geometrically finite groups are divergent and have finite BMS measure. In Section 10, we prove Theorem 1.6. As in Roblin [41, Th. 5.2], the proof is based on estimating the measure of "cusps" in U_Γ with respect to the measure m_R appearing in the statement of the theorem. One of the main ingredients is a version of the classical Shadow Lemma, which we verified for GPS systems in [8].

Finally, in Section 11 we explain how to apply our results in the setting of transverse and relatively Anosov subgroups of semisimple Lie groups.

Acknowledgements: We have been informed that the ongoing work of Kim and Oh [29] contains a proof, using different techniques, that the BMS measure associated to a relatively Anosov group is finite and that the flow is mixing with respect to BMS measure in this setting.

2. Background

2.1. **Convergence group actions.** When M is a compact metrizable space, a subgroup $\Gamma \subset \mathsf{Homeo}(M)$ is called a (discrete) *convergence group* if for every sequence $\{\gamma_n\}$ of distinct elements in Γ, there exist points $x, y \in M$ and a subsequence $\{\gamma_{n_j}\}$ such that $\gamma_{n_j}|_{M \smallsetminus \{y\}}$ converges locally uniformly to x. This notion was first introduced in [24]. Recall the following classification of elements of Γ.

Lemma 2.1 ([47, Th. 2B])**.** *If $\Gamma \subset \mathsf{Homeo}(M)$ is a convergence group, then every element $\gamma \in \Gamma$ is either*

- loxodromic*: it has two fixed points γ^+ and γ^- in M such that $\gamma^{\pm n}|_{M \smallsetminus \{\gamma^\mp\}}$ converges locally uniformly to $\gamma^\pm$,*
- parabolic*: it has one fixed point $p \in M$ such that $\gamma^{\pm n}|_{M \smallsetminus \{p\}}$ converges locally uniformly to p, or*
- elliptic*: it has finite order.*

Given a convergence group, as in [11, 47], we define the following:

(1) The *limit set* $\Lambda(\Gamma)$ is the set of points $x \in M$ where there exist $y \in M$ and a sequence $\{\gamma_n\}$ in Γ so that $\gamma_n|_{M \smallsetminus \{y\}}$ converges locally uniformly to x. (Note that fixed points of elements of Γ are in the limit set.)
(2) A point $x \in \Lambda(\Gamma)$ is a *conical limit point* if there exist distinct points $a, b \in M$ and a sequence of elements $\{\gamma_n\}$ in Γ where $\lim_{n\to\infty} \gamma_n(x) = a$ and $\lim_{n\to\infty} \gamma_n(y) = b$ for all $y \in M \smallsetminus \{x\}$.
(3) A point $p \in \Lambda(\Gamma)$ is a *bounded parabolic point* if no element of
$$\Gamma_p := \mathrm{Stab}_\Gamma(p)$$
is loxodromic and Γ_p acts cocompactly on $\Lambda(\Gamma) \smallsetminus \{p\}$.

We say that a convergence group Γ is *non-elementary* if $\Lambda(\Gamma)$ contains at least 3 points. In this case $\Lambda(\Gamma)$ is the smallest Γ-invariant closed subset of M (see [47, Th. 2S]). Finally, we say that a non-elementary convergence group Γ is *geometrically finite* if every point in $\Lambda(\Gamma)$ is either a conical limit point or a bounded parabolic point. The stabilizers of the bounded parabolic points are called the *maximal parabolic subgroups* of Γ.

The rest of this section recalls several results about convergence groups that we will need later. We first recall a closing lemma due to Tukia.

Lemma 2.2 (Tukia [47, Cor. 2E])**.** *Suppose $\Gamma \subset \mathsf{Homeo}(M)$ is a convergence group. If $\{\gamma_n\} \subset \Gamma$ is a sequence where $\gamma_n|_{M\smallsetminus\{b\}}$ converges locally uniformly to a and $a \neq b$, then for n sufficiently large γ_n is loxodromic, $\gamma_n^+ \to a$ and $\gamma_n^- \to b$.*

Lemmas 2.1 and 2.2 have the following immediate consequence.

Lemma 2.3 ([11, Prop. 3.2])**.** *If $\Gamma \subset \mathsf{Homeo}(M)$ is a convergence group, the stabiliser of $x \in M$ is infinite and contains no loxodromic elements, then x is not a conical limit point. In particular, parabolic fixed points are not conical limit points.*

If Γ is geometrically finite, there are only finitely many conjugacy classes of subgroups stabilizing a parabolic fixed point.

Lemma 2.4 ([48, Th. 1B])**.** *If $\Gamma \subset \mathsf{Homeo}(M)$ is a geometrically finite convergence group, then there are finitely many Γ-orbits of parabolic fixed points.*

In [8], we observed that the space M can be used to compactify Γ, see also [11, 13].

Definition 2.5. Given a convergence group $\Gamma \subset \mathsf{Homeo}(M)$, a *compactifying topology* on $\Gamma \sqcup M$ is a topology such that:

- $\Gamma \sqcup M$ is a compact metrizable space.
- The inclusions $\Gamma \hookrightarrow \Gamma \sqcup M$ and $M \hookrightarrow \Gamma \sqcup M$ are embeddings (where in the first embedding Γ has the discrete topology).
- Γ acts as a convergence group on $\Gamma \sqcup M$.

A metric d on $\Gamma \sqcup M$ is called *compatible* if it induces a compactifying topology.

In [8], we observed that compactifying topologies exist, are unique, and have the following properties.

Proposition 2.6 ([8, Prop. 2.3])**.** *If $\Gamma \subset \mathsf{Homeo}(M)$ is a convergence group, then there exists a unique compactifying topology on $\Gamma \sqcup M$. Moreover, with respect to this topology the following hold:*

(1) If $\{\gamma_n\} \subset \Gamma$ is a sequence where $\gamma_n \to a \in M$ and $\gamma_n^{-1} \to b \in M$, then $\gamma_n|_{M\smallsetminus\{b\}}$ converges locally uniformly to a.
(2) A sequence $\{\gamma_n\} \subset \Gamma$ converges to $a \in M$ if and only if for every subsequence $\{\gamma_{n_j}\}$ there exist $b \in M$ and a further subsequence $\{\gamma_{n_{j_k}}\}$ such that $\gamma_{n_{j_k}}|_{M\smallsetminus\{b\}}$ converges locally uniformly to a.
(3) Γ is open in $\Gamma \sqcup M$ and its closure is $\Gamma \sqcup \Lambda(\Gamma)$.

We will also use the following result about this compactification.

Proposition 2.7. *Suppose $\Gamma \subset \mathsf{Homeo}(M)$ is a convergence group. If $p \in M$ is a bounded parabolic point and $\Gamma_p := \mathsf{Stab}_\Gamma(p)$, then Γ_p acts cocompactly on $(\Gamma \sqcup \Lambda(\Gamma)) \smallsetminus \{p\}$.*

Proof. Fix a compact set $K_0 \subset \Lambda(\Gamma) \smallsetminus \{p\}$ such that $\Gamma_p \cdot K_0 = \Lambda(\Gamma) \smallsetminus \{p\}$. Then fix an open set $U \supset K_0$ in $\Gamma \sqcup \Lambda(\Gamma)$ such that $p \notin \overline{U}$. Fix an enumeration $\Gamma = \{\gamma_k\}$ so that $\gamma_1 = \mathrm{id}$ and let $F_n := \{\gamma_1, \ldots, \gamma_n\}$. Then $K_n := \overline{U} \cup F_n$ is compact in $(\Gamma \sqcup \Lambda(\Gamma)) \smallsetminus \{p\}$. We claim that for n sufficiently large, $\Gamma_p \cdot K_n = (\Gamma \sqcup \Lambda(\Gamma)) \smallsetminus \{p\}$.

Suppose not. Since $\Lambda(\Gamma) \smallsetminus \{p\} = \Gamma_p \cdot K_0$, then for each n we can find $\gamma_{k_n} \in \Gamma - \Gamma_p \cdot K_n$. Since $\gamma_{k_n} \notin \Gamma_p$, we must have $\gamma_{k_n}(p) \neq p$. Then for each $n \geq 1$, there exists $\alpha_n \in \Gamma_p$ such that $\alpha_n \gamma_{k_n}(p) \in K_0$. Since $\gamma_{k_n} \notin \Gamma_p \cdot F_n$, we see that $\{\alpha_n \gamma_{k_n}\}$ is an escaping sequence in Γ. So passing to a subsequence we can suppose that there exists $b^+, b^- \in M$ such that $\alpha_n \gamma_{k_n}(x) \to b^+$ for all $x \in M \smallsetminus \{b^-\}$. We can further suppose that $\alpha_n\gamma_{k_n}(p) \to a \in K_0$. If $a \neq b^+$, then $p = b^-$ and p is a conical limit point, which is impossible by Lemma 2.3. So we must have $b^+ = a \in K_0$. So for n large, $\alpha_n\gamma_{k_n} \in U$, which implies that $\gamma_{k_n} \in \Gamma \cdot K_n$. So we have a contradiction. $\square$

2.2. **Cocycles and Patterson–Sullivan measures.** In [8] we introduced the notion of an expanding cocycle and proved that the cocycles in a GPS system are expanding. In this section we recall the definition and their basic properties.

Recall from the introduction that given a cocycle σ, the *σ-magnitude* of $\gamma \in \Gamma$ is

$$\|\gamma\|_\sigma := \max_{x \in M} \sigma(\gamma, x).$$

Definition 2.8. Suppose $\Gamma \subset \mathsf{Homeo}(M)$ is a convergence group and d is a compatible distance on $\Gamma \sqcup M$. A continuous cocycle $\sigma\colon \Gamma \times M \to \mathbb{R}$ is *expanding* if

(1) σ is proper, i.e. $\|\gamma_n\|_\sigma \to \infty$ whenever $\{\gamma_n\}$ is an escaping sequence in Γ.
(2) For any $\epsilon > 0$ there exists $C > 0$ such that: if $\gamma \in \Gamma$, $x \in M$ and $\mathrm{d}(\gamma^{-1}, x) \geq \epsilon$, then

$$\sigma(\gamma, x) \geq \|\gamma\|_\sigma - C.$$

In previous work, we established the following results about expanding cocycles.

Proposition 2.9 ([8, Prop. 3.2]). *Suppose $\Gamma \subset \mathsf{Homeo}(M)$ is a convergence group, d is a compatible distance on $\Gamma \sqcup M$ and σ is an expanding cocycle, then:*

(1) $\ell_\sigma(\gamma) = \sigma(\gamma, \gamma^+) > 0$ for all loxodromic $\gamma \in \Gamma$ and $\sigma(\gamma, p) = 0$ for any parabolic element $\gamma \in \Gamma$ with fixed point p.
(2) If $\{\gamma_n\}$ is a divergent sequence in Γ and

$$\liminf_{n \to \infty} \sigma(\gamma_n, y_n) > -\infty,$$

then $\mathrm{d}(\gamma_n y_n, \gamma_n) \to 0$.
(3) For any $\epsilon > 0$ there exists $C > 0$ such that: if $\alpha, \beta \in \Gamma$ and $\mathrm{d}(\alpha^{-1}, \beta) \geq \epsilon$, then

$$\|\alpha\|_\sigma + \|\beta\|_\sigma - C \leq \|\alpha\beta\|_\sigma.$$

(4) For any compact subset $K \subset M^{(2)}$ there exists $C > 0$ such that: if $\gamma \in \Gamma$ is loxodromic and $(\gamma^-, \gamma^+) \in K$, then

$$\ell_\sigma(\gamma) \geq \|\gamma\|_\sigma - C.$$

Proof. The only claim that doesn't appear in [8, Prop. 3.2] is (4). Let

$$\Gamma_K := \{\gamma \in \Gamma : \gamma \text{ is loxodromic and } (\gamma^-, \gamma^+) \in K\}.$$

Since K is a compact subset of $M^{(2)}$, there exists $\epsilon > 0$ so that if $\gamma \in \Gamma_K$, then $\mathrm{d}(\gamma^+, \gamma^-) \geq \epsilon$. The set F_K of elements of Γ_K such that $\mathrm{d}(\gamma^{-1}, \gamma^-) \geq \frac{\epsilon}{2}$ is finite (since $\gamma_n - \to z$ if and only if $\gamma_n^{-1} \to z$). If $\gamma \in \Gamma_K \smallsetminus F_K$, then $\mathrm{d}(\gamma^+, \gamma^{-1}) \geq \frac{\epsilon}{2}$, so the expanding property implies that there exists $C_0 > 0$ such that

$$\ell_\sigma(\gamma_n) = \sigma(\gamma_n, \gamma_n^+) \geq \|\gamma_n\|_\sigma - C_0.$$

Claim (4), then follows if we take $C_1 := \max_{\gamma \in F_k} \|\gamma\|_\sigma$ and $C := \max(C_0, C_1)$. $\square$

Suppose $\Gamma \subset \mathsf{Homeo}(M)$ is a non-elementary convergence group and $\sigma\colon \Gamma \times M \to \mathbb{R}$ is a continuous cocycle. A probability measure μ on M is called a

σ-Patterson–Sullivan measure of dimension β if for every $\gamma \in \Gamma$ the measures $\gamma_*\mu$, μ are absolutely continuous and

$$\frac{d\gamma_*\mu}{d\mu} = e^{-\beta\sigma(\gamma^{-1},\cdot)}.$$

Recall from the introduction that the *σ-critical exponent* is

$$\delta_\sigma(\Gamma) = \limsup_{R\to\infty} \frac{1}{R} \log \# \{\gamma \in \Gamma : \|\gamma\|_\sigma \leq R\} \in [0,\infty].$$

Equivalently, $\delta_\sigma(\Gamma)$ is the critical exponent of the series

$$Q_\sigma(s) := \sum_{\gamma\in\Gamma} e^{-s\|\gamma\|_\sigma},$$

that is $Q_\sigma(s)$ diverges when $0 < s < \delta_\sigma(\Gamma)$ and converges when $s > \delta_\sigma(\Gamma)$.

In previous work, we showed if σ is an expanding cocycle for a convergence group action $\Gamma \subset \mathsf{Homeo}(M)$ with finite critical exponent $\delta_\sigma(\Gamma)$, then there is a σ-Patterson–Sullivan measure of dimension $\delta_\sigma(\Gamma)$. We also proved that this measure is unique and ergodic when the σ-Poincaré series diverges at its critical exponent.

Theorem 2.10 ([8, Th. 1.3 and Prop. 6.3])**.** *If σ is an expanding cocycle for a convergence group $\Gamma \subset \mathsf{Homeo}(M)$ and $\delta := \delta_\sigma(\Gamma) < +\infty$, then there exists a σ-Patterson–Sullivan measure of dimension δ, which is supported on the limit set $\Lambda(\Gamma)$. Moreover, if*

$$Q_\sigma(\delta) = \sum_{\gamma\in\Gamma} e^{-\delta\|\gamma\|_\sigma} = +\infty,$$

then:

1. *there is a unique σ-Patterson–Sullivan measure μ of dimension δ,*
2. *μ has no atoms, and*
3. *the action of Γ on (M,μ) is ergodic.*

2.3. **GPS systems and flow spaces.** We recall from [8] properties of the cocycles in a GPS system and the flow space associated to them.

Proposition 2.11 ([8, Prop. 3.3])**.** *Suppose $(\sigma, \bar\sigma, G)$ is a continuous GPS system for a convergence group $\Gamma \subset \mathsf{Homeo}(M)$.*

1. *There exists $C > 0$ such that*
$$\left\|\gamma^{-1}\right\|_{\bar\sigma} - C \leq \|\gamma\|_\sigma \leq \left\|\gamma^{-1}\right\|_{\bar\sigma} + C$$
for all $\gamma \in \Gamma$.
2. *$\delta_\sigma(\Gamma) = \delta_{\bar\sigma}(\Gamma)$.*
3. *σ and $\bar\sigma$ are expanding cocycles.*

We also established a version of the Hopf–Tsuji–Sullivan dichotomy.

Theorem 2.12. [8, Th. 1.7] *Suppose $(\sigma, \bar\sigma, G)$ is a continuous GPS system and $\delta_\sigma(\Gamma) < +\infty$. Let μ and $\bar\mu$ be Patterson–Sullivan measures of dimension δ for σ and $\bar\sigma$. Then $\nu := e^{\delta G}\bar\mu \otimes \mu$ is a locally finite Γ-invariant measure on $M^{(2)}$, and we have the following dichotomy:*

1. *If $\sum_{\gamma\in\Gamma} e^{-\delta\|\gamma\|_\sigma} = +\infty$, then:*
   - (a) *$\delta = \delta_\sigma(\Gamma)$.*
   - (b) *$\mu(\Lambda^{\rm con}(\Gamma)) = 1 = \bar\mu(\Lambda^{\rm con}(\Gamma))$.*

(c) *The* Γ *action on* $(M^{(2)}, \nu)$ *is ergodic and conservative.*

(2) *If* $\sum_{\gamma\in\Gamma} e^{-\delta\|\gamma\|_\sigma} < +\infty$*, then:*
   (a) $\delta \geq \delta_\sigma(\Gamma)$*.*
   (b) $\mu(\Lambda^{\rm con}(\Gamma)) = 0 = \bar{\mu}(\Lambda^{\rm con}(\Gamma))$*.*
   (c) *The* Γ *action on* $(M^{(2)}, \nu)$ *is non-ergodic and dissipative.*

Next we carefully describe the flow space associated to a GPS system $(\sigma, \bar{\sigma}, G)$, which was briefly described in the introduction. In the case when $M = \partial_\infty X$ is the geodesic boundary of a Hadamard manifold X (simply connected with pinched negative curvature), the group $\Gamma \subset \mathsf{Isom}(X)$ is discrete, and σ is the Busemann cocycle, this flow space is topologically conjugate to the non wandering part of the geodesic flow on the quotient $\Gamma\backslash T^1X$ of the unit tangent bundle T^1X of X.

For the rest of the section suppose $(\sigma, \bar{\sigma}, G)$ is a continuous GPS system for a convergence group $\Gamma \subset \mathsf{Homeo}(M)$. As in the introduction, let $\Lambda(\Gamma)^{(2)} \subset \Lambda(\Gamma) \times \Lambda(\Gamma)$ denote the set of distinct pairs and let $\tilde{U}_\Gamma := \Lambda(\Gamma)^{(2)} \times \mathbb{R}$.

By Proposition 2.11, the cocycles σ and $\bar{\sigma}$ are expanding. Hence, by [8, Prop. 10.2], the action of Γ on $\tilde{U}_\Gamma$ given by

$$\gamma(x, y, t) = (\gamma x, \gamma y, t + \sigma(\gamma, y))$$

is properly discontinuous. Therefore $U_\Gamma := \Gamma\backslash\tilde{U}_\Gamma$ is a locally compact metrizable space. Further the flow $\psi^t\colon \tilde{U}_\Gamma \to \tilde{U}_\Gamma$ defined by

$$\psi^t(x, y, s) = (x, y, s + t)$$

descends to a flow on U_Γ, which we also denote by ψ^t.

Now suppose, in addition, that $\delta := \delta_\sigma(\Gamma) < +\infty$ and $Q_\sigma(\delta) = +\infty$. By Theorem 2.10 and Proposition 2.11 there is a unique σ-Patterson–Sullivan measure μ of dimension δ and a unique $\bar{\sigma}$-Patterson–Sullivan measure $\bar{\mu}$ of dimension δ. By Equation (1), the measure $\tilde{m}$ on $\tilde{U}_\Gamma = \Lambda(\Gamma)^{(2)} \times \mathbb{R}$ defined by

$$\tilde{m} := e^{\delta G(x,y)} d\bar{\mu}(x) \otimes d\mu(y) \otimes dt$$

is flow-invariant and Γ-invariant. So $\tilde{m}$ descends to a flow-invariant measure m_Γ on the quotient $U_\Gamma = \Gamma\backslash\tilde{U}_\Gamma$ (see Section 2.4 below for the definition of quotient measures). We refer to m_Γ as the *Bowen–Margulis–Sullivan (BMS) measure* associated to $(\sigma, \bar{\sigma}, G)$. In [8, Th. 11.1] we used Theorem 2.12 to show that the flow is ergodic with respect to its Bowen-Margulis-Sullivan measure in this setting.

2.4. **Quotient measures.** In this short expository section we review properties of quotient measures. Suppose X is a proper metric space and $\tilde{\nu}$ is a locally finite Borel measure on X. Assume Γ is a discrete group which acts properly on X and preserves the measure $\tilde{\nu}$. Given a measurable function $f\colon X \to [0, +\infty]$, define $P(f)\colon \Gamma\backslash X \to [0, +\infty]$ by

$$P(f)([x]) = \sum_{\gamma\in\Gamma} f(\gamma x).$$

Then the quotient space $\Gamma\backslash X$ has a unique Borel measure ν such that

$$\int_{\Gamma\backslash X} P(f) d\nu = \int_X f d\tilde{\nu}. \tag{2}$$

for all measurable functions $f : X \to [0, +\infty]$. The existence of such a measure is classical and also follows from the discussion in [8, Appendix A].

Next suppose that $\phi^t\colon X \to X$ is a measurable flow which commutes with the Γ action and preserves the measure $\tilde{\nu}$. Then ϕ^t descends to a flow on the quotient $\Gamma\backslash X$ which we also denote by ϕ^t and preserves ν. We recall that $\phi^t\colon (\Gamma\backslash X, \nu) \to (\Gamma\backslash X, \nu)$ is *mixing* if $\|\nu\| := \nu(\Gamma\backslash X)$ is finite and whenever A and B are measurable subsets of $\Gamma\backslash X$, we have

$$\lim_{t\to\infty} \nu(A \cap \gamma\phi^t(B)) = \frac{\nu(A)\nu(B)}{\|\nu\|}.$$

The following observation interprets this fact in terms of the flow on X.

Observation 2.13. Suppose $\|\nu\| = \nu(\Gamma\backslash X) < +\infty$ and $\phi^t\colon (\Gamma\backslash X, \nu) \to (\Gamma\backslash X, \nu)$ is mixing. If $A, B \subset X$ have finite $\tilde{\nu}$-measure, then

$$\lim_{t\to\infty} \sum_{\gamma\in\Gamma} \tilde{\nu}(A \cap \gamma\phi^t(B)) = \frac{\tilde{\nu}(A)\tilde{\nu}(B)}{\|\nu\|}.$$

Proof. Notice that

$$P(f) \cdot P(g) = \sum_{\gamma\in\Gamma} P\big(f \cdot (g \circ \gamma)\big).$$

Hence

$$\begin{aligned}\sum_{\gamma\in\Gamma} \tilde{\nu}(A \cap \gamma\phi^{-t}(B)) &= \sum_{\gamma\in\Gamma} \int 1_A \cdot (1_B \circ \phi^t \circ \gamma^{-1}) d\tilde{\nu} \\ &= \sum_{\gamma\in\Gamma} \int P\big(1_A \cdot (1_B \circ \phi^t \circ \gamma^{-1})\big) d\nu = \int P(1_A) \cdot (P(1_B) \circ \phi^t) d\nu\end{aligned}$$

which tends to $\dfrac{\int P(1_A)d\nu \int P(1_B)d\nu}{\|\nu\|} = \dfrac{\tilde{\nu}(A)\tilde{\nu}(B)}{\|\nu\|}$ as $t \to \infty$. $\square$

3. Stable and unstable manifolds

Given a flow $\phi^t\colon X \to X$ on a metric space, the *strongly stable manifold of* $v \in X$ is

$$W^{ss}(v) := \left\{ w \in X : \lim_{t\to\infty} \mathrm{d}(\phi^t(v), \phi^t(w)) = 0 \right\}$$

and the *strongly unstable manifold of* v is

$$W^{su}(v) := \left\{ w \in X : \lim_{t\to-\infty} \mathrm{d}(\phi^t(v), \phi^t(w)) = 0 \right\}.$$

In this section we study these sets for the flow associated to a GPS system. (In our general setting we do not expect these to be manifolds, but the terminology is conventional.)

Fix a GPS system $(\sigma, \bar{\sigma}, G)$ for a convergence group action $\Gamma \subset \mathsf{Homeo}(M)$ and let

$$p\colon \tilde{U}_\Gamma \to U_\Gamma$$

denote the quotient map. As observed in Section 2.3, the quotient $U_\Gamma = \Gamma\backslash\tilde{U}_\Gamma$ is a locally compact metrizable space. Hence the one-point compactification $U_\Gamma \sqcup \{\infty\}$ of U_Γ admits a metric $\mathrm{d}_\star$, see [33]. Then for $v \in U_\Gamma$, let $W^{ss}(v)$ and $W^{su}(v)$ denote the strongly stable and unstable manifolds for the metric $\mathrm{d}_\star$ restricted to U_Γ.

We first show that the strongly stable manifold of $p(v^-, v^+, t_0)$ contains quotients of all elements with the same forward endpoint v^+ and time parameter t_0.

Proposition 3.1. *If $(v^-, v^+, t_0) \in \tilde{U}_\Gamma$ and $v := p(v^-, v^+, t_0) \in U_\Gamma$, then*

$$p(y, v^+, t_0) \in W^{ss}(v)$$

for all $y \in \Lambda(\Gamma) \smallsetminus \{v^+\}$.

Proof. Fix $y \in \Lambda(\Gamma) \smallsetminus \{v^+\}$ and let $w := p(y, v^+, t_0)$. Then fix a sequence $\{t_n\}$ where $t_n \to \infty$ and

$$\limsup_{t\to\infty} \mathrm{d}_\star(\psi^t(v), \psi^t(w)) = \lim_{n\to\infty} \mathrm{d}_\star(\psi^{t_n}(v), \psi^{t_n}(w)).$$

Passing to a subsequence we can assume that one of the following cases hold:

Case 1: Assume $\{\psi^{t_n}(v)\}$ is relatively compact in U_Γ. Passing to a further subsequence, we can find a sequence $\{\gamma_n\}$ in Γ such that

$$\lim_{n\to\infty} \gamma_n(v^-, v^+, t_n + t_0) = \lim_{n\to\infty} (\gamma_n(v^-), \gamma_n(v^+), t_n + t_0 + \sigma(\gamma_n, v^+))$$

exists in $\tilde{U}_\Gamma$. Passing to another subsequence we can suppose that $\gamma_n \to a \in \Lambda(\Gamma)$, $\gamma_n^{-1} \to b \in \Lambda(\Gamma)$. Then $\gamma_n(z) \to a$ uniformly on compact subsets of $M \smallsetminus \{b\}$.

Notice that, since σ is a cocycle,

$$0 = \sigma(\mathrm{id}, v^+) = \sigma(\gamma_n^{-1}\gamma_n, \gamma_n^{-1}(\gamma_n(v^+)) = \sigma(\gamma_n^{-1}, \gamma_n(v^+)) + \sigma(\gamma_n, v^+).$$

Since $t_n \to +\infty$, we must have

$$\lim_{n\to\infty} \sigma(\gamma_n^{-1}, \gamma_n(v^+)) = \lim_{n\to\infty} -\sigma(\gamma_n, v^+) = +\infty.$$

So Proposition 2.9(2) implies that $\mathrm{d}(\gamma_n^-(\gamma_n(v^+)), \gamma_n^{-1}) = \mathrm{d}(v^+, \gamma_n^{-1}) \to 0$. Since $\gamma_n^{-1} \to b$, we see that $b = v^+$. Therefore, $\gamma_n(z) \to a$ for all $z \in \Lambda(\Gamma) \smallsetminus \{v^+\}$. So

$$\lim_{n\to\infty} \gamma_n(y) = a = \lim_{n\to\infty} \gamma_n(v^-).$$

Thus

$$\lim_{n\to\infty} \gamma_n\psi^{t_n}(y, v^+, t_0) = \lim_{n\to\infty} \gamma_n\psi^{t_n}(v^-, v^+, t_0),$$

which implies that

$$\lim_{n\to\infty} \mathrm{d}_\star(\psi^{t_n}(v), \psi^{t_n}(w)) = 0$$

and hence that $w \in W^{ss}(v)$.

Case 2: Assume $\{\psi^{t_n}(w)\}$ is relatively compact in U_Γ. By Case 1, $v \in W^{ss}(w)$ so $w \in W^{ss}(v)$.

Case 3: Assume $\{\psi^{t_n}(v)\}$ and $\{\psi^{t_n}(w)\}$ both converge to ∞ in $U_\Gamma \sqcup \{\infty\}$. Then

$$\lim_{n\to\infty} \mathrm{d}_\star(\psi^{t_n}(v), \psi^{t_n}(w)) = \mathrm{d}_\star(\infty, \infty) = 0$$

and hence $w \in W^{ss}(v)$. ◻

We establish the analogous result for the strongly unstable manifold.

Proposition 3.2. *If $(v^-, v^+, t_0) \in \tilde{U}_\Gamma$ and $v := p(v^-, v^+, t_0) \in U_\Gamma$, then*

$$p\left(v^-, x, t_0 + G(v^-, x) - G(v^-, v^+)\right) \in W^{su}(v)$$

for all $x \in \Lambda(\Gamma) \smallsetminus \{v^-\}$.

Proof. Fix $x \in \Lambda(\Gamma) \smallsetminus \{v^-\}$ and let

$$w := p\left(v^-, x, t_0 + G(v^-, x) - G(v^-, v^+)\right).$$

Then fix a sequence $\{t_n\}$ where $t_n \to -\infty$ and

$$\limsup_{t \to -\infty} \mathrm{d}_\star(\psi^t(v), \psi^t(w)) = \lim_{n \to \infty} \mathrm{d}_\star(\psi^{t_n}(v), \psi^{t_n}(w)).$$

Passing to a subsequence we can assume that one of the following cases hold:

Case 1: Assume $\{\psi^{t_n}(v)\}$ is relatively compact in U_Γ. Then passing to a further subsequence, we can find a sequence $\{\gamma_n\}$ in Γ such that

$$\lim_{n \to \infty} \gamma_n(v^-, v^+, t_n + t_0) = \lim_{n \to \infty} (\gamma_n(v^-), \gamma_n(v^+), t_n + t_0 + \sigma(\gamma_n, v^+))$$

exists in $\tilde{U}_\Gamma$. Passing to another subsequence we can suppose that $\gamma_n \to a \in \Lambda(\Gamma)$, $\gamma_n^{-1} \to b \in \Lambda(\Gamma)$. Then $\gamma_n(z) \to a$ uniformly on compact subsets of $M \smallsetminus \{b\}$.

Since $t_n \to -\infty$, we must have

$$\lim_{n \to \infty} \sigma(\gamma_n, v^+) = +\infty.$$

So Proposition 2.9(2) implies that $\gamma_n(v^+) \to a$. Let $b' := \lim_{n \to \infty} \gamma_n(v^-)$. Since

$$\lim_{n \to \infty} \gamma_n(v^-, v^+) = (b', a) \in \Lambda(\Gamma)^{(2)}$$

and $\gamma_n(z) \to a$ for all $z \in M \smallsetminus \{b\}$, we must have $v^- = b$. Then $\gamma_n(x) \to a$ since $x \in \Lambda(\Gamma) \smallsetminus \{v^-\} = \Lambda(\Gamma) \smallsetminus \{b\}$.

Then, by Equation (1),

$$\begin{aligned}
\lim_{n \to \infty} \sigma(\gamma_n, v^+) - \sigma(\gamma_n, x) &= \lim_{n \to \infty} \Big(\bar{\sigma}(\gamma_n, v^-) + \sigma(\gamma_n, v^+)\Big) - \Big(\bar{\sigma}(\gamma_n, v^-) + \sigma(\gamma_n, x)\Big) \\
&= \lim_{n \to \infty} G(\gamma_n(v^-), \gamma_n(v^+)) - G(v^-, v^+) - G(\gamma_n(v^-), \gamma_n(x)) + G(v^-, x) \\
&= G(b', a) - G(v^-, v^+) - G(b', a) + G(v^-, x) \\
&= G(v^-, x) - G(v^-, v^+).
\end{aligned}$$

Thus

$$\lim_{n \to \infty} t_n + t_0 + \sigma(\gamma_n, v^+) = \lim_{n \to \infty} t_n + t_0 + G(v^-, x) - G(v^-, v^+) + \sigma(\gamma_n, x)$$

and so

$$\lim_{n \to \infty} \gamma_n \psi^{t_n}(v^-, v^+, t_0) = \lim_{n \to \infty} \gamma_n \psi^{t_n}(v^-, x, t_0 + G(v^-, x) - G(v^-, v^+)),$$

which implies that

$$\lim_{n \to \infty} \mathrm{d}_\star(\psi^{t_n}(v), \psi^{t_n}(w)) = 0$$

and hence that $w \in W^{su}(v)$.

Case 2: Assume $\{\psi^{t_n}(w)\}$ is relatively compact in U_Γ. By Case 1, $v \in W^{su}(w)$ so $w \in W^{su}(v)$.

Case 3: Assume $\{\psi^{t_n}(v)\}$ and $\{\psi^{t_n}(w)\}$ both converge to ∞ in $U_\Gamma \sqcup \{\infty\}$. Then

$$\lim_{n \to \infty} \mathrm{d}_\star(\psi^{t_n}(v), \psi^{t_n}(w)) = \mathrm{d}_\star(\infty, \infty) = 0$$

and hence $w \in W^{su}(v)$. $\square$

4. Mixing

In this section we establish mixing for the flow associated to a GPS system when the Bowen–Margulis is finite and the length spectrum is non-arithmetic. In fact, we can slightly weaken the assumption of non-arithmetic length spectrum and instead assume that the "cross ratios" generate a dense subgroup of $\mathbb{R}$.

For the rest of the section suppose that $(\sigma, \bar{\sigma}, G)$ is a continuous GPS system for a convergence group action $\Gamma \subset \mathsf{Homeo}(M)$ where $\delta := \delta_\sigma(\Gamma) < +\infty$ and $Q_\sigma(\delta) = +\infty$. Let m_Γ denote the BMS measure on U_Γ constructed in Section 2.3.

We define a cross ratio

$$B(x, x', y, y') := G(x, y) + G(x', y') - G(x', y) - G(x, y')$$

for $x, x', y, y' \in M$ such that $\{x, x'\}$ and $\{y, y'\}$ are disjoint. We then define the *cross ratio spectrum*

$$\mathcal{CR} := \left\{B(x, x', y, y') : y, y', x, x' \in \Lambda(\Gamma) \text{ and } \{x, x'\} \cap \{y, y'\} = \varnothing\right\}.$$

We say that the cross ratio spectrum is non-arithmetic if it generates a dense subgroup of $\mathbb{R}$.

The next lemma shows that the cross ratio spectrum $\mathcal{CR}$ contains the length spectrum

$$\mathcal{L}(\sigma, \bar{\sigma}) = \{\ell_\sigma(\gamma) + \ell_{\bar{\sigma}}(\gamma) : \gamma \in \Gamma \text{ loxodromic}\}.$$

Lemma 4.1. *If $\gamma \in \Gamma$ is loxodromic and $x \in M - \{\gamma^+, \gamma^-\}$, then*

$$B(x, \gamma x, \gamma^-, \gamma^+) = \ell_\sigma(\gamma) + \ell_{\bar{\sigma}}(\gamma).$$

Proof. Notice that

$$\bar{\sigma}(\gamma, \gamma^+) + \sigma(\gamma, \gamma^-) = G(\gamma^+, \gamma^-) - G(\gamma^+, \gamma^-) = 0.$$

So, by Equation (1),

$$\begin{aligned} B(x, \gamma x, \gamma^-, \gamma^+) &= G(x, \gamma^-) + G(\gamma x, \gamma^+) - G(\gamma x, \gamma^-) - G(x, \gamma^+) \\ &= G(\gamma x, \gamma^+) - G(x, \gamma^+) - G(\gamma x, \gamma^-) + G(x, \gamma^-) \\ &= \bar{\sigma}(\gamma, x) + \sigma(\gamma, \gamma^+) - \bar{\sigma}(\gamma, x) - \sigma(\gamma, \gamma^-) \\ &= \sigma(\gamma, \gamma^+) + \bar{\sigma}(\gamma, \gamma^+) = \ell_\sigma(\gamma) + \ell_{\bar{\sigma}}(\gamma). \end{aligned}$$

□

By Lemma 4.1, the following theorem is a (slight) extension of Theorem 1.3

Theorem 4.2. *If the BMS measure m_Γ is finite and the cross ratio spectrum $\mathcal{CR}$ is non-arithmetic, then the flow $\psi^t : (U_\Gamma, m_\Gamma) \to (U_\Gamma, m_\Gamma)$ is mixing.*

Our proof is inspired by earlier work of Blayac [6] in the setting of rank-one convex projective manifolds with compact convex core. This strategy of proof goes back to work of Babillot [1]. In particular, to establish mixing, we will use a criterion due to Coudène. To state his result we need a preliminary definition.

Suppose is a measurable flow $\phi^t : X \to X$ on a metric space and μ is a flow-invariant measure, then a function $f : X \to \mathbb{R}$ is W^{ss}*-invariant* if there exists a full μ-measure subset $X' \subset X$ such that if $v, w \in X'$ and $w \in W^{ss}(v)$, then $f(v) = f(w)$. Likewise, f is W^{su}*-invariant* if there exists a full μ-measure subset $X'' \subset X$ such that if $v, w \in X''$ and $w \in W^{su}(w)$, then $f(v) = f(w)$.

Proposition 4.3 (Coudène [17])**.** *Let X be a metric space, μ be a finite Borel measure on X, and $\phi^t\colon X \to X$ a measure-preserving flow on X. If any measurable function which is W^{ss}-invariant and W^{su}-invariant is constant almost everywhere, then $\phi^t\colon (X,\mu) \to (X,\mu)$ is mixing.*

We are now ready to prove the theorem.

Proof of Theorem 4.2. As in Section 3, we fix a metric $\mathrm{d}_\star$ on the one-point compactification of U_Γ and consider the stable/unstable manifolds relative to this metric.

By Coudène's result, it suffices to show that every measurable function f on U_Γ which is W^{ss}-invariant and W^{su}-invariant is m_Γ-almost everywhere constant. Let f be such a function, and let $\tilde{f}$ denote the lift of f to $\tilde{U}_\Gamma$. Let $A \subset U_\Gamma$ be a full measure subset such that for all $v, v' \in A$, if $v' \in W^{ss}(v)$ or $W^{su}(v)$ then $f(v') = f(v)$. Let $\tilde{A} \subset \tilde{U}_\Gamma$ denote the preimage of A.

By Theorem 2.10, the measures μ and $\bar{\mu}$ have no atoms and so $\tilde{A}$ is a full measure set for the product measure $\bar{\mu} \otimes \mu \otimes dt$ on $\Lambda(\Gamma) \times \Lambda(\Gamma) \times \mathbb{R}$.

For $(x,y) \in \Lambda(\Gamma)^{(2)}$ and $y' \in \Lambda(\Gamma) \smallsetminus \{x\}$, let

$$\rho_{x,y}(y') = G(x,y') - G(x,y).$$

Notice that $\rho_{x,y}(y') + \rho_{x',y'}(y) = -B(x,x',y,y')$.

Since f is W^{ss}- and W^{su}-invariant, by Propositions 3.1 and 3.2 we have

$$\tilde{f}(x,y,t) = \tilde{f}(x',y,t) = \tilde{f}(x,y',t+\rho_{x,y}(y'))$$

for $\bar{\mu}^2 \otimes \mu^2 \otimes dt$-almost any (x,x',y,y',t). Since, for $\bar{\mu}^2 \otimes \mu^2 \otimes dt$-almost any (x,x',y,y',t) we have $(x,y,t) \in \tilde{A}$ and $(x',y,t) \in \tilde{A}$ and $(x,y',t+\rho_{x,y}(y')) \in \tilde{A}$, and the projection of (x',y,t) in U_Γ is in the strong stable manifold of the projection of (x,y,t) (by Proposition 3.1) while $(x,y',t+\rho_{x,y}(y'))$ is in the strong unstable manifold of (x,y,t) (by Proposition 3.2).

Hence, by Fubini's Theorem, we can find $x_0, y_0 \in \Lambda(\Gamma)$ such that

$$\tilde{f}(x_0,y_0,t) = \tilde{f}(x_0,y,t+\rho_{x_0,y_0}(y)) = \tilde{f}(x,y,t+\rho_{x_0,y_0}(y))$$

for $\bar{\mu} \otimes \mu \otimes dt$-almost any (x,y,t).

In particular it suffices to show that $g(t) := \tilde{f}(x_0,y_0,t)$ is Lebesgue-almost everywhere constant. Consider the additive subgroup

$$\mathsf{H} := \{\tau \in \mathbb{R} : g(t+\tau) = g(t) \text{ for Lebesgue-almost any } t \in \mathbb{R}\}.$$

The following classical result says that H is a closed subgroup of $\mathbb{R}$. For the reader's convenience we recall its proof after we finish the current proof.

Lemma 4.4. *If $g\colon \mathbb{R} \to \mathbb{R}$ be a measurable function and $\mathsf{H}(g) := \{\tau \in \mathbb{R} : g(t+\tau) = g(t)$ for Lebesgue-almost any $t \in \mathbb{R}\}$, then $\mathsf{H}(g)$ is a closed subgroup of $\mathbb{R}$.*

We now claim that $\mathcal{CR} \subset \mathsf{H}$. The assumption that $\mathcal{CR}$ generate a dense subgroup of $\mathbb{R}$ then implies that $\mathsf{H} = \mathbb{R}$, and hence that g is Lebesgue-almost everywher constant, as desired.

To this end, we observe that, for $\bar{\mu}$-almost all x and x', μ-almost all y and y' and Lebesgue-almost any t, the four points x, x', y, y' are distinct (since μ and $\bar{\mu}$ do not have atoms by Theorem 2.10) and

$$\begin{aligned} g(t) &= \tilde{f}(x, y, t + \rho_{x_0,y_0}(y)) \\ &= \tilde{f}(x', y', t + \rho_{x_0,y_0}(y) + \rho_{x,y}(y')) \\ &= \tilde{f}(x, y, t + \rho_{x_0,y_0}(y) + \rho_{x,y}(y') + \rho_{x',y'}(y)) \\ &= \tilde{f}(x, y, t + \rho_{x_0,y_0}(y) - B(x, x', y, y')) = g(t - B(x, x', y, y')). \end{aligned}$$

So $B(x, x', y, y') \in \mathsf{H}$.

Since the Patterson–Sullivan measures μ and $\bar{\mu}$ have full support in $\Lambda(\Gamma)$, B is continuous, and H is closed, we obtain that $\mathcal{CR} \subset \mathsf{H}$. Therefore, g is Lebesgue-almost everywhere constant, so $\tilde{f}$ is $\tilde{m}$-almost everywhere constant, so f is m_Γ-almost everywhere constant, which completes the proof. ☐

Proof of Lemma 4.4. The fact that $\mathsf{H}(g)$ is a subgroup of $\mathbb{R}$ is an immediate consequence of the invariance of the Lebesgue measure under translation. It remains to check $\mathsf{H}(g)$ is closed.

Note that

$$\mathsf{H}(g) = \bigcap_{R>0} \mathsf{H}(g \cdot 1_{|g| \leq R}),$$

so we can assume ϕ is bounded by some $R > 0$.

For any compactly supported continuous function $\alpha \colon \mathbb{R} \to \mathbb{R}$, let

$$\mathsf{H}_\alpha(g) := \left\{ \tau : \int \alpha(t) g(t + \tau) dt = \int \alpha(t) g(t) dt \right\}.$$

It is an easy exercise in measure theory that

$$\mathsf{H}(g) = \bigcap_{\alpha \in C_c(\mathbb{R})} \mathsf{H}_\alpha(g).$$

So it suffices to prove $\mathsf{H}_\alpha(g)$ is closed. This a consequence of the fact that

$$\int \alpha(t) g(t + \tau) dt = \int \alpha(t - \tau) g(t) dt$$

is continuous in τ, which follows from the fact that α is uniformly continuous. ☐

Remark 4.5. (1) It follows from Theorem 1.5, that if m_Γ is mixing then $\{\ell_\sigma(\gamma) : \gamma \in \Gamma \text{ loxodromic}\}$ generates a dense subgroup of $\mathbb{R}$. However this does not imply that the length spectrum $\mathcal{L}(\sigma, \bar{\sigma})$ is non-arithmetic, unless σ is symmetric (i.e. if $\sigma = \bar{\sigma}$). If σ is symmetric, we have an equivalence between m_Γ being mixing, the length spectrum being non-arithmetic, and the set of cross ratios generating a dense subgroup of $\mathbb{R}$.

(2) There exist cases when m_Γ is finite but not mixing (and the length spectrum generates a discrete group). For instance, take M to be the Gromov boundary of an infinite 4-regular tree with all edges of length 1, take Γ to be the nonabelian free group with two generators acting on M, and define σ and $\bar{\sigma}$ to be the Busemann cocycles on this CAT(-1) space. Then the free group action is a uniform convergence action, hence m_Γ is finite, but the length spectrum generates a discrete additive subgroup which is contained in $\mathbb{Z}$, the measure m_Γ is not mixing, and $Re^{-\delta_\Gamma R} \#\{[\gamma] \in [\Gamma] : \ell_\sigma(\gamma) \leq R\}$ does not converge as $R \to \infty$.

5. Non-arithmeticity of the cross ratio spectrum

In this section, we investigate when the length spectrum or cross ratio spectrum is non-arithmetic. Our results and arguments are very similar to earlier work in the context of Riemannian manifolds, see Dal'bo [18, §II].

When Γ contains a parabolic element, the length spectrum itself is always non-arithmetic.

Proposition 5.1. *Suppose $(\sigma, \bar{\sigma}, G)$ is a continuous GPS system for a convergence group $\Gamma \subset \mathsf{Homeo}(M)$. If Γ contains a parabolic element, then the length spectrum $\mathcal{L}(\sigma, \bar{\sigma})$ is non-arithmetic.*

Proof. Let $\alpha \in \Gamma$ be a parabolic element with fixed point $p \in M$ and choose $\beta \in \Gamma$ that does not fix p. Set $\gamma_n := \beta\alpha^n$ for each $n \geq 0$. Note that $\gamma_n \to \beta(p)$ while $\gamma_n^{-1} = \alpha^{-n}\beta^{-1} \to p \neq \beta(p)$. So, by Lemma 2.2, for n large enough γ_n is loxodromic with $\gamma_n^+ \to \beta(p)$ and $\gamma_n^- \to p$. Hence it suffices to prove that

$$\lim_{n\to\infty} \ell_\sigma(\gamma_{n+1}) - \ell_\sigma(\gamma_n) = 0 = \lim_{n\to\infty} \ell_{\bar{\sigma}}(\gamma_{n+1}) - \ell_{\bar{\sigma}}(\gamma_n).$$

By definition,

$$\ell_\sigma(\gamma_n) = \sigma(\beta\alpha^n, \gamma_n^+) = \sigma(\beta, \alpha^n\gamma_n^+) + \sigma(\alpha^n, \gamma_n^+).$$

Notice that $\sigma(\beta, \alpha^n\gamma_n^+) \to \sigma(\beta, p)$. Further $\sigma(\alpha^{n+1}, \gamma_{n+1}^+) = \sigma(\alpha, \alpha^n(\gamma_{n+1}^+)) + \sigma(\alpha^n, \gamma_{n+1}^+)$ and $\sigma(\alpha, \alpha^n(\gamma_{n+1}^+)) \to \sigma(\alpha, p) = 0$. So,

$$\limsup_{n\to\infty} |\ell_\sigma(\gamma_{n+1}) - \ell_\sigma(\gamma_n)| = \limsup_{n\to\infty} \left|\sigma(\alpha^n, \gamma_{n+1}^+) - \sigma(\alpha^n, \gamma_n^+)\right|.$$

Since $\alpha^{-n}\beta(p) \to p$, we see that

$$\begin{aligned}\sigma(\alpha^n, &\gamma_{n+1}^+) - \sigma(\alpha^n, \gamma_n^+) = \bar{\sigma}(\alpha^n, \alpha^{-n}\beta(p)) + \sigma(\alpha^n, \gamma_{n+1}^+) - \bar{\sigma}(\alpha^n, \alpha^{-n}\beta(p)) - \sigma(\alpha^n, \gamma_n^+)\\ &= G(\beta(p), \alpha^n(\gamma_{n+1}^+)) - G(\alpha^{-n}\beta(p), \gamma_{n+1}^+) - G(\beta(p), \alpha^n\gamma_n^+) + G(\alpha^{-n}\beta(p), \gamma_n^+)\\ &\to G(\beta(p), p) - G(p, \beta(p)) - G(\beta(p), p) + G(p, \beta(p)) = 0.\end{aligned}$$

The proof that $\ell_{\bar{\sigma}}(\gamma_{n+1}) - \ell_{\bar{\sigma}}(\gamma_n) \to 0$ is completely analogous. □

We verify that the cross ratio spectrum is non-arthimetic in the following cases.

Proposition 5.2. *Suppose $(\sigma, \bar{\sigma}, G)$ is a continuous GPS system for a convergence group $\Gamma \subset \mathsf{Homeo}(M)$. If any one of the following hold, then the cross ratio spectrum $\mathcal{CR}$ is non-arithmetic:*

1. *there exists $x \in \Lambda(\Gamma)$ so that the path component of $\Lambda(\Gamma)$ containing x is infinite and $\lim_{y\in\Lambda(\Gamma)\to x} G(x, y) = +\infty$,*
2. *there exists a conical limit point $x \in \Lambda^{\mathrm{con}}(\Gamma)$ so that the path component of $\Lambda(\Gamma)$ containing x is infinite, or*
3. *Γ is uniform convergence group (i.e. every limit point is conical) and is not virtually free.*

The rest of the section is devoted to the proof of Proposition 5.2. We will see that (3) is a particular case of (2), which is a particular case of (1).

Proof of non-arithmeticity given (1). Fix a continuous path $c\colon [0,1] \to \Lambda(\Gamma)$ such that $c(0) = x$ and $c(0) \neq c(t)$ for any $t \in (0, 1]$ and $c(\frac{1}{2}) \neq c(1)$. The function

$$\begin{aligned}t \in (0, \epsilon) \mapsto &B(c(0), c(\tfrac{1}{2}), c(t), c(1))\\ &= G(c(0), c(t)) + G(c(\tfrac{1}{2}), c(1)) - G(c(\tfrac{1}{2}), c(t)) - G(c(0), c(1))\end{aligned}$$

is well-defined and continuous for $\epsilon > 0$ small enough, and goes to infinity as $t \to 0$ because $G(c(0), c(t)) \to \infty$ by assumption, while $G(c(\frac{1}{2}), c(t)) \to G(c(\frac{1}{2}), c(0))$. By the intermediate value theorem, the image of that function contains a whole nontrivial segment of $\mathbb{R}$ and hence $\mathcal{CR}$ generates $\mathbb{R}$ as a group. $\square$

The next two lemmas show that the condition $\lim_{y\in\Lambda(\Gamma)\to x} G(x,y) = +\infty$ is automatically satisfied at conical and parabolic limit points. It follows from the previous part and the next lemma that (2) implies non-arithmetic cross ratio spectrum.

Lemma 5.3. *If $x \in \Lambda(\Gamma)$ is conical, then*

$$\lim_{(y,z)\in M^{(2)}\to(x,x)} G(y,z) = +\infty.$$

Proof. Since x is conical there exist $\{\gamma_k\} \subset \Gamma$ and $a \neq b \in \Lambda(\Gamma)$ such that $\gamma_k x \to a$ and $\gamma_k y \to b$ for any $y \neq x$, which implies $\gamma_k \to b$.

Then by the expanding property there exists a constant $C > 0$ such that for any k we have

$$\sigma(\gamma_k^{-1}, \gamma_k x) \geq ||\gamma_k^{-1}||_\sigma - C \quad \text{and} \quad \bar{\sigma}(\gamma_k^{-1}, \gamma_k x) \geq ||\gamma_k^{-1}||_{\bar{\sigma}} - C.$$

Using Equation (1), the non-negativity of G, and the continuity of $\sigma(\gamma_k, \cdot)$ we obtain

$$\begin{aligned}
\liminf_{y\neq z\to x} G(y,z) &= \liminf_{y\neq z\to x} G(\gamma_k y, \gamma_k z) + \bar{\sigma}(\gamma_k^{-1}, \gamma_k y) + \sigma(\gamma_k^{-1}, \gamma_k z) \\
&\geq \bar{\sigma}(\gamma_k^{-1}, \gamma_k x) + \sigma(\gamma_k^{-1}, \gamma_k x) \\
&\geq \left\|\gamma_k^{-1}\right\|_{\bar{\sigma}} - C + \left\|\gamma_k^{-1}\right\|_\sigma - C.
\end{aligned}$$

Letting $k \to \infty$ we get $\liminf_{y\neq z\to x} G(y,z) = +\infty$. $\square$

Lemma 5.4. *If $x \in \Lambda(\Gamma)$ is bounded parabolic point, then*

$$\lim_{y\in\Lambda(\Gamma)\to x} G(x,y) = +\infty = \lim_{y\in\Lambda(\Gamma)\to x} G(y,x).$$

Proof. Since x is a bounded parabolic point there exists a compact subset K of $\Lambda(\Gamma) \smallsetminus \{x\}$ such that $\Gamma_x(K) = \Lambda(\Gamma) \smallsetminus \{x\}$.

Fix a sequence $\{y_n\} \subset \Lambda(\Gamma) \smallsetminus \{x\}$ converging to x. For each n, choose $\gamma_n \in \Gamma_x$ so that $\gamma_n(y_n) \in K$. Since σ is expanding and $\gamma_n \to x$, there exists $C > 0$ such that

$$\sigma(\gamma_n^{-1}, \gamma_n y_n) \geq ||\gamma_n||_\sigma - C$$

for all $n \geq 1$. Since $\bar{\sigma}(\gamma_n^{-1}, x) = 0$, see Proposition 2.9(1), Equation (1) implies that

$$G(x, y_n) = G(x, \gamma_n(y_n)) + \sigma(\gamma_n^{-1}, \gamma_n(y_n)) \geq ||\gamma_n^{-1}||_\sigma - C.$$

Since $\gamma_n \to x$, we see that $||\gamma_n^{-1}||_\sigma \to +\infty$, so $G(x, y_n) \to +\infty$. A similar argument shows that $G(y_n, x) \to +\infty$, where we use $\bar{\sigma}$ in place of σ. Since $\{y_n\} \subset \Lambda(\Gamma) \smallsetminus \{x\}$ was an arbitrary sequence converging to x, the lemma follows. $\square$

Finally we verify (3).

Proof of non-arithmetic length spectrum given (3). Since Γ is a uniform convergence group, Γ is word hyperbolic and there exists a equivariant homeomorphism between the Gromov boundary and the limit set $\Lambda(\Gamma)$. Then, since Γ is not virtually free, a result of Bonk–Kleiner [10] implies that the limit set contains an embedded circle. It then follows from (2) that the group generated by $\mathcal{CR}$ is dense in $\mathbb{R}$. $\square$

6. Equidistribution

We are now ready to prove Theorem 1.5, which we restate here. The proof follows a classical strategy that goes back to Margulis [34]. Our particular implementation of this strategy is influenced by Roblin [41], although, unlike Roblin, we directly establish equidistribution of closed geodesics, without first establishing double equidistribution of orbit points in $\tilde{U}_\Gamma$.

Theorem 6.1. *Suppose $(\sigma, \bar{\sigma}, G)$ is a continuous GPS system for a convergence group $\Gamma \subset \mathsf{Homeo}(M)$ where $\delta := \delta_\sigma(\Gamma) < +\infty$ and $Q_\sigma(\delta) = +\infty$. Let μ be the unique σ-Patterson–Sullivan measure of dimension δ and let $\bar{\mu}$ be the unique $\bar{\sigma}$-Patterson–Sullivan measure of dimension δ.*

If the BMS measure m_Γ is finite and mixing, then

$$\lim_{T\to\infty} \delta e^{-\delta T} \sum_{\substack{\gamma \in \Gamma_{\mathrm{lox}} \\ \ell_\sigma(\gamma) \leq T}} \mathcal{D}_{\gamma^-} \otimes \mathcal{D}_{\gamma^+} = \frac{1}{\|m_\Gamma\|} e^{\delta G(x,y)} d\bar{\mu}(x) \otimes d\mu(y)$$

in the dual of compactly supported continuous functions.

The rest of the section is devoted to the proof of the theorem. Reformulating in terms of measures on the flow space, the conclusion of the theorem is equivalent to

$$\|m_\Gamma\| \delta e^{-\delta T} \sum_{\substack{\gamma \in \Gamma_{\mathrm{lox}} \\ \ell_\sigma(\gamma) \leq T}} \mathcal{D}_{\gamma^-} \otimes \mathcal{D}_{\gamma^+} \otimes dt \xrightarrow[T\to\infty]{} \tilde{m} = e^{\delta G(x,y)} d\bar{\mu}(x) \otimes d\mu(y) \otimes dt.$$

For ease of notation, let

$$\nu_T := \sum_{\substack{\gamma \in \Gamma_{\mathrm{lox}} \\ \ell_\sigma(\gamma) \leq T}} \mathcal{D}_{\gamma^-} \otimes \mathcal{D}_{\gamma^+} \otimes dt.$$

Lemma 6.2. *For any compact set $K \subset M^{(2)}$ and bounded interval $I \subset \mathbb{R}$,*

$$\sup_{T \geq 0} e^{-\delta T} \nu_T(K \times I) < +\infty.$$

Proof. By Proposition 2.9(4), there exists $C_1 > 0$ such that: if $\gamma \in \Gamma_{\mathrm{lox}}$ and $(\gamma^-, \gamma^+) \in K$, then

$$\ell_\sigma(\gamma) \geq \|\gamma\|_\sigma - C_1.$$

Further, by [8, Prop. 6.3] there exists $C_2 > 0$ such that

$$\#\{\gamma \in \Gamma : \|\gamma\|_\sigma \leq R\} \leq C_2 e^{\delta R}.$$

Hence

$$\sup_{T \geq 0} e^{-\delta T} \nu_T(K \times I) \leq C_2 e^{\delta C_1} \mathrm{Leb}(I). \qquad \square$$

The above lemma implies that the family of measures $\{e^{-\delta T}\nu_T\}$ is relatively compact. As a consequence, it is enough to fix an accumulation point

$$\tilde{m}' := \lim_{n\to\infty} \|m_\Gamma\| \, \delta e^{-\delta T_n} \nu_{T_n}$$

(where $T_n \to +\infty$) and prove that $\tilde{m} = \tilde{m}'$.

The heart of the proof consists of the following lemmas that give measure estimates for rectangles $A \times B \times I \subset M^{(2)} \times \mathbb{R}$ which satisfy

(3) $|G(a,b) - G(a',b')| \leq \epsilon \quad$ and $\quad |G(b,a) - G(b',a')| \leq \epsilon \quad \forall (a,b),(a',b') \in A \times B$

for some $\epsilon > 0$. We will postpone the proof of these lemmas until after we conclude the proof of Theorem 1.5.

Lemma 6.3. *For any $\epsilon > 0$ and any relatively compact subset $A\times B\times I \subset M^{(2)}\times\mathbb{R}$ with $\tilde m, \tilde m'$-null boundary satisfying (3) and $\operatorname{diam} I \leq \epsilon$,*

$$\tilde m(A\times B\times I) \leq e^{7\epsilon\delta}\tilde m'(A\times B\times I).$$

Lemma 6.4. *For any $\epsilon > 0$ and any relatively compact subset $A\times B\times I \subset M^{(2)}\times\mathbb{R}$ with $\tilde m, \tilde m'$-null boundary satisfying (3) and $\operatorname{diam} I \leq \epsilon$,*

$$\tilde m(A\times B\times I) \geq e^{-6\epsilon\delta}\tilde m'(A\times B\times I).$$

6.1. **Proof of Theorem 6.1.** Assuming Lemmas 6.3 and 6.4 we prove Theorem 6.1. We start with a general observation that shows that null boundary sets are abundant.

Observation 6.5. If X is a metric space and λ is a locally finite Borel measure on X, then for any $x_0 \in X$ there is $r_0 > 0$ such that the set

$$\{r < r_0 : \lambda(\partial B_r(x_0)) > 0\}$$

is countable.

Proof. Fix $r_0 > 0$ such that $\lambda(B_{r_0}(x_0)) < \infty$. Then the function

$$r \in [0, r_0] \mapsto F(r) = \lambda(B_r(x_0)) \in \mathbb{R}$$

is monotone increasing. Since a monotone increasing function can have only countably many points of discontinuity, we see that

$$\{r < r_0 : F \text{ discontinuous at } r\} \supset \{r < r_0 : \lambda(\partial B_r(x_0)) > 0\}$$

is countable. □

Next we use Lemmas 6.3 and 6.4 to prove the following.

Lemma 6.6. *If $A\times B\times I \subset M^{(2)}\times\mathbb{R}$ is relatively compact and has $\tilde m, \tilde m'$-null boundary, then*

$$\tilde m'(A\times B\times I) = \tilde m(A\times B\times I).$$

Proof. Fix $\epsilon > 0$. By the relative compactness of $A\times B\times I$, the continuity of G, and Observation 6.5, we can find finite covers $A \subset \bigcup_i A_i$, $B \subset \bigcup_j B_j$ and $I \subset \bigcup_k I_k$ such that for all i, j, k, we have

- $\bar\mu(\partial A_i) = 0$, $\mu(\partial B_j) = 0$, and $\operatorname{Leb}(\partial I_k) = 0$;
- $A_i \times B_j \times I_k$ is relatively compact in $M^{(2)}\times\mathbb{R}$ and satisfies (3);
- $\operatorname{diam} I_k \leq \epsilon$.

Set $A_i' := A \cap A_i - (A_{i-1}\cup\cdots\cup A_1)$, $B_j' := B\cap B_j - (B_{j-1}\cup\cdots\cup B_1)$ and $I_k' := I \cap I_k - (I_{k-1}\cup\cdots\cup I_1)$ for all i, j, k. Then

$$A\times B\times I = \bigsqcup_{i,j,k} A_i'\times B_j'\times I_k'.$$

Further, each $A_i'\times B_j'\times I_k'$ has $\tilde m, \tilde m'$-null boundary, hence by Lemmas 6.3 and 6.4 we have

$$e^{-6\epsilon\delta}\tilde m(A_i'\times B_j'\times I_k') \leq \tilde m'(A_i'\times B_j'\times I_k') \leq e^{7\epsilon\delta}\tilde m(A_i'\times B_j'\times I_k').$$

So

$$e^{-6\epsilon\delta}\tilde m(A\times B\times I) \leq m'(A\times B\times I) \leq e^{7\epsilon\delta}\tilde m(A\times B\times I).$$

Since $\epsilon > 0$ was arbitrary, $\tilde{m}(A \times B \times I) = \tilde{m}'(A \times B \times I)$. $\square$

Proof of Theorem 6.1 . The collection of relatively compact subset $A \times B \times I \subset M^{(2)} \times \mathbb{R}$ with $\tilde{m}, \tilde{m}'$-null boundary is a π-system (i.e. it is closed under finite intersection) and by Observation 6.5 it generates the Borel sigma-algebra. This completes the proof, since two measures are equivalent if they agree on a π-system which generates the Borel sigma-algebra. $\square$

6.2. **Proofs of Lemma 6.3 and 6.4.** In the proofs of Lemma 6.3 and 6.4 we will need the following uniform closing lemma for GPS systems. Recall that ψ^t denotes the flow on $\psi^t(x, y, s) = (x, y, s + t)$ on $\tilde{U}_\Gamma$.

Lemma 6.7. *Suppose that $A \times B \times I \subset M^{(2)} \times \mathbb{R}$ is open and relatively compact, and that $A' \times B' \times I'$ is a compact subset of $A \times B \times I$. Then there exists T such that: if $\gamma \in \Gamma$, $t \geq T$ and*

$$A' \times B' \times I' \cap \psi^{-t}\gamma(A' \times B' \times I') \neq \varnothing,$$

then

(1) $\gamma^{-1}(A) \times \gamma(B) \subset A \times B$,
(2) γ is loxodromic with $(\gamma^-, \gamma^+) \in A \times B$.

Proof. If not, then there exist sequences $\{\gamma_n\} \subset \Gamma$, $\{t_n\} \subset \mathbb{R}$ and

$$(x_n, y_n, s_n) \in A' \times B' \times I' \cap \psi^{-t_n}\gamma_n(A' \times B' \times I'),$$

such that $t_n \to +\infty$ and each γ_n fails the conclusion of the lemma.

Since $\psi^{t_n}\gamma_n^{-1}(x_n, y_n, s_n) \in A' \times B' \times I'$, the sequence $\{s_n + t_n + \sigma(\gamma_n^{-1}, y_n)\}$ is bounded, so $\sigma(\gamma_n^{-1}, y_n) \to -\infty$ and hence $\{\gamma_n\}$ is an escaping sequence. Then, passing to a subsequence, we may assume that the sequences $\gamma_n^{\pm 1} \to p^\pm \in M$.

We claim that $(p^-, p^+) \in A' \times B'$. Since $\sigma(\gamma_n, \gamma_n^{-1}y_n) = -\sigma(\gamma_n^{-1}, y_n) \to +\infty$, Proposition 2.9(2) implies that $y_n \to p^+$. Therefore, $p^+ \in B'$. If K is any compact subset of $M \smallsetminus \{p^+\}$, then $\gamma_n^{-1}(K) \to p^-$. Since A' is a compact subset of $M \smallsetminus B \subset M \smallsetminus \{p^+\}$ and $x_n \in A'$ for all n, we see that $\gamma_n^{-1}(x_n) \to p^-$ and so $p^- \in A'$.

Now since $\bar{A}$ is a compact subset of $M \smallsetminus \overline{B} \subset M \smallsetminus \{p^+\}$, $p^- \in A$, and A is open, we see that $\gamma_n^{-1}(A) \subset A$ for all large enough n. Similarly, $\gamma_n(B) \subset B$ for n large. Further, since $p^- \neq p^+$, Lemma 2.2 implies that for n large γ_n is loxodromic and $(\gamma_n^-, \gamma_n^+) \to (p^-, p^+) \in A \times B$. Since $A \times B$ is open, then $(\gamma_n^-, \gamma_n^+) \in A \times B$ for n large. We have achieved a contradiction. $\square$

Proof of Lemma 6.3. It suffices to fix $\epsilon' > 0$ and show that

$$\tilde{m}(A \times B \times I) \leq e^{4\epsilon'} e^{7\epsilon\delta} \tilde{m}'(A \times B \times I).$$

We start by reducing to the setting of Lemma 6.7. By considering its interior (which has full $\tilde{m}, \tilde{m}'$-measure since the boundary has zero $\tilde{m}, \tilde{m}'$-measure), we can assume that $A \times B \times I$ is open. By inner regularity, we find a compact subset $A' \times B' \times I' \subset A \times B \times I$ such that

$$\tilde{m}(A \times B \times I) \leq e^{\epsilon'} \tilde{m}(A' \times B' \times I'). \tag{4}$$

Then by Lemma 6.7 there exists T_1 such that: if $\gamma \in \Gamma$, $t \geq T_1$ and

$$A' \times B' \times I' \cap \psi^{-t}\gamma(A' \times B' \times I') \neq \varnothing,$$

then $\gamma^{-1}A \times \gamma B \subset A \times B$, γ is loxodromic and $(\gamma^-, \gamma^+) \in A \times B$. We will show that in addition

(a) $|\ell_\sigma(\gamma) - \sigma(\gamma, b)| \leq \epsilon$ for every $b \in B$,
(b) $|\ell_\sigma(\gamma) - t| \leq 2\epsilon$.

In particular,

(c) $\gamma \in \Theta^{t+2\epsilon} := \{\gamma \in \Gamma_{\rm lox} : \ \gamma^{-1}A \times \gamma B \subset A \times B, \ (\gamma^-, \gamma^+) \in A \times B, \ \ell_\sigma(\gamma) \leq t + 2\epsilon\}$.

Indeed (a) is due to assumption (3) and the fact that

$$\begin{aligned} \ell_\sigma(\gamma) - \sigma(\gamma, b) &= \bar\sigma(\gamma, \gamma^-) + \sigma(\gamma, \gamma^+) - (\bar\sigma(\gamma, \gamma^-) + \sigma(\gamma, b)) \\ &= G(\gamma\gamma^-, \gamma\gamma^+) - G(\gamma^-, \gamma^+) - G(\gamma\gamma^-, \gamma b) + G(\gamma^-, b) \\ &= G(\gamma^-, b) - G(\gamma^-, \gamma b). \end{aligned} \tag{5}$$

Moreover, if $(x, y, s) \in A' \times B' \times I' \cap \psi^{-t}\gamma(A' \times B' \times I')$ then both s and $s - t + \sigma(\gamma, y)$ are in I' so

$$|t - \sigma(\gamma, y)| \leq \operatorname{diam} I \leq \epsilon,$$

so (b) and (c) hold.

By Observation 2.13 (mixing) we may choose $T_2 > T_1$ such that for any $t \geq T_2$

$$\frac{\tilde m(A' \times B' \times I')^2}{\|m_\Gamma\|} \leq e^{\epsilon'} \sum_{\gamma \in \Gamma} \tilde m(A' \times B' \times I' \cap \psi^{-t}\gamma(A' \times B' \times I')). \tag{6}$$

Finally, we may choose $T_3 > T_2$, so that if $T > T_3$, then

$$1 \leq e^{\epsilon'} \delta e^{-\delta T} \int_{T_2}^T e^{\delta t} dt. \tag{7}$$

We now fix $(a_0, b_0) \in A \times B$ and $T > T_3$. By Equations (4) and (7),

$$\frac{\tilde m(A \times B \times I)^2}{\|m_\Gamma\|} \leq e^{2\epsilon'} \frac{\tilde m(A' \times B' \times I')^2}{\|m_\Gamma\|} \leq e^{3\epsilon'} \delta e^{-\delta T} \int_{t=T_2}^T \frac{\tilde m(A' \times B' \times I')^2}{\|m_\Gamma\|} e^{\delta t} dt.$$

So by Equation (6) and (c),

$$\begin{aligned} \frac{\tilde m(A \times B \times I)^2}{\|m_\Gamma\|} &\leq e^{4\epsilon'} \delta e^{-\delta T} \sum_{\gamma \in \Gamma} \int_{t=T_2}^T \tilde m(A' \times B' \times I' \cap \psi^{-t}\gamma(A' \times B' \times I')) e^{\delta t} dt \\ &\leq e^{4\epsilon'} \delta e^{-\delta T} \sum_{\gamma \in \Theta^{T+2\epsilon}} \int_{t=T_2}^T \tilde m(A' \times B' \times I' \cap \psi^{-t}\gamma(A' \times B' \times I')) e^{\delta t} dt. \end{aligned}$$

Notice that assumption (3) implies that the integrand satisfies

$$\begin{aligned} &\tilde m(A' \times B' \times I' \cap \psi^{-t}\gamma(A' \times B' \times I')) \leq \tilde m(A \times \gamma B \times I) \\ &\quad \leq e^{\delta\epsilon} e^{\delta G(a_0, b_0)} \bar\mu(A)\mu(\gamma B) \mathrm{Leb}(I). \end{aligned}$$

Further, by (a),

$$\mu(\gamma B) = \int_B e^{-\delta\sigma(\gamma, b)} d\mu \leq e^{\epsilon\delta} e^{-\delta \ell_\sigma(\gamma)} \mu(B).$$

By (b),

$$\{t : A' \times B' \times I' \cap \psi^{-t}\gamma(A' \times B' \times I') \neq \varnothing\} \subset [\ell_\sigma(\gamma) - 2\epsilon, \ell_\sigma(\gamma) + 2\epsilon]$$

and has Lebesgue measure at most $\mathrm{Leb}(I)$. Hence

$$\frac{\tilde m(A \times B \times I)^2}{\|m_\Gamma\|} \leq e^{4\epsilon'} e^{4\epsilon\delta} \cdot e^{\delta G(a_0, b_0)} \bar\mu(A)\mu(B) \mathrm{Leb}(I) \cdot \delta e^{-\delta T} \mathrm{Leb}(I) \#\big(\Theta^{T+2\epsilon}\big).$$

So by assumption (3) and the definition of $\nu_{T+2\epsilon}$,

$$\frac{\tilde m(A\times B\times I)^2}{\|m_\Gamma\|} \le e^{4\epsilon'}e^{5\epsilon\delta}\tilde m(A\times B\times I)\delta e^{-\delta T}\nu_{T+2\epsilon}(A\times B\times I).$$

By applying the above estimate to the sequence $\{T_n-2\epsilon\}$, we see that, for all sufficiently large n,

$$\tilde m(A\times B\times I)^2 \le e^{4\epsilon'}e^{5\epsilon\delta}\tilde m(A\times B\times I)\cdot e^{2\delta\epsilon}\left(\|m_\Gamma\|\,\delta e^{-\delta T_n}\nu_{T_n}(A\times B\times I)\right).$$

Letting $n\to\infty$, we see that

$$\tilde m(A\times B\times I) \le e^{4\epsilon'}e^{7\epsilon\delta}\tilde m'(\bar A\times\bar B\times\bar I) = e^{4\epsilon'}e^{7\epsilon\delta}\tilde m'(A\times B\times I).$$

Notice that the last equality used the fact that $A\times B\times I$ has $\tilde m'$-null boundary. $\square$

Proof of Lemma 6.4. The proof is very similar to the proof of Lemma 6.3.

It suffices to fix $\epsilon'>0$ and show that

$$\tilde m'(A\times B\times I)\le e^{2\epsilon'}e^{6\epsilon\delta}\tilde m(A\times B\times I).$$

As in the proof of Lemma 6.3, we can assume that $A\times B\times I$ is open and find a compact subset $A'\times B'\times I'\subset A\times B\times I$ with $\tilde m'$-null boundary such that

$$\tilde m'(A\times B\times I)\le e^{\epsilon'}\tilde m'(A'\times B'\times I'). \tag{8}$$

For $S<T$, define

$$\Gamma_S^T := \{\gamma\in\Gamma_{\mathrm{lox}}:\ (\gamma^-,\gamma^+)\in A'\times B',\ S\le \ell_\sigma(\gamma)\le T\}.$$

Notice that if $\gamma\in\Gamma_{\mathrm{lox}}$ and $(\gamma^-,\gamma^+)\in A'\times B'$, then

$$A'\times B'\times I'\cap\psi^{-\ell_\sigma(\gamma)}\gamma(A'\times B'\times I')\neq\varnothing.$$

So by Lemma 6.7 there exists T_1 such that if $\gamma\in\Gamma_{T_1}^\infty$, then

(d) $\gamma^{-1}A\times\gamma B\subset A\times B$.

Moreover:

(e) $|\ell_\sigma(\gamma)-\sigma(\gamma,b)|\le\epsilon$ for every $b\in B$.
(f) If $A\times B\times I\cap\psi^{-t}\gamma(A\times B\times I)\neq\varnothing$ for some t, then $|t-\ell_\sigma(\gamma)|\le 2\epsilon$.

Notice that (e) follows from the computation in (5) and (f) is an immediate consequence of (e) and the fact that $\operatorname{diam} I\le\epsilon$.

By Observation 2.13 (mixing) we may choose $T_2>T_1$ such that for any $t\ge T_2$

$$\frac{\tilde m(A\times B\times I)^2}{\|m_\Gamma\|}\ge e^{-\epsilon'}\sum_{\gamma\in\Gamma}\tilde m(A\times B\times I\cap\psi^{-t}\gamma(A\times B\times I)).$$

We now fix $(a_0,b_0)\in A\times B$ and $T>T_2$. Using the fact that $\delta e^{-\delta T}\int_{T_2}^T e^{\delta t}dt<1$ and (f),

$$\frac{\tilde m(A\times B\times I)^2}{\|m_\Gamma\|}\ge \delta e^{-\delta T}\int_{t=T_2}^T\frac{\tilde m(A\times B\times I)^2}{\|m_\Gamma\|}e^{\delta t}\,dt$$
$$\ge e^{-\epsilon'}\delta e^{-\delta T}e^{-2\delta\epsilon}\int_{t=T_2}^T\sum_{\gamma\in\Gamma_{T_2+2\epsilon}^{T-2\epsilon}}\tilde m(A\times B\times I\cap\psi^{-t}\gamma(A\times B\times I))e^{\delta\ell_\sigma(\gamma)}\,dt.$$

By (d), in the integrand we have $A \cap \gamma A = A$ and $B \cap \gamma B = \gamma B$, so by assumption (3) we have

$$\tilde{m}(A \times B \times I \cap \psi^{-t}\gamma(A \times B \times I)) \geq e^{-\delta\epsilon} e^{\delta G(a_0,b_0)} \bar{\mu}(A) \int_{\gamma B} \mathrm{Leb}(I \cap (I - t + \sigma(\gamma, b))) d\mu(b).$$

Also, since $\gamma \in \Gamma_{T_2+2\epsilon}^{T-2\epsilon}$, property (e) implies that

$$\begin{aligned}\int_{t=T_2}^{T} \mathrm{Leb}(I \cap (I - t + \sigma(\gamma, b)))dt &= \int_{s \in I} \mathrm{Leb}([T_2, T] \cap (I - s + \sigma(\gamma, b)))ds \\ &= \int_{s \in I} \mathrm{Leb}(I - s + \sigma(\gamma, b))ds = \mathrm{Leb}(I)^2.\end{aligned}$$

Combining the above estimates we have,

$$\frac{\tilde{m}(A \times B \times I)^2}{\|m_\Gamma\|} \geq e^{-\epsilon'} \delta e^{-\delta T} e^{-3\delta\epsilon} e^{\delta G(a_0,b_0)} \bar{\mu}(A) \mathrm{Leb}(I)^2 \sum_{\gamma \in \Gamma_{T_2+2\epsilon}^{T-2\epsilon}} e^{\delta \ell_\sigma(\gamma)} \mu(\gamma B).$$

By (e), we have

$$\mu(\gamma B) = \int_B e^{-\delta\sigma(\gamma, b)} d\mu(b) \geq e^{-\delta\epsilon} e^{-\delta \ell_\sigma(\gamma)} \mu(B).$$

Hence

$$\frac{\tilde{m}(A \times B \times I)^2}{\|m_\Gamma\|} \geq e^{-\epsilon'} e^{-3\delta\epsilon} e^{\delta G(a_0,b_0)} \bar{\mu}(A) \mu(B) \mathrm{Leb}(I) \cdot \delta e^{-\delta T} \# \left(\Gamma_{T_2+2\epsilon}^{T-2\epsilon}\right) \mathrm{Leb}(I).$$

By assumption (3),

$$e^{\delta G(a_0,b_0)} \bar{\mu}(A) \mu(B) \mathrm{Leb}(I) \geq e^{-\delta\epsilon} \tilde{m}(A \times B \times I)$$

and by definition

$$\delta e^{-\delta T} \# \left(\Gamma_{T_2+2\epsilon}^{T-2\epsilon}\right) \mathrm{Leb}(I) = \delta e^{-\delta T} \left(\nu_{T-2\epsilon} - \nu_{T_2+2\epsilon}\right)(A' \times B' \times I).$$

So by applying the above estimates to the sequence $\{T_n + 2\epsilon\}$, we see that, for all sufficiently large n,

$$\tilde{m}(A \times B \times I)^2 \geq e^{-\epsilon'} e^{-4\epsilon\delta} \tilde{m}(A \times B \times I) \cdot e^{-2\delta\epsilon} \left(\|m_\Gamma\| \delta e^{-\delta T_n} (\nu_{T_n} - \nu_{T_2+2\epsilon})(A' \times B' \times I)\right).$$

Sending $n \to \infty$ and using Equation (8) and that $A' \times B' \times I'$ has $\tilde{m}'$-null boundary, we obtain

$$\tilde{m}(A \times B \times I) \geq e^{-\epsilon'} e^{-6\epsilon\delta} \tilde{m}'(A' \times B' \times I') \geq e^{-2\epsilon'} e^{-6\epsilon\delta} \tilde{m}'(A \times B \times I),$$

which completes the proof. $\square$

7. Dirichlet domains for flow spaces

For the rest of the section fix a GPS system $(\sigma, \bar{\sigma}, G)$ for a convergence group $\Gamma \subset \mathsf{Homeo}(M)$. In this section we describe how the space $M^{(2)} \times \mathbb{R}$ can be compactified using M and then construct an analogue of the classical Dirichlet fundamental domain in hyperbolic geometry for the action of Γ on $M^{(2)} \times \mathbb{R}$.

Proposition 7.1. *The set $\overline{M^{(2)} \times \mathbb{R}} := \left(M^{(2)} \times \mathbb{R}\right) \sqcup M$ has a topology with the following properties:*

(1) $\overline{M^{(2)} \times \mathbb{R}}$ is a compact metrizable space.
(2) The inclusions $M^{(2)} \times \mathbb{R} \hookrightarrow \overline{M^{(2)} \times \mathbb{R}}$ and $M \hookrightarrow \overline{M^{(2)} \times \mathbb{R}}$ are embeddings, and $M^{(2)} \times \mathbb{R}$ is open.

(3) *The action of* Γ *on* $\overline{M^{(2)} \times \mathbb{R}}$ *is by homeomorphisms.*
(4) *A sequence* $\{(x_n, y_n, t_n)\}$ *in* $\overline{M^{(2)} \times \mathbb{R}}$ *converges to* $z \in M$ *if and only if either*
   (a) $x_n \to z$ *and* $t_n \to -\infty$,
   (b) $y_n \to z$ *and* $t_n \to +\infty$, *or*
   (c) $x_n \to z$ *and* $y_n \to z$.
(5) *For any* $v_0 \in M^{(2)} \times \mathbb{R}$, *the map*

$$x \in \Gamma \sqcup M \mapsto \begin{cases} x(v_0) & \textit{if } x \in \Gamma \\ x & \textit{if } x \in M \end{cases}.$$

is continuous.

The proof is an exercise in point set topology (namely showing that the convergence in (4) is compatible with a topology) and is sketched at the end of the section.

Next we define an analogue of the Dirichlet domain for the action of Γ on $M^{(2)} \times \mathbb{R}$. Define a function

$$\chi \colon M^{(2)} \times \mathbb{R} \to [0, \infty),$$

which plays the role of the distance to the basepoint in the traditional construction, by setting

$$\chi(x, y, t) := G(y, x) + |t|.$$

We then define our Dirichlet domain

$$\mathcal{D} := \{v \in M^{(2)} \times \mathbb{R} : \chi(v) \leq \chi(\gamma v) \quad \forall \gamma \in \Gamma\}.$$

We will prove that these domains have the following properties.

Proposition 7.2.
(1) $\Gamma(\mathcal{D}) = M^{(2)} \times \mathbb{R}$ *and each* $v \in M^{(2)} \times \mathbb{R}$ *is contained in finitely many* Γ*-translates of* $\mathcal{D}$.
(2) *The closure of* $\mathcal{D}$ *in* $\overline{M^{(2)} \times \mathbb{R}}$ *does not contain any conical limit points.*

Remark 7.3. Unlike the classical setting of hyperbolic geometry, it is likely not always true that Γ-iterates of the interior of $\mathcal{D}$ are disjoint.

Both parts of the proposition follow from a quantitative lower bound on χ along an orbit of Γ. Fix a distance d on $\overline{M^{(2)} \times \mathbb{R}}$ which generates the topology in Proposition 7.1.

Lemma 7.4. *For any* $v_0 \in M^{(2)} \times \mathbb{R}$ *and* $\epsilon > 0$ *there exists* $C > 0$ *such that: if* $\gamma \in \Gamma$, $v \in M^{(2)} \times \mathbb{R}$ *and* $\mathrm{d}(v, \gamma^{-1}(v_0)) > \epsilon$, *then*

$$\chi(\gamma(v)) \geq \chi(v) + \min\left(\|\gamma\|_\sigma, \left\|\gamma^{-1}\right\|_\sigma\right) - C.$$

Delaying the proof of the lemma, we prove the proposition.

Proof of Proposition 7.2. (1): Lemma 7.4 implies that for fixed $v \in M^{(2)} \times \mathbb{R}$, the map

$$\gamma \in \Gamma \mapsto \chi(\gamma(v)) \in [0, \infty)$$

is proper and hence has a minimum, which is realized only finitely many times. Thus $\Gamma(\mathcal{D}) = M^{(2)} \times \mathbb{R}$ and each $v \in M^{(2)} \times \mathbb{R}$ is contained in finitely many Γ-translates of $\mathcal{D}$.

(2): Suppose $x \in M$ is a conical limit point and $\{v_n\} \subset M^{(2)} \times \mathbb{R}$ converges to x. We will show that $v_n \notin \mathcal{D}$ when n is large.

Since x is a conical limit point, there exist $\{\gamma_m\}$ in Γ and distinct $a, b \in M$ such that $\gamma_m(x) \to a$ and $\gamma_m(y) \to b$ for any $y \in M \smallsetminus \{x\}$. Thus $\gamma_m \to b$ and $\gamma_m^{-1} \to x$ in the topology on $\Gamma \sqcup M$.

Let $\epsilon := \frac{1}{2}\,\mathrm{d}(a,b)$ and fix $v_0 \in M^{(2)} \times \mathbb{R}$. Let $C = C(v_0, \epsilon) > 0$ be as in Lemma 7.4. Then Proposition 7.1(5) implies that $\gamma_m(v_0) \to b$. Since $\gamma_m(x) \to a$, we can fix $m \geq 1$ sufficiently large that

$$\mathrm{d}(\gamma_m(x), \gamma_m(v_0)) > \epsilon.$$

Since σ is proper, by increasing m we can also assume that

$$\min\left(\|\gamma_m\|_\sigma, \left\|\gamma_m^{-1}\right\|_\sigma\right) > C + 1.$$

Now since $v_n \to x$, for n sufficiently large we have $\mathrm{d}(\gamma_m(v_n), \gamma_m(v_0)) > \epsilon$. Then, for any such n, Lemma 7.4 implies that

$$\chi(v_n) = \chi(\gamma_m^{-1}\gamma_m v_n) \geq \chi(\gamma_m v_n) + \min\left(\|\gamma_m\|_\sigma, \left\|\gamma_m^{-1}\right\|_\sigma\right) - C > \chi(\gamma_m v_n) + 1.$$

Hence $v_n \notin \mathcal{D}$ when n is sufficiently large. $\square$

7.1. **Proof of Lemma 7.4.** Fix $v_0 \in M^{(2)} \times \mathbb{R}$ and $\epsilon > 0$. Suppose for a contradiction that no such C exists. Then for every $n \geq 1$ there exist $\gamma_n \in \Gamma$ and $v_n \in M^{(2)} \times \mathbb{R}$ where $\mathrm{d}(v_n, \gamma_n^{-1}(v_0)) > \epsilon$ and

$$\chi(\gamma_n(v_n)) \leq \chi(v_n) + \min\left(\|\gamma_n\|_\sigma, \left\|\gamma_n^{-1}\right\|_\sigma\right) - n. \tag{9}$$

If $v_n = (x_n, y_n, t_n)$, then by Equation (1),

$$\begin{aligned} \chi(\gamma_n(v_n)) - \chi(v_n) &= G(\gamma(x_n), \gamma(y_n)) - G(x_n, y_n) + |t_n + \sigma(\gamma_n, y_n)| - |t_n| \\ &= \bar{\sigma}(\gamma_n, x_n) + \sigma(\gamma_n, y_n) + |t_n + \sigma(\gamma_n, y_n)| - |t_n|. \end{aligned} \tag{10}$$

Passing to a subsequence we can suppose that $t_n \to t \in [-\infty, \infty]$ and that $x_n \to x$, $y_n \to y$ and $\gamma_n^{-1} \to b$ in $\Gamma \sqcup M$. Notice that (9) and (10) imply that

$$\begin{aligned} n &\leq \chi(v_n) - \chi(\gamma_n(v_n)) + \min\left(\|\gamma_n\|_\sigma, \left\|\gamma_n^{-1}\right\|_\sigma\right) \\ &\leq |\bar{\sigma}(\gamma_n, x_n)| + 2\,|\sigma(\gamma_n, y_n)| + \min\left(\|\gamma_n\|_\sigma, \left\|\gamma_n^{-1}\right\|_\sigma\right). \end{aligned}$$

So $\{\gamma_n\}$ is an escaping sequence. Hence $b \in M$. Then Proposition 7.1(5) implies that $\gamma_n^{-1}(v_0) \to b$.

Since $\mathrm{d}(v_n, \gamma_n^{-1}(v_0)) > \epsilon$, the sequence $\{v_n = (x_n, y_n, t_n)\}$ does not converge to b. Recall that $x_n \to x$, that $y_n \to y$ and that $t_n \to t$. By Proposition 7.1(4) this means that x and y cannot be both equal to b. If $x = b$ then $v_n \not\to b$ forces $t \neq -\infty$ (in addition to $y \neq b$), while if $y = b$ then $v_n \not\to b$ forces $t \neq +\infty$. Otherwise, we have $x \neq b$ and $y \neq b$. Thus we are in one of the following cases.

$$(y \neq b \text{ and } t \neq -\infty) \text{ or } (x \neq b \text{ and } t \neq +\infty) \text{ or } (x \neq b \text{ and } y \neq b).$$

Case 1: Assume $y \neq b$ and $t \neq -\infty$. Since σ is expanding, there exists $C_1 > 0$ such that

$$\sigma(\gamma_n, y_n) \geq \|\gamma_n\|_\sigma - C_1$$

for all $n \geq 1$. Then for n sufficiently large

$$|t_n + \sigma(\gamma_n, y_n)| - |t_n| = t_n + \sigma(\gamma_n, y_n) - |t_n| \geq \|\gamma_n\|_\sigma - C_1 + (t_n - |t_n|).$$

By the definition of $\|\cdot\|_{\bar\sigma}$ and Proposition 2.11(1) there exists $C_2 > 0$ such that

$$\bar\sigma(\gamma_n, x_n) = -\bar\sigma(\gamma_n^{-1}, \gamma_n x) \geq -\left\|\gamma_n^{-1}\right\|_{\bar\sigma} \geq -\left\|\gamma_n\right\|_{\sigma} - C_2.$$

for all $n \geq 1$. So by (10),

$$\begin{aligned}\chi(\gamma_n(v_n)) - \chi(v_n) &\geq (-\left\|\gamma_n\right\|_{\sigma} - C_2) + (\left\|\gamma_n\right\|_{\sigma} - C_1) + (\left\|\gamma_n\right\|_{\sigma} - C_1 + (t_n - |t_n|)) \\ &= \left\|\gamma_n\right\|_{\sigma} + (t_n - |t_n|) - 2C_1 - C_2.\end{aligned}$$

However, since $t_n \to t > -\infty$, this estimate contradicts Equation (9).

Case 2: Assume $x \neq b$ and $t \neq \infty$. Since $\bar\sigma$ is expanding, there exists $C_1 > 0$ such that

$$\bar\sigma(\gamma_n, x_n) \geq \left\|\gamma_n\right\|_{\bar\sigma} - C_1$$

for all $n \geq 1$. Then by Proposition 2.11(1), there exists $C_2 > 0$ such that

$$\bar\sigma(\gamma_n, x_n) \geq \left\|\gamma_n^{-1}\right\|_{\sigma} - C_2$$

for all $n \geq 1$. So by (10),

$$\chi(\gamma_n(v_n)) - \chi(v_n) \geq \left(\left\|\gamma_n^{-1}\right\|_{\sigma} - C_2\right) + \sigma(\gamma_n, y_n) + \left(|t_n + \sigma(\gamma_n, y_n)| - |t_n|\right).$$

Notice that

$$\sigma(\gamma_n, y_n) + |t_n + \sigma(\gamma_n, y_n)| - |t_n| = \begin{cases} -(t_n + |t_n|) & \text{if } \sigma(\gamma_n, y_n) \leq -t_n \\ 2\sigma(\gamma_n, y_n) + t_n - |t_n| & \text{if } \sigma(\gamma_n, y_n) \geq -t_n \end{cases}$$

is bounded below by $-(t_n + |t_n|)$. So

$$\chi(\gamma_n(v_n)) - \chi(v_n) \geq \left\|\gamma_n^{-1}\right\|_{\sigma} - C_2 - (t_n + |t_n|).$$

However, since $t_n \to t < +\infty$, this estimate contradicts Equation (9).

Case 3: Assume $x \neq b$ and $y \neq b$. Since σ and $\bar\sigma$ are expanding, there exists $C_1 > 0$ such that

$$\bar\sigma(\gamma_n, x_n) \geq \left\|\gamma_n\right\|_{\bar\sigma} - C_1 \quad \text{and} \quad \sigma(\gamma_n, y_n) \geq \left\|\gamma_n\right\|_{\sigma} - C_1$$

for all $n \geq 1$. Then by Proposition 2.11(1), there exists $C_2 > 0$ such that

$$\bar\sigma(\gamma_n, x_n) \geq \left\|\gamma_n^{-1}\right\|_{\sigma} - C_2$$

for all $n \geq 1$. Also,

$$\sigma(\gamma_n, y_n) \geq \inf_{\gamma \in \Gamma} \left\|\gamma\right\|_{\sigma} - C_1$$

and so

$$\sigma(\gamma_n, y_n) \geq |\sigma(\gamma_n, y_n)| - 2\inf_{\gamma \in \Gamma} \left\|\gamma\right\|_{\sigma} - 2C_1$$

for all $n \geq 1$. So by (10),

$$\begin{aligned}\chi(\gamma_n(v_n)) - \chi(v_n) &\geq \left(\left\|\gamma_n^{-1}\right\|_{\sigma} - C_2\right) + \sigma(\gamma_n, y_n) + \left(|t_n| - |\sigma(\gamma_n, y_n)| - |t_n|\right) \\ &\geq \left\|\gamma_n^{-1}\right\|_{\sigma} - C_2 - 2\inf_{\gamma \in \Gamma} \left\|\gamma\right\|_{\sigma} - 2C_1,\end{aligned}$$

which contradicts Equation (9).

7.2. **Proof of Proposition 7.1.** For an open set $U \subset M$, let

$$I_{U,n} := \{(x, y, t) : (x, t) \in U \times (-\infty, -n) \text{ or } (y, t) \in U \times (n, \infty) \text{ or } x, y \in U\}.$$

Fix a countable basis $\mathcal{B}_0$ of the topology on $M^{(2)} \times \mathbb{R}$ and a countable basis $\mathcal{B}_1$ of the topology on M. Let

$$\mathcal{B} := \mathcal{B}_0 \cup \{I_{U,n} \cup U : U \in \mathcal{B}_1, n \in \mathbb{N}\}$$

and endow $\overline{M^{(2)} \times \mathbb{R}}$ with the topology generated by $\mathcal{B}$, which is a countable basis. One can check the topology is regular, so by Urysohn's metrization theorem it is metrizable. By the definition of the $I_{U,n}$, this topology satisfies (4), which implies it is compact. Properties (2) and (3) follow in a straightforward way from the definition of $\mathcal{B}$.

To prove (5) , it suffices to fix $v_0 \in M^{(2)} \times \mathbb{R}$ and a sequence $\{\gamma_n\} \subset \Gamma$ where $\gamma_n \to a \in M$ in the topology of $\Gamma \sqcup M$, then show that $\gamma_n(v_0) \to a$ in the topology of $\overline{M^{(2)} \times \mathbb{R}}$. Since $\overline{M^{(2)} \times \mathbb{R}}$ is compact, it suffices to consider the case where $\gamma_n(v_0) \to a' \in \overline{M^{(2)} \times \mathbb{R}}$ and then show that $a' = a$. Passing to a subsequence, we can suppose that $\gamma_n^{-1} \to b \in M$ in the topology of $\Gamma \sqcup M$.

Suppose $v_0 = (x, y, t)$. Then $\gamma_n(v_0) = (\gamma_n x, \gamma_n y, t + \sigma(\gamma_n, y))$.

Case 1: Assume $y \neq b$. Then $\gamma_n y \to a$ and by the expanding property $t+\sigma(\gamma_n, y) \to +\infty$. So by (4), $\gamma_n v_0 \to a$.

Case 2: Assume $y = b$. Then $x \neq b$ and so $\gamma_n x \to a$. If $\gamma_n y \to a$, then (4) implies that $\gamma_n v_0 \to a$. If $\gamma_n y \not\to a$, then the expanding property implies that

$$t + \sigma(\gamma_n, y) = t - \sigma(\gamma_n^{-1}, \gamma_n y)$$

has a subsequence which converges to $-\infty$. Hence (4) implies that $\gamma_n v_0 \to a$.

8. Horoballs and a decomposition of the flow space

In this section, we construct horoballs about bounded parabolic points. In the geometrically finite case, we will show that Γ acts cocompactly on the complement of an equivariant collection of horoballs about each point. This gives a useful decomposition of our flow space, which allows us to prove Theorem 1.4 in the next section.

For the rest of the section fix a GPS system $(\sigma, \bar{\sigma}, G)$ for a convergence group $\Gamma \subset \mathsf{Homeo}(M)$.

Given $p \in M$ and $S, T \in \mathbb{R}$ the associated *horoball* $H_{p,S,T} \subset M^{(2)} \times \mathbb{R}$ is the set of all $v = (x, y, t)$ where either

- $y = p$ and $t > S$,
- $x = p$ and $t < G(p, y) - T$, or
- $x \neq p$, $y \neq p$ and $G(x, y) - G(x, p) + S < t < G(p, y) - T$.

Remark 8.1. (1) If one uses the convention that $G(z, z) = \infty$ for any $z \in M$ (this would not necessarily be a continuous extension because we do not assume G is "proper"), then $H_{p,S,T}$ is the set of (x, y, t) such that $x \neq y$ and

$$G(x, y) - G(x, p) + S < t < G(p, y) - T.$$

(2) If $S' \geq S$ and $T' \geq T$ then $H_{p,S',T'} \subset H_{p,S,T}$.

In what follows let $\mathcal{F} \subset M$ denote the set of bounded parabolic points and for $p \in \mathcal{F}$ let $\Gamma_p \subset \Gamma$ denote the stabilizer of p in Γ. The definition of our horoballs leads to the following equivariance property.

Proposition 8.2. *If $\gamma \in \Gamma$ and $p \in M$ is a bounded parabolic point, then*

$$\gamma(H_{p,S,T}) = H_{\gamma p, S+\sigma(\gamma,p), T+\bar{\sigma}(\gamma,p)}.$$

In particular, if $\gamma \in \Gamma_p$, then $\gamma(H_{p,S,T}) = H_{p,S,T}$ by Proposition 2.9(1).

In the geometrically finite case, we will prove the following structure theorem for the quotient space $\Gamma \backslash \tilde{U}_\Gamma$.

Theorem 8.3. *Suppose Γ acts geometrically finitely on M (i.e. every limit point is either a bounded parabolic point or a conical limit point). Then:*

(1) There exists an Γ-equivariant disjoint set of horoballs $\{H_{p,S_p,T_p}\}_{p\in\mathcal{F}}$, i.e.

$$\gamma H_{p,S_p,T_p} = H_{\gamma p, S_{\gamma p}, T_{\gamma p}}$$

for all $p \in \mathcal{F}$ and $\gamma \in \Gamma$.

(2) If $\{H_{p,S_p,T_p}\}_{p\in\mathcal{F}}$ is any Γ-equivariant disjoint set of horoballs, then the quotient

$$\Gamma \backslash \left(\tilde{U}_\Gamma - \bigcup_p H_p \right)$$

is compact (recall that $\tilde{U}_\Gamma = \Lambda(\Gamma)^{(2)} \times \mathbb{R}$).

As a corollary to the proof we observe the following discreteness results.

Corollary 8.4. *Suppose Γ acts geometrically finitely on M. If $T > 0$, then*

$$\{[\gamma] \in [\Gamma_{\mathrm{lox}}] : \ell_\sigma(\Gamma) \le T\}$$

is finite. In particular, the systole *defined by*

$$\mathrm{sys}(\Gamma, \sigma) = \min\{\ell_\sigma(\gamma) : \gamma \in \Gamma_{\mathrm{lox}}\} > 0$$

is well-defined and positive.

8.1. **Proof of Proposition 8.2.** By symmetry it suffices to show that

$$\gamma H_{p,S,T} \subset H_{\gamma p, S+\sigma(\gamma,p), T+\bar{\sigma}(\gamma,p)}.$$

Fix $v = (x, y, t) \in H_{p,S,T}$. Then

$$\gamma v = (\gamma x, \gamma y, t + \sigma(\gamma, y))$$

Case 1: Assume $y = p$ and $t > S$. Then clearly $\gamma v \in H_{\gamma p, S+\sigma(\gamma,p), T+\bar{\sigma}(\gamma,p)}$.

Case 2: Assume $x = p$ and $t < G(p, y) - T$. Then Equation (1) implies

$$t + \sigma(\gamma, y) < G(p, y) + \sigma(\gamma, y) - T = G(\gamma y, \gamma p) - \bar{\sigma}(\gamma, p) - T.$$

So $\gamma v \in H_{\gamma p, S+\sigma(\gamma,p), T+\bar{\sigma}(\gamma,p)}$.

Case 3: Assume $x \neq p$, $y \neq p$ and $G(x, y) - G(x, p) + S < t < G(p, y) - T$. Then by Equation (1),

$$\begin{aligned} G(\gamma x, \gamma y) - G(\gamma x, \gamma p) + (S + \sigma(\gamma, p)) &= G(x, y) - G(x, p) + \sigma(\gamma, y) + S \\ &< t + \sigma(\gamma, y) \end{aligned}$$

and

$$G(\gamma p, \gamma y) - (T + \bar{\sigma}(\gamma, p)) = G(p, y) + \sigma(\gamma, y) - T > t + \sigma(\gamma, y).$$

So $\gamma v \in H_{\gamma p, S+\sigma(\gamma,p), T+\bar{\sigma}(\gamma,p)}$. □

8.2. **Constructing an equivariant set of horoballs.** In this subsection we prove part (1) of Theorem 8.3.

Fix a distance d on $\overline{M^{(2)} \times \mathbb{R}}$ which generates the topology in Proposition 7.1 and for $p \in \overline{M^{(2)} \times \mathbb{R}}$ and $r > 0$ let $B_r(p)$ denote the associated metric ball. We first show that horoballs can be made arbitrary small, in a uniform way.

Lemma 8.5. *For any $\epsilon > 0$ there exists $T > 0$ such that $H_{p,T,T} \subset B_\epsilon(p)$ for all $p \in M$.*

Proof. Suppose not. Then for every $n \geq 1$ there exist $p_n \in M$ and $v_n = (x_n, y_n, t_n) \in H_{p_n,n,n}$ with $\mathrm{d}(v_n, p_n) > \epsilon$. Passing to a subsequence we can suppose that $p_n \to p$. Passing to a further subsequence, it suffices to consider the following cases.

Case 1: Assume $y_n = p_n$ and $t_n > n$ for all n. Then Proposition 7.1(4) implies that $v_n \to p$. Thus $\mathrm{d}(v_n, p_n) < \epsilon$ for large n, which is a contradiction.

Case 2: Assume $x_n = p_n$ and $t_n < G(p_n, y_n) - n$ for all n. Passing to a subsequence, we can suppose that $y_n \to y$. If $y = p$, then $x_n \to p$ and $y_n \to p$. If $y \neq p$, then

$$\liminf_{n\to\infty} t_n \leq G(p, y) + \liminf_{n\to\infty} -n = -\infty.$$

So in all cases, Proposition 7.1(4) implies that $v_n \to p$. Thus $\mathrm{d}(v_n, p_n) < \epsilon$ for large n, which is a contradiction.

Case 3: Assume $x_n \neq p_n$, $y_n \neq p_n$ and

$$G(x_n, y_n) - G(x_n, p_n) + n < t_n < G(p_n, y_n) - n$$

for all n. Passing to a subsequence we can suppose that $x_n \to x$ and $y_n \to y$. Since

$$G(x_n, p_n) + G(p_n, y_n) > 2n + G(x_n, y_n) \geq 2n$$

and G is locally bounded on $M^{(2)}$, at least one of x or y must equal p. If $x = p$ and $y \neq p$, then

$$\liminf_{n\to\infty} t_n \leq \liminf_{n\to\infty} G(p_n, y_n) - n = G(p, y) + \liminf_{n\to\infty} -n = -\infty.$$

If $y = p$ and $x \neq p$, then

$$\liminf_{n\to\infty} t_n \geq \liminf_{n\to\infty} G(x_n, y_n) - G(x_n, p_n) + n = G(x, y) - G(x, p) + \liminf_{n\to\infty} n = +\infty.$$

So in all cases, Proposition 7.1(4) implies that $v_n \to p$. Thus $\mathrm{d}(v_n, p_n) < \epsilon$ for large n, which is a contradiction. □

Next we show that images under elements $\gamma \in \Gamma$ of a fixed small horoball at a bounded parabolic point can not become too big.

Lemma 8.6. *If $p \in M$ is a bounded parabolic point, then there exists $C > 0$ such that $\sigma(\gamma, p) \geq -C$ and $\bar{\sigma}(\gamma, p) \geq -C$ for all $\gamma \in \Gamma$.*

In particular $\gamma H_{p,T,T} \subset H_{\gamma p, T-C, T-C}$ for all $\gamma \in \Gamma$ and T.

Proof. Suppose not. Then there exists $\{\gamma_n\} \subset \Gamma$ such that $\sigma(\gamma_n, p) \to -\infty$ (the case $\bar{\sigma}(\gamma_n, p) \to -\infty$ is similar). By Proposition 2.7, there exists $\{\alpha_n\} \subset \Gamma_p$ such that $\{\alpha_n \gamma_n^{-1}\}$ stays in a compact set of $\Gamma \sqcup \Lambda(\Gamma) \smallsetminus \{p\}$. Proposition 2.9(1) implies that $\sigma(\alpha_n^{-1}, p) = 0$, so, since σ is a cocycle, $\sigma(\gamma_n \alpha_n^{-1}, p) = \sigma(\gamma_n, p) \to -\infty$. Therefore, since σ is expanding, $\alpha_n \gamma_n^{-1} = (\gamma_n \alpha_n^{-1})^{-1} \to p$, which is a contradiction. $\square$

The following lemma will allow us to choose the size of our horoballs.

Lemma 8.7. *If $p, q \in M$ are bounded parabolic points, then there exists $T = T(p,q) > 0$ such that $H_{p,T,T} \cap \gamma H_{q,T,T} \neq \varnothing$ if and only if $\gamma(q) = p$.*

Proof. ($\Leftarrow$): Suppose $\gamma(q) = p$. Let us check that $H_{p,T,T} \cap \gamma H_{q,T,T} \neq \varnothing$ for any T:

$$(x, p, t) \in H_{p,T,T} \cap \gamma H_{q,T,T} = H_{p,T,T} \cap H_{p,T+\sigma(\gamma,q),T+\bar{\sigma}(\gamma,q)}$$

if $x \in M - \{p\}$ and $t > T + \max\{0, \sigma(\gamma, q)\}$.

($\Rightarrow$): Fix a compact set $K \subset \Lambda(\Gamma) \smallsetminus \{p\}$ such that $\Gamma_p K = \Lambda(\Gamma)$ and let $\epsilon := \mathrm{d}(p, K)$. By Lemma 8.6 there exists $C > 0$ such that $\sigma(\gamma, q) \geq -C$ and $\bar{\sigma}(\gamma, q) \geq -C$ for all $\gamma \in \Gamma$. Lemma 8.5 then implies that there exists T such that $H_{x,T-C,T-C} \subset B_{\epsilon/2}(x)$ for any $x \in M$.

Fix $\gamma \in \Gamma$ such that $\gamma(q) \neq p$. Then fix $\alpha \in \Gamma_p$ such that $\alpha\gamma(q) \in K$. Since $\alpha H_{p,T,T} = H_{p,T,T}$ by Proposition 8.2, it suffices to prove that $H_{p,T,T}$ and $\alpha\gamma H_{q,T,T}$ are disjoint. On one hand,

$$\alpha\gamma H_{q,T,T} = H_{\alpha\gamma q, T+\sigma(\alpha\gamma,q), T+\bar{\sigma}(\alpha\gamma,q)} \subset H_{\alpha\gamma q, T-C, T-C} \subset B_{\epsilon/2}(\alpha\gamma q).$$

On the other hand, $H_{p,T,T} \subset H_{p,T-C,T-C} \subset B_{\epsilon/2}(p)$, and these two balls of radius $\epsilon/2$ are disjoint since $\mathrm{d}(p, \alpha\gamma q) \geq \mathrm{d}(p, K) = \epsilon$, so $H_{p,T,T}$ and $H_{q,T,T}$ are disjoint. $\square$

By Lemma 2.4 there are finitely many Γ-orbits of parabolic points. Let $\{p_1, \ldots, p_k\}$ contain one representative of each orbit. Applying Lemma 8.7 to each pair in $\{p_1, \ldots, p_k\}$ we can find $T > 0$ such that

$$\gamma_1 H_{p_i,T,T} \cap \gamma_2 H_{p_j,T,T} = \gamma_1 \left(H_{p_i,T,T} \cap \gamma_1^{-1}\gamma_2 H_{p_j,T,T} \right) = \varnothing$$

when $\gamma_1(p_i) \neq \gamma_2(p_j)$. Then for $p \in \mathcal{F}$, let $H_{p,S_p,T_p} := \gamma H_{p_i,T,T}$ where $p = \gamma p_i$. This completes the proof of part (1) of Theorem 8.3. $\square$

8.3. **Cocompactness.** In this subsection we prove part (2) of Theorem 8.3.

Notice that Proposition 7.1(4) implies that $\tilde{U}_\Gamma \sqcup \Lambda(\Gamma)$ is a closed subset of $\overline{M^{(2)} \times \mathbb{R}}$.

Lemma 8.8. *If $p \in M$ is a bounded parabolic point, then Γ_p acts cocompactly on $(\tilde{U}_\Gamma \sqcup \Lambda(\Gamma)) - (H_{p,S,T} \sqcup \{p\})$ for all $S, T \in \mathbb{R}$.*

Proof. Since p is bounded parabolic point, there exists a compact set $K \subset \Lambda(\Gamma) - \{p\}$ such that $\Gamma_p(K) = \Lambda(\Gamma) \smallsetminus \{p\}$. Define subsets

$$A_1 := \{(x, y, t) \in \tilde{U}_\Gamma : x \in K \text{ and } t \leq G(x, y) - G(x, p) + S\}$$

and

$$A_2 := \{(x, y, t) \in \tilde{U}_\Gamma : y \in K \text{ and } t \geq G(p, y) - T\}.$$

Notice that A_1 and A_2 are closed in $\tilde{U}_\Gamma$. Further, since $G \colon M^{(2)} \to \mathbb{R}$ is continuous and hence bounded on compact subsets, Proposition 7.1(4) implies that these sets only accumulate on $K \subset \Lambda(\Gamma)$. Hence, $A_1 \cup A_2 \cup K$ is a compact subset of $\tilde{U}_\Gamma \sqcup \Lambda(\Gamma)$. So to prove that Γ_p acts cocompactly, it suffices to fix

$$v = (x, y, t) \in \tilde{U}_\Gamma - H_{p,S,T}$$

and then show that there exists $\gamma \in \Gamma_p$ with

$$\gamma(v) = (\gamma(x), \gamma(y), t + \sigma(\gamma, y)) \in A_1 \cup A_2.$$

Since $v \notin H_{p,S,T}$, by the definition of horoballs we have three cases. If $y = p$, then $v \notin H_{p,S,T}$ which forces $t \leq S$. If $x = p$, then $v \notin H_{p,S,T}$ which forces $t \geq G(p, y) - T$. The last case where $x \neq p$ and $y \neq p$ splits into two subcases: then $v \notin H_{p,S,T}$ forces either $t \leq G(x, y) - G(x, p) + S$ or $t \geq G(p, y) - T$. Let us deal with each of these four cases separately.

Case 1: Assume $y = p$. Then $t \leq S$. Choose $\gamma \in \Gamma_p$ so that $\gamma(x) \in K$. Then

$$\gamma(v) = \gamma(x, p, t) = (\gamma(x), p, t) \in A_1,$$

since $t \leq S = G(\gamma x, p) - G(\gamma x, p) + S$.

Case 2: Assume $x = p$. Then $t \geq G(p, y) - T$. Choose $\gamma \in \Gamma_p$ so that $\gamma(y) \in K$. Then

$$G(p, \gamma(y)) - T = G(p, y) + \bar{\sigma}(\gamma, p) + \sigma(\gamma, y) - T \leq t + \sigma(\gamma, y)$$

since $\bar{\sigma}(\gamma, p) = 0$. Hence

$$\gamma(v) = \gamma(p, y, t) = (p, \gamma(y), t + \sigma(\gamma, y)) \in A_2$$

Case 3: Assume $x \neq p$, $y \neq p$ and $t \leq G(x, y) - G(x, p) + S$. Then by Equation (1),

$$G(\gamma(x), \gamma(y)) - G(x, y) = \bar{\sigma}(\gamma, x) + \sigma(\gamma, y)$$

and

$$G(\gamma(x), p) - G(x, p) = \bar{\sigma}(\gamma, x) + \sigma(\gamma, p) = \bar{\sigma}(\gamma, x).$$

So

$$G(\gamma(x), \gamma(y)) - G(\gamma(x), p) + S = G(x, y) - G(x, p) + \sigma(\gamma, y) + S \geq t + \sigma(\gamma, y).$$

Thus

$$\gamma(v) = (\gamma(x), \gamma(y), t + \sigma(\gamma, y)) \in A_1.$$

Case 4: Assume $x \neq p$, $y \neq p$ and $t \geq G(p, y) - T$. Arguing as in Case 2, there exists $\gamma \in \Gamma_p$ where $\gamma(v) \in A_2$. $\square$

Now let $\chi \colon M^{(2)} \times \mathbb{R} \to [0, \infty)$ with $\chi(x, y, t) = G(y, x) + |t|$ and

$$\mathcal{D} = \{v \in M^{(2)} \times \mathbb{R} : \chi(v) \leq \chi(\gamma v) \text{ for all } \gamma \in \Gamma\}$$

be as in Section 7. We show that if p is a bounded parabolic point and H is a horoball based at p, then $\mathcal{D} \cap \tilde{U}_\Gamma \smallsetminus H$ does not accumulate at p.

Lemma 8.9. *If $p \in M$ is a bounded parabolic point and H is a horoball based at p, then $\mathcal{D} \cap \tilde{U}_\Gamma \smallsetminus H$ does not accumulate on p.*

Proof. Suppose for a contradiction that there exists a sequence $\{v_n\}$ in $\mathcal{D} \cap \tilde{U}_\Gamma \smallsetminus H$ converging to p. Using Lemma 8.8 and passing to a subsequence we can find $\{\gamma_n\} \subset \Gamma_p$, such that

$$\gamma_n(v_n) \to v \in (\tilde{U}_\Gamma \sqcup \Lambda(\Gamma)) - (H_{p,S,T} \sqcup \{p\}).$$

Since $\gamma_n(p) = p$ for all n, the sequence $\{\gamma_n\}$ must be escaping (otherwise we would have $\gamma_n(v_n) \to p$ since $v_n \to p$). Since $\{\gamma_n\} \subset \Gamma_p$, we must have $\gamma_n^{-1} \to p$ and $\gamma_n \to p$.

Then, since $\gamma_n(v_n) \to v \neq p$, by Lemma 7.4 there exists $C > 0$ such that

$$\chi(v_n) = \chi(\gamma_n^{-1}\gamma_n v_n) \geq \chi(\gamma_n v_n) + \min\left(\left\|\gamma_n^{-1}\right\|_\sigma, \|\gamma_n\|_\sigma\right) - C$$

for all n. Since $\left\|\gamma_n^{\pm 1}\right\|_\sigma \to +\infty$, we have $\chi(v_n) > \chi(\gamma_n v_n)$ when n is sufficiently large. Hence $v_n \notin \mathcal{D}$ when n is sufficiently large, which is a contradiction. □

Finally, we conclude that Γ acts cocompactly on the complement of the horoballs about all the bounded parabolic points.

Lemma 8.10. *If $\{H_{p,S_p,T_p}\}_{p \in \mathcal{F}}$ is any Γ-equivariant disjoint set of horoballs, then the quotient*

$$\Gamma \backslash \left(\tilde{U}_\Gamma - \bigcup_p H_p\right)$$

is compact.

Proof. Let

$$L := \mathcal{D} \cap \tilde{U}_\Gamma - \bigcup_p H_p.$$

By Proposition 7.2,

$$\Gamma(L) = \tilde{U}_\Gamma - \bigcup_p H_p.$$

So it suffices to check that L is compact. Since L is a closed subset of $\tilde{U}_\Gamma$, it suffices to check that L does not accumulate at any point of M in $\overline{M^{(2)} \times \mathbb{R}}$. By Proposition 7.1(4) the closure of $\tilde{U}_\Gamma = \Lambda(\Gamma)^{(2)} \times \mathbb{R}$ in $\overline{M^{(2)} \times \mathbb{R}}$ is $\Lambda(\Gamma)$. So it suffices to show that L does not accumulate at any point of $\Lambda(\Gamma)$.

Proposition 7.2 implies that $\mathcal{D}$ does not accumulate at any conical point in $\Lambda(\Gamma)$, while Lemma 8.9 implies that $\mathcal{D} - \bigcup_p H_p$ does not accumulates at any bounded parabolic point in $\Lambda(\Gamma)$. Since Γ acts geometrically finitely, every point of $\Lambda(\Gamma)$ is either a conical limit point or a bounded parabolic point, so this concludes the proof. □

8.4. **Proof of Corollary 8.4.** Using the notation in the proof of Lemma 8.10, the set $L := \mathcal{D} \cap \tilde{U}_\Gamma - \cup_p H_p$ is compact. By Proposition 2.9(4), there exists $C > 0$ so that if the axis $(\gamma^-, \gamma^+) \times \mathbb{R}$ of $\gamma \in \Gamma_{\mathrm{lox}}$ intersects L, then

$$\ell_\sigma(\gamma) \geq \|\gamma\|_\sigma - C.$$

Now suppose that $\gamma \in \Gamma_{\mathrm{lox}}$, then its axis cannot be entirely contained in any H_p, since otherwise $\gamma(H_p) \cap H_p = H_{\gamma(p)} \cap H_p$ would intersect. Therefore, there exists an element $\hat{\gamma}$ which is conjugate to γ and intersects L. It follows that

$$\ell_\sigma(\gamma) = \ell_\sigma(\hat{\gamma}) \geq \|\hat{\gamma}\|_\sigma - C.$$

Since σ is proper, we conclude that the set of conjugacy classes of loxodromic elements of length at most T must be finite. □

9. Finiteness criteria for BMS measures

We obtain a finiteness criterion for BMS measures in the geometrically finite case, which is the natural analogue of the criterion of Dal'bo–Otal–Peigné [20]. In fact, our development of the theory of horoballs was designed so that we can emulate their proof. This criterion was generalized to geometrically infinite groups by Pit–Schapira [37, Th. 1.4, 1.6, 1.8], in the context of negatively curved Riemannian manifolds.

Theorem 9.1. *Suppose $(\sigma, \bar{\sigma}, G)$ is a continuous GPS system for a geometrically finite convergence group $\Gamma \subset \mathrm{Homeo}(M)$ where $\delta = \delta_\sigma(\Gamma) < +\infty$. If $\delta_\sigma(P) < \delta$ for any maximal parabolic subgroup P of Γ, then $Q_\sigma(\delta) = +\infty$ and the BMS measure is finite.*

In [8, Th. 4.2] we showed that if the σ-Poincaré series diverges for a maximal parabolic subgroup Γ_p of Γ, then $\delta_\sigma(\Gamma_p) < \delta_\sigma(\Gamma)$. So Theorem 9.1 implies the following criterion.

Corollary 9.2. *Suppose $(\sigma, \bar{\sigma}, G)$ is a GPS system for a geometrically finite convergence group $\Gamma \subset \mathsf{Homeo}(M)$ with $\delta := \delta_\sigma(\Gamma) < +\infty$. If*

$$\sum_{\gamma \in \Gamma_p} e^{-\delta_\sigma(\Gamma_p)\|\gamma\|_\sigma} = +\infty$$

whenever p is a bounded parabolic point of Γ, then $Q_\sigma(\delta) = +\infty$ and the BMS measure is finite.

The rest of the section is devoted to the proof of Theorem 9.1. Fix a GPS system $(\sigma, \bar{\sigma}, G)$ for a geometrically finite convergence group $\Gamma \subset \mathsf{Homeo}(M)$ with $\delta := \delta_\sigma(\Gamma) < +\infty$.

The theorem will follow from the next two propositions and Theorem 8.3 above. The first provides a condition for the "cusps" to have finite measure and the second verifies this condition when the hypotheses of Theorem 9.1 are satisfied.

Proposition 9.3. *Suppose p is a bounded parabolic point of Γ, μ is a σ-Patterson–Sullivan measure of dimension δ, and $\bar{\mu}$ is a $\bar{\sigma}$-Patterson–Sullivan measure of dimension δ. Let m_Γ denote the quotient measure on U_Γ associated to the measure $\tilde{m} = e^{\delta G(x,y)} d\bar{\mu}(x) \otimes d\mu(y) \otimes dt$ on $\tilde{U}_\Gamma$.*

If $\mu(\{p\}) = \bar{\mu}(\{p\}) = 0$ and

$$\sum_{\gamma \in \Gamma_p} ||\gamma||_\sigma e^{-\delta ||\gamma||_\sigma} < +\infty,$$

then for any horoball $H_p = H_{p,S,T}$ based at p the measure m_Γ is finite on the image of $H_p \cap \tilde{U}_\Gamma$ in $U_\Gamma = \Gamma \backslash \tilde{U}_\Gamma$.

Proposition 9.4. *If $\delta_\sigma(\Gamma_p) < \delta$ whenever p is a bounded parabolic point of Γ, then there exist a σ-Patterson–Sullivan-measure μ of dimension δ and a $\bar{\sigma}$-Patterson–Sullivan-measure $\bar{\mu}$ of dimension δ such that*

$$\mu(\{p\}) = \bar{\mu}(\{p\}) = 0$$

whenever p is a bounded parabolic point.

Assuming the propositions we prove Theorem 9.1.

Proof of Theorem 9.1. Let $\mathcal{F} \subset \Lambda(\Gamma)$ denote the set of bounded parabolic points. By Theorem 8.3 there exists an Γ-equivariant collection $\{H_p = H_{p,S_p,T_p}\}_{p\in\mathcal{F}}$ of disjoint horoballs so that the action of Γ on $\tilde{U}_\Gamma - \bigcup H_p$ is cocompact. Proposition 9.4 implies that there exist Patterson–Sullivan measures μ and $\bar{\mu}$ of dimension δ with the property that $\mu(\{p\}) = \bar{\mu}(\{p\}) = 0$ for any $p \in \mathcal{F}$. Since Γ is geometrically finite, there are countably many bounded parabolic points by Lemma 2.4, and the rest are conical, so the conical limit set has full measure for both μ and $\bar{\mu}$. Theorem 2.12 then implies that $Q_\sigma(\delta) = Q_{\bar{\sigma}}(\delta) = +\infty$.

Let m_Γ denote the BMS measure constructed in Section 2.3. By definition this measure is locally finite and so the m_Γ-measure of the quotient of $\tilde{U}_\Gamma - \bigcup H_p$ is finite. Since $\delta_\sigma(\Gamma_p) < \delta$ whenever p is a bounded parabolic point of Γ, we see that

$$\sum_{\gamma\in\Gamma_p} \|\gamma\|_\sigma \, e^{-\delta\|\gamma\|_\sigma} < +\infty.$$

Hence Proposition 9.3 then implies that m_Γ is finite on the image of $H_p \cap \tilde{U}_\Gamma$ in U_Γ for all $p \in \mathcal{F}$. Since there are only finitely many orbits of bounded parabolic points by Lemma 2.4, we conclude that m_Γ is finite on U_Γ. $\square$

9.1. **Proof of Proposition 9.3.** Our proof closely follows a classical argument (see [21, Th. IV.2] in the case of negatively curved Riemannian manifolds). Fix a bounded parabolic point $p \in \Lambda(\Gamma)$ and Patterson–Sullivan measures μ, $\bar{\mu}$ as in the statement of the proposition.

Since p is a bounded parabolic point, there exists a compact subset $K \subset \Lambda(\Gamma) \smallsetminus \{p\}$ such that $\Gamma_p(K) = \Lambda(\Gamma) \smallsetminus \{p\}$. Since Γ_p only accumulates on p, the expanding property implies that there exists a constant $D > 0$ such that

$$\|\gamma\|_\sigma - D \leq \sigma(\gamma, z) \leq \|\gamma\|_\sigma \tag{11}$$

for all $\gamma \in \Gamma_p$ and $z \in K$.

Given $x, y \in M \smallsetminus \{p\}$ distinct, let $I_{x,y} \subset \mathbb{R}$ be the interval satisfying

$$H_p \cap \big((x, y) \times \mathbb{R}\big) = (x, y) \times I_{x,y}.$$

If $I_{x,y} \neq \varnothing$, then by definition

$$I_{x,y} := \big(G(x, y) - G(x, p) + S, G(p, y) - T\big).$$

We record the following estimate on the Lebesgue measure of $I_{x,y}$ as a lemma since it will also be used later (in the proof of Proposition 10.3).

Lemma 9.5. *There exists $C > 0$ (independent of S, T) such that: if $\gamma \in \Gamma_p$, $x \in K$, $y \in \gamma(K)$ and $I_{x,y} \neq \varnothing$, then*

$$G(x, y) \leq C + \max(0, -S - T),$$

and

$$\|\gamma\|_\sigma - \max(0, S + T) - C \leq \mathrm{Leb}(I_{x,y}) \leq \|\gamma\|_\sigma - (S + T) + C.$$

Proof. Let

$$R := \max_{z\in K} \max\left(G(z, p), G(p, z)\right).$$

Fix an open neighborhood $\mathcal{O}$ of p in M such that

$$\sup_{z_1\in\mathcal{O}, z_2\in K} \max\left(G(z_1, z_2), G(z_2, z_1)\right) \leq R + 1.$$

Since Γ_p only accumulates on p, there exists $N>0$ such that $\gamma(K)\subset\mathcal{O}$ whenever $\gamma\in\Gamma_p$ and $\|\gamma\|_\sigma\geq N$. Let

$$R' := \max_{\gamma\in\Gamma_p, \|\gamma\|_\sigma<N, z\in\gamma K} \max\left(G(z,p),G(p,z)\right).$$

Now suppose $\gamma\in\Gamma_p$, $x\in K$, $y\in\gamma(K)$, $I_{x,y}\neq\varnothing$.

If $\|\gamma\|_\sigma\geq N$, then $G(x,y)\leq R+1$ since $x\in K$ and $y\in\mathcal{O}$. If $\|\gamma\|_\sigma<N$, then $I_{x,y}\neq\varnothing$ implies that

$$G(x,y)\leq G(x,p)+G(p,y)-T-S\leq 2R'-T-S.$$

Hence the first inequality of the lemma is true for any $C\geq\max(R+1,2R')$.

By Equation (1) and Proposition 2.9(1),

$$G(p,y)=G(p,\gamma^{-1}y)+\bar{\sigma}(\gamma,p)+\sigma(\gamma,\gamma^{-1}y)=G(p,\gamma^{-1}y)+\sigma(\gamma,\gamma^{-1}y).$$

Combining the above equation, the definition of $I_{x,y}$, the non-negativity of G, and Equation (11) we obtain

$$\begin{aligned}
\mathrm{Leb}(I_{x,y})-(\|\gamma\|_\sigma-S-T)&=G(p,y)-G(x,y)+G(x,p)-\|\gamma\|_\sigma\\
&=G(p,\gamma^{-1}y)+\sigma(\gamma,\gamma^{-1}y)-G(x,y)+G(x,p)-\|\gamma\|_\sigma\\
&\leq G(p,\gamma^{-1}y)+\sigma(\gamma,\gamma^{-1}y)+G(x,p)-\|\gamma\|_\sigma\\
&\leq R+R,
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Leb}(I_{x,y})-(\|\gamma\|_\sigma-S-T)&=G(p,\gamma^{-1}y)+\sigma(\gamma,\gamma^{-1}y)-G(x,y)+G(x,p)-\|\gamma\|_\sigma\\
&\geq G(p,\gamma^{-1}y)+\sigma(\gamma,\gamma^{-1}y)-G(x,y)-\|\gamma\|_\sigma\geq -G(x,y)-D\\
&\geq -\max(R+1,2R'-S-T)-D.
\end{aligned}$$

□

As in the statement of Proposition 9.3, let

$$d\tilde{m}:=e^{\delta G(x,y)}d\bar{\mu}(x)\otimes d\mu(y)\otimes dt$$

and let m_Γ be the quotient measure on U_Γ. Also let $\pi\colon\tilde{U}_\Gamma\to U_\Gamma$ be the projection map. It follows from Equation (2) that

$$m_\Gamma(\pi(A))\leq\tilde{m}(A)$$

for all measurable subsets $A\subset\tilde{U}_\Gamma$ (since $P(1_A)\geq 1_{\pi(A)}$). (Observe that $\pi(A)$ is measurable since $\pi^{-1}(\pi(A))=\Gamma\cdot A$ is a countable union of measurable sets.)

Let

$$H_\gamma:=\bigcup_{x\in K,y\in\gamma K}(x,y)\times I_{x,y}.$$

By Lemma 9.5 there is $C>0$ dependent on S,T such that if $\gamma\in\Gamma_p$, $x\in K$, $y\in\gamma(K)$ and $I_{x,y}\neq\varnothing$, then

$$G(x,y)\leq C\quad\text{and}\quad\mathrm{Leb}(I_{x,y})\leq\|\gamma\|_\sigma+C.$$

Then by Equation (11) we have

$$\begin{aligned}\tilde{m}(H_\gamma) &= \int_{\substack{x\in K\\ y\in\gamma K}} \mathrm{Leb}(I_{x,y})e^{\delta G(x,y)}d\bar{\mu}(x)d\mu(y) \leq (C+||\gamma||_\sigma)\, e^{\delta C}\bar{\mu}(K)\mu(\gamma K)\\ &= (C+||\gamma||_\sigma)\, e^{\delta C}\bar{\mu}(K)\int_K e^{-\delta\sigma(\gamma,y)}d\mu(y)\\ &\leq (C+||\gamma||_\sigma)e^{\delta C}\bar{\mu}(K)e^{\delta D}e^{-\delta||\gamma||_\sigma}\mu(K).\end{aligned}$$

Let $\widehat{H}_p$ denote the image of $H_p \cap \tilde{U}_\Gamma$ in U_Γ. Notice that $\widehat{H}_p$ is contained in the image of

$$\Big(\{p\}\times(\Lambda(\Gamma)\smallsetminus\{p\})\times\mathbb{R}\Big)\cup\Big((\Lambda(\Gamma)\smallsetminus\{p\})\times\{p\}\times\mathbb{R}\Big)\cup\bigcup_{\gamma\in\Gamma_p}H_\gamma.$$

Since μ and $\bar{\mu}$ do not have atoms at p, the measure of the set of geodesic segments in H_p with one endpoint at p has $\tilde{m}$-measure zero. Therefore,

$$m_\Gamma(\widehat{H}_p) \leq \sum_{\gamma\in\Gamma_p}\tilde{m}(H_\gamma) \leq e^{\delta(D+C)}\sum_{\gamma\in\Gamma_p}\bar{\mu}(K)\mu(K)e^{-\delta||\gamma||_\sigma}\,(C+||\gamma||_\sigma)\,.$$

This implies that $m_{\mu,\bar{\mu}}(\widehat{H}_p)$ is finite, since σ is proper and we assumed that

$$\sum_{\gamma\in\Gamma_p}||\gamma||_\sigma e^{-\delta||\gamma||_\sigma} < +\infty.$$

9.2. **Proof of Proposition 9.4.** By symmetry, it suffices to construct a σ-Patterson–Sullivan measure μ of dimension δ such that

$$\mu(\{p\}) = 0$$

whenever p is a bounded parabolic point.

We will use the standard construction for Patterson–Sullivan measures. Endow $M \sqcup \Gamma$ with the topology described in Proposition 2.6.

Fix a non-decreasing function $h : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 1}$ such that

(a) For every $\epsilon > 0$ there exists $R > 0$ such that $h(r+t) \leq e^{\epsilon t}h(r)$ for any $r \geq R$ and $t \geq 0$,

(b) $\sum_{\gamma\in\Gamma} h(\|\gamma\|_\sigma)e^{-\delta\|\gamma\|_\sigma} = +\infty$.

(When $\sum_{\gamma\in\Gamma} e^{-\delta\|\gamma\|_\sigma} = +\infty$, we can take $h \equiv 1$).

For $s > \delta$, consider the probability measure

$$\mu_s := \frac{1}{Q(s)}\sum_{\gamma\in\Gamma} h(\|\gamma\|_\sigma)e^{-s\|\gamma\|_\sigma}\mathcal{D}_\gamma,$$

where $Q(s)$ denotes the modified Poincaré series $Q(s) := \sum_{\gamma\in\Gamma} h(\|\gamma\|_\sigma)e^{-s\|\gamma\|_\sigma}$ and D_γ denotes the Dirac measure centered on $\gamma \in \Gamma \sqcup M$.

Fix $s_m \searrow \delta$ such that $\mu_{s_m} \to \mu$ in the weak-$*$ topology. Then one can show that μ is a σ-Patterson–Sullivan measure of dimension δ, see the proof of [8, Th. 4.1] for details.

By Proposition 2.7, there exists a compact subset K of $(\Gamma \sqcup M) \smallsetminus \{p\}$ such that $\Gamma_p(K) = (\Gamma \sqcup M) \smallsetminus \{p\}$. Then let $\Gamma_0 := K \cap \Gamma$ and notice that $\Gamma_p(\Gamma_0) = \Gamma$.

Next enumerate $\Gamma_p = \{\alpha_n\}$. Then consider the decreasing sequence of open neighborhoods of p given by

$$V_n := (\Gamma \sqcup M) \smallsetminus (\alpha_1(K) \cup \cdots \cup \alpha_n(K))\,.$$

We will show that $\mu(V_n) \to 0$ as $n \to \infty$.

Since $\alpha_k^{-1} \to p$ as $k \to \infty$, Proposition 2.9(3) implies that there exists $D > 0$ such that

$$\|\alpha_k\|_\sigma + \|\gamma\|_\sigma - D \leq \|\alpha_k \gamma\|_\sigma$$

for all $k \geq 1$ and $\gamma \in \Gamma_0$. Further, by definition,

$$\|\alpha_k\gamma\|_\sigma = \max_{x \in M} \sigma(\alpha_k \gamma, x) = \max_{x \in M} \sigma(\alpha_k, \gamma x) + \sigma(\gamma, x) \leq \|\alpha_k\|_\sigma + \|\gamma\|_\sigma$$

for all $k \geq 1$ and $\gamma \in \Gamma_0$.

Set $\epsilon := (\delta_\sigma(\Gamma) - \delta_\sigma(P))/2$. By property (a) of the function h and its monotonicity,

$$h(\|\alpha_k\gamma\|_\sigma) \leq h(\|\alpha_k\|_\sigma + \|\gamma\|_\sigma) \leq e^{\epsilon\|\alpha_k\|_\sigma} h(\|\gamma\|_\sigma)$$

for k sufficiently large and for all $\gamma \in \Gamma_0$.

Then for n sufficiently large we obtain for any $s > \delta$

$$\begin{aligned}
\mu_s(V_n) &\leq \frac{1}{Q(s)} \sum_{k>n} \sum_{\gamma \in \Gamma_0} h(\|\alpha_k\gamma\|_\sigma) e^{-s\|\alpha_k\gamma\|_\sigma} \\
&\leq \frac{e^{sD}}{Q(s)} \sum_{k>n} \sum_{\gamma \in \Gamma_0} e^{\epsilon\|\alpha_k\|_\sigma} h(\|\gamma\|_\sigma) e^{-s(\|\alpha_k\|_\sigma + \|\gamma\|_\sigma)} \\
&\leq \frac{e^{sD}}{Q(s)} \sum_{k>n} e^{-(s-\epsilon)\|\alpha_k\|_\sigma} \sum_{\gamma \in \Gamma_0} h(\|\gamma\|_\sigma) e^{-s\|\gamma\|_\sigma} \\
&\leq e^{sD} \sum_{k>n} e^{-(\delta-\epsilon)\|\alpha_k\|_\sigma}.
\end{aligned}$$

Since V_n is open, we can take the limit $s_m \searrow \delta$ to get

$$\begin{aligned}
\mu(\{p\}) &\leq \lim_{n\to\infty} \mu(V_n) \leq \lim_{n\to\infty} \liminf_{m\to\infty} \mu_{s_m}(V_n) \\
&\leq \lim_{n\to\infty} e^{\delta D} \sum_{k>n} e^{-(\delta-\epsilon)||\alpha_k||_\sigma} = 0,
\end{aligned}$$

where in the last equality we use the fact that $\sum_{k=1}^\infty e^{-(\delta-\epsilon)||\alpha_k||_\sigma} < +\infty$.

10. Equidistribution for geometrically finite actions

In this section we establish Theorem 1.6 and Theorem 1.1, which we restate below. For the rest of the section suppose that:

- $(\sigma, \bar\sigma, G)$ is a GPS system for a geometrically finite convergence group $\Gamma \subset \mathsf{Homeo}(M)$ with $\delta := \delta_\sigma(\Gamma) < +\infty$,
- $\delta_\sigma(\Gamma_p) < \delta$ whenever p is a bounded parabolic point of Γ. Hence by Theorem 9.1 the Poincaré series of Γ diverges at its critical exponent and the BMS measure m_Γ is finite.
- The flow $\psi^t \colon (U_\Gamma, m_\Gamma) \to (U_\Gamma, m_\Gamma)$ is mixing.

The method we will use to obtain counting result naturally provides counting for closed ψ^t-orbits. In general (when one allows torsion), these orbits do not correspond to conjugacy classes, instead they correspond to slightly bigger classes called weak conjugacy classes, that we introduce now.

We say two loxodromic elements $\gamma_1, \gamma_2 \in \Gamma$ are *weakly conjugate* if they have the same length and there exists $g \in \Gamma$ such that γ_1 and $g\gamma_2 g^{-1}$ have the same attracting fixed point and the same repelling fixed point. We denote by $[\Gamma_{\rm lox}]^w$ the set of weak conjugacy classes. Notice that if Γ is torsion-free, then the set of weak conjugacy classes is simply the set of conjugacy classes. Moreover, the number of conjugacy classes in a given weak conjugacy class $[\gamma]^w$ is bounded above by the size of the stabilizer of the axis of γ (see [9, Eq. (7.1)]).

For every $R > 0$, let m_R be the sum of Lebesgue measures on closed ψ^t-orbits of length at most R in U_Γ (counted with multiplicity), more precisely m_R is the quotient measure associated to the measure $\tilde{m}_R$ on $\tilde{U}_\Gamma$ defined by

$$\tilde{m}_R = \sum_{\substack{\gamma\in\Gamma_{\rm lox}\\ \ell_\sigma(\gamma)\leq R}} \mathcal{D}_{\gamma^-}\otimes\mathcal{D}_{\gamma^+}\otimes dt.$$

Notice that m_R is a finite measure by Corollary 8.4. Further, m_R does indeed "count multiplicity": if $\gamma \in \Gamma$ is loxodromic and c_γ is the image of $(\gamma^-, \gamma^+) \times \mathbb{R}$ in U_Γ, then by Equation (2) we have

$$m_R(c_\gamma) = \sum_{\substack{\alpha\in\Gamma_{\rm lox},\\ \alpha^\pm=\gamma^\pm,\\ \ell_\sigma(\alpha)\leq R}} \ell_\sigma(\alpha).$$

Theorem 10.1. *For any bounded continuous function f on $\Gamma\backslash(M^{(2)}\times\mathbb{R})$ we have*

$$\lim_{R\to\infty} \delta e^{-\delta R}\int f\,dm_R = \int f\,\frac{dm_\Gamma}{\|m_\Gamma\|}.$$

As a corollary, we will obtain Theorem 1.1 from the introduction.

Corollary 10.2.

$$\#\left\{[\gamma]\in[\Gamma_{\rm lox}]^w : 0<\ell_\sigma(\gamma)\leq R\right\}\sim\frac{e^{\delta R}}{\delta R},$$

i.e. the ratio of the two sides goes to 1 as $R\to+\infty$.

10.1. **Proof of Theorem 10.1.** The main idea in the proof is to obtain an estimate for the decay of the measure m_R on "cusps" based at a bounded parabolic point. More precisely, given a bounded parabolic fix point $p \in M$ and a horoball $H_{p,S,T}$ based at p, let $\widehat{H}_{p,S,T}$ denote the image of $H_{p,S,T}\cap\tilde{U}_\Gamma$ in U_Γ. Then we will prove the following estimate.

Proposition 10.3. *If $p \in M$ is a bounded parabolic point, then there exist $C > 0$ and $T_0 \in \mathbb{R}$ such that for all $R > 0$ and $T > T_0$, we have*

$$m_R(\widehat{H}_{p,T,T})\leq Ce^{\delta R}\sum_{\substack{\alpha\in\Gamma_p\\ ||\alpha||_\sigma\geq 2T-C}}||\alpha||_\sigma e^{-\delta||\alpha||_\sigma}.$$

Delaying the proof of the proposition, we prove Theorem 10.1.

Proof of Theorem 10.1. Fix representatives $p_1, \ldots, p_n$ of the Γ-orbits of bounded parabolic points (recall there are finitely many such orbits by Lemma 2.4). Given $T > 0$, let

$$C_T := \widehat{H}_{p_1,T,T}\cup\cdots\cup\widehat{H}_{p_n,T,T}.$$

Notice that $C_{T_2}\subset C_{T_1}$ when $T_2 \geq T_1$.

Fix $\epsilon > 0$. Proposition 10.3 implies there exists $T > 0$ such that for any $R > 0$

$$\delta e^{-\delta R}m_R(C_T)\leq\tfrac{\epsilon}{2}.$$

Theorem 9.1 implies that m_Γ is finite, hence there is a compact set $K \subset \tilde{U}_\Gamma$ such that its projection $\widehat{K}$ to U_Γ satisfies

$$m_\Gamma(U_\Gamma \smallsetminus \widehat{K}) \leq \tfrac{\epsilon}{2} \|m_\Gamma\|.$$

Lemma 10.4. *After possibly increasing $T > 0$, we can assume that C_T is disjoint from $\widehat{K}$.*

Proof. It suffices to find $T > 0$ such that $K \cap \gamma H_{p_i,T,T} = \varnothing$ for all i and $\gamma \in \Gamma$. Recall that $\gamma H_{p_i,T,T} = H_{\gamma p, T+\sigma(\gamma,p_i), T+\bar{\sigma}(\gamma,p_i)}$. By Lemma 8.6 there exists $C > 0$ such that $\gamma H_{p_i,T,T} \subset H_{\gamma p_i, T-C, T-C}$ for any T and all i and $\gamma \in \Gamma$.

Fix a metric d compatible with the topology on $\overline{M^{(2)} \times \mathbb{R}}$ from Proposition 7.1. Then fix $\epsilon > 0$ so that $\mathrm{d}(v, x) \geq \epsilon$ for all $v \in K$ and $x \in M$. By Lemma 8.5 there exists T such that $H_{x,T-C,T-C} \subset B_\epsilon(x)$ for any $x \in M$. Then $\gamma H_{p_i,T,T} \subset H_{\gamma p_i, T-C, T-C} \subset B_\epsilon(\gamma p_i)$ which is disjoint from K for all i and γ. $\square$

The above lemma implies that

$$m_\Gamma(C_T) \leq m_\Gamma(U_\Gamma \smallsetminus \widehat{K}) \leq \tfrac{\epsilon}{2} \|m_\Gamma\|.$$

Recall $U_\Gamma \smallsetminus C_T$ is compact by Theorem 8.3, so we can find a compactly supported continuous function $\chi \colon U_\Gamma \to [0, 1]$ which is equal to 1 on $U_\Gamma \smallsetminus C_T$. By Theorem 1.5,

$$\lim_{R\to\infty} \left| \delta e^{-\delta R} \int \chi f \, dm_R - \int \chi f \, \frac{dm_\Gamma}{\|m_\Gamma\|} \right| = 0.$$

So,

$$\begin{aligned}
\limsup_{R\to\infty} &\left| \delta e^{-\delta R} \int f \, dm_R - \int f \, \frac{dm_\Gamma}{\|m_\Gamma\|} \right| \\
&\leq \limsup_{R\to\infty} \left| \int (1-\chi) f \, \frac{dm_\Gamma}{\|m_\Gamma\|} \right| + \left| \delta e^{-\delta R} \int (1-\chi) f \, dm_R \right| \\
&\leq \|f\|_\infty \frac{m_\Gamma(C_T)}{\|m_\Gamma\|} + \|f\|_\infty \, \delta e^{-\delta R} m_R(C_T) \\
&\leq \|f\|_\infty \, \epsilon.
\end{aligned}$$

Since $\epsilon > 0$ was arbitrary,

$$\lim_{R\to\infty} \delta e^{-\delta R} \int f \, dm_R = \int f \, \frac{dm_\Gamma}{\|m_\Gamma\|}. \qquad \square$$

10.2. **Proof of Proposition 10.3.** To prove the proposition we use the shadows studied in our earlier work [8], for which we proved a Shadow Lemma. The proof follows a classical strategy (see [41, Th. 5.2]) but in our generality becomes long and technical. We first explain briefly sketch the proof, omitting certain technical details.

We fix a fundamental domain $K \subset \Lambda(\Gamma) \smallsetminus \{p\}$ for the action of Γ_p. We need to estimate for every $\gamma \in \Gamma_p$ the number of loxodromics $g \in \Gamma$ with axis from K to γK and the time spent by this axis in the horoball about p. First we prove all ψ^t-orbits from K to γK intersect the horoball at p in an interval of length roughly $\|\gamma\|_\sigma$. Then we prove all loxodromics $g \in \Gamma$ of length roughly R with $g^- \in K$ and $g^+ \in \gamma K$ have pairwise disjoint shadows $\mathcal{S}_\epsilon(g)$ that are all contained in the shadow $\mathcal{S}_\epsilon(\gamma)$ of γ. The Shadow Lemma gives an estimate for the measure of $\mathcal{S}_\epsilon(g)$, independent of g, and that of $\mathcal{S}_\epsilon(\gamma)$, and the number of these loxodromics is roughly the ratio of these measures.

Fix a distance d on $\Gamma \sqcup M$ which generates the topology in Proposition 2.6. Then given $\gamma \in \Gamma$ and $\epsilon > 0$, the associated *shadow* is

$$\mathcal{S}_\epsilon(\gamma) := \gamma\left(M - B_\epsilon(\gamma^{-1})\right).$$

We will use the following properties of shadows.

Proposition 10.5 ([8, Prop. 5.1])**.** *For any $\epsilon > 0$, there exist $C_1 = C_1(\epsilon) > 0$ and $\epsilon' = \epsilon'(\epsilon) \in (0, \epsilon)$ such that: if $\alpha, \beta \in \Gamma$, $\|\alpha\|_\sigma \leq \|\beta\|_\sigma$, and $\mathcal{S}_\epsilon(\alpha) \cap \mathcal{S}_\epsilon(\beta) \neq \varnothing$, then*

$$\|\beta\|_\sigma \geq \|\alpha\|_\sigma + \left\|\alpha^{-1}\beta\right\|_\sigma - C_1$$

and

$$\mathcal{S}_\epsilon(\beta) \subset \mathcal{S}_{\epsilon'}(\alpha).$$

In [8], we also established a version of the Shadow Lemma. Let μ be the σ-Patterson–Sullivan measure of dimension δ (which is unique by Theorem 2.10).

Proposition 10.6 ([8, Th. 6.1])**.** *For any $\epsilon > 0$ sufficiently small, there exists $C_2 = C_2(\epsilon) > 1$ such that:*

$$\frac{1}{C_2} e^{-\delta\|\gamma\|_\sigma} \leq \mu\left(\mathcal{S}_\epsilon(\gamma)\right) \leq C_2 e^{-\delta\|\gamma\|_\sigma}$$

for all $\gamma \in \Gamma$.

Fix a compact set $K \subset \Lambda(\Gamma) \smallsetminus \{p\}$ such that $\Gamma_p(K) = \Lambda(\Gamma) \smallsetminus \{p\}$.

Given $\alpha \in \Gamma_p$, let

$$A_\alpha := \{\gamma \in \Gamma_{\mathrm{lox}} : \gamma^- \in K,\ \gamma^+ \in \alpha K\}.$$

As in Section 9.1, given $x, y \in M \smallsetminus \{p\}$ distinct, for any T let $I^T_{x,y} \subset \mathbb{R}$ be the interval satisfying

$$H_{p,T,T} \cap \big((x,y) \times \mathbb{R}\big) = (x,y) \times I^T_{x,y}.$$

The measure $\tilde{m}_R|_{H_{p,T,T}}$ is concentrated on the union of intervals $(\gamma^-, \gamma^+) \times I^T_{\gamma^-,\gamma^+}$ with $\gamma \in \Gamma_{\mathrm{lox}}$ (and $\gamma^\pm \neq p$ since p is a bounded parabolic point), which is contained in the Γ_p-orbit of the union of intervals $(\gamma^-, \gamma^+) \times I^T_{\gamma^-,\gamma^+}$ with $\gamma \in A_\alpha$ for some $\alpha \in \Gamma_p$. This implies

$$m_R\left(\widehat{H}_{p,T,T}\right) \leq \sum_{\alpha \in \Gamma_p} \sum_{\substack{\gamma \in A_\alpha \\ \ell_\sigma(\gamma) \leq R}} \mathrm{Leb}\left(I^T_{\gamma^-,\gamma^+}\right). \tag{12}$$

So we need to control $\mathrm{Leb}\left(I^T_{\gamma^-,\gamma^+}\right)$ and the number of loxodromics in A_α with length at most R.

First we use Lemma 8.7 to fix $T_0 > 0$ such that

$$H_{p,T_0,T_0} \cap \gamma(H_{p,T_0,T_0}) \neq \varnothing \quad \text{if and only if} \quad \gamma \in \Gamma_p, \tag{13}$$

This choice has the following consequence: if γ is loxodromic, then $\gamma(\gamma^-, \gamma^+, t) = (\gamma^-, \gamma^+, t + \ell_\sigma(\gamma))$ for any t and so

$$\|\gamma\|_\sigma \geq \ell_\sigma(\gamma) \geq \mathrm{Leb}\left(I^{T_0}_{\gamma^-,\gamma^+}\right). \tag{14}$$

Using this and Lemma 9.5 we deduce the following estimates.

Lemma 10.7. *There exists $C_3 > 0$ such that: if $\alpha \in \Gamma_p$, $\gamma \in A_\alpha$, $T \geq T_0$ and $I^T_{\gamma^-,\gamma^+} \neq \varnothing$, then*

$$\|\alpha\|_\sigma - 2T - C_3 \leq \mathrm{Leb}\left(I^T_{\gamma^-,\gamma^+}\right) \leq \|\alpha\|_\sigma - 2T + C_3 \tag{15}$$

and

$$2T - C_3 \leq \|\alpha\|_\sigma \leq \|\gamma\|_\sigma + 2T_0 + C_3.$$

Proof. Lemma 9.5 implies Equation (15) for some $C_3 > 0$. Then $2T - C_3 \leq \|\alpha\|_\sigma$ comes from the fact that $\mathrm{Leb}\left(I^T_{\gamma^-,\gamma^+}\right) \geq 0$.

Notice that $I^T_{\gamma^-,\gamma^+} \neq \varnothing$ implies that $I^{T_0}_{\gamma^-,\gamma^+} \neq \varnothing$. By (14) we have

$$\|\gamma\|_\sigma \geq \ell_\sigma(\gamma) \geq \mathrm{Leb}\left(I^{T_0}_{\gamma^-,\gamma^+}\right) \geq \|\alpha\|_\sigma - 2T_0 - C_3. \qquad \square$$

Before we can use the previous lemma to produce estimates for $m_R(\widehat{H}_{p,T,T})$ we need to settle a few technical details.

First fix $\epsilon > 0$ small enough so that $B_{3\epsilon}(p) \cap K = \varnothing$ and Proposition 10.6 (the Shadow Lemma) holds for any $\epsilon' \leq \epsilon$, that is

$$\frac{1}{C_2(\epsilon')} e^{-\delta\|\gamma\|_\sigma} \leq \mu\left(\mathcal{S}_{\epsilon'}(\gamma)\right) \leq C_2(\epsilon') e^{-\delta\|\gamma\|_\sigma}$$

for all $\gamma \in \Gamma$.

Since Γ_p only accumulates on p, one can find $D_0 > 0$ large enough so that for any $\alpha \in \Gamma_p$, if $\|\alpha\|_\sigma \geq D_0$ then

(1) $\alpha K \subset B_\epsilon(p)$, so that $\mathrm{d}(x, y) \geq 2\epsilon$ for all $x \in K$ and $y \in \alpha K$, and
(2) $\alpha^{-1} \in B_\epsilon(p)$, so that $K \subset M - B_\epsilon(\alpha^{-1})$, i.e. αK is contained in the shadow $\mathcal{S}_\epsilon(\alpha) = \alpha(M - B_\epsilon(\alpha^{-1}))$.

Now let $T_1 := \max(T_0, D_0 + C_3)$. Then by Lemma 10.7: if $\alpha \in \Gamma_p$, $\gamma \in A_\alpha$, and $I^{T_1}_{\gamma^-,\gamma^+} \neq \varnothing$, then

$$\|\alpha\|_\sigma \geq 2T_1 - C_3 \geq D_0.$$

Finally, by Proposition 2.9(4) there exists $C_4 > 0$ such that: if $\gamma \in \Gamma_{\mathrm{lox}}$ and $\mathrm{d}(\gamma^-, \gamma^+) \geq \epsilon$, then

$$\ell_\sigma(\gamma) \geq \|\gamma\|_\sigma - C_4.$$

Then combining Lemma 10.7 with Equation (12), we obtain:

$$m_R\left(\widehat{H}_{p,T,T}\right) \leq \sum_{\substack{\alpha \in \Gamma_p \\ \|\alpha\|_\sigma \geq 2T - C_3}} \sum_{\substack{\gamma \in A_\alpha \\ \|\alpha\|_\sigma - 2T_0 - C_3 \leq \|\gamma\|_\sigma \leq R + C_4}} \left(\|\alpha\|_\sigma + C_3\right) \tag{16}$$

when $T \geq T_1$.

Let

$$A^0_\alpha := \{\gamma \in A_\alpha : \|\alpha\|_\sigma - 2T_0 - C_3 \leq \|\gamma\|_\sigma \leq \|\alpha\|_\sigma\}$$

and for $n \geq 1$ let

$$A^n_\alpha := \{\gamma \in A_\alpha : \|\alpha\|_\sigma + (n-1) \leq \|\gamma\|_\sigma \leq \|\alpha\|_\sigma + n\}.$$

Then by Equation (16),

$$m_R\left(\widehat{H}_{p,T,T}\right) \leq \sum_{\substack{\alpha \in \Gamma_p \\ \|\alpha\|_\sigma \geq 2T - C_3}} \left(\|\alpha\|_\sigma + C_3\right) \sum_{n=0}^{R + C_4 + 1 - \|\alpha\|_\sigma} \# A^n_\alpha \tag{17}$$

when $T \geq T_1$.

We will estimate the size of each A_α^n by studying shadows.

Lemma 10.8. *Given $\epsilon > 0$, there exists $D_1 > 0$ such that: if $\gamma \in \Gamma_{\rm lox}$, $\mathrm{d}(\gamma^-, \gamma^+) \geq 2\epsilon$ and $\|\gamma\|_\sigma \geq D_1$, then $\gamma^+ \in \mathcal{S}_\epsilon(\gamma)$.*

Proof. Suppose not. Then for every $n \geq 1$ there exists $\gamma_n \in \Gamma_{\rm lox}$ where $\mathrm{d}(\gamma_n^-, \gamma_n^+) \geq 2\epsilon$, $\|\gamma_n\|_\sigma \geq n$ and $\gamma_n^+ \notin \mathcal{S}_\epsilon(\gamma_n)$. Passing to a subsequence we can suppose that $\gamma_n \to a \in M$ and $\gamma_n^{-1} \to b \in M$. Arguing as in the proof of Proposition 2.9(4) we see that $(\gamma_n^-, \gamma_n^+) \to (b, a)$. Hence $\mathrm{d}(b, a) \geq 2\epsilon$. So for n large we have $\mathrm{d}(\gamma_n^+, \gamma_n^{-1}) > \epsilon$, thus

$$\gamma_n^+ = \gamma_n \gamma_n^+ \in \gamma_n \left(M - B_\epsilon(\gamma_n^{-1})\right) = \mathcal{S}_\epsilon(\gamma_n).$$

So we have a contradiction. □

For $r \geq 0$, let

$$\tau(r) := \#\{\gamma \in \Gamma : \|\gamma\|_\sigma \leq r\}.$$

Lemma 10.9. *If $x \in M$ and $R, r > 0$, then*

$$\#\{\gamma \in \Gamma : R - r \leq \|\gamma\|_\sigma \leq R \text{ and } x \in \mathcal{S}_\epsilon(\gamma)\} \leq \tau(r + C_1)$$

where C_1 is the constant from Proposition 10.5.

Proof. Fix an element $\gamma_{\max} \in A := \{\gamma \in \Gamma : R - r \leq \|\gamma\|_\sigma \leq R \text{ and } x \in \mathcal{S}_\epsilon(\gamma)\}$ with maximal magnitude $\|\gamma_{\max}\|_\sigma$. Then if $\gamma \in A$, Proposition 10.5 implies that

$$\|\gamma\|_\sigma + \left\|\gamma^{-1}\gamma_{\max}\right\|_\sigma - C_1 \leq \|\gamma_{\max}\|_\sigma,$$

which implies that

$$\left\|\gamma^{-1}\gamma_{\max}\right\|_\sigma \leq r + C_1.$$

So $\#A \leq \tau(r + C_1)$. □

Lemma 10.10. *There exists $C_5 > 0$ such that: if $\alpha \in \Gamma_p$ and $\|\alpha\|_p \geq D_1 + 2T_1 + C_3$, then*

$$\#A_\alpha^n \leq C_5 e^{\delta n}.$$

Proof. Suppose $\gamma \in A_\alpha^n$. By property (2) (from the choice of D_0) and Lemma 10.8,

$$\gamma^+ \in \mathcal{S}_\epsilon(\alpha) \cap \mathcal{S}_\epsilon(\gamma).$$

If $n = 0$, then Proposition 10.5 implies that

$$\|\gamma\|_\sigma + \left\|\gamma^{-1}\alpha\right\|_\sigma - C_1 \leq \|\alpha\|_\sigma,$$

which implies that

$$\left\|\gamma^{-1}\alpha\right\|_\sigma \leq C_1 + 2T_1 + C_3.$$

Hence

$$\#A_\alpha^0 \leq \tau(C_1 + 2T_1 + C_3).$$

If $n \geq 1$, then Proposition 10.5 implies that there exists $\epsilon' \in (0, \epsilon)$ (which only depends on ϵ) such that

$$\bigcup_{\gamma \in A_\alpha^n} \mathcal{S}_\epsilon(\gamma) \subset \mathcal{S}_{\epsilon'}(\alpha).$$

Then, using Proposition 10.6 (the Shadow Lemma) twice, and Lemma 10.9,

$$\begin{aligned}\# A_\alpha^n &\leq \sum_{\gamma \in A_\alpha^n} C_2(\epsilon) e^{\delta \|\gamma\|_\sigma} \mu(\mathcal{S}_\epsilon(\gamma)) \\ &\leq C_2(\epsilon)\tau(1+C_1) e^{\delta\|\alpha\|_\sigma} e^{\delta n} \mu\left(\bigcup_{\gamma \in A_\alpha^n} \mathcal{S}_\epsilon(\gamma)\right) \\ &\leq C_2(\epsilon) C_2(\epsilon')\tau(1+C_1) e^{\delta n}.\end{aligned}$$

So $C_5 := \max\{\tau(C_1 + 2T_1 + C_3), C_2(\epsilon)C_2(\epsilon')\tau(1+C_1)\}$ suffices. □

Combining Equation (17) with the previous lemma gives a constant $C_6 > 0$ such that

$$m_R\left(\widehat{H}_{p,T,T}\right) \leq C_6 e^{\delta R} \sum_{\substack{\alpha \in \Gamma_p \\ \|\alpha\|_\sigma \geq 2T - C_3}} \|\alpha\|_\sigma \, e^{-\delta\|\alpha\|_\sigma}$$

for all $T \geq T_1 + C_3 + \frac{D_1}{2}$.

10.3. **Proof of Corollary 10.2.** In our proof of our counting corollary, it will be convenient to use a slightly modified equidistribution statement which concerns a measure m_R' which is closely related to m_R.

For every $R > 0$, let m_R' be the sum of normalized Lebesgue measures on closed orbits of length at most R in U_Γ (counted with multiplicity), more precisely m_R' is the quotient measure associated to the measure

$$\tilde{m}_R' = \sum_{\substack{\gamma \in \Gamma_{\rm lox} \\ \ell_\sigma(\gamma) \leq R}} \frac{1}{\ell_\sigma(\gamma)} \mathcal{D}_{\gamma^-} \otimes \mathcal{D}_{\gamma^+} \otimes dt$$

on $\tilde{U}_\Gamma$. Notice that if $\gamma \in \Gamma$ is loxodromic and c_γ is the image of $(\gamma^-, \gamma^+) \times \mathbb{R}$ in U_Γ, then

$$m_R'(c_\gamma) = \#\left\{\alpha \in \Gamma_{\rm lox} : \ \alpha^\pm = \gamma^\pm, \text{ and } \ell_\sigma(\alpha) \leq R\right\}.$$

This follows from Equation (2).

One expects that m_R is close to $R m_R'$ since we expect most geodesics of length at most R have length close to R. The next result makes this intuition precise.

Proposition 10.11. *If $f \colon \Gamma \backslash (M^{(2)} \times \mathbb{R}) \to \mathbb{R}$ is a bounded continuous function, then*

$$\lim_{R \to \infty} \delta R e^{-\delta R} \int f \, dm_R' = \int f \frac{dm_\Gamma}{\|m_\Gamma\|}.$$

Proof. For any real number $R > 0$, let $\mathcal{G}_R = \{[\gamma] \in [\Gamma_{\rm lox}]^w : \ell_\sigma(\gamma) \leq R\}$ and for $c \in [\Gamma_{\rm lox}]^w$ let Leb_c be the quotient measure associated to $\sum_{\gamma \in c} \mathcal{D}_{\gamma^-} \otimes \mathcal{D}_{\gamma^+} \otimes dt$. Note that

$$R\, m_R' - m_R = \sum_{c \in \mathcal{G}_R} \left(\frac{R}{\ell_\sigma(c)} - 1\right) \mathrm{Leb}_c \geq 0.$$

In particular,

$$\liminf_{R \to \infty} \delta R e^{-\delta R} \int f \, dm_R' \geq \int f \frac{m_\Gamma}{\|m_\Gamma\|}.$$

Let

$$s := \mathrm{sys}(\Gamma, \sigma) = \min\{\ell_\sigma(\gamma) : \gamma \in \Gamma_{\rm lox}\}$$

be the systole of (Γ, σ), which is positive by Corollary 8.4. For any $r > 0$ and $R \geq 2e^r$, we have

$$\begin{aligned} e^{-\delta R} \int f \, dm_R &\geq e^{-\delta R} \sum_{\gamma \in \mathcal{G}_R - \mathcal{G}_{e^{-r}R}} \frac{e^{-r}R}{\ell_\sigma(\gamma)} \int f \, d\mathrm{Leb}_\gamma \\ &= e^{-\delta R} e^{-r} \int fR \, dm'_R - e^{-r} R e^{-\delta R} \sum_{\gamma \in \mathcal{G}_{e^{-r}R}} \frac{1}{\ell_\sigma(\gamma)} \int f \, d\mathrm{Leb}_\gamma \\ &\geq e^{-\delta R} e^{-r} \int fR \, dm'_R - e^{-r} R e^{-\delta R} \sum_{\gamma \in \mathcal{G}_{e^{-r}R}} \frac{1}{s} \int f \, d\mathrm{Leb}_\gamma \\ &= e^{-\delta R} e^{-r} \int fR \, dm'_R - e^{-r-\delta R} \frac{R}{s} \int f \, dm_{e^{-r}R}. \end{aligned}$$

So

$$e^{-\delta R} \int fR \, dm'_R \leq e^r e^{-\delta R} \int f \, dm_R + e^{-\delta R} \frac{R}{s} \int f \, dm_{e^{-r}R}.$$

Theorem 10.1 implies that $e^{-\delta e^{-r}R} \int f \, dm_{e^{-r}R}$ remains bounded as $R \to \infty$, so $e^{-\delta R} \frac{R}{s} \int f \, dm_{e^{-r}R} \to 0$ as $R \to \infty$. Hence,

$$\limsup_{R \to \infty} \delta R e^{-\delta R} \int f \, dm'_R \leq e^r \int f \frac{dm_\Gamma}{\|m_\Gamma\|}$$

for any $r > 0$. We may then conclude by taking $r \to 0$. □

By integrating the bounded constant 1 function against both sides of the equation in Proposition 10.11, we obtain the desired counting statement (Corollary 10.2). □

11. Transverse and relatively Anosov subgroups of semisimple Lie groups

In this section we will apply our results to the particular case of relatively Anosov subgroups of semisimple Lie groups. To state these results in full generality requires a number of definitions, for a more detailed discussion using the same notation see [13].

Let G be a connected semisimple Lie group without compact factors and with finite center. Fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ of the Lie algebra of G, a Cartan subspace $\mathfrak{a} \subset \mathfrak{p}$ and a Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$. Let $\mathsf{K} \subset \mathsf{G}$ denote the maximal compact subgroup with Lie algebra $\mathfrak{k}$. Then let $\kappa \colon \mathsf{G} \to \mathfrak{a}^+$ denote the *Cartan projection*, that is $\kappa(g) \in \mathfrak{a}^+$ is the unique element where

$$g = k_1 e^{\kappa(g)} k_2$$

for some $k_1, k_2 \in \mathsf{K}$. Let $\iota \colon \mathfrak{a} \to \mathfrak{a}$ be the opposite involution, which has the defining property that

$$\iota(\kappa(g)) = \kappa(g^{-1}). \tag{18}$$

for all $g \in \mathsf{G}$.

Let $\Delta \subset \mathfrak{a}^*$ denote the system of simple restricted roots corresponding to the choice of $\mathfrak{a}^+$. Given a subset $\theta \subset \Delta$, we let $\mathsf{P}_\theta \subset \mathsf{G}$ denote the associated parabolic subgroup and let $\mathcal{F}_\theta := \mathsf{G}/\mathsf{P}_\theta$ denote the associated flag manifold.

A discrete subgroup $\Gamma \subset \mathsf{G}$ is P_θ-*divergent* if whenever $\{\gamma_n\}$ is a sequence of distinct elements of Γ and $\alpha \in \theta$, then

$$\lim_{n\to\infty} \alpha(\kappa(\gamma_n)) = +\infty.$$

Next we describe the limit set of a P_θ-divergent group. For every $g \in \mathsf{G}$, fix a Cartan decomposition

$$g = m_g e^{\kappa(g)} \ell_g$$

Then following the notation in [23], define a map

$$U_\theta \colon \mathsf{G} \to \mathcal{F}_\theta$$

by letting $U_\theta(g) := m_g \mathsf{P}_\theta$. One can show that if $\alpha(\kappa(g)) > 0$ for all $\alpha \in \theta$, then $U_\theta(g)$ is independent of the choice of Cartan decomposition, see [25, Chap. IX, Th. 1.1]. In particular, if Γ is P_θ-divergent, then $U_\theta(\gamma)$ is well-defined for all but finitely many $\gamma \in \Gamma$. The limit set of a P_θ-divergent group $\Gamma \subset \mathsf{G}$ is given by

$$\Lambda_\theta(\Gamma) := \{F \in \mathcal{F}_\theta : \exists \{\gamma_n\} \subset \Gamma \text{ distinct such that } U_\theta(\gamma_n) \to F\}.$$

One motivation for this definition comes from the dynamical behavior described in Proposition 11.4 below.

For the rest of the section, we assume that θ is symmetric (i.e. $\iota^*(\theta) = \theta$ where $\iota \colon \mathfrak{a} \to \mathfrak{a}$ is the involution associated to $\mathfrak{a}^+$). In this case there is a natural notion of transversality for pairs in $\mathcal{F}_\theta$ and a P_θ-divergent subgroup Γ is called P_θ-*transverse* if any two flags in $\Lambda_\theta(\Gamma)$ are transverse. We say Γ is *non-elementary* if $\#\Lambda_\theta(\Gamma) \geq 3$. Every non-elementary P_θ-transverse group acts on its limit set as a convergence group action, see [28, Section 15] or Observation 11.5 below. For a more in-depth discussion of transverse groups and their dynamical properties see [13, 30, 31].

Associated to a subset $\theta \subset \Delta$ is a natural subspace of $\mathfrak{a}$ defined by

$$\mathfrak{a}_\theta := \{a \in \mathfrak{a} : \beta(a) = 0 \text{ for all } \beta \in \Delta - \theta\}.$$

One can show that $\mathfrak{a}_\theta^*$ is generated by $\{\omega_\alpha|_{\mathfrak{a}_\theta} : \alpha \in \theta\}$ where ω_α is the fundamental weight associated to α. Hence we can identify $\mathfrak{a}_\theta^*$ as a subspace of $\mathfrak{a}^*$. Then given $\phi \in \mathfrak{a}_\theta^*$ and a P_θ-divergent subgroup Γ, we define a Poincaré series

$$Q_\Gamma^\phi(s) = \sum_{\gamma\in\Gamma} e^{-s\phi(\kappa(\gamma))},$$

which has a critical exponent $\delta^\phi(\Gamma) := \inf\{s > 0 : Q_\Gamma^\phi(s) < +\infty\} \in [0, \infty]$. Moreover, there exists a well-defined θ-Cartan projection

$$\kappa_\theta : \mathsf{G} \to \mathfrak{a}_\theta$$

so that $\kappa_\theta = p_\theta \circ \kappa$ where $p_\theta : \mathfrak{a} \to \mathfrak{a}_\theta$ is the unique projection map so that $\omega_\alpha|_{\mathfrak{a}_\theta} = \omega_\alpha \circ p_\theta$ for all $\alpha \in \theta$. Hence

$$\phi \circ \kappa = \phi \circ \kappa_\theta$$

for all $\phi \in \mathfrak{a}_\theta$.

The action of G on $\mathcal{F}_\theta$ preserves a smooth vector-valued cocycle

$$B_\theta \colon \mathsf{G} \times \mathcal{F}_\theta \to \mathfrak{a}_\theta,$$

so that $B_\theta(g, U_\theta(g)) = \kappa_\theta(g)$ for all g, see Quint [39] or Benoist–Quint [5, §6.7.5]. Let $\mathcal{F}_\theta^{(2)} \subset \mathcal{F}_\theta \times \mathcal{F}_\theta$ denote the subset of transverse flags. There is also a smooth map $G_\theta \colon \mathcal{F}_\theta^{(2)} \to \mathfrak{a}_\theta$ which satisfies

$$G_\theta(gF, gF') - G_\theta(F, F') = \iota \circ B_\theta(g, F) + B_\theta(g, F'), \tag{19}$$

for any $g \in G$, see [43, Lem. 4.12] or [31, pg. 11].

We will verify the following.

Proposition 11.1. *Suppose Γ is non-elementary P_θ-transverse, $\phi \in \mathfrak{a}_\theta^*$ and $\delta^\phi(\Gamma) < +\infty$. Define cocycles $\sigma_\phi, \bar{\sigma}_\phi \colon \Gamma \times \Lambda_\theta(\Gamma) \to \mathbb{R}$ by*

$$\sigma_\phi(\gamma, F) = \phi(B_\theta(\gamma, F)) \quad \text{and} \quad \bar{\sigma}_\phi(\gamma, F) = \iota^*(\phi)(B_\theta(\gamma, F)).$$

Then $(\sigma_\phi, \bar{\sigma}_\phi, \phi \circ G)$ is a continuous GPS system for the action of Γ on $\Lambda_\theta(\Gamma)$. Moreover:

(1) If $\lambda \colon \mathsf{G} \to \mathfrak{a}^+$ is the Jordan projection, then

$$\phi(\lambda(\gamma)) = \ell_{\sigma_\phi}(\gamma)$$

for all $\gamma \in \Gamma$.

(2) There exists $C > 0$ such that

$$\left| \phi(\kappa(\gamma)) - \|\gamma\|_{\sigma_\phi} \right| < C$$

for all $\gamma \in \Gamma$. In particular, $\delta^\phi(\Gamma) = \delta_{\sigma_\phi}(\Gamma)$.

We will also show that the length spectrum is always non-arithmetic.

Proposition 11.2. *If Γ is non-elementary P_θ-transverse, $\phi \in \mathfrak{a}_\theta^*$ and $\delta^\phi(\Gamma) < +\infty$, then*

$$\{\phi(\lambda(\gamma)) + \iota^*(\phi)(\lambda(\gamma)) : \gamma \in \Gamma\}$$

generates a dense subgroup of $\mathbb{R}$.

Next we define relatively Anosov subgroups. There are several equivalent definitions, see the discussion in [51, Section 4], and the one we use comes from [27].

If $\Gamma \subset \mathsf{G}$ is relatively hyperbolic (as an abstract group) with respect to a finite collection $\mathcal{P}$ of finitely generated subgroups of Γ, then Γ is P_θ*-relatively Anosov* if it is P_θ-transverse and there exists a Γ-equivariant homeomorphism from the Bowditch boundary $\partial(\Gamma, P)$ to the limit set $\Lambda_\theta(\Gamma)$.

Corollary 11.3. *Suppose $\Gamma \subset \mathsf{G}$ is P_θ-relatively Anosov with respect to $\mathcal{P}$. If $\phi \in \mathfrak{a}_\theta^*$ and $\delta := \delta^\phi(\Gamma) < +\infty$, then*

$$\#\{[\gamma]^w \in [\Gamma_{\mathrm{lox}}]^w : 0 < \phi(\lambda(\gamma)) \leq R\} \sim \frac{e^{\delta R}}{\delta R}.$$

Proof. In [14, Cor. 7.2] it was shown that $\delta^\phi(H) < \delta^\phi(\Gamma)$ for any maximal parabolic subgroup of Γ. Combining this with Proposition 11.2 and Proposition 11.1 shows that the hypothesis of Theorem 1.1 is satisfied. $\square$

11.1. **Proof of Proposition 11.1.** We first explain why Γ acts as a convergence group on $\Lambda_\theta(\Gamma)$. This observation appears in [28, Section 15], but since the proof is short and [28] use different terminology we include it here.

For $F \in \mathcal{F}_\theta$, let

$$\mathcal{Z}_F := \{F' \in \mathcal{F}_\theta : F \text{ is not transverse to } F'\}.$$

We use the following description of the action of G on $\mathcal{F}_\theta$ (see for instance [13, Prop. 2.6]).

Proposition 11.4. *Suppose $\theta \subset \Delta$ is symmetric, $F^\pm \in \mathcal{F}_\theta$ and $\{g_n\}$ is a sequence in G. The following are equivalent:*

1. *$U_\theta(g_n) \to F^+$, $U_\theta(g_n^{-1}) \to F^-$ and $\lim_{n\to\infty} \alpha(\kappa(g_n)) = \infty$ for every $\alpha \in \theta$,*
2. *$g_n(F) \to F^+$ for all $F \in \mathcal{F}_\theta \smallsetminus \mathcal{Z}_{F^-}$, and this convergence is uniform on compact subsets of $\mathcal{F}_\theta \smallsetminus \mathcal{Z}_{F^-}$.*
3. *$g_n^{-1}(F) \to F^-$ for all $F \in \mathcal{F}_\theta \smallsetminus \mathcal{Z}_{F^+}$, and this convergence is uniform on compact subsets of $\mathcal{F}_\theta \smallsetminus \mathcal{Z}_{F^+}$.*

Proposition 11.4 immediately implies that a transverse group is a convergence group.

Observation 11.5. (see also [28, Section 15]) If $\Gamma \subset \mathsf{G}$ is P_θ-transverse, then Γ acts on $\Lambda_\theta(\Gamma)$ as a convergence group.

Proof. Suppose $\{\gamma_n\} \subset \Gamma$ is a sequence of distinct elements. Since Γ is P_θ-divergent, $\lim_{n\to\infty} \alpha(\kappa(g_n)) = \infty$ for every $\alpha \in \theta$. Since $\mathcal{F}_\theta$ is compact, we can pass to a subsequence so that $U_\theta(g_n) \to F^+$ and $U_\theta(g_n^{-1}) \to F^-$. Then Proposition 11.4 implies that $g_n(F) \to F^+$ for all $F \in \mathcal{F}_\theta \smallsetminus \mathcal{Z}_{F^-}$. Since $\Lambda_\theta(\Gamma)$ is transverse, this implies that $g_n(F) \to F^+$ for all $F \in \Lambda_\theta(\Gamma) \smallsetminus \{F^-\}$. $\square$

We also use the following estimate from [39]. Let $\|\cdot\|$ denote some fixed norm on $\mathfrak{a}$.

Lemma 11.6 (Quint [39, Lem. 6.5])**.** *For any $\epsilon > 0$ and distance $\mathrm{d}_{\mathcal{F}_\theta}$ on $\mathcal{F}_\theta$ induced by a Riemannian metric there exists $C = C(\epsilon, \mathrm{d}_{\mathcal{F}_\theta}) > 0$ such that: if $g \in \mathsf{G}$, $F \in \mathcal{F}_\theta$ and $\mathrm{d}_{\mathcal{F}_\theta}\left(F, \mathcal{Z}_{U_\theta(g^{-1})}\right) > \epsilon$, then*

$$\|B_\theta(g, F) - \kappa_\theta(g)\| < C.$$

We now start the proof of Proposition 11.1. So fix a non-elementary P_θ-transverse subgroup Γ and $\phi \in \mathfrak{a}_\theta^*$ with $\delta^\phi(\Gamma) < +\infty$. Define cocycles $\sigma_\phi, \bar\sigma_\phi \colon \Gamma \times \Lambda_\theta(\Gamma) \to \mathbb{R}$ by

$$\sigma_\phi(\gamma, F) = \phi(B_\theta(\gamma, F)) \quad \text{and} \quad \bar\sigma_\phi(\gamma, F) = \iota^*(\phi)(B_\theta(\gamma, F)).$$

Lemma 11.7. *There exists $C_1 > 0$ such that:*

$$\|\gamma\|_{\sigma_\phi} \geq \phi(\kappa_\theta(\gamma)) - C_1$$

for all $\gamma \in \Gamma$. In particular, σ_ϕ is a proper cocycle.

Proof. Suppose not. Then for every $n \geq 1$ there exist $\gamma_n \in \Gamma$ such that

$$\|\gamma_n\|_{\sigma_\phi} \leq \phi(\kappa_\theta(\gamma_n)) - n.$$

Passing to a subsequence we can suppose that $U_\theta(\gamma_n^{-1}) \to F^- \in \Lambda_\theta(\Gamma)$. Since Γ is non-elementary, there exists $F \in \Lambda_\theta(\Gamma) \smallsetminus \{F^-\}$. Then by Lemma 11.6 there exists $C > 0$ such that

$$\|B_\theta(\gamma_n, F) - \kappa_\theta(\gamma_n)\| < C$$

for all sufficiently large n. Then

$$\|\gamma_n\|_{\sigma_\phi} \geq \sigma_\phi(\gamma_n, F) = \phi(B_\theta(\gamma_n, F)) \geq \phi(\kappa_\theta(\gamma_n)) - C\,\|\phi\|$$

for all n. So we have a contradiction. Thus the first assertion is true. For the second, notice that since $\delta^\phi(\Gamma) < +\infty$, we have

$$\lim_{n\to\infty} \phi(\kappa_\theta(\gamma_n)) = +\infty$$

for any sequence $\{\gamma_n\} \subset \Gamma$ of distinct elements. Hence the same is true $\|\cdot\|_{\sigma_\phi}$ and so σ_ϕ is a proper cocycle. □

Equation (18) implies that $\delta^{\bar{\phi}}(\Gamma) = \delta^\phi(\Gamma) < +\infty$, so the proof of Lemma 11.7 can be used to show that $\bar{\sigma}_\phi$ is also a proper cocycle. Thus, by Equation (19), the triple $(\sigma_\phi, \bar{\sigma}_\phi, \phi \circ G_\theta)$ is a GPS-system.

Item (1) in the "moreover" part of Proposition 11.1 follows immediately from the fact that if γ is loxodromic, then $\phi(\lambda(\gamma)) = B_\theta(\gamma, \gamma^+)$ (see the discussion at the start of Section 9.2 in [5]). Item (2) follows from Lemma 11.7 and the next result.

Lemma 11.8. *There exists $C_2 > 0$ such that:*

$$\|\gamma\|_{\sigma_\phi} \leq \phi(\kappa_\theta(\gamma)) + C_2$$

for all $\gamma \in \Gamma$.

Remark 11.9. In the case when ϕ has positive coefficients relative to the basis $\{\omega_\alpha\}_{\alpha\in\theta}$ of $\mathfrak{a}_\theta^*$ one can use [5, Lem. 6.33] to show that $\|\gamma\|_{\sigma_\phi} \leq \phi(\kappa_\theta(\gamma))$.

Proof. Suppose not. Then for every $n \geq 1$ there exist $\gamma_n \in \Gamma$ such that

$$\|\gamma_n\|_{\sigma_\phi} \geq \phi(\kappa_\theta(\gamma_n)) + n.$$

Passing to a subsequence we can suppose that $U_\theta(\gamma_n^{-1}) \to F^- \in \Lambda_\theta(\Gamma)$. Then by Proposition 11.4, we have also $\gamma_n^{-1} \to F^-$ in the topology on $\Gamma \sqcup \Lambda_\theta(\Gamma)$ defined in Proposition 2.6.

Since Γ is non-elementary, there exists $F \in \Lambda_\theta(\Gamma) \smallsetminus \{F^-\}$. Since σ_ϕ is expanding (see Proposition 2.11) there exists $C > 0$ such that

$$\sigma_\phi(\gamma_n, F) \geq \|\gamma_n\|_\sigma - C$$

for al sufficiently large n. Also, by applying Lemma 11.6, and possibly increasing $C > 0$, we can assume that

$$\|B_\theta(\gamma_n, F) - \kappa_\theta(\gamma_n)\| < C$$

for all n. Then

$$\|\gamma_n\|_{\sigma_\phi} \leq \sigma_\phi(\gamma_n, F) + C = \phi(B_\theta(\gamma_n, F)) + C \leq \phi(\kappa_\theta(\gamma_n)) + C\,\|\phi\| + C$$

for all n. So we have a contradiction. □

11.2. **Proof of Proposition 11.2.** We will deduce the proposition from the following general result.

Proposition 11.10. *Suppose $\Gamma \subset \mathsf{G}$ is a subgroup which is not virtually solvable. If $\phi \in \mathfrak{a}^*$ is positive on the cone*

$$\overline{\bigcup_{\gamma\in\Gamma} \mathbb{R}_{>0} \cdot \lambda(\gamma)} \smallsetminus \{0\},$$

then the subgroup of $\mathbb{R}$ generated by $\ell_\phi(\Gamma)$ is dense in $\mathbb{R}$.

Remark 11.11. Previously, Benoist [3] proved that if Γ is a Zariski-dense *semigroup* of G, then the additive subgroup of $\mathfrak{a}$ generated by $\lambda(\Gamma)$ is dense. In particular, the subgroup of $\mathbb{R}$ generated by $\ell_\phi(\Gamma) = (\phi \circ \lambda)(\Gamma)$ is dense in $\mathbb{R}$. As we will explain in Example 11.12 below, Proposition 11.10 is not true when Γ is only assumed to be a semigroup.

Proof. By replacing G with $\mathrm{Ad}(\mathsf{G})$ we can assume that G is an algebraic subgroup of $\mathsf{SL}(d,\mathbb{R})$ and thus speak of Zariski closures of subgroups of G.

Let $\rho\colon \Gamma \hookrightarrow \mathsf{G}$ be the inclusion representation. Let $\mathsf{G}' \subset \mathsf{G}$ be the Zariski closure of $\rho(\Gamma)$, and $\mathsf{G}' = \mathsf{H} \ltimes \mathrm{R}_u(\mathsf{G}')$ be a Levi decomposition of G', where $\mathrm{R}_u(\mathsf{G}')$ is the unipotent radical and H is a Levi factor, which is a reductive group.

Following [23, §2.5.4], let ρ^{ss} be the semisimplification of ρ, obtained by composing ρ with the projection on the Levi factor H, so that the Zariski closure of $\rho^{ss}(\Gamma)$ is exactly H. Moreover, ρ^{ss} has the same length function $\ell_\phi \circ \rho^{ss} = \ell_\phi \circ \rho$ as ρ by [23, Lem. 2.40].

The Lie algebra $\mathfrak{h} \subset \mathfrak{g}$ of $\mathsf{H} \subset \mathsf{G}$ splits as $\mathfrak{h} = \mathfrak{h}' + \mathfrak{z}$ where $\mathfrak{h}'$ is semisimple and $\mathfrak{z}$ is the center of $\mathfrak{h}$. Note that the semisimple part $\mathfrak{h}'$ cannot be trivial, otherwise the Lie algebra of G' would be solvable, so the identity component of G' would be solvable, and hence the intersection of this component with $\Gamma = \rho(\Gamma)$ would be a solvable finite-index subgroup of Γ, which contradicts the assumption that Γ is not virtually solvable.

Let $\mathfrak{a}'_{\mathsf{H}}$ be a Cartan subspace of $\mathfrak{h}'$. Then $\mathfrak{a}_{\mathsf{H}} := \mathfrak{a}'_{\mathsf{H}} + \mathfrak{z}$ is a Cartan subspace of $\mathfrak{h}$. Up to conjugating by an element of G, we may assume that $\mathfrak{a}_{\mathsf{H}} \subset \mathfrak{a}$.

Let $\lambda_{\mathsf{H}}\colon \mathsf{H} \to \mathfrak{a}'_{\mathsf{H}} + \mathfrak{z}$ denote the Jordan projection of H. Despite the inclusion $\mathfrak{a}_{\mathsf{H}} \subset \mathfrak{a}$, we note that λ_{H} may not equal $\lambda|_{\mathsf{H}}$. However, we have the following: If $h \in \mathsf{H}$, then

$$\lambda(h) = \lambda\left(e^{\lambda_{\mathsf{H}}(h)}\right). \tag{20}$$

To see this, let $h = h_e h_u h_{ss}$ denote the Jordan decomposition of h. Then h_{ss} is conjugate to $e^{\lambda_{\mathsf{H}}(h)}$ and so

$$\lambda(h) = \lambda(h_{ss}) = \lambda\left(e^{\lambda_{\mathsf{H}}(h)}\right).$$

Define $f\colon \mathfrak{a}_{\mathsf{H}} \to \mathbb{R}$ by

$$f(X) = \ell_\phi(e^X) = \phi\left(\lambda\left(e^X\right)\right).$$

Then f is piecewise linear, more precisely $\mathfrak{a}_{\mathsf{H}} = \bigcup_j W_j$ is a finite union of closed convex cones each with non-empty interior where $f|_{W_j}$ is linear (each W_j has the form $\mathfrak{a}_{\mathsf{H}} \cap w_j \mathfrak{a}^+$ where w_j is the Weyl group of $\mathfrak{a}$).

By [5, Prop. 9.7], the intersection of

$$\mathcal{C} := \overline{\bigcup_{\gamma \in \rho^{ss}(\Gamma)} \mathbb{R}_{>0} \cdot \lambda_{\mathsf{H}}(\gamma)}$$

and $\mathfrak{a}'_{\mathsf{H}}$ contains some non-zero X_0 (there is a typo in the reference, the intersection part of the result does not hold for semigroups, but it does hold for groups). Since ϕ is positive on $\overline{\bigcup_{\gamma\in\Gamma} \mathbb{R}_{>0} \cdot \lambda(\gamma)} \smallsetminus \{0\}$, Equation (20) implies that $f(X_0) > 0$.

Fix W_j such that $\mathcal{C} \cap W_j$ has non-empty interior in $\mathcal{C}$ and $X_0 \in W_j$. Then by [2, §5.1] there exists a Zariski-dense semigroup $S \subset \rho^{ss}(\Gamma)$ where

$$\mathcal{C}_S := \overline{\bigcup_{\gamma \in S} \mathbb{R}_{>0} \cdot \lambda_{\mathsf{H}}(\gamma)} \subset \mathcal{C} \cap W_j.$$

Let $f' \colon \mathfrak{a}_{\mathsf{H}} \to \mathbb{R}$ be the linear map with $f'|_{W_j} = f|_{W_j}$. By [5, Prop. 9.8] the closure of the additive group generated by $\lambda_{\mathsf{H}}(S)$ in $\mathfrak{a}_{\mathsf{H}}$ contains $\mathfrak{a}'_{\mathsf{H}}$. Thus the closure of the additive subgroup of $\mathbb{R}$ generated by

$$\ell_\phi(S) = f \circ \lambda_{\mathsf{H}}(S) = f' \circ \lambda_{\mathsf{H}}(S)$$

is the image under f' (which is linear) of the closure of the additive group generated by $\lambda_{\mathsf{H}}(S)$, which contains $\mathfrak{a}'_{\mathsf{H}}$ and hence the line spanned by X_0. Therefore the closure of the additive subgroup of $\mathbb{R}$ generated by $\ell_\phi(\Gamma)$ contains $f'(\mathbb{R} X_0) = \mathbb{R} f'(X_0) = \mathbb{R}$ as $f'(X_0) = f(X_0) > 0$, which is what we wanted to prove. □

It seems unlikely that in the context of transverse groups, finite critical exponent implies the positivity on the limit cone hypothesis needed to use Proposition 11.10. However, this implication is known to be true for Anosov (or more generally relatively Anosov) groups, and such groups can always be found as subgroups of transverse groups.

Proof of Proposition 11.2. Suppose Γ is a P_θ-transverse group, $\phi \in \mathfrak{a}_\theta^*$ and $\delta^\phi(\Gamma) < +\infty$.

Let $\psi := \phi + \iota^*(\phi)$. Notice that

$$Q_\Gamma^\psi(s) = \sum_{\gamma \in \Gamma} e^{-s(\phi+\iota^*(\phi))(\kappa(\gamma))} \leq \sqrt{Q_\Gamma^\phi(2s) \cdot Q_\Gamma^{\iota^*(\phi)}(2s)}$$

by Hölder's inequality. Further, Equation (18) implies that $\delta^{\iota^*(\phi)}(\Gamma) = \delta^\phi(\Gamma) < +\infty$ and so

$$\delta^\psi(\Gamma) \leq \frac{1}{2}\delta^\phi(\Gamma) < +\infty.$$

Using ping-pong we can fix a free subgroup $\Gamma' \subset \Gamma$ such that there is Γ'-equivariant homeomorphism $\partial_\infty \Gamma' \to \Lambda_\theta(\Gamma') \subset \Lambda_\theta(\Gamma)$ of the Gromov boundary and the limit set. Then Γ' is P_θ-Anosov and

$$\delta^\psi(\Gamma') \leq \delta^\psi(\Gamma) < +\infty.$$

By a result of Sambarino [44, Lem. 3.4.2], see [14, Th. 10.1(4)] for the relatively Anosov case, ψ is positive on

$$\overline{\bigcup_{\gamma \in \Gamma'} \mathbb{R}_{>0} \cdot \lambda(\gamma)} \smallsetminus \{0\},$$

Then Proposition 11.10 implies that

$$\{\phi(\lambda(\gamma)) + \iota^*(\phi)(\lambda(\gamma)) : \gamma \in \Gamma\} \supset \{\psi(\lambda(\gamma)) : \gamma \in \Gamma'\}$$

generates a dense subgroup of $\mathbb{R}$. □

Example 11.12. Here we provide an example showing that Proposition 11.10 fails when Γ is only assumed to be a semigroup.

Let Γ be a convex cocompact free subgroup of $\mathsf{SL}_2(\mathbb{R})$ with two generators $a, b \in \Gamma$. Let $S \subset \Gamma$ be the semigroup generated by a and b. There exists $C \geq 1$ large enough so that for all nonnegative $n_1, m_1, \ldots, n_k, m_k$, if $s = a^{n_1} b^{m_1} \cdots a^{n_k} b^{m_k} \in S$

then $\log \|s\| \leq \frac{C}{2}|s|$ where $|s| = (n_1 + m_1 + \cdots + n_k + m_k)$ is word-length with respect to the generating set $\{a, b\}$. Now set

$$\rho(g) = \begin{pmatrix} g & 0 & 0 \\ 0 & e^{C|s|} & 0 \\ 0 & 0 & e^{-C|s|} \end{pmatrix} \in \mathsf{G} = \mathsf{SL}_4(\mathbb{R}).$$

If $\phi(\mathrm{diag}(x, y, z, w)) = x$ for any $\mathrm{diag}(x, y, z, w) \in \mathfrak{a}$, then $\ell_\phi(s) \in C\,\mathbb{Z}_{\geq 1}$ for any $s \in \rho(S)$. Thus the length spectrum is arithmetic, even though ϕ is positive on the limit cone of $\rho(S)$.

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