

Topological mixing of the geodesic flow on convex projective manifolds

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Abstract

We introduce a natural subset of the unit tangent bundle of an irreducible convex projective manifold, which is closed and invariant under the geodesic flow, and we prove that the geodesic flow is topologically mixing on it. We also show that, for higher-rank compact convex projective manifolds, the geodesic flow is topologically mixing on each connected component of the non-wandering set.

1 Introduction

This article is concerned with *convex projective manifolds*, namely quotients $M = \Omega/\Gamma$ of a properly convex open subset Ω of a finite-dimensional real projective space $\mathbb{P}(V)$ by a torsion-free discrete subgroup Γ of $\mathrm{PGL}(V)$ preserving Ω . Recall that *properly convex* means that Ω is convex and bounded in some affine chart of $\mathbb{P}(V)$. These manifolds are generalisations of hyperbolic manifolds, to whom they bring a new diversity of geometric features. When M is compact, we say that Γ *divides* Ω and that Ω is a *divisible convex set* (see [Ben08]).

Convex projective manifolds are endowed with a natural Finsler metric, which is not necessarily Riemannian. This metric defines a *geodesic flow* $(\phi_t)_{t \in \mathbb{R}}$ on the unit tangent bundle $T^1M = T^1\Omega/\Gamma$, obtained by following geodesics which are projective lines (see Section 2.1).

There is a dichotomy depending on whether Ω is *strictly convex* (meaning there is no non-trivial segment in the boundary $\partial\Omega$ of Ω in $\mathbb{P}(V)$, see Section 2.2), or not.

On the one hand, when Ω is strictly convex and Ω/Γ is compact, Benoist [Ben04, Th. 1.1] proved that many dynamical properties of the geodesic flow on hyperbolic manifolds still hold. Then Crampon and Marquis [CM14] generalised this to the case when Ω is strictly convex but Ω/Γ is not necessarily compact.

On the other hand, Benoist proved that when Ω is *not* strictly convex, one of the key properties of the classical geodesic flow, namely uniform hyperbolicity, is never satisfied. Still, Bray [Bra20b, Th. 5.7] managed to recover some classical dynamical properties of the geodesic flow in this context: namely, he established that the geodesic flow is *topologically mixing* when Ω/Γ is compact and 3-dimensional, not necessarily strictly convex, and Γ is strongly irreducible (i.e. Γ does not preserve any finite union of proper projective subspaces). Recall that a continuous flow $(f_t)_{t \in \mathbb{R}}$ on a topological space X is called topologically mixing if for any non-empty open subsets $U, V \subset X$ and any large enough time $t > 0$, the set $f_t(U)$ meets V . In order to prove his theorem, Bray used — and this is where the assumption that Ω/Γ is compact and 3-dimensional is crucial — another paper of Benoist [Ben06, Th. 1.1], which gives a precise and beautiful description of these compact 3-manifolds.

In this paper, we generalise Bray's result on topological mixing to the setting where Ω/Γ is not necessarily compact, in arbitrary dimension, as we explain below.

More refined dynamical properties of the geodesic flow on convex projective manifolds will be established in the forthcoming paper [Bla20]. In particular, we shall generalise [Bra20a] and prove the existence of a unique flow-invariant measure of maximal entropy (*Bowen–Margulis measure*) on the unit tangent bundle of *rank-one* (Definition 1.3) compact convex projective manifolds.

1.1 Main result

Recall that an element of $\mathrm{PGL}(V)$ is said to be *proximal* if it has an attracting fixed point in $\mathbb{P}(V)$. The *proximal limit set* $\Lambda_\Gamma \subset \mathbb{P}(V)$ of Γ is the closure of the set of attracting fixed points of proximal

elements of Γ ; it is Γ -invariant. We denote by $\text{Aut}(\Omega)$ the group of elements of $\text{PGL}(V)$ preserving Ω . We introduce the following subset of T^1M .

Definition 1.1. Let $\Omega \subset \mathbb{P}(V)$ be a properly convex open set and $\Gamma \subset \text{Aut}(\Omega)$ a discrete subgroup; denote by M the quotient Ω/Γ . The *biproximal unit tangent bundle* of M is

$$T^1M_{bip} := \{v \in T^1\Omega : \phi_{\pm\infty}v \in \Lambda_\Gamma\}/\Gamma \subset T^1M,$$

where $\phi_{\pm\infty}v = \lim_{t \rightarrow \pm\infty} \pi\phi_tv$ are the intersection points of the projective line generated by v with the boundary $\partial\Omega$.

Note that the biproximal unit tangent bundle is closed and invariant under the action of the geodesic flow. It is contained in the *non-wandering set* $\text{NW}(T^1M, (\phi_t)_{t \in \mathbb{R}})$ (Corollary 2.9), which consists of those vectors v whose neighbourhoods contain a geodesic which comes back close to v infinitely often (see Definition 2.10); we write $\text{NW}(T^1M)$ for short when the context is clear. The main result of this paper is the following.

Theorem 1.2. *Let $\Omega \subset \mathbb{P}(V)$ be a properly convex open set and $\Gamma \subset \text{Aut}(\Omega)$ a discrete subgroup which is strongly irreducible. Denote by M the quotient Ω/Γ . Then the geodesic flow on T^1M_{bip} is topologically mixing.*

The assumption that Γ is strongly irreducible is mild, and one can always restrict to it in the divisible case (see [Vey70, Th. 3] and [Ben08, Sec. 5.1]). When $\dim(\Omega) = 3$ and Ω/Γ is compact, we recover Bray's result [Bra20b, Th. 5.7] because [Ben06, Th. 1.1] implies $T^1M_{bip} = T^1M$ in that case. When Ω is strictly convex, one can see that $T^1M_{bip} = \text{NW}(T^1M)$ (see [CM14, §3.3] or Observation 2.12), and Crampon–Marquis [CM14, Prop. 6.1] showed that in this case the geodesic flow is topologically mixing on $\text{NW}(T^1M)$, if $\partial\Omega$ is smooth (for $\partial\Omega$ smooth means \mathcal{C}^1 , see Section 2.2). Thus, the point of Theorem 1.2 is to treat the non-strictly convex case, where in general we can only prove that T^1M_{bip} is contained in $\text{NW}(T^1M)$, see Remark 2.11.

Our strategy of proof for Theorem 1.2 is similar to that of Bray in [Bra20b], but we manage to work without Benoist's geometric description [Ben06, Th. 1.1] of 3-dimensional compact convex projective manifolds. Furthermore, we slightly shorten the proofs by using more algebraic arguments inspired by [Ben00a, Ben00b] (see Section 4): they allow us to prove topological mixing directly, without establishing first *topological transitivity* — a slightly weaker property than topological mixing — and then using a closing lemma and a weak-orbit gluing lemma as in [Bra20b, Th. 4.4 & Lem. 5.3].

Another strategy of proof for Theorem 1.2, in the compact case, could be to find a good geometric description that generalises Benoist's [Ben06] to arbitrary dimension. This is an interesting question, and recent work suggests that such a description could exist: Benoist [Ben06], Marquis [Mar10], Ballas–Danciger–Lee [BDL18], and Choi–Lee–Marquis [CLM20] constructed non-strictly convex, compact convex projective manifolds that share a number of nice geometric features, in dimensions 3 to 7; recent work of Bobb [Bob] extends some results of [Ben06] to all dimensions.

1.2 The biproximal unit tangent bundle

Topological mixing and topological transitivity belong to a family of transitivity properties which have been investigated for geodesic flows on non-positively curved Riemannian manifolds X for many decades, see for instance the classical surveys [Hed39, EHS93]. We now briefly relate Theorem 1.2 to older results for non-positively curved manifolds.

The topological transitivity of $(\phi_t)_{t \in \mathbb{R}}$ on $\text{NW}(T^1M)$ for $M = \Omega/\Gamma$, when Ω is strictly convex, $\partial\Omega$ is smooth and $\pi_1(M)$ is non-elementary [CM14, Prop. 6.1] is analogous to that of $(\phi_t)_{t \in \mathbb{R}}$ on $\text{NW}(T^1X)$ when X is negatively curved and $\pi_1(X)$ is non-elementary, which was proved by Eberlein [Ebe72, Th. 3.11].

When Ω is not necessarily strictly convex and T^1M_{bip} is non-empty, the situation is analogous to X being non-positively curved and *rank-one*, i.e. having a *rank-one* periodic vector. This notion was introduced by Ballmann–Brin–Eberlein [BBE85, Def. p. 1]. Ballmann [Bal82, Th. 3.5] proved that if X is rank-one and $\text{NW}(T^1X) = T^1X$ (e.g. if X is rank-one and compact), then $(\phi_t)_{t \in \mathbb{R}}$ is topologically mixing on T^1X . Coudène–Schapira studied the action of $(\phi_t)_{t \in \mathbb{R}}$ on $\text{NW}(T^1X)$ without assuming that $\text{NW}(T^1X) = T^1X$; they established [CS10, Th. 5.2] the topological transitivity of $(\phi_t)_{t \in \mathbb{R}}$ on some

invariant subset $NW_1(T^1X)$ of $NW(T^1X)$, defined in [CS10, §5.1], consisting of rank-one vectors with an extra condition.

We wish to interpret T^1M_{bip} as an analogue of $NW_1(T^1X)$. The definitions of rank-one convex projective manifolds and their rank-one periodic geodesics are now available thanks to the very recent work of Islam [Isl, Def. 1.3 & 6.2] and A. Zimmer [Zim, Def. 1.1]. Here we adopt Islam's definition, which we reformulate as follows.

Definition 1.3 ([Isl]). Let $\Omega \subset P(V)$ be a properly convex open set, $\Gamma \subset \text{Aut}(\Omega)$ a discrete subgroup, and $M := \Omega/\Gamma$. A periodic vector $v \in T^1M$ is said to be *rank-one* if the endpoints $\phi_{\pm\infty}\tilde{v}$ of any of its lift $\tilde{v} \in T^1\Omega$ are smooth and *strongly extremal* (meaning not contained in any non-trivial segment of the boundary $\partial\Omega$); in this case, any torsion-free element of Γ which preserves the orbit of a lift \tilde{v} is said to be *rank-one*. The convex projective manifold (or orbifold) M is *rank-one* if T^1M contains a rank-one periodic vector.

The classical Fact 2.6 below ensures that rank-one periodic vectors are contained in T^1M_{bip} , which is then non-empty whenever M is rank-one. Furthermore, Proposition 3.4 tells us that, when they exist, periodic rank-one vectors are dense in T^1M_{bip} .

Further evidence for thinking that T^1M_{bip} is analogous to $NW_1(T^1X)$, is the fact that if M is compact and *higher-rank* (i.e. not rank-one), then T^1M_{bip} is empty (see Remark 7.2). In this case the proximal limit set is non-empty (Fact 2.8), but any segment between two distinct points of Λ_Γ is contained in $\partial\Omega$.

To conclude this section, we ask the following question: if T^1M_{bip} is non-empty, is it the whole non-wandering set $NW(T^1M)$? In the Riemannian setting, if X is compact and rank-one, then $NW_1(T^1X)$ is dense in $NW(T^1X) = T^1X$ [CS10, §5]. In the paper in preparation [Bla20], we prove that this is also true in the convex projective setting: if M is compact and rank-one, then $T^1M_{bip} = NW(T^1M) = T^1M$.

When the manifold is non-compact, the situation is more subtle. In the Riemannian setting, Coudène–Schapira [CS10, §5.2] constructed an example where X is non-compact and $NW_1(T^1X)$ is non-empty and not dense in $NW(T^1X)$. In the convex projective setting, we show in [Bla20] that T^1M_{bip} may be non-empty and smaller than $NW(T^1M)$ for non-compact M , even when M is convex cocompact in the sense of [DGK].

Observe that when M is higher-rank and compact, $NW(T^1M)$ is different from T^1M_{bip} , since, contrary to the latter, the former is non-empty.

1.3 The higher-rank compact case

When M is compact and higher-rank, Theorem 1.2 does not tell us anything since T^1M_{bip} is empty. However, in this case, the investigation of dynamical properties of the geodesic flow happens to be easier, thanks to the recent work of Zimmer [Zim, Th. 1.4], which classifies higher-rank compact convex projective manifolds (this is similar to a classification of compact higher-rank non-positively curved Riemannian manifolds by Ballmann [Bal85, Cor. 1] and Burns–Spatzier [BS87, Th. 5.1]). More precisely, he proves that universal covers in $P(V)$ of higher-rank compact convex projective manifolds belong to a narrow and explicit list of properly convex open sets, called *symmetric* (see Section 7). We use this to establish the following.

Proposition 1.4. *Let M be a higher-rank compact convex projective manifold. Then the non-wandering set of the geodesic flow on T^1M has several (more than one) connected components, and the geodesic flow is topologically mixing on each of them.*

Proposition 1.4 is a direct consequence of Proposition 7.4, where the connected components of the non-wandering set are described more precisely.

Organisation of the paper In Section 2 we recall some basic definitions and properties in convex projective geometry. In Section 3 we investigate the regularity of endpoints of biproximal periodic geodesics. In Section 4 we prove that, when $T^1M_{bip} \neq \emptyset$, the length spectrum is locally non-arithmetic; in other words, for every non-empty open subset U of T^1M_{bip} , the additive subgroup of \mathbb{R} generated by lengths of biproximal periodic geodesics through U is dense in \mathbb{R} . In Section 5 we prove that a geodesic, which has the same endpoint in $\partial\Omega$ as a biproximal periodic geodesic γ , must in the

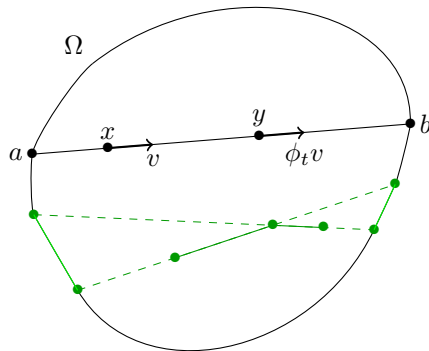


Figure 1: The Hilbert metric and the geodesic flow ($t = d_\Omega(x, y)$).

quotient wrap around closer and closer to γ (see Figure 4). In Section 6 we prove Theorem 1.2 using Sections 3, 4 and 5, and classical dynamical arguments. In Section 7 we study the non-wandering set of the geodesic flow on higher-rank compact convex projective manifolds, and prove Proposition 7.4. In Appendix A we fill in a missing detail in Crampon’s original proof of a useful technical lemma in convex projective geometry.

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2 Reminders

2.1 Properly convex open subsets of $\mathbb{P}(\mathbb{R}^{d+1})$ and their geodesic flow

In the whole paper we fix a real vector space $V = \mathbb{R}^{d+1}$. Let $\Omega \subset \mathbb{P}(V)$ be a properly convex open set. Recall that Ω admits an $\text{Aut}(\Omega)$ -invariant proper metric called the *Hilbert metric* and defined by the following formula: for $(a, x, y, b) \in \partial\Omega \times \Omega \times \Omega \times \partial\Omega$ aligned in this order (see Figure 1),

$$d_\Omega(x, y) = \frac{1}{2} \log([a, x, y, b]),$$

where $[a, x, y, b]$ is the cross-ratio of the four points, normalised so that $[0, 1, t, \infty] = t$.

Recall that if Ω is an ellipsoid, then (Ω, d_Ω) is the Klein model of the hyperbolic space of dimension d , and if Ω is a d -simplex, then (Ω, d_Ω) is isometric to \mathbb{R}^d endowed with a hexagonal norm.

Any discrete subgroup $\Gamma \subset \text{PGL}(V)$ of automorphisms of Ω preserves d_Ω , hence must act properly discontinuously on Ω and therefore the quotient $M = \Omega/\Gamma$ is an orbifold. Furthermore, M is a manifold if the action is free (i.e. if Γ is torsion-free, by Brouwer’s fixed point theorem, applied to the convex hull of a finite orbit of a torsion element). Note that by Selberg’s lemma [Sel60], Γ always has a torsion-free finite-index subgroup. We will work in general with Γ not necessarily torsion-free, so we set the notation $T^1M = T^1\Omega/\Gamma$.

The intersections of Ω with projective lines can be parametrised to be geodesics, which are said to be *straight*. However, an interesting feature in the non-strictly convex case is that when there are two coplanar non-trivial segments in the boundary $\partial\Omega$, one can construct geodesics which are not straight, see for instance the broken green segment in Figure 1. In order to define the geodesic flow we only take into account straight geodesics: for v in $T^1\Omega$, let $t \mapsto c(t)$ be the parametrisation of the projective line tangent to v such that c is an isometric embedding from \mathbb{R} to Ω and $c'(0) = v$. For $t \in \mathbb{R}$ we set $\phi_t(v) = c'(t) \in T^1\Omega$. See Figure 1.

The geodesic flow on T^1M is well defined because the two actions of $\text{Aut}(\Omega)$ and $(\phi_t)_{t \in \mathbb{R}}$ on $T^1\Omega$ commute.

Denoting by $\pi : T^1M \rightarrow M$ and $\pi : T^1\Omega \rightarrow \Omega$ the projections, we consider the following metrics:

$$\begin{aligned} \forall x, y \in M, \quad d_M(x, y) &= \min\{d_\Omega(\tilde{x}, \tilde{y}) : \tilde{x}, \tilde{y} \in \Omega \text{ lifts of } x, y\}, \\ \forall v, w \in T^1\Omega, \quad d_{T^1\Omega}(v, w) &= \max_{0 \leq t \leq 1} d_\Omega(\pi\phi_t v, \pi\phi_t w), \\ \forall v, w \in T^1M, \quad d_{T^1M}(v, w) &= \min\{d_{T^1\Omega}(\tilde{v}, \tilde{w}) : \tilde{v}, \tilde{w} \in T^1\Omega \text{ lifts of } v, w\}. \end{aligned}$$

The following remark is a direct consequence of the definition of the Hilbert metric.

Remark 2.1. Let $\Omega \subset P(V)$ be a properly convex open set, and fix an affine chart containing $\bar{\Omega}$. Then

$$\bar{B}_\Omega(x, r) \subset (1 - e^{-2r})(\bar{\Omega} - x) + x$$

for all $x \in \Omega$ and $r > 0$, where $(1 - e^{-2r})(\bar{\Omega} - x) + x$ is the image of $\bar{\Omega}$ under the homothety (of the affine chart) centred at x and with ratio $1 - e^{-2r}$, and $\bar{B}_\Omega(x, r)$ is the closed ball of radius r , centred at x , for the metric d_Ω .

2.2 Smooth and extremal points of the boundary

We recall here some terminology on convex sets. Let $\Omega \subset P(V)$ be a properly convex open set. Let $\xi \in \partial\Omega$ be a point of the boundary.

- A *supporting hyperplane* of Ω at ξ is a hyperplane which contains ξ but does not intersect Ω . Note that there always exists such a hyperplane.
- The point ξ is said to be a *smooth* point of $\partial\Omega$ if there is only one supporting hyperplane of Ω at ξ , which we then denote by $T_\xi\partial\Omega$.
- The point ξ is said to be *extremal* if it is not contained in the relative interior of a non-trivial segment contained in the boundary $\partial\Omega$.
- As it was defined in the introduction, Ω is said to be *strictly convex* when all points of $\partial\Omega$ are extremal.
- As in Definition 1.3, we shall say that ξ is *strongly extremal* if it is not contained any non-trivial segment contained in the boundary $\partial\Omega$.

2.3 Proximal linear transformations

In this section we recall the notion of a proximal linear transformation, which was used in the definition of the proximal limit set Λ_Γ and the biproximal unit tangent bundle T^1M_{bip} in Section 1.1.

Notation 2.2. If W_1 and W_2 are two subspaces of V such that $W_1 \cap W_2 = \{0\}$, we write $W_1 \oplus W_2 \subset V$ for their direct sum and $P(W_1) \oplus P(W_2) = P(W_1 \oplus W_2)$ for its projectivisation. In particular, if $x, y \in P(V)$ are two distinct points, we write $x \oplus y$ for the projective line through x and y .

Definition 2.3. A linear transformation $g \in \text{End}(V)$ is *proximal* if it has exactly one complex eigenvalue with maximal modulus among all eigenvalues, and if this eigenvalue has multiplicity 1. The associated eigenline in $P(V)$ is the attracting fixed point of g and is denoted by x_g^+ .

An invertible linear transformation $g \in \text{GL}(V)$ is said to be *biproximal* if g and g^{-1} are proximal. The attracting fixed point of g^{-1} is the repelling fixed point of g and is denoted by x_g^- . The projective line $x_g^+ \oplus x_g^-$ (see Notation 2.2) is the *axis* of g and is denoted by $\text{axis}(g)$. The g -invariant complementary subspace to the axis of g is denoted by x_g^0 . Note that the notions of biproximality, attracting/repelling fixed point, and axis, are well defined for the image of g in $\text{PGL}(V)$.

Remark 2.4. The set of proximal linear transformations is open in $\text{End}(V)$, and the map sending a proximal linear transformation to the pair (attracting fixed point, maximal eigenvalue) is continuous.

Remark 2.5. As observed by Benoist [Ben97, Lem. 3.6.ii], for any irreducible subgroup $\Gamma \subset \mathrm{PGL}(V)$ which contains a proximal element, the proximal limit set is the smallest closed Γ -invariant non-empty subset of $\mathrm{P}(V)$; in particular, the action of Γ on Λ_Γ is minimal (i.e. any orbit is dense). Indeed, consider any proximal element $\gamma \in \Gamma$, and let $\mathrm{P}(W) \subset \mathrm{P}(V)$ be the γ -invariant complementary subspace to x_γ^+ . By irreducibility, any closed Γ -invariant non-empty subset $X \subset \mathrm{P}(V)$ contains a point x outside $\mathrm{P}(W)$, and then x_γ^+ , which is the limit of the sequence $(\gamma^n x)_{n \in \mathbb{N}}$, belongs to X .

2.4 Periodic geodesics and automorphisms of Γ

In this section we recall the link between periodic geodesics in $T^1\Omega/\Gamma$ and conjugacy classes of Γ . Let $\Omega \subset \mathrm{P}(V)$ be a properly convex open set. Let $g \in \mathrm{GL}(V)$. We denote by $\lambda_1(g) \geq \dots \geq \lambda_{d+1}(g)$ the non-increasing sequence of logarithms of moduli of g ; we set

$$\ell(g) := \frac{1}{2}(\lambda_1(g) - \lambda_{d+1}(g)). \quad (2.1)$$

Observe that $\ell(g)$ only depends on the class of g in $\mathrm{PGL}(V)$. If g preserves Ω , then

$$\ell(g) = \inf\{d_\Omega(x, g \cdot x) : x \in \Omega\} \geq 0. \quad (2.2)$$

The right-hand side of (2.2) is called the *translation length* of g . See [CLT15, Prop. 2.1] for a proof.

Combined with an elementary computation, (2.2) yields:

Fact 2.6. *Let $\Omega \subset \mathrm{P}(V)$ be a properly convex open set, let $\Gamma \subset \mathrm{Aut}(\Omega)$ be a discrete subgroup, and let $M = \Omega/\Gamma$. Then for any lift in Ω of any periodic straight geodesic of M , there is an automorphism $\gamma \in \Gamma$ which preserves it and acts by positive translation on it. Let $\tilde{\gamma} \in \mathrm{GL}(V)$ be a lift of γ . The endpoints in $\partial\Omega$ of the geodesic are fixed by γ , the associated eigenvalues of $\tilde{\gamma}$ are $\lambda_1(\tilde{\gamma})$ and $\lambda_{d+1}(\tilde{\gamma})$, and the length of the geodesic in M is the translation length of γ . If furthermore these endpoints are extremal, then γ is biproximal.*

By Fact 2.6, rank-one elements of Γ (Definition 1.3) are biproximal. Hence rank-one periodic vectors of T^1M belong to T^1M_{bip} .

Definition 2.7. Let $\Omega \subset \mathrm{P}(V)$ be a properly convex open set and $\Gamma \subset \mathrm{Aut}(\Omega)$ a discrete subgroup. Let $\gamma \in \Gamma$ be a biproximal element whose axis meets Ω . Then the periodic geodesic associated to γ is said to be *biproximal*, and the unit tangent vectors along this geodesic are said to be *biproximal* periodic.

There are cases where $\gamma \in \Gamma$ is biproximal but its axis does not intersect Ω (e.g. when Ω is a triangle, or is symmetric as in Section 7). Then we cannot make sense of a straight periodic geodesic associated to γ .

2.5 Density of biproximal geodesics

We gather here two results of Benoist which imply that biproximal periodic vectors are dense in T^1M_{bip} .

Fact 2.8 ([Ben00a, Prop. 1.1] & [Ben97, Lem. 3.6.iv]). *Let $\Gamma \subset \mathrm{PGL}(V)$ be a strongly irreducible subgroup.*

1. *If Γ preserves a properly convex open set $\Omega \subset \mathrm{P}(V)$, then it contains a proximal element.*
2. *If Γ contains a proximal element, then then the following subset is dense in $\Lambda_\Gamma \times \Lambda_\Gamma$:*

$$\{(x_\gamma^+, x_\gamma^-) \in \Lambda_\Gamma \times \Lambda_\Gamma : \gamma \in \Gamma \text{ biproximal}\}.$$

Corollary 2.9. *Let $\Omega \subset \mathrm{P}(V)$ be a properly convex open set and $\Gamma \subset \mathrm{Aut}(\Omega)$ a strongly irreducible discrete subgroup. Denote by M the quotient Ω/Γ , and suppose that T^1M_{bip} is non-empty. Then biproximal periodic geodesics exist and are dense in T^1M_{bip} .*

2.6 The non-wandering set

In this section we recall the definition of the non-wandering set and the link between the non-wandering set of the geodesic flow on T^1M and the non-wandering set of the actions of Γ and $\text{Aut}(\Omega)$ on the space of geodesics of Ω . This will be used in Section 7.

Definition 2.10. Let X be a locally compact topological space equipped with a continuous action by a locally compact group G . The *non-wandering set* $\text{NW}(X, G)$ is the set of points x in X such that for any compact neighbourhood U of x , the set $\{g \in G : gU \cap U \neq \emptyset\}$ is non-compact.

In other words, it is the set of points all of whose neighbourhoods come back infinitely often under the action; we call such points *non-wandering*. The non-wandering set is closed and G -invariant. Note that if X is compact but G is not, then the non-wandering set is non-empty. When G is \mathbb{R} , i.e. when we have a flow $(\phi_t)_{t \in \mathbb{R}}$ on X , observe that given a non-wandering point $x \in X$ and a neighbourhood U of x , one can find arbitrarily large *positive* times t such that $\phi_t U \cap U \neq \emptyset$; indeed, if $\phi_t U \cap U \neq \emptyset$, then $\phi_{-t} U \cap U \neq \emptyset$.

In our setting, there are three non-wandering sets of interest for us. Let $\Omega \subset \text{P}(V)$ be a properly convex open set, let $\Gamma \subset \text{Aut}(\Omega)$ be a closed subgroup of automorphisms, and let M be the quotient Ω/Γ . One can consider the non-wandering set $\text{NW}(T^1M, (\phi_t)_{t \in \mathbb{R}})$ of the geodesic flow.

Remark 2.11. Any vector of T^1M which is tangent to a periodic straight geodesic belongs to the non-wandering set $\text{NW}(T^1M, (\phi_t)_{t \in \mathbb{R}})$. As a consequence, if Γ is strongly irreducible, then T^1M_{bip} is contained in $\text{NW}(T^1M, (\phi_t)_{t \in \mathbb{R}})$ by Corollary 2.9.

Let us denote by $\text{Geod}(\Omega) = T^1\Omega/(\phi_t)_{t \in \mathbb{R}}$ the set of straight geodesics of Ω : it is an open subset of $\partial\Omega^2$, consisting of the pairs (x, y) such that $x \neq y$ and the projective line through x and y meets Ω . The group Γ naturally acts on $\text{Geod}(\Omega)$ and one can consider its non-wandering set $\text{NW}(\text{Geod}(\Omega), \Gamma)$. Finally, one can consider the two commutative and proper actions of Γ and \mathbb{R} (by the geodesic flow) on $T^1\Omega$, it yields the non-wandering set $\text{NW}(T^1\Omega, \Gamma \times \mathbb{R})$. All three of these non-wandering sets are actually identified in the following sense. Denote the canonical projections by $\pi_{\mathbb{R}} : T^1\Omega \rightarrow \text{Geod}(\Omega)$ and $\pi_{\Gamma} : T^1\Omega \rightarrow T^1M$. Then

$$\pi_{\mathbb{R}}^{-1}(\text{NW}(\text{Geod}(\Omega), \Gamma)) = \text{NW}(T^1\Omega, \Gamma \times \mathbb{R}) = \pi_{\Gamma}^{-1}(\text{NW}(T^1M, (\phi_t)_{t \in \mathbb{R}})).$$

We will use this while studying symmetric properly convex open sets in Section 7.

To end this section, we observe that the non-wandering set $\text{NW}(T^1M, (\phi_t)_{t \in \mathbb{R}})$ is contained in another $(\phi_t)_{t \in \mathbb{R}}$ -invariant subset T^1M , defined similarly to T^1M_{bip} , but using another limit set in the boundary. Recall that Danciger, Guéritaud, and Kassel [DGK, Def. 1.10] defined the *full orbital limit set* $\Lambda_{\Gamma}^{orb} \subset \partial\Omega$ as the union, over all $x \in \Omega$, of the set of accumulation points of the orbit $\Gamma \cdot x$; the full orbital limit set always contains the proximal limit set. Similarly to T^1M_{bip} , we can consider

$$T^1M_{core} := \{v \in T^1\Omega : \phi_{\pm\infty}v \in \Lambda_{\Gamma}^{orb}\}/\Gamma \subset T^1M.$$

Observation 2.12. Let $\Omega \subset \text{P}(V)$ be a properly convex open set, let Γ be a discrete group of automorphisms of Ω , and denote by M the quotient Ω/Γ . Then

$$\text{NW}(T^1M, (\phi_t)_{t \in \mathbb{R}}) \subset T^1M_{core}.$$

Proof. Consider a vector $v \in T^1\Omega$ whose projection in T^1M is non-wandering. Let x be the footpoint of v . We want to show that $\phi_{\infty}v$ is an accumulation point of $\Gamma \cdot x$. Since the projection of v in T^1M is non-wandering, we can find sequences of vectors $(v_n)_n$ in $T^1\Omega$ converging to v , of positive times $(t_n)_n$ going to infinity, and of automorphisms $(\gamma_n)_n$ in Γ such that $(d_{T^1\Omega}(\phi_{t_n}v_n, \gamma_nv))_n$ tends to zero. Since $(v_n)_n$ tends to v and $(t_n)_n$ goes to infinity, $(\pi\phi_{t_n}v_n)_n$ must converge to $\phi_{\infty}v$. By Remark 2.1, the fact that $(d_{\Omega}(\pi\phi_{t_n}v_n, \gamma_nx))_n$ tends to zero implies that $(\gamma_nx)_n$ also converges to $\phi_{\infty}v$ in $\text{P}(V)$. \square

The full orbital limit set Λ_{Γ}^{orb} and Observation 2.12 will not be used in the remainder of the paper.

3 Endpoints of biproximal periodic geodesics are smooth

This section contains an elementary result (Lemma 3.1), which will be used in the proof of topological mixing in Section 6. Furthermore, as a more immediate consequence, we also use it to justify a claim of the introduction: that rank-one periodic geodesics are dense in the biproximal unit tangent bundle of rank-one manifolds (Proposition 3.4).

3.1 On the regularity of endpoints of biproximal periodic geodesics

The main consequence of the following lemma is that we will be able to apply Proposition 5.1 to biproximal periodic vectors.

Lemma 3.1. *Let $\Omega \subset P(V)$ be a properly convex open set. Let $g \in \text{PGL}(V)$ be a biproximal automorphism of Ω . Then $\text{axis}(g) \cap \Omega$ is non-empty if and only if x_g^+ is smooth; in this case $T_{x_g^+} \partial\Omega = x_g^+ \oplus x_g^0$.*

Proof. Assume that x_g^+ is not smooth. Then there is a supporting hyperplane H of Ω at x_g^+ which is different from $x_g^+ \oplus x_g^0$. Let $x \in H \setminus (x_g^+ \oplus x_g^0)$, so that the sequence $(g^{-n}x)_n$ tends to x_g^- . The sequence of projective lines through $g^{-n}x$ and x_g^+ must converge to the axis of g , and they are all contained in $P(V) \setminus \Omega$ which is closed. Therefore $P(V) \setminus \Omega$ must contain the axis of g as well.

Conversely if $P(V) \setminus \Omega$ contains $\text{axis}(g)$, then x_g^+ has a supporting hyperplane which contains $\text{axis}(g)$, and which is thereby different from the supporting hyperplane $x_g^0 \oplus x_g^+$. \square

3.2 Rank-one periodic geodesics and their dual

In this section we give several equivalent conditions for an automorphism of a properly convex open set to be rank-one (Definition 1.3), which follow from Lemma 3.1. It will be used in Section 3.3, and may be interesting in its own right.

Let us recall the notion of duality for properly convex open sets. We identify the dual projective space $P(V^*)$ with the set of projective hyperplanes of $P(V)$. Let Ω be a properly convex open subset of $P(V)$. The dual of Ω , denoted by Ω^* , is the properly convex open subset of $P(V^*)$ defined as the set of projective hyperplanes which do not intersect Ω . We naturally identify $\text{PGL}(V)$ and $\text{PGL}(V^*)$, then $\text{Aut}(\Omega)$ identifies with $\text{Aut}(\Omega^*)$, and the attracting (resp. repelling) fixed point of the action on $P(V^*)$ of any biproximal element $g \in \text{PGL}(V)$ is $x_g^+ \oplus x_g^0$ (resp. $x_g^- \oplus x_g^0$).

Lemma 3.2. *Let $\Omega \subset P(V)$ be a properly convex open set. Let $g \in \text{PGL}(V)$ be a biproximal automorphism of Ω . Then the following are equivalent:*

- (a) g is rank-one;
- (b) $x_g^+, x_g^- \in \partial\Omega$ are smooth and strongly extremal points;
- (c) x_g^+ is strongly extremal;
- (d) x_g^+ is smooth and x_g^0 does not intersect $\partial\Omega$;
- (e) the element g seen as an automorphism of Ω^* is rank-one;
- (f) the axis of g in $P(V)$ intersects Ω , and the axis of g in $P(V^*)$ intersects Ω^* .

We will need in the proof two elementary facts concerning duality of properly convex open sets. Recall that the canonical isomorphism between V and V^{**} identifies Ω with Ω^{**} . By definition of Ω^* , the boundary $\partial\Omega^*$ is the set of supporting hyperplanes of Ω ; by duality $\partial\Omega = \partial\Omega^{**}$ is the set of supporting hyperplanes of Ω^* .

Fact 3.3. *Let $\Omega \subset P(V)$ be a properly convex open set.*

- (i) *A smooth point $x \in \partial\Omega$ is strongly extremal if and only if its tangent space $T_x \partial\Omega$ is a smooth point of $\partial\Omega^*$; in this case $T_x \partial\Omega$ is strongly extremal.*
- (ii) *For any $H, H' \in \partial\Omega^*$, the segment $[H, H'] \subset \overline{\Omega^*}$ is contained in $\partial\Omega^*$ if and only if $H \cap H' \cap \partial\Omega$ is non-empty.*

Proof of Lemma 3.2. • The equivalence between (a) and (b) is a consequence of Fact 2.6.

- The fact that (b) and (e) are equivalent is a direct consequence of Fact 3.3(i).
- Let us see why (d) and (f) are equivalent. By Lemma 3.1, the axis of g in $P(V)$ intersects Ω if and only if x_g^+ is smooth. By Fact 3.3(ii), the axis of g in $P(V^*)$ intersects Ω^* if and only if $x_g^0 \cap \partial\Omega = \emptyset$.

- That (b) implies (c) is immediate.
- Let us prove that (c) implies (f). Assume that x_g^+ is strongly extremal. Then $[x_g^+, x_g^-]$ is not contained in $\partial\Omega$, so the axis of g in $P(V)$ intersects Ω . Furthermore, $x_g^0 \cap \partial\Omega$ is contained in $(x_g^0 \oplus x_g^+) \cap \partial\Omega \setminus \{x_g^+\}$ which is empty. By Fact 3.3(ii) this exactly means that the axis of g in $P(V^*)$ intersects Ω^* .
- Let us prove that (f) implies (b). Assume that the axis of g in $P(V)$ intersects Ω , and that the axis of g in $P(V^*)$ intersects Ω^* . By Lemma 3.1, the points $x_g^+, x_g^- \in \partial\Omega$ and $(x_g^+ \oplus x_g^0), (x_g^- \oplus x_g^0) \in \partial\Omega^*$ are smooth. By Fact 3.3(i), this implies that x_g^+ and x_g^- are strongly extremal. \square

3.3 Density of rank-one periodic geodesics

The following proposition justifies a claim of the introduction, and it will not be used in the remainder of this paper.

Proposition 3.4. *Let $\Omega \subset P(V)$ be a properly convex open set. Then*

1. *for any pair $(\xi, \eta) \in \text{Geod}(\Omega)$ such that ξ is strongly extremal, there exist neighbourhoods U of ξ and V of η such that for any biproximal automorphism $g \in \text{Aut}(\Omega)$, if $(x_g^+, x_g^-) \in U \times V$, then g is rank-one;*
2. *if $\Gamma \subset \text{Aut}(\Omega)$ is a strongly irreducible discrete subgroup such that $M = \Omega/\Gamma$ is rank-one, then rank-one periodic geodesics are dense in T^1M_{bip} . In particular T^1M_{bip} is not empty.*

Proof. 1. Let us assume by contradiction that there is a sequence of biproximal automorphisms $(g_n)_{n \in \mathbb{N}}$ which are not rank-one, and such that $(x_{g_n}^+)_{n \in \mathbb{N}}$ and $(x_{g_n}^-)_{n \in \mathbb{N}}$ respectively converge to ξ and η . For n large enough, $(x_{g_n}^-, x_{g_n}^+) \in \text{Geod}(\Omega)$, hence by Lemma 3.2(d), there exists $\xi_n \in x_{g_n}^0 \cap \partial\Omega$. Up to extraction we can assume that $(\xi_n)_{n \in \mathbb{N}}$ converges to some $\xi' \in \partial\Omega$. Since

$$[\xi_n, x_{g_n}^+] \subset (x_{g_n}^0 \oplus x_{g_n}^+) \cap \bar{\Omega} \subset \partial\Omega$$

for all n , passing to the limit we obtain that $[\xi', \xi] \subset \Omega$, which implies that $\xi' = \xi$, because ξ is strongly extremal. Similarly, $[\xi_n, x_{g_n}^-] \subset \partial\Omega$ for all n , so $[\xi, \eta] \subset \partial\Omega$, which is a contradiction.

2. We denote by $\partial_{\text{sse}}\Omega$ the set of smooth and extremal points of $\partial\Omega$. By assumption, the Γ -invariant subset $\Lambda_\Gamma \cap \partial_{\text{sse}}\Omega$ is non-empty, hence dense in Λ_Γ because the action of Γ on Λ_Γ is minimal (Remark 2.5). Therefore it is enough to show that attracting/repelling pairs of rank-one biproximal elements of Γ are dense in $(\Lambda_\Gamma \cap \partial_{\text{sse}}\Omega)^2$. Consider a pair of points $\xi \neq \eta$ in $\Lambda_\Gamma \cap \partial_{\text{sse}}\Omega$, and neighbourhoods U and V of respectively ξ and η . By point 1. above, there are smaller neighbourhoods $U' \subset U$ and $V' \subset V$ such that any biproximal automorphism of Ω with attracting/repelling pair in $U' \times V'$ must be rank-one. By Fact 2.8.2, there is such an automorphism in Γ . \square

4 Non-arithmeticity of the length spectrum

In this section we prove the following.

Proposition 4.1. *Let $\Omega \subset P(V)$ be a properly convex open set and $\Gamma \subset \text{Aut}(\Omega)$ a strongly irreducible discrete subgroup. Denote by M the quotient Ω/Γ . Let $U \subset T^1M_{\text{bip}}$ be a non-empty open set. Then the additive group generated by the lengths of biproximal periodic geodesics through U is dense in \mathbb{R} .*

In other words, not only are biproximal periodic geodesics dense in T^1M_{bip} , but moreover the length spectrum is *locally* non-arithmetic.

Our proofs are heavily influenced by the work of Benoist [Ben00a, Ben00b]; see also [BQ16, Ch. 7].

4.1 Density of the group generated by Jordan projections

We gather here two results which imply that any strongly irreducible semi-group of automorphisms of a properly convex open set has a non-arithmetic length spectrum.

A proof of the following result can be found in [CM14, Prop. 6.5].

Fact 4.2 ([Ben00a, Rem. p. 17]). *Let $\Omega \subset \mathbb{P}(V)$ a properly convex open set, $\Gamma \subset \mathrm{SL}(V)$ a irreducible discrete strongly subgroup preserving Ω . Then the Zariski-closure of Γ is semi-simple and non-compact.*

The proof of the following result uses the language of semi-simple Lie groups, see e.g. [BQ16, Ch. 6] for definitions.

Proposition 4.3. *Let $\Gamma \subset \mathrm{SL}(V)$ be a sub-semi-group whose Zariski-closure is irreducible, semi-simple and non-compact. Then*

$$\overline{\langle \ell(\gamma), \gamma \in \Gamma \rangle} = \mathbb{R},$$

where $\ell(\gamma)$ is given by (2.1).

Note that Proposition 4.3 can (and will) be applied, not only to subgroups, but to sub-semi-groups. This is the key observation that allows us to slightly shorten Bray's proof of the topological mixing. Proposition 4.3 is a consequence of the following famous theorem of Benoist.

Fact 4.4 ([Ben00b, Prop. p.2]). *Let G be a connected real semi-simple linear Lie group. Let \mathfrak{a}_G be a Cartan subspace of its Lie algebra, let $\mathfrak{a}_G^+ \subset \mathfrak{a}_G$ be a closed Weyl chamber, and let $\lambda_G : G \rightarrow \mathfrak{a}_G^+$ be the associated Jordan projection. Let $\Gamma \subset G$ be a Zariski-dense sub-semi-group. Then the additive group generated by $\lambda_G(\Gamma)$ is dense in \mathfrak{a}_G .*

Proof of Proposition 4.3. Let G be the Zariski closure of Γ in $\mathrm{SL}(V)$, with associated Cartan subspace \mathfrak{a}_G , Weyl chamber \mathfrak{a}_G^+ , and Jordan projection $\lambda_G : G \rightarrow \mathfrak{a}_G^+$. Up to replacing Γ by a finite-index subgroup, we may assume that G is connected. We denote by χ_1 (resp. χ_d) the highest weight of the representation $\rho : G \hookrightarrow \mathrm{SL}(V)$ (resp. the dual representation in $\mathrm{SL}(V^*)$), which are linear maps defined on \mathfrak{a}_G . By definition, for any element a in \mathfrak{a}_G^+ , the numbers $\chi_1(a)$ and $\chi_d(a)$ are respectively the highest and lowest eigenvalues of $d_{\mathrm{id}}\rho(a)$. Furthermore, the moduli of the eigenvalues of $\rho(g)$, for $g \in G$, are the eigenvalues of $\exp(d_{\mathrm{id}}\rho \circ \lambda_G(g))$, as a consequence $\ell(\rho(g))$ is $(\chi_1 \circ \lambda_G(g) - \chi_d \circ \lambda_G(g))/2$. By linearity, the additive group generated by the translation lengths of the elements of Γ is the image by $(\chi_1 - \chi_d)/2$ of the additive subgroup of \mathfrak{a}_G generated by the Jordan projections of the elements of Γ . We conclude thanks to Fact 4.4. \square

4.2 Construction of a suitable free sub-semi-group

Let $\Omega \subset \mathbb{P}(V)$ be a properly convex open set, let $\Gamma \subset \mathrm{Aut}(\Omega)$ be a strongly irreducible subgroup, and denote by M the quotient Ω/Γ . In order to prove Proposition 4.1 we will apply Fact 4.2 and Proposition 4.3 to suitable free sub-semi-groups of Γ . Recall that given a non-empty open subset $U \subset T^1\Omega$, we are looking for lots of biproximal periodic geodesics through U . To this end, we are going to construct, using a ping-pong argument, a strongly irreducible free sub-semi-group of Γ all of whose elements are biproximal with axis going through U (this is what suitable means here).

The first lemma gives us sufficient conditions for a family of automorphisms to generate a suitable free sub-semi-group.

Lemma 4.5. *Let $k \geq 2$ and $\gamma_1, \dots, \gamma_k \in \mathrm{PGL}(V)$ be biproximal such that:*

- (a) $\mathrm{Span}(x_{\gamma_i}^\pm, 1 \leq i \leq k) = V$,
- (b) $\bigcap_i x_{\gamma_i}^0 = 0$,
- (c) and $x_{\gamma_i}^\alpha \not\subset x_{\gamma_j}^\beta \oplus x_{\gamma_j}^0$ for any $1 \leq i \neq j \leq k$ and $\alpha, \beta \in \{\pm\}$.

Let $\Gamma = \langle \gamma_i, 1 \leq i \leq k \rangle$ be the group generated by the γ_i 's. Then the action of Γ on V is strongly irreducible.

Furthermore, if we are given a family $\{U_i^\alpha : 1 \leq i \leq k \text{ and } \alpha = \pm\}$ of disjoint compact neighbourhoods in $P(V)$ of the points $\{x_{\gamma_i}^\alpha : 1 \leq i \leq k \text{ and } \alpha = \pm\}$, such that $U_i^\alpha \cap (x_{\gamma_j}^\beta \oplus x_{\gamma_j}^0)$ and $U_i^\alpha \cap (x_{\gamma_i}^{-\alpha} \oplus x_{\gamma_i}^0)$ are empty whenever $i \neq j$ and $\alpha, \beta \in \{\pm\}$, then for any large enough integer $N > 0$, denoting by Γ' the group generated by $\gamma_1^N, \dots, \gamma_k^N$,

- (1) Γ' is strongly irreducible,
- (2) Γ' is freely generated by $\gamma_1^N, \dots, \gamma_k^N$,
- (3) every element of Γ' is biproximal,
- (4) for any non-trivial cyclically reduced word $\gamma = \gamma_{i_1}^{\alpha_1 N} \dots \gamma_{i_n}^{\alpha_n N}$ of Γ' , its attracting fixed point lies in $U_{i_1}^{\alpha_1}$ while its repelling fixed point lies in $U_{i_n}^{-\alpha_n}$,
- (5) and, denoting by Γ^+ the sub-semi-group generated by $\gamma_1^N, \dots, \gamma_k^N$, every word $\gamma \in \Gamma^+$ is cyclically reduced and its attracting fixed point lies in $\bigcup_i U_i^+$ while the repelling fixed point lies in $\bigcup_i U_i^-$.

To prove this we need a technical fact:

Fact 4.6. Consider a sequence $(g_n)_n$ in $PGL(V)$, a point x in $P(V)$ and a compact neighbourhood U of x such that the sequence $(g_n(U))_n$ converges to x . Then the accumulation points of $(g_n)_n$ in $P(\text{End}(V))$ are rank-1 projectors on x whose kernel does not intersect the interior of U .

Proof. Recall that $V = \mathbb{R}^{d+1}$, so that we can use the usual Cartan decomposition in $GL(V)$: for any n we can write $g_n = [k_n a_n l_n]$, where the elements k_n and l_n are in the (maximal compact) classical orthogonal subgroup $K = O(d+1)$ of $GL_{d+1}(\mathbb{R})$, and a_n is diagonal with positive non-increasing entries. Assume without loss of generality that $(k_n)_n$ and $(l_n)_n$ converge in K with respective limits k and l . That $(g_n)_n$ contracts an open set to a point implies that $[a_n]_n$ must converge to $[p]$, where p is the projector onto the first vector e_1 of the canonical basis. Assume without loss of generality that $(a_n)_n$ converges to p , so that $(k_n a_n l_n)_n$ converges to kpl , a rank-1 matrix whose image is $k(e_1)$ and kernel is $l^{-1} \text{Ker}(p)$. The assumption that $(k_n a_n l_n(U))_n$ tends to x implies that $(a_n l(U))_n$ goes to $k^{-1}(x)$, which in turn implies that $l(U) \cap \text{Ker}(p) = \emptyset$ and $k^{-1}x = e_1$. \square

Proof of Lemma 4.5. Let us first prove that the action of Γ on V is irreducible. Consider a non-zero subspace $W \subset V$ which is stable under Γ , and a non-zero vector $w \in W$. Using the assumption (b), we can find i such that $w \notin x_{\gamma_i}^0$, and then $\alpha = \pm$ such that $w \notin x_{\gamma_i}^{-\alpha} \oplus x_{\gamma_i}^0$, so that the sequence $(\gamma_i^{\alpha n}[w])_n$ converges to $x_{\gamma_i}^\alpha$. This means that $x_{\gamma_i}^\alpha \subset W$. Similarly for any $j \neq i$ and $\beta = \pm$, because $x_{\gamma_i}^\alpha \notin x_{\gamma_j}^{-\beta} \oplus x_{\gamma_j}^0$ (assumption (c)), we have $x_{\gamma_j}^\beta \subset W$. Since $k \geq 2$ we deduce that $x_{\gamma_i}^{-\alpha} \subset W$. By assumption (a) this means that $W = V$.

Now let $\Gamma_1 \subset \Gamma$ be a finite-index subgroup. There exists an integer $N > 0$ such that $\Gamma_2 = \langle \gamma_i^N, 1 \leq i \leq k \rangle \subset \Gamma_1$. The group Γ_2 verifies the same hypotheses than Γ , hence its action on V is irreducible, and so is that of Γ_1 : we have proved that Γ is strongly irreducible.

Let us prove the second part of the lemma. Let U be a compact set with non-empty interior, disjoint from the U_i^α 's and the $x_{\gamma_i}^\alpha \oplus x_{\gamma_i}^0$'s. Let N be large enough so that for any $i = 1, \dots, k$ and $\alpha = \pm$,

$$\gamma_i^{\alpha N}(U \cup U_i^\alpha \cup \bigcup_{\substack{j \neq i \\ \beta = \pm}} U_j^\beta) \subset U_i^\alpha.$$

Then $\Gamma' = \langle \gamma_i^N, 1 \leq i \leq k \rangle$ is free thanks to a ping-pong argument: prove by induction on length that for any non-trivial reduced word:

$$\gamma_{i_1}^{\alpha_1 N} \dots \gamma_{i_p}^{\alpha_p N}(U \cup U_{i_p}^{\alpha_p} \cup \bigcup_{\substack{j \neq i_p \\ \beta = \pm}} U_j^\beta) \subset U_{i_1}^{\alpha_1}.$$

Assume for the rest of the proof that this N is actually 1.

Now if by contradiction we could not find N large enough to have the conclusions (3) and (4) of the lemma, there would exist a sequence of cyclically reduced words

$$\gamma_n = \gamma_{i_1^{(n)}}^{\alpha_1^{(n)N_n}} \gamma_{i_1^{(n)}}^{\alpha_1^{(n)}} \gamma_{i_2^{(n)}}^{\alpha_2^{(n)}} \cdots \gamma_{i_{p_n}^{(n)}}^{\alpha_{p_n}^{(n)}} \gamma_{i_{p_n}^{(n)}}^{\alpha_{p_n}^{(n)N_n}}$$

with $(N_n)_n$ going to infinity and $(i_1^{(n)}, \alpha_1^{(n)}) \neq (i_{p_n}^{(n)}, -\alpha_{p_n}^{(n)})$ for every n , and which are not biproximal with attracting/repelling pair in $U_{i_1^{(n)}}^{\alpha_1^{(n)}} \times U_{i_{p_n}^{(n)}}^{-\alpha_{p_n}^{(n)}}$. Up to extracting assume that $i_1^{(n)} = i$, $i_{p_n}^{(n)} = j$, $\alpha_1^{(n)} = \alpha$ and $\alpha_{p_n}^{(n)} = \beta$ do not depend on n .

Finally see that the sequences $(\gamma_n)_n$ and $(\gamma_n^{-1})_n$ respectively contract U_i^α into $x_{\gamma_i}^\alpha$ and $U_j^{-\beta}$ into $x_{\gamma_j}^{-\beta}$, and apply Fact 4.6 to them. Up to passing to a subsequence we can assume that $(\gamma_n)_n$ and $(\gamma_n^{-1})_n$ converge to rank-1 projectors on respectively $x_{\gamma_i}^\alpha$ and on $x_{\gamma_j}^{-\beta}$. Finally, by Remark 2.4, γ_n and γ_n^{-1} are proximal with attracting fixed points respectively in U_i^α and $U_j^{-\beta}$: this is a contradiction. \square

The next proposition uses the strong irreducibility of our group of automorphisms to find a family of automorphisms which verifies the conditions of Lemma 4.5, hence which generates a suitable free sub-semi-group.

Proposition 4.7. *Let $\Omega \subset \mathbb{P}(V)$ be a properly convex open set and $\Gamma \subset \text{GL}(V)$ be a strongly irreducible discrete subgroup preserving Ω . Denote by M the quotient Ω/Γ . Assume that $T^1M_{\text{bip}} \neq \emptyset$. Let $U \subset T^1\Omega$ be a non-empty open subset whose projection in T^1M intersects T^1M_{bip} . Then we can construct a discrete subgroup $\Gamma' \subset \Gamma$ generated as a group by a sub-semi-group Γ^+ such that:*

- Γ' acts strongly irreducibly on V ,
- Γ' is free,
- every element of Γ' is biproximal,
- for every γ in Γ' , $\text{axis}(\gamma) \cap \Omega \neq \emptyset$,
- and for every γ in Γ^+ , $\text{axis}(\gamma) \cap U \neq \emptyset$.

Proof. By Corollary 2.9 we can consider a biproximal automorphism $\gamma \in \Gamma$ whose axis intersect U . We are going to construct $\gamma_1, \dots, \gamma_k \in \Gamma$ which verify the assumptions of Lemma 4.5 (with $\gamma_1 = \gamma$) and such that $\text{axis}(\gamma_i) \cap \Omega \neq \emptyset$ for any i . Then for all $(i, \alpha) \neq (j, \beta) \in \{1, \dots, k\} \times \{\pm\}$, the fact that $x_{\gamma_i}^\alpha \notin x_{\gamma_j}^\beta \oplus x_{\gamma_i}^0$ implies that $(x_{\gamma_i}^\alpha \oplus x_{\gamma_j}^\beta) \cap \Omega \neq \emptyset$, because $x_{\gamma_j}^\beta$ is a smooth point of the boundary (see Lemma 3.1) and $x_{\gamma_j}^\beta \oplus x_{\gamma_j}^0$ is its tangent hyperplane. At this point we can consider neighbourhoods U_i^α 's of the $x_{\gamma_i}^\alpha$'s small enough to satisfy the hypotheses of Lemma 4.5 and such that, for all $(i, \alpha) \neq (j, \beta)$ and for each pair (ξ, η) in $U_i^\alpha \times U_j^\beta$, the intersection $(\xi \oplus \eta) \cap \Omega$ is non-empty, and such that for each pair (ξ, η) in $U_1^+ \times U_1^-$, the intersection $(\xi \oplus \eta) \cap U$ is non-empty. Then by Lemma 4.5, for N large enough, considering the sub-semi-group generated by $\gamma_1^N \gamma_i^N \gamma_1^N$, $i = 1, \dots, k$, is a possible way to conclude the proof of Proposition 4.7.

We are going to construct the γ_i 's inductively, taking conjugates of γ . First set $\gamma_1 = \gamma$. Let $\gamma_1, \dots, \gamma_k$ be constructed such that for any $1 \leq i \neq j \leq k$ and $\alpha, \beta \in \{\pm\}$, $x_{\gamma_i}^\alpha \notin x_{\gamma_j}^\beta \oplus x_{\gamma_i}^0$.

If $\text{Span}(x_{\gamma_i}^\pm) = V$ and $\bigcap_i x_{\gamma_i}^0 = 0$ then we are done.

Otherwise, we are looking for $\gamma_{k+1} = g\gamma g^{-1}$ where $g \in \Gamma$ is such that:

- for all $i \leq k$ and α, β in $\{\pm\}$,

$$x_{g\gamma g^{-1}}^\alpha = g(x_\gamma^\alpha) \notin x_{\gamma_i}^\beta \oplus x_{\gamma_i}^0,$$

- for all $i \leq k$ and α, β in $\{\pm\}$,

$$x_{\gamma_j}^\beta \notin g(x_\gamma^\alpha \oplus x_\gamma^0),$$

- if $\text{Span}(x_{\gamma_i}^\alpha, i \leq k, \alpha = \pm) \neq V$ then

$$gx_\gamma^\pm \notin \text{Span}(x_{\gamma_i}^\alpha, i \leq k, \alpha = \pm),$$

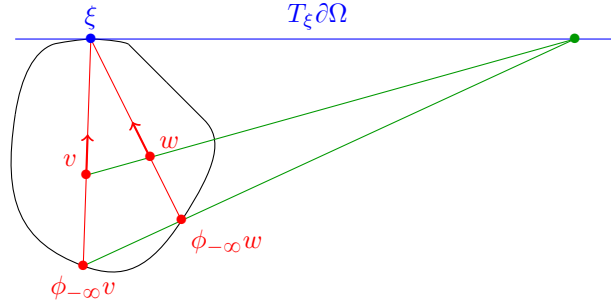


Figure 2: v and w in the same strong stable manifold

- and if $\bigcap_{i \leq k} x_{\gamma_i}^0 \neq 0$ then

$$\bigcap_{i \leq k} x_{\gamma_i}^0 \not\subset g(x_\gamma^0).$$

Call Γ_0 the irreducible component of the identity in Γ for the Zariski topology. Then it is well known (see e.g. [BQ16, Lem. 6.21]) that Γ_0 is a normal finite-index subgroup of Γ . Hence its action on V is also strongly irreducible. Note that each condition above is Zariski-open with respect to $g \in \Gamma_0$, and non-empty because of irreducibility of the action of Γ_0 . Since Γ_0 is Zariski-irreducible, it implies that the (finite) intersection of all conditions is still non-empty, and we can thus construct γ_{k+1} . Notice that the two last conditions ensure that the process will eventually stop. \square

4.3 Proof of Proposition 4.1

Up to taking a finite-index subgroup we can assume that $\Gamma \subset \mathrm{PSL}(V)$, and consider its (full) preimage $\tilde{\Gamma} \subset \mathrm{SL}(V)$. Let $\tilde{U} \subset T^1\Omega$ be the preimage of U . Proposition 4.7 gives a strongly irreducible subgroup $\Gamma' \subset \tilde{\Gamma}$ generated by a sub-semi-group $\Gamma^+ \subset \Gamma'$. By Fact 4.2, the Zariski-closure of Γ^+ (which is also that of Γ') is semi-simple and non-compact. By Proposition 4.3, the additive group generated by $\{\ell(\gamma) : \gamma \in \Gamma^+\}$, is dense in \mathbb{R} . But by Proposition 4.7, each element γ of Γ^+ is biproximal with axis *through* \tilde{U} ; by definition this axis projects on a biproximal periodic geodesic *through* U , whose length is $\ell(\gamma)$.

5 Strong stable manifolds

The *strong stable manifold* of a vector $v \in T^1\Omega$ is a classical notion in the theory of dynamical systems; it is defined as the set of vectors $w \in T^1\Omega$ such that $d_{T^1\Omega}(\phi_t v, \phi_t w)$ goes to zero as t goes to infinity. The goal of this section is to establish the following geometric description of the strong stable manifolds centred at smooth points.

Proposition 5.1. *Let $\Omega \subset \mathbb{P}(V)$ be a properly convex open set, let $v \in T^1\Omega$ and let $\xi := \phi_\infty v \in \partial\Omega$. Then*

1. *For any $w \in T^1\Omega$ such that $\phi_\infty w = \xi$, the function $t \mapsto d_{T^1\Omega}(\phi_t v, \phi_t w)$ is non-increasing.*
2. *Suppose ξ is smooth. Then for any $w \in T^1\Omega$ with $\phi_\infty w = \xi$, there is a unique time $t_0 \in \mathbb{R}$ for which the lines $\phi_{-\infty} v \oplus \phi_{-\infty} w$ and $\pi v \oplus \pi \phi_{t_0} w$ intersect on $T_\xi \partial\Omega$ (see Figure 2); moreover t_0 is the unique time for which v and $\phi_{t_0} w$ are on the same strong stable manifold, i.e.*

$$d_{T^1\Omega}(\phi_t v, \phi_{t+t_0} w) \xrightarrow[t \rightarrow \infty]{} 0.$$

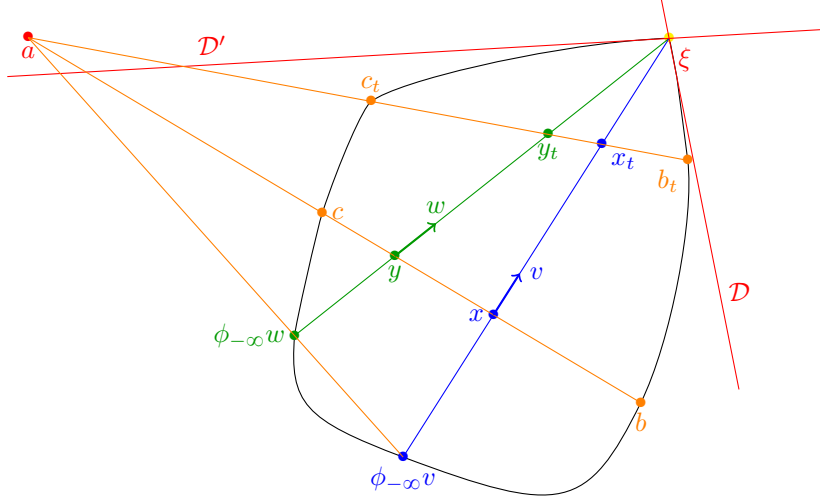


Figure 3: Illustration for the notations in Proposition 5.3

5.1 Crampon's Lemma

Let us first state a useful lemma about convex projective geometry, which in particular implies the first part of Proposition 5.1. We will give a proof of Lemma 5.2 in the appendix, to clarify a missing detail in the original proof.

Lemma 5.2 ([Cra09, Lem. 8.3]). *Let Ω be a properly convex open subset of $\mathbb{P}(V)$. Let c_1 and c_2 be two straight geodesics parametrised with constant speed, but not necessarily with the same speed. Then for all $0 \leq t \leq T$,*

$$d_{\Omega}(c_1(t), c_2(t)) \leq d_{\Omega}(c_1(0), c_2(0)) + d_{\Omega}(c_1(T), c_2(T)).$$

5.2 An explicit computation of $\lim_{t \rightarrow \infty} d_{T^1\Omega}(\phi_t v, \phi_t w)$

We now prove a proposition from which the geometric description of strong stable manifolds will be a corollary.

Proposition 5.3. *Let $\Omega \subset \mathbb{P}(V)$ be a properly convex open set. Take $v \neq w$ in $T^1\Omega$ both pointing at $\xi \in \partial\Omega$, let a be the intersection point of $\pi v \oplus \pi w$ and $\phi_{-\infty} v \oplus \phi_{-\infty} w$, and suppose that the projective line $a \oplus \xi$ does not intersect Ω . Let P be the projective plane spanned by $x = \pi v$, $y = \pi w$, and ξ , let \mathcal{D} and \mathcal{D}' be the two lines of P starting at ξ , tangent to $\partial\Omega$, such that the four lines \mathcal{D} , $(\xi \oplus x)$, $(\xi \oplus y)$, \mathcal{D}' lie in this order around ξ , and let $\delta > 0$ be half the logarithm of the cross-ratio of these four lines (see Figure 3). Then:*

$$d_{\Omega}(\pi\phi_t v, \pi\phi_t w) \xrightarrow[t \rightarrow \infty]{} \delta \text{ and } d_{T^1\Omega}(\phi_t v, \phi_t w) \xrightarrow[t \rightarrow \infty]{} \delta.$$

Proof. We consider $x_t = \pi\phi_t v$ and $y_t = \pi\phi_t w$. Since $d_{\Omega}(x, x_t) = t = d_{\Omega}(y, y_t)$ and by definition of the cross-ratio, we see that $y_t \in (y \oplus \xi) \cap (a \oplus x_t)$. Let b_t and $c_t \in \partial\Omega$ be such that the four points b_t, x_t, y_t, c_t are aligned in this order. We consider $\mathcal{D}_t = (\xi \oplus b_t)$ and $\mathcal{D}'_t = (\xi \oplus c_t)$. By definition of the tangent lines, the two sequences $(\mathcal{D}_t)_{t \rightarrow \infty}$ and $(\mathcal{D}'_t)_{t \rightarrow \infty}$ converge respectively to \mathcal{D} and \mathcal{D}' . By definition $d_{\Omega}(x_t, y_t)$ is half the logarithm of the cross-ratio of the four lines $\mathcal{D}_t, (\xi \oplus x), (\xi \oplus y), \mathcal{D}'_t$, which converges to the cross-ratio of the four lines $\mathcal{D}, (\xi \oplus x), (\xi \oplus y), \mathcal{D}'$. \square

5.3 Proof of Proposition 5.1

By definition of $d_{T^1\Omega}$, in order to prove that $t \mapsto d_{T^1\Omega}(\phi_t v, \phi_t w)$ is non-increasing, it is enough to prove that $t \mapsto d_{\Omega}(\pi\phi_t v, \pi\phi_t w)$ is non-increasing. Observe that it will also have as a consequence $d_{T^1\Omega}(v, w) \leq d_{\Omega}(\pi v, \pi w)$. We fix $t \geq 0$. We consider a sequence $(x_n)_{n \in \mathbb{N}} \in \Omega^{\mathbb{N}}$ converging to ξ ,

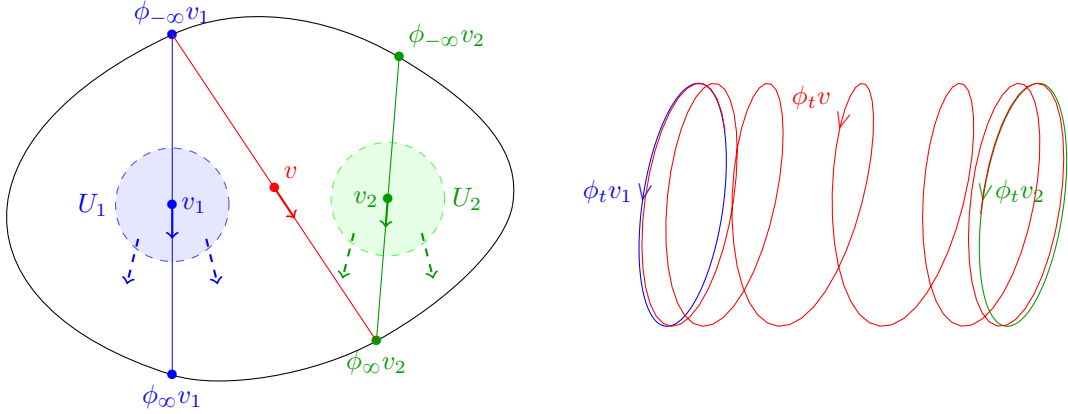


Figure 4: Proof of mixing. On the left: In Ω . On the right: In the quotient $M = \Omega/\Gamma$.

and the sequences of vectors $(v_n)_{n \in \mathbb{N}}$ and $(w_n)_{n \in \mathbb{N}}$ in $T^1\Omega$ such that $\pi v_n = \pi v$, while $\pi w_n = \pi w$ and $\pi \phi_{d_\Omega(\pi v, x_n)} v_n$ and $\pi \phi_{d_\Omega(\pi w, x_n)} w_n$ are equal to x_n for all n . It is then easy to see that $\phi_t(v) = \lim_{n \rightarrow \infty} \phi_t(v_n)$ and that $\phi_t(w) = \lim_{n \rightarrow \infty} \phi_t(w_n)$. By Lemma 5.2, we obtain

$$d_\Omega(\pi \phi_t v_n, \pi \phi_{t \frac{d_\Omega(\pi w, x_n)}{d_\Omega(\pi v, x_n)}} w_n) \leq d_\Omega(\pi v, \pi w),$$

and from this we let n go to infinity, and get the desired inequality since $\frac{d_\Omega(\pi w, x_n)}{d_\Omega(\pi v, x_n)}$ converges to 1.

If ξ is a smooth point of $\partial\Omega$ and the lines $\phi_{-\infty} v \oplus \phi_{-\infty} w$ and $\pi v \oplus \pi w$ intersect on $T_\xi \partial\Omega$, then the fact that $d_{T^1\Omega}(\phi_t v, \phi_t w)$ goes to zero as t goes to infinity, is an immediate corollary of Proposition 5.3.

6 Proof of Theorem 1.2

Let U_1 and U_2 be two open subsets of $T^1 M_{bip}$. Let us prove that there exists $T > 0$ such that $\phi_t(U_1) \cap U_2 \neq \emptyset$ for all $t \geq T$.

Since the map $(t, v) \mapsto \phi_t(v)$ is continuous, we can find an open subset $\emptyset \neq U'_2 \subset U_2$, and $\epsilon > 0$ such that $\phi_{[-\epsilon, \epsilon]}(U'_2) \subset U_2$. As a consequence, for any time $t \in \mathbb{R}$, if $\phi_t(U_1)$ and U'_2 intersect, then for all s which are ϵ -close to t , the sets $\phi_s(U_1)$ and U_2 intersect.

Let $\pi_\Gamma : T^1\Omega \rightarrow T^1M$ be the natural projection. Let us find small non-empty open subsets $\tilde{V}_1 \subset \pi_\Gamma^{-1}U_1$ and $\tilde{V}_2 \subset \pi_\Gamma^{-1}U'_2$ such that for any $(\tilde{u}_1, \tilde{u}_2) \in \tilde{V}_1 \times \tilde{V}_2$, the line $(\phi_{-\infty} \tilde{u}_1 \oplus \phi_{\infty} \tilde{u}_2)$ intersects Ω . We consider $\tilde{u}_1 \in \pi_\Gamma^{-1}(U_1)$ and $\tilde{u}_2 \in \pi_\Gamma^{-1}(U'_2)$ biproximal periodic, they exist thanks to Proposition 4.1. By Lemma 3.1, $\phi_{\pm\infty} \tilde{u}_i$, where $i = 1, 2$, are smooth. By irreducibility of Γ there is an automorphism $\gamma \in \Gamma$ such that $\gamma T_{\phi_{\infty} \tilde{u}_2} \partial\Omega \neq T_{\phi_{-\infty} \tilde{u}_1} \partial\Omega$. This implies that $\phi_{-\infty} \tilde{u}_1 \oplus \gamma \phi_{\infty} \tilde{u}_2$ intersect Ω , and since this condition is open, as well as $\pi_\Gamma^{-1}U_1$ and $\pi_\Gamma^{-1}U'_2$, we can take for \tilde{V}_1 and \tilde{V}_2 small neighbourhoods of \tilde{u}_1 and $\gamma \tilde{u}_2$.

For $i = 1, 2$, pick $\tilde{v}_i \in \tilde{V}_i$ biproximal with period τ_i such that $\tau_1 \mathbb{Z} + \tau_2 \mathbb{Z}$ is $\frac{\epsilon}{2}$ -dense in \mathbb{R} ; this is possible thanks to Proposition 4.1 and Observation 6.1 below.

We know that $(\phi_{-\infty} \tilde{v}_1, \phi_{\infty} \tilde{v}_2) \subset \Omega$. Let us consider $\tilde{w} \in T^1\Omega$ tangent to this line: it is pointing forward at $\phi_{\infty} \tilde{v}_2$ and backward at $\phi_{-\infty} \tilde{v}_1$. By Lemma 3.1, the points $\phi_{\pm\infty} \tilde{v}_i$ are smooth. Using Proposition 5.1, we can find t_1 and $t_2 \in \mathbb{R}$ which verify $\lim_{t \rightarrow \infty} d_{T^1\Omega/\Gamma}(\phi_{-t} v_1, \phi_{t_1-t} w) = 0$ and $\lim_{t \rightarrow \infty} d_{T^1\Omega/\Gamma}(\phi_t v_2, \phi_{t_2+t} w) = 0$, where for $i = 1, 2$, $v_i = \pi_\Gamma \tilde{v}_i$, and $w = \pi_\Gamma \tilde{w}$. This implies that there is an integer $N \geq 0$ such that for any $n \geq N$, $\phi_{t_1-n\tau_1} w \in V_1 := \pi_\Gamma \tilde{V}_1$ and $\phi_{t_2+n\tau_2} w \in V_2 := \pi_\Gamma \tilde{V}_2$. We deduce that $\phi_t(V_1) \cap V_2 \neq \emptyset$ for any t in $\{-t_1 + t_2 + n_1\tau_1 + n_2\tau_2 : n_1, n_2 \geq N\}$.

We conclude by observing that the ϵ -neighbourhood of $\{-t_1 + t_2 + n_1\tau_1 + n_2\tau_2 : n_1, n_2 \geq N\}$ contains a real interval of the form $[T, \infty)$ for T large enough. Indeed, there exists $N' > 0$ such that $[0, \tau_1]$ is contained in the ϵ -neighbourhood of $\{n_1\tau_1 + n_2\tau_2 : |n_i| \leq N'\}$. Then the ϵ -neighbourhood of $\{-t_1 + t_2 + n_1\tau_1 + n_2\tau_2 : n_1, n_2 \geq N\}$ contains $[-t_1 + t_2 + (N + N')\tau_1 + (N + N')\tau_2, \infty[$. This ends the proof of Theorem 1.2.

Observation 6.1. *Let A be a subset of \mathbb{R} which generates a dense additive subgroup G of \mathbb{R} . Let $x, \epsilon > 0$. Then there is $g \in A$ such that $x\mathbb{Z} + g\mathbb{Z}$ is ϵ -dense in \mathbb{R} (i.e. any point in \mathbb{R} is at distance at most ϵ from $x\mathbb{Z} + g\mathbb{Z}$).*

Proof. Up to replacing A by A/x and ϵ by ϵ/x , we can assume that $x = 1$. Then there are two possibilities.

- The set A contains an irrational number g . Then $\mathbb{Z} + g\mathbb{Z}$ is dense in \mathbb{R} .
- The set A is contained in \mathbb{Q} . Let $q_0 \in \mathbb{N}^*$ be such that $\frac{1}{q_0} < \epsilon$. The subgroup $\frac{1}{q_0!}\mathbb{Z}$ is not dense in \mathbb{R} , so A must contain an element outside of it, of the form $\frac{p}{q}$ with p and q coprime and $q > q_0$. The group $\mathbb{Z} + \frac{p}{q}\mathbb{Z} = \frac{1}{q}\mathbb{Z}$ is ϵ -dense in \mathbb{R} . \square

7 The geodesic flow in the higher-rank compact case

The goal of this section is to prove Proposition 1.4. We are actually going to prove a finer statement: that the connected components of the non-wandering set of the geodesic flow are quotients of homogeneous spaces whose Haar measure is mixing.

We denote by \mathbb{H} the classical division algebra of quaternions, and by \mathbb{O} the classical non-associative division algebra of octonions. Fix an integer $N \geq 3$ and the algebra $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, or, if $N = 3$, \mathbb{O} . We shall use the following notation. In the case $\mathbb{K} = \mathbb{R}$, conjugation is the identity and we abusively say Hermitian instead of symmetric.

- For $x \in \mathbb{K}$, the element $\bar{x} \in \mathbb{K}$ is the conjugate of x .
- We consider the Hermitian bilinear form on \mathbb{K}^N given by $\langle x, y \rangle = \sum_{i=1}^N \bar{x}_i y_i$.
- The real vector space $V = V_{N, \mathbb{K}}$ consists of the Hermitian matrices of size N with entries in \mathbb{K} .
- The cone $C = C_{N, \mathbb{K}} \subset V$ consists of the positive-definite Hermitian matrices.
- The properly convex open set $\Omega = \Omega_{N, \mathbb{K}} \subset P(V)$ is the projectivisation of C .
- The group $\text{Aut}(C) \subset \text{GL}(V)$ consists of the transformations preserving C .
- The group $G = G_{N, \mathbb{K}} := \text{Aut}(\Omega) = \text{Aut}(C)/\mathbb{R}^*$, where \mathbb{R}^* is seen as the group of homotheties of $\text{GL}(V)$, is the automorphism group of Ω .
- The group $K \subset \text{Aut}(C)$ is the stabiliser of the identity matrix; note that the map $K \rightarrow G$ is an embedding, and that K is a maximal compact subgroup of G . In particular it means that Ω identifies as a G -space with the Riemannian symmetric space of the simple real Lie group G .
- Finally the group A consists of the diagonal matrices of size N with entries in $\mathbb{R}_{>0}$; we see it embedded in $\text{Aut}(C)$, acting on V by the following formula: $a \cdot X = aXa$ for $a \in A$ and $X \in V$.

Let us be more explicit about the case $\mathbb{K} = \mathbb{R}$. The group $\text{Aut}(C_{N, \mathbb{R}})$ can be identified with $\text{GL}_N(\mathbb{R})/\{\pm 1\}$, acting on $V_{N, \mathbb{R}}$ by the formula $g \cdot X = gX^t g$; the group $G_{N, \mathbb{R}}$ is identified with $\text{PGL}_N(\mathbb{R})$; the group K is identified with $\text{O}(N)/\{\pm 1\}$.

We come back to the general case. The spectral theorem (see [FK94, Th. V.2.5]) ensures that for every $X \in V$ there exists $k \in K$ such that $k \cdot X$ is diagonal with real entries. This, using the action of A , has two consequences: $\text{Aut}(C)$ acts transitively on C , and can be written as the product $KAK = \{k_1 a k_2 : k_1, k_2 \in K, a \in A\}$. Then, the quotient group G acts transitively on Ω , and can be written $K(A/\mathbb{R}_{>0})K$ — actually, the element of A/\mathbb{R}^* in the decomposition can be taken with non-increasing entries on the diagonal, and this yields the *Cartan decomposition* of G . The Lie algebra of G is $\mathfrak{sl}(N, \mathbb{K})$ when $\mathbb{K} \neq \mathbb{O}$, and $\mathfrak{e}_{6(-26)}$ if $\mathbb{K} = \mathbb{O}$ (see [FK94, p. 97]), therefore G is a real simple Lie group, with finitely many connected components, and with finite center.

Since $\Omega = G/K$, a discrete subgroup $\Gamma \subset G$ acts cocompactly on Ω if and only if G/Γ is compact, i.e. Γ is a uniform lattice of G ; uniform lattices exist by a theorem of Borel [Bor63, Th. C]. The family of properly convex open sets $\Omega_{N, \mathbb{K}}$ are called the *symmetric divisible convex sets*. Zimmer [Zim, Th. 1.4] recently proved that the higher-rank closed convex projective manifolds are exactly the quotients of the form $\Omega_{N, \mathbb{K}}/\Gamma$, where $N \geq 3$, the field $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, or \mathbb{O} (if $N = 3$), and Γ is a uniform lattice of $G_{N, \mathbb{K}}$.

7.1 The non-wandering set of $G \times \mathbb{R}$ on $T^1\Omega$

In this section we describe $\text{NW}(T^1\Omega, G \times \mathbb{R})$ and prove that G acts transitively on each of its connected components.

The boundary of Ω is the projectivisation of the cone of positive-semi-definite Hermitian matrices. For $1 \leq i, j \leq N-1$, we denote by $T^1\Omega_{i,j}$ the set of unit tangent vectors $v \in T^1\Omega$ such that the respective ranks of $\phi_{-\infty}v$ and $\phi_{\infty}v$ (meaning the rank of any representative in V) are i and j . Note that $T^1\Omega_{i,j}$ is non-empty if and only if $i + j \geq N$ (see Proposition 7.1). The subsets $T^1\Omega_{i,j}$, for $1 \leq i, j \leq N$ and $i + j \geq N$, are invariant under the automorphism group $\text{Aut}(\Omega)$ and the geodesic flow $(\phi_t)_{t \in \mathbb{R}}$. They stratify $T^1\Omega$ in the following way:

- $T^1\Omega$ is the disjoint union of the $T^1\Omega_{i,j}$,
- the closure of $T^1\Omega_{i,j}$ is the union of the $T^1\Omega_{k,l}$ for $1 \leq k \leq i$ and $1 \leq l \leq j$,
- in particular, $T^1\Omega_{i,N-i}$ is closed for $1 \leq i \leq N-1$,
- $T^1\Omega_{N-1,N-1}$ is open and dense in $T^1\Omega$.

When $\mathbb{K} = \mathbb{R}$ we compute $\dim(T^1\Omega_{i,j}) = i(N-i) + \frac{i(i+1)}{2} + j(N-j) + \frac{j(j+1)}{2} - 1$.

We denote by $\text{Geod}(\Omega)_{i,j}$ the quotient $T^1\Omega_{i,j}/(\phi_t)_{t \in \mathbb{R}}$; observe that $\text{Geod}(\Omega) := T^1\Omega/(\phi_t)_{t \in \mathbb{R}}$ identifies with the set of pairs (x, y) in $\partial\Omega^2$ such that $\text{Ker}(x) \cap \text{Ker}(y) = \emptyset$. We are going to prove that $\text{NW}(\text{Geod}(\Omega), G)$ is the union $\bigcup_{1 \leq i \leq N-1} \text{Geod}(\Omega)_{i,N-i}$. This exactly means, according to Section 2.6, that $\text{NW}(T^1\Omega, G \times \mathbb{R})$ is $\bigcup_{1 \leq i \leq N-1} T^1\Omega_{i,N-i}$. We choose a basepoint $v_{i,N-i} \in T^1\Omega_{i,N-i}$, such that $\pi v_{i,N-i}$, $\phi_{-\infty}v_{i,N-i}$ and $\phi_{\infty}v_{i,N-i}$ are the projectivisations of, respectively, the identity matrix, the orthogonal projection onto $\mathbb{K}^i \times \{0\}$ and the orthogonal projection onto $\{0\} \times \mathbb{K}^{N-i}$. We set

$$A_{i,N-i} := \left\{ a_t := \begin{bmatrix} e^{t/2}I_i & 0 \\ 0 & e^{-t/2}I_j \end{bmatrix} : t \in \mathbb{R} \right\} \subset A,$$

where I_k is the identity matrix of size k , and we observe that for any time $t \in \mathbb{R}$, the image $a_t \cdot v_{i,N-i}$ is exactly $\phi_t v_{i,N-i}$. We denote by G_0 the identity component of G and by $K_{i,N-i}$ the stabilizer in G_0 of $v_{i,N-i}$; they are normalised by $A_{i,N-i} \subset G_0$.

Proposition 7.1. *Consider $N \geq 3$, the algebra $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, or \mathbb{O} (if $N = 3$), the vector space $V = V_{N,\mathbb{K}}$, the properly convex open set $\Omega = \Omega_{N,\mathbb{K}}$, and the group $G = G_{N,\mathbb{K}}$, with identity component G_0 .*

- (1) For $1 \leq i, j \leq N-1$, the set $T^1\Omega_{i,j}$ is non-empty if and only if $i + j \geq N$.
- (2) For $1 \leq i \leq N-1$, the group G_0 acts transitively on $T^1\Omega_{i,N-i}$. If we identify $T^1\Omega_{i,N-i}$ with $G_0/K_{i,N-i}$, then the geodesic flow identifies with the action by right multiplication of $A_{i,N-i}$ on $G_0/K_{i,N-i}$.
- (3) The non-wandering set of G on $\text{Geod}(\Omega)$ is

$$\text{NW}(\text{Geod}(\Omega), G) = \bigcup_{1 \leq i \leq N-1} \text{Geod}(\Omega)_{i,N-i}.$$

Proof. (1) Suppose there exists $v \in T^1\Omega_{i,j}$. Because G_0 acts transitively on Ω we can find $g_1 \in G_0$ such that $g_1\pi v$ is the projectivisation of the identity matrix. Then by the spectral theorem there exists an automorphism $g_2 \in K$ (i.e. fixing $g_1\pi v$) such that $\text{Ker}(g_2g_1\phi_{-\infty}v) = \{0\} \times \mathbb{K}^{N-i}$; since the space of $(N-i)$ -dimensional right \mathbb{K} -sub-modules of V is connected, we can take g_2 in G_0 . We note that the subspaces $\text{Ker}(g_2g_1\phi_{-\infty}v)$ and $\text{Ker}(g_2g_1\phi_{\infty}v)$ are orthogonal. (Indeed, if T and T' are representatives in V of $g_2g_1\phi_{-\infty}v$ and $g_2g_1\phi_{\infty}v$ such that $T + T'$ is the identity matrix, and if $x \in \text{Ker}(T)$ while $y \in \text{Ker}(T')$, then we compute

$$\langle x, y \rangle = \langle x, Ty + T'y \rangle = \langle x, Ty \rangle = \langle Tx, y \rangle = 0.)$$

This implies that $i + j \geq N$.

- (2) Let $1 \leq i \leq N-1$, let us show that there exists $g \in G_0$ such that $g \cdot v$ is the basepoint $v_{i,N-i}$ of $T^1\Omega_{i,N-i}$. We have already seen that there are $g_1, g_2 \in G$ such that $\pi g_2 g_1 v$ is the projectivisation of the identity matrix, and $\text{Ker}(g_2 g_1 \phi_{-\infty} v) = \{0\} \times \mathbb{K}^{N-i}$. Then $\text{Ker}(g_2 g_1 \phi_{\infty} v) = \mathbb{K}^i \times \{0\}$, since $\text{Ker}(g_2 g_1 \phi_{-\infty} v)$ and $\text{Ker}(g_2 g_1 \phi_{\infty} v)$ are orthogonal. Moreover $g_2 g_1 \phi_{-\infty} v$ and $g_2 g_1 \phi_{\infty} v$ are the projectivisations of the orthogonal projections onto $\mathbb{K}^i \times \{0\}$ and $\{0\} \times \mathbb{K}^{N-i}$ (Indeed, consider representatives T and T' of $\phi_{-\infty} g_2 g_1 v$ and $\phi_{\infty} g_2 g_1 v$ in V such that $T + T'$ is the identity matrix, then T' and T are the orthogonal projections onto $\mathbb{K}^i \times \{0\}$ and $\{0\} \times \mathbb{K}^{N-i}$).
- (3) The stabiliser of $(\phi_{-\infty} v_{i,N-i}, \phi_{\infty} v_{i,N-i}) \in \text{Geod}(\Omega)_{i,N-i}$ contains $A_{i,N-i}$, therefore the stabilisers of points in $\text{Geod}(\Omega)_{i,N-i}$ are non-compact, hence $\bigcup_{1 \leq i \leq N-1} \text{Geod}(\Omega)_{i,N-i}$ is contained in $\text{NW}(\text{Geod}(\Omega), G)$. As a consequence, in order to prove (2), we only need to prove the other inclusion.

By contradiction, we may assume the existence of sequences of positive semi-definite Hermitian matrices $(S_n)_{n \in \mathbb{N}}, (T_n)_{n \in \mathbb{N}}$ in V , of automorphisms $(g_n)_{n \in \mathbb{N}}$ in $\text{Aut}(C)$ and of positive scalars $(\lambda_n)_{n \in \mathbb{N}}, (\mu_n)_{n \in \mathbb{N}}$, such that

- $(S_n)_{n \in \mathbb{N}}, (\lambda_n g_n S_n)_{n \in \mathbb{N}}, (T_n)_{n \in \mathbb{N}}$, and $(\mu_n g_n T_n)_{n \in \mathbb{N}}$ respectively converge to S, S', T , and T' ,
- the rank of S and S' is i , the rank of T and T' is i , with $1 \leq i, j \leq N-1$ and $i+j > N$,
- $\text{Ker}(S) \cap \text{Ker}(T) = \text{Ker}(S') \cap \text{Ker}(T') = 0$,
- $[g_n] \in G$ leaves every compact subset.

Using $\text{Aut}(C) = KAK$ and extracting, we may assume that $g_n = a_n \in A$ converge to a non-invertible non-zero matrix $a \in A$.

Since $\text{Ker}(S) \cap \text{Ker}(T) = 0$, up to exchanging S and T , we can assume that $aSa \neq 0$, and that $aSa = S'$, which means a has rank greater or equal than i . Since $i+j > N$, necessarily $aTa \neq 0$, and we can assume that $aTa = T'$. But now the kernel of a is contained in $\text{Ker}(S') \cap \text{Ker}(T') = 0$, this is a contradiction. \square

7.2 The non-wandering set of $(\phi_t)_{t \in \mathbb{R}}$ on T^1M

Let Γ be a lattice of G , not necessarily uniform. We set $M = \Omega/\Gamma$. For $1 \leq i, j \leq N-1$, we denote by $T^1M_{i,j}$ the quotient $T^1\Omega_{i,j}/\Gamma$.

Remark 7.2. The biproximal unit tangent bundle T^1M_{bip} is empty. To see this, recall that the attracting fixed point of a proximal automorphism of Ω is always an extremal point of Ω , so the proximal limit set of Γ is contained in the closure of the set of extremal points of Ω .

Here, since Ω is symmetric, the set of extremal points is closed and consists of projectivisations of rank-1 positive-semi-definite Hermitian matrices, so the set of straight geodesics between to extremal points is $\text{Geod}(\Omega)_{1,1}$, which is empty, since $N \geq 3$.

In this section we use the Howe–Moore theorem to study the action of the geodesic flow on each $T^1M_{i,N-i}$, with $1 \leq i \leq N$. Proposition 7.1 and Section 2.6 imply that the non-wandering sets $\text{NW}(\text{Geod}(\Omega), \Gamma)$, $\text{NW}(T^1\Omega, \Gamma \times \mathbb{R})$, and $\text{NW}(T^1M, (\phi_t)_{t \in \mathbb{R}})$ are respectively contained in the unions $\bigcup_{1 \leq i \leq N-1} \text{Geod}(\Omega)_{i,N-i}$, $\bigcup_{1 \leq i \leq N-1} T^1\Omega_{i,N-i}$ and $\bigcup_{1 \leq i \leq N-1} T^1M_{i,N-i}$. We are now going to see that we actually have equalities. Recall that a finite measure μ preserved by a measurable flow $(\phi_t)_{t \in \mathbb{R}}$ is called *mixing* if, for any two functions $f, g \in L^2(\mu)$ with zero integral, we have

$$\int f \cdot (g \circ \phi_t) d\mu \xrightarrow{t \rightarrow \infty} 0.$$

Recall also that a continuous flow is topologically mixing on the support of a mixing invariant measure, therefore Proposition 1.4 is an immediate consequence of Proposition 7.4 below, and of Zimmer's rigidity theorem [Zim, Th. 1.4].

Fact 7.3 ([HM79], see e.g. [Zim84, Th. 2.2.20]). *Let G be a connected non-compact simple Lie group with finite center, let π be a unitary representation of G in a separable Hilbert space, without any non-zero G -invariant vector. Let x, y be two vectors in the Hilbert space. Then*

$$\langle x, gy \rangle \xrightarrow{g \rightarrow \infty} 0.$$

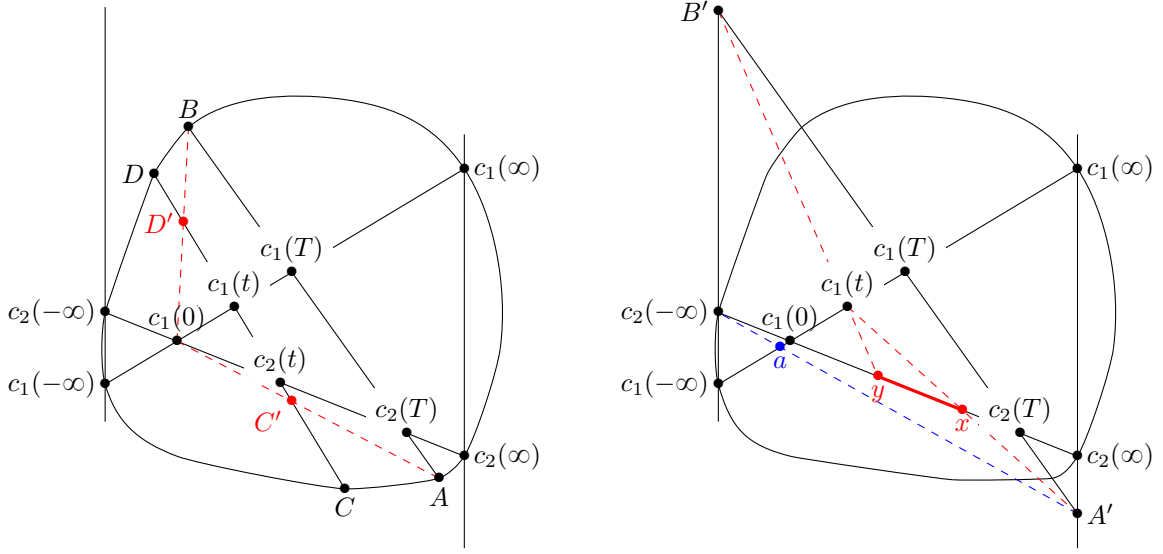


Figure 5: Proof of Crampon's Lemma 5.2

Proposition 7.4. *Consider $N \geq 3$, the algebra $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, or \mathbb{O} (if $N = 3$), the vector space $V = V_{N, \mathbb{K}}$, the properly convex open set $\Omega = \Omega_{N, \mathbb{K}}$, and the group $G = G_{N, \mathbb{K}}$. Take a lattice Γ of G , not necessarily uniform, and denote by M the quotient Ω/Γ . Then for any $1 \leq i \leq N - 1$, the (finite and fully supported) Haar measure on $T^1 M_{i, N-i}$ is mixing under the geodesic flow; as a consequence the geodesic flow is topologically mixing on $T^1 M_{i, N-i}$. Furthermore, $\text{NW}(T^1 M, (\phi_t)_{t \in \mathbb{R}})$ has exactly $N - 1$ connected components, which are $\{T^1 M_{i, N-i} : i = 1, \dots, N - 1\}$.*

Proof. Up to replacing Γ by a finite-index subgroup, we can assume that Γ is contained in G_0 . Since Γ is a lattice, the Haar measure on $\Gamma \backslash G_0$ is finite; the Howe–Moore theorem (Fact 7.3) classically implies that it is mixing under the action of the non-compact subgroup $A_{i, N-i} \subset G_0$. According to Proposition 7.1, it immediately follows that, for each $1 \leq i \leq N - 1$, the induced Haar measure on $T^1 M_{i, N-i} = \Gamma \backslash G/K_{i, N-i}$ is mixing under the action of the geodesic flow. Since the Haar measure is fully supported, the geodesic flow on $T^1 M_{i, N-i}$ is topologically mixing, and its non-wandering set is $T^1 M_{i, N-i}$. □

A Proof of Crampon's Lemma 5.2

It is enough to establish Lemma 5.2 when $c_1(0) = c_2(0)$. Indeed, suppose the lemma true in this case. Consider two straight geodesics c_1 and c_2 , each parametrised with constant speed. Let c_3 be the straight geodesic, parametrised with constant speed, such that $c_3(0) = c_1(0)$ and $c_3(T) = c_2(T)$. For $t \leq T$ we have

$$\begin{aligned} d_\Omega(c_1(t), c_2(t)) &\leq d_\Omega(c_1(t), c_3(t)) + d_\Omega(c_3(t), c_2(t)) \\ &\leq d_\Omega(c_1(T), c_3(T)) + d_\Omega(c_3(0), c_2(0)) \\ &\leq d_\Omega(c_1(T), c_2(T)) + d_\Omega(c_1(0), c_2(0)). \end{aligned}$$

We now assume $c_1(0) = c_2(0)$ (and that c_1 and c_2 are not constant, otherwise the proof is trivial). We can then assume that Ω has dimension 2, and we can consider an affine chart in which both projective lines $(c_1(-\infty) \oplus c_2(-\infty))$ and $(c_1(\infty) \oplus c_2(\infty))$ are vertical. Fix $0 < t < T$. We draw Figure 5 (left-hand side) which contains the following points:

- A and B are the intersection points of the line $(c_2(T) \oplus c_1(T))$ with $\partial\Omega$;
- C and D are the intersection points of the line $(c_2(t) \oplus c_1(t))$ with $\partial\Omega$;

- C' and D' are the intersection points of the line $(c_2(t) \oplus c_1(t))$ with the lines $(c_1(0) \oplus A)$ and $(c_1(0) \oplus B)$.

If we are in the case, as in Figure 5, where the lines $(c_1(t) \oplus c_2(t))$ and $(c_1(T) \oplus c_2(T))$ do not intersect inside Ω , then by convexity of Ω the point C' lies between C and $c_2(t)$ and D' lies between D and $c_1(t)$. Therefore by definition of the cross-ratio we deduce that

$$\begin{aligned} d_{\Omega}(c_1(T), c_2(T)) &= d_{(C', D')}(c_1(t), c_2(t)) \\ &\geq d_{(C, D)}(c_1(t), c_1(t)) \\ &\geq d_{\Omega}(c_1(t), c_2(t)). \end{aligned}$$

It remains to prove that the lines $(c_1(t) \oplus c_2(t))$ and $(c_1(T) \oplus c_2(T))$ do not cross inside Ω (this is the missing explanation in Crampon's original proof). We draw Figure 5 (right-hand side) which contains the points:

- A' and B' are the intersection points of the line $(c_2(T) \oplus c_1(T))$ with the lines $(c_1(\infty) \oplus c_2(\infty))$ and $(c_1(-\infty) \oplus c_2(-\infty))$.
- x and y are the intersection points of the line $(c_2(-\infty) \oplus c_2(\infty))$ with the lines $(c_1(t) \oplus A')$ and $(c_1(t) \oplus B')$.
- a is the intersection point of the line $(c_1(-\infty) \oplus c_1(\infty))$ with the line $(c_2(-\infty) \oplus A')$.

And we observe that it is enough to prove that $c_2(t)$ is on the segment $[x, y]$. In other words we want to establish:

$$\frac{d_{\Omega}(c_2(0), y)}{d_{\Omega}(c_2(0), c_2(T))} \leq \frac{d_{\Omega}(c_2(0), c_2(t))}{d_{\Omega}(c_2(0), c_2(T))} = \frac{t}{T} = \frac{d_{\Omega}(c_1(0), c_1(t))}{d_{\Omega}(c_1(0), c_1(T))} \leq \frac{d_{\Omega}(c_2(0), x)}{d_{\Omega}(c_2(0), c_2(T))}.$$

For example, if we want to establish the inequality on the right, we see by definition of the cross-ratio that it is enough to prove:

$$\frac{d_{\Omega}(c_1(0), c_1(t))}{d_{\Omega}(c_1(0), c_1(T))} \leq \frac{d_{(a, c_1(\infty))}(c_1(0), c_1(t))}{d_{(a, c_1(\infty))}(c_1(0), c_1(T))}.$$

It is a consequence of the following lemma. This, and a similar argument for the inequality on the left, conclude the proof of Lemma 5.2.

Lemma A.1. *For all $a < a' < x < y < z < b \in \mathbb{R}$,*

$$\frac{d_{(a, b)}(x, y)}{d_{(a, b)}(x, z)} \leq \frac{d_{(a', b)}(x, y)}{d_{(a', b)}(x, z)}.$$

Proof. Up to acting by a projective transformation we can assume that $x = 0$, $y = 1$ and $b = \infty$. For $z > 1$ we consider the function:

$$a \mapsto f_z(a) = \frac{d_{(a, \infty)}(0, 1)}{d_{(a, \infty)}(0, z)}.$$

on $(-\infty, 0)$. We have to check that this function f_z is non-decreasing. This follows immediately from the fact that, for every $a < 0$,

$$f_z(a) = \frac{\log(1 + \frac{-1}{a})}{\log(1 + \frac{-z}{a})}.$$

and from the computation of the derivative. □

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