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Aspects dynamiques des structures projectives convexes

Dynamical aspects of convex projective structures

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Pierre-Louis Blayac

Composition du jury

Mme Françoise Dal'bo Professeure des universités, Université de Rennes 1	Présidente
M. Gilles Courtois Directeur de recherche, Sorbonne Université	Rapporteur & Examinateur
M. Boris Hasselblatt Professeur des universités, Tufts University	Rapporteur & Examinateur
M. Yves Benoist Directeur de recherche, Université Paris-Saclay	Examinateur
Mme Ursula Hamenstädt Professeure des universités, Rheinische Friedrich-Wilhelms-Universität Bonn	Examinatrice

Direction de la thèse

Mme Fanny Kassel Directrice de recherche, Université Paris-Saclay	Directrice de thèse

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La vie vaut la peine d'être vécue,
Avec une sœur et deux frères qui vous rendent déchus !*

TB

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Contents

Introduction

Cette thèse a pour sujet principal les variétés projectives convexes. Plus précisément, la majeure partie des travaux présentés ici (i.e. les chapitres 5 à 9) est consacrée à l'étude de la dynamique du flot géodésique de ces variétés. Les variétés projectives convexes généralisent les variétés hyperboliques réelles, du point de vue de leur modèle projectif, aussi appelé modèle de Beltrami–Klein. Considérons une surface hyperbolique orientée S décrite à l'aide du modèle projectif, c'est-à-dire que $S = \mathbb{H}^2/\Delta$, où $\mathbb{H}^2 := \{[x_1 : x_2 : x_3] \in \mathrm{P}(\mathbb{R}^3) : x_1^2 + x_2^2 < x_3^2\}$ (on note $\mathrm{P}(\mathbb{R}^3)$ l'espace projectif associé à $\mathbb{R}^3$) et Δ est un sous-groupe discret et sans torsion de la composante neutre $\mathrm{PO}(2,1)_0$ du stabilisateur de $\mathbb{H}^2$ dans $\mathrm{PSL}(\mathbb{R}^3)$. On voit là le prototype d'une structure géométrique uniformisable $\mathcal{U}/\Lambda$, où $\mathcal{U}$ est un ouvert d'une variété X et Λ un sous-groupe discret du stabilisateur de $\mathcal{U}$ dans un groupe de Lie G qui agit transitivement sur X ; c'est dans ce cadre abstrait que s'incrivent également les variétés projectives convexes. Un autre point de vue important sur S, qui permet de définir le flot géodésique qui lui est naturellement associé, est de la voir comme une surface munie d'une métrique riemannienne de courbure constante égale à -1 ; cette métrique induit un flot géodésique sur les fibrés unitaires tangents $T^1\mathbb{H}^2$ et T^1S, et les orbites de ce flot dans $\mathbb{H}^2$ sont des portions de droites projectives.

Fixons un espace vectoriel réel $\mathbf{V}$ de dimension finie. Les variétés projectives convexes sont des variétés munies d'une structure géométrique uniformisable modelée sur $\mathrm{P}(\mathbf{V})$, sur lequel agit le groupe de Lie $\mathrm{PGL}(\mathbf{V})$. Un sous-ensemble de $\mathrm{P}(\mathbf{V})$ est dit *proprement convexe* s'il est convexe et borné dans une carte affine. Les *orbivariétés* (resp. *variétés*) *projectives convexes* sont les quotients de la forme $M = \Omega/\Gamma$ où $\Omega \subset \mathrm{P}(\mathbf{V})$ un ouvert proprement convexe et Γ est un sous-groupe discret (resp. discret et sans torsion) du stabilisateur $\mathrm{Aut}(\Omega)$ de Ω dans $\mathrm{PGL}(\mathbf{V})$.

Rappelons rapidement pourquoi les terminologies d'«orbivariété» et de «variété» sont justifiées ; cela nous donne l'occasion d'introduire la célèbre *distance de Hilbert* d_Ω sur les ouverts proprement convexes $\Omega \subset \mathrm{P}(\mathbf{V})$. Celle-ci est donnée par la formule $d_\Omega(x,y) = \frac{1}{2}\log[a,x,y,b]$, où $x, y \in \Omega$ et $a, b \in \partial\Omega$ sont tels que a, x, y, b sont alignés dans cet ordre (voir la figure 1), et $[a,x,y,b]$ est le birapport projectif, tel que $[0,1,t,\infty] = t$ pour tout $t \in \mathbb{R}$ sous l'identification $\mathrm{P}(\mathbb{R}^2) = \mathbb{R} \cup \{\infty\}$. La fonction d_Ω définit une distance, qui de surcroît est $\mathrm{Aut}(\Omega)$-invariante et propre (les boules fermées sont compactes). Ceci implique que tout sous-groupe discret $\Gamma \subset \mathrm{Aut}(\Omega)$ agit proprement discontinûment sur Ω, de sorte que M est une orbivariété ; si de plus Γ est sans torsion alors il agit aussi librement sur Ω, et M est une variété.

Nous travaillerons autant que faire se peut sur des orbivariétés, notamment car il existe une classe importante constituée d'orbivariétés projectives convexes Ω/Γ qui ne sont pas des variétés, Γ étant un groupe de Coxeter (voir la section 0.1.3 et le chapitre 11). Dans les chapitres 7 et 9, nous aurons plusieurs fois recours à des arguments qui ne fonctionnent que sur des variétés ; toutefois on pourra étendre aux orbivariétés les résultats ainsi obtenus grâce au célèbre lemme de Selberg [Sel60], qui dit que tout sous-groupe de type fini de

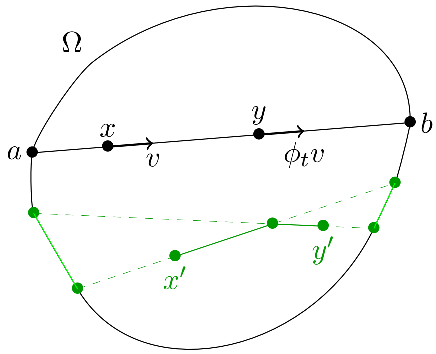

Figure 1: La distance de Hilbert et le flot géodésique ($t = d_\Omega(x, y)$)

GL(**V**) ou PGL(**V**) admet un sous-groupe d'indice fini sans torsion.

Toute droite projective intersecte Ω en une géodésique de celui-ci, dite *rectiligne*, ce qui permet de définir un flot géodésique naturel $(\phi_t)_t$ sur le fibré unitaire tangent $T^1\Omega$, tel que pour tout $v \in T^1\Omega$, la trajectoire $(\pi\phi_t v)_t$ paramétrise la géodésique rectiligne tangente à v, où l'on a noté $\pi : T^1\Omega \to \Omega$ la projection naturelle. Le flot $(\phi_t)_t$ commute avec l'action de $\mathrm{Aut}(\Omega)$, par conséquent il induit un flot, encore noté $(\phi_t)_t$, sur le quotient $T^1M := (T^1\Omega)/\Gamma$, appelé fibré unitaire tangent de M. Notons que d_Ω provient d'une métrique finslérienne, qui, lorsque $\Omega = \mathbb{H}^2$, coïncide avec la métrique riemannienne évoquée plus haut. Remarquons aussi qu'il existe des cas où Ω contient des géodésiques non rectilignes (voir le chemin vert de x' à y' dans la figure 1, ainsi que le fait 2.1.13), contrairement au cas où $\Omega = \mathbb{H}^2$.

Les premiers travaux sur les variétés projectives convexes qui ne sont pas nécessairement des variétés hyperboliques réelles remontent à Kuiper [Kui54] dans les années 50. Ils abordent des problématiques orientées vers la théorie générale des structures géométriques, et ont connu de considérables développements, avec un intérêt particulier porté sur le cas des variétés compactes, qui ont reçu une terminologie spéciale. Lorsque Γ agit cocompactement sur Ω, on dit qu'il *divise* Ω, et que Ω est *divisible*. Benoist a écrit en 2008 un article de survol [Ben08] sur les convexes divisibles, qui approfondit beaucoup de sujets traités dans cette introduction. Une autre référence importante sur la géométrie des ouverts proprement convexes de P(**V**), plus récente, est [PT14].

Les aspects dynamiques du flot géodésique sur les orbivariétés projectives convexes (autres que les orbivariétés hyperboliques) constituent un sujet d'étude plutôt récent : il a été initié par Benoist en 2006 [Ben06a]. Avant d'expliquer en détails les avancées effectuées ces dernières années sur ce thème, nous allons faire un rappel historique succinct à propos du flot géodésique sur les variétés hyperboliques, centré sur les propriétés dynamiques considérées dans cette thèse ; pour des définitions précises de ces propriétés on pourra se reporter au chapitre 1.

0.1 Contexte

0.1.1 Le flot géodésique sur les variétés hyperboliques

Les premiers travaux sur le sujet sont dus à Hadamard [Had98] en 1898, et concernent en fait la dynamique du flot géodésique sur des surfaces riemanniennes à courbure négative variable, plongées dans $\mathbb{R}^3$. Une des motivations de Hadamard était d'ordre dynamique : il s'agissait de trouver et comprendre un système physique (dans son cas une bille bien huilée glissant sur une surface aimantée, en l'absence de force extérieure) qui possède, sous une forme plus accessible, la complexité de ceux de la mécanique céleste ; voir l'article de vulgarisation [Buz13] écrit par Buzzi à propos du papier d'Hadamard. Si, aujourd'hui, les flots considérés par Hadamard jouent un rôle moins proéminent dans la théorie des systèmes dynamiques, dans la mesure où l'on dispose d'une panoplie plus diverse d'exemples, nous pensons que les motivations d'Hadamard sont encore d'actualité et s'étendent aux flots géodésiques des orbivariétés projectives convexes. Notons par ailleurs que les flots géodésiques possèdent un autre point commun, pas encore connu en 1898, avec les problèmes de la mécanique céleste : selon la théorie de la relativité d'Einstein, les mouvements de particules soumises à la seule force de la gravitation sont donnés par des lignes géodésiques sur une certaine variété pseudo-riemannienne de dimension quatre.

Bien que la notion de groupe fondamental fût encore à l'époque en travaux, Hadamard a établi la correspondance entre les orbites périodiques du flot géodésique $(\phi_t)_t$ et les classes de conjugaison du groupe fondamental, et en utilisant la structure de groupe de ce dernier, il a mis en évidence une abondance d'orbites périodiques, mais aussi et surtout de trajectoires récurrentes plus compliquées.

Quelques années plus tard, les idées d'Hadamard furent reprises et développées, avec un recentrage sur le cas des surfaces hyperboliques réelles, par un grand nombre de mathématiciens (tels que Birkhoff, Hopf, Morse, Hedlund...), voir l'article de survol de Hedlund [Hed39]. Ces auteurs ont affiné les propriétés de récurrence de nature topologiques de $(\phi_t)_t$, en développant entre autres les concepts d'orbite récurrente, transitivité topologique, mélange topologique, point non-errant. De plus, ils ont introduit dans ce domaine de recherche des questions et des outils de nature probabiliste provenant de la physique statistique, comme les notions d'ergodicité, conservativité, mélange (probabiliste).

Dans les années 60, Anosov [Ano67] a défini une classe très générale de systèmes dynamiques, appelés flots et difféomorphismes d'Anosov, qui englobe les flots géodésiques sur les variétés riemanniennes fermées à courbure négative. Rappelons qu'un flot sur une variété compacte est d'Anosov s'il préserve dans le fibré tangent deux distributions supplémentaires à la direction du flot, l'une (dite stable) qu'il contracte, et l'autre (dite instable) qu'il étire. C'est dans ce cadre que Margulis [Mar70] et Bowen [Bow71] ont donné une description de la répartition des orbites périodiques (sous l'hypothèse du mélange topologique). Leur description fait notamment intervenir la notion d'entropie, qui provient de la physique et avait été introduite quelques années plus tôt par Kolmogorov en théorie des systèmes dynamiques (à chaque mesure invariante est associée un nombre réel appelé entropie). Margulis et Bowen ont démontré qu'il existe une unique mesure d'entropie maximale, appelée mesure de Bowen–Margulis, pour laquelle les orbites périodiques de période inférieure à T se répartissent de façon de plus en plus dense et uniforme à mesure que T grandit. Une référence classique concernant les flots d'Anosov, et plus généralement les systèmes dynamiques dits hyperboliques, est le livre de Katok–Hasselblatt [HK95].

Transposer le résultat de Bowen et Margulis au flot géodésique des orbivariétés projectives convexes constitue l'objectif principal de cette thèse. Ce résultat avait déjà été généralisé à de nombreux flots géodésiques dans des contextes géométriques différents. Citons deux auteurs qui ont particulièrement influencé cette thèse : Knieper [Kni98] a étudié le flot géodésique de certaines variétés riemanniennes compactes à courbure négative ou nulle, dites de rang 1 (dont le flot géodésique n'est pas d'Anosov en général), et Roblin [Rob03] a étudié le cadre très général du flot géodésique sur des quotients non compacts d'espaces CAT(-1). Knieper et Roblin ont tous deux eu recours à un outil conçu par Patterson [Pat76] et Sullivan [Sul79] : les *densités conformes*, qui permettent de construire des mesures invariantes par le flot géodésique, appelées ici *mesures de Sullivan*, possédant de bonnes propriétés (par exemple l'une d'elles est dans beaucoup de cas intéressants l'unique mesure d'entropie maximale). Citons quelques travaux très récents sur les flots géodésiques en géométrie riemannienne et CAT(0) qui font usage des densités conformes ou de généralisations de ces dernières : [BCFT18, BAPP19, Lin20, Ric, LWW20]. Notons que les travaux de Bowen et Margulis évoqués ci-dessus ne font pas appel aux densités conformes, en particulier ils construisent différemment l'unique mesure d'entropie maximale ; il existe encore d'autres façons de construire cette mesure : citons par exemple une construction due à Hamenstädt [Ham89] dans le cadre des variétés riemanniennes compactes de courbure négative.

Pour finir, mentionnons une application de la théorie des flots d'Anosov à l'étude des sous-groupes discrets des groupes de Lie, due à A. Sambarino [Sam14] ; précisons qu'on a restreint pour simplifier le cadre des travaux cités dans ce paragraphe. Considérons à nouveau une surface hyperbolique fermée $S = \mathbb{H}^2/\Delta$. Labourie [Lab06] a défini une large classe de représentations de Δ dans $\mathrm{PGL}(\mathbf{V})$ (où $\mathbf{V}$ est de dimension finie quelconque), appelées représentations *anosoviennes* car leur définition fait explicitement intervenir un flot d'Anosov, lié au flot géodésique $(\phi_t)_t$ sur T^1S. Fixons une représentation projectivement anosovienne ρ de Δ. Alors l'image $\rho(g)$ de chaque élément non trivial $g \in \Delta$ possède dans $\mathrm{P}(\mathbf{V})$ un point fixe attracteur $x^+_{\rho(g)}$ et un point fixe répulseur $x^-_{\rho(g)}$. Pour tout $g \in \mathrm{PGL}(\mathbf{V})$, notons $\|g\|$ la norme d'opérateur de n'importe quel relevé de g dans $\mathrm{GL}(\mathbf{V})$ de déterminant ± 1.

Sambarino donne une description de la répartition dans $\mathrm{P}(\mathbf{V})^2$, lorsque T tend vers l'infini, de l'ensemble des couples $(x^-_{\rho(g)}, x^+_{\rho(g)})$, où $g \in \Delta$ est tel que $\|\rho(g)\| \leq T$. Ces couples correspondent à des couples de points fixes attracteur/répulseur dans $\partial\mathbb{H}^2$, qui correspondent à des géodésiques de $\mathbb{H}^2$, et passent au quotient en des géodésiques fermées de S. Mais compter les éléments $g \in \Delta$ tels que $\|\rho(g)\| \leq T$ ne revient pas à compter les $(\phi_t)_t$-orbites périodiques de période inférieure à T : d'après les travaux de Sambarino, il faut en fait considérer une reparamétrisation (elle aussi d'Anosov) bien choisie de $(\phi_t)_t$, à laquelle s'appliquent par exemple les travaux de Bowen et Margulis.

Dans le cas où $\mathbf{V} = \mathbb{R}^3$, l'ensemble des représentations ρ étudiées par Sambarino incluent celles dont l'image $\rho(\Delta)$ divise un ouvert proprement convexe Ω de $\mathrm{P}(\mathbf{V})$, et les flots d'Anosov considérés par Sambarino incluent en particulier le flot géodésique sur $T^1\Omega/\rho(\Delta)$. Les travaux de Sambarino montrent que l'étude du flot géodésique sur les orbivariétés projectives convexes peut aussi être motivée par la théorie des sous-groupes discrets des groupes de Lie.

0.1.2 Le flot géodésique des variétés projectives convexes compactes de type "courbure négative"

Un des principaux résultats de Benoist [Ben04, Th. 1.1] concernant le flot géodésique $(\phi_t)_t$ sur les variétés projectives convexes compactes $M = \Omega/\Gamma$ est le suivant, établissant un lien entre la dynamique de $(\phi_t)_t$ et la régularité du bord $\partial\Omega$. Benoist a démontré que les assertions suivantes sont équivalentes :

- Ω est strictement convexe (i.e. son bord $\partial\Omega$ ne contient pas de segment non trivial);
- $\partial\Omega$ est une sous-variété de $\mathrm{P}(\mathbf{V})$ de classe $\mathcal{C}^1$ (on dira simplement que $\partial\Omega$ est $\mathcal{C}^1$);
- Γ est Gromov-hyperbolique pour la métrique des mots;

- le flot géodésique sur T^1M est d'Anosov.

Ainsi, les résultats de Bowen et Margulis dont on a parlé à la section précédente s'appliquent à $(\phi_t)_t$ si et seulement si Ω est strictement convexe. Par analogie avec les variétés riemanniennes, on dira qu'une orbivariété projective convexe $M = \Omega/\Gamma$ (pas nécessairement compacte) est *de type "courbure négative"* si Ω est strictement convexe et $\partial\Omega$ est $\mathcal{C}^1$. Par la suite, on portera un intérêt tout particulier au flot géodésique sur les orbivariétés projectives convexes qui ne sont pas de type "courbure négative", flot qui a pour l'instant été peu étudié. De plus, nous essaierons autant que faire se peut d'étendre nos résultats à des orbivariétés non compactes.

0.1.3 Exemples de convexes divisibles

Dans cette section nous récapitulons les exemples connus de convexes divisibles. L'ouvert proprement convexe $\mathbb{H}^2 \subset \mathrm{P}(\mathbb{R}^3)$ est bien sûr un convexe divisible ; une méthode célèbre et commode pour construire des surfaces hyperboliques fermées est de décomposer en pantalons une surface topologique de genre supérieur à 2, puis de mettre une structure hyperbolique sur chaque pantalon, et enfin de recoller les pantalons (voir par exemple [IT92, Chap. 3]). L'existence en toute dimension de variétés hyperboliques compactes est due à Siegel [Sie51], qui a construit par des méthodes arithmétiques des réseaux (i.e. sous-groupes discrets de covolume fini) cocompacts de $\mathrm{PSO}(\mathtt{d}, 1)$ pour tout $\mathtt{d} \geq 2$.

En dehors des espaces hyperboliques réels $(\mathbb{H}^n)_{n\geq 2}$, on trouve, parmi les convexes divisibles, ceux dits *symétriques, irréductibles et de rang supérieur*, qui sont les modèles projectifs des espaces symétriques de $\mathrm{PGL}_n(\mathbb{K})$, où $n \geq 3$ et $\mathbb{K}$ est le corps des réels, des complexes, ou l'algèbre à division des quaternions (ou des octonions pour $n = 3$) (voir la section 5.5 pour une description plus précise), grâce à un théorème de Borel [Bor63] qui généralise celui de Siegel en construisant des sous-groupes discrets cocompacts dans tout groupe de Lie semi-simple réel. Le flot géodésique des quotients des convexes divisibles symétriques irréductibles de rang supérieur intervient dans la section 5.5, où l'on voit apparaître un lien entre ces flots et des flots plus classiques : des sous-groupes à un paramètre de $\mathrm{PGL}_n(\mathbb{K})$ (formés de matrices diagonales) agissant sur des quotients de $\mathrm{PGL}_n(\mathbb{K})$ par des sous-groupes discrets. On pourra comparer les résultats obtenus en section 5.5 avec un article de Dang–Glorieux [DG] et un autre plus récent de Dang [Dan].

Il existe un moyen, étant donnés deux convexes divisibles $\Omega_1 \subset \mathrm{P}(V_1)$ et $\Omega_2 \subset \mathrm{P}(V_2)$, de produire un convexe divisible plus gros : le *joint de* Ω_1 *et* Ω_2 est défini comme l'ensemble des points de $\mathrm{P}(V_1 \oplus V_2)$ de la forme $[v_1 + v_2]$ où $v_i \in V_i \smallsetminus \{0\}$ est tel que $[v_i] \in \Omega_i$ pour $i = 1, 2$; il est facile de voir que le joint est lui aussi divisible, et l'on dit qu'il est *réductible* (les convexes divisibles qui ne sont pas des joints sont dits *irréductibles*). L'espace projectif réel de dimension zéro $\mathrm{P}(\mathbb{R}^1)$, qui est réduit à un point, est un convexe divisible; en particulier, pour $n \geq 3$, le joint de n copies de $\mathrm{P}(\mathbb{R}^1)$ est le simplexe $S = \{[x_1e_1 + \cdots + x_ne_n] \in \mathrm{P}(\mathbb{R}^n) : x_1, \ldots, x_n > 0\} \subset \mathrm{P}(\mathbb{R}^n)$, où $(e_1, \ldots, e_n)$ est la base canonique de $\mathbb{R}^n$. Avec sa distance de Hilbert, S est isométrique à un espace vectoriel réel de dimension $n - 1$ muni d'une norme (dont les sphères sont des polytopes — des hexagones en dimension deux), de sorte que les translations de cet espace correspondent à l'action sur S par les classes de matrices diagonales. À ma connaissance, il n'existe pas d'étude de la dynamique du flot géodésique sur les quotients de joints d'ouverts proprement convexes.

Une méthode plus sophistiquée pour construire des convexes divisibles est celle dite du *pliage* (bending en anglais), développée par Johnson–Millson [JM87]. Étant donnée une variété hyperbolique compacte $M = \mathbb{H}^n/\Gamma$ (où $n \geq 2$) qui possède une hypersurface totalement géodésique, cette technique produit une famille à un paramètre de représentations

$(\rho_t)_{t\geq 0}$ de Γ dans $\mathrm{PGL}(\mathbb{R}^{n+1})$ telles que ρ_0 est l'inclusion et $\rho_t(\Gamma)$ est Zariski-dense dans $\mathrm{PGL}(\mathbb{R}^{n+1})$. Un théorème de Koszul [Kos68, Cor. p. 103] affirme qu'alors, pour t assez petit, $\rho_t(\Gamma)$ divise un ouvert proprement convexe Ω_t, qui n'est pas un ellipsoïde. En raffinant la méthode de pliage, Kapovich [Kap07, Th. 1.1] a construit des variétés convexes projectives compactes Ω/Γ où Γ est Gromov-hyperbolique mais pas abstraitement isomorphe à un réseau de $\mathrm{PO}(n,1)$ pour $n\geq 2$.

Enfin, on peut construire des convexes divisibles Ω en recollant les images d'un polytope P de $\mathrm{P}(\mathbf{V})$ par le groupe Γ engendré par les réflexions projectives le long des faces du polytope. C'est d'ailleurs de cette façon que Poincaré [Poi82] a construit les premiers exemples de sous-groupes discrets cocompacts de $\mathrm{PSL}_2(\mathbb{R})$ (voir aussi [Mas88, §4.H]). Vinberg a par la suite considérablement développé l'idée : il explique dans [Vin71] sous quelles conditions l'intérieur Ω de $\Gamma\cdot P$ est un ouvert proprement convexe de $\mathrm{P}(\mathbf{V})$ tel que $P\cap\Omega$ soit un domaine fondamental pour l'action de Γ (qui est dans ce cas un groupe de Coxeter en tant que groupe abstrait) ; il montre de plus que Γ divise Ω si et seulement si $\Gamma\cdot P=\Omega$, si et seulement si pour chaque facette de P, le sous-groupe de Γ engendré par les réflexions le long des faces attenantes à la facette est fini.

Benoist [Ben06a, Prop. 1.3] a utilisé les techniques de Vinberg pour construire des convexes divisibles non strictement convexes, irréductibles, et non symétriques (voir aussi [Mar10, BDL18, CLM20]) ; les exemples ainsi construits sont tous de dimension inférieure à 7. Danciger–Guéritaud–Kassel–Lee–Marquis [DGKLM] ont récemment utilisé les techniques de Vinberg pour décrire tous les polytopes P et groupes de réflexions associés Γ tels que Γ agit *convexe cocompactement* sur $\Omega=\Gamma\cdot P$, et démontrer qu'il en existe beaucoup pour lesquels Γ n'est pas Gromov-hyperbolique (notamment en toute dimension). Nous revenons sur la notion de convexe cocompacité en section 0.1.5, son intérêt dans la thèse étant que pour comprendre la dynamique du flot géodésique sur les quotients d'actions convexes cocompactes, il suffit de la comprendre sur une partie compacte, où l'on peut souvent raisonner comme sur les orbivariétés projectives convexes fermées.

0.1.4 Les orbivariétés de Benoist

Soit $M=\Omega/\Gamma$ une orbivariété projective convexe compacte irréductible telle que Ω n'est pas strictement convexe, ni symétrique. L'orbivariété M ne peut être une surface d'après un théorème de Benzécri [Ben60, Prop. 5.3.9]. Par contre, il existe des exemples en dimension 3, construits par Benoist à l'aide de groupes de Coxeter. Supposons $\dim M=3$. Benoist a donné une description géométrique très riche [Ben06a, Th. 1.1] de M, et l'on dira donc que M est une *orbivariété de Benoist*. Cette description fait intervenir des *triangles proprement plongés* dans Ω, qui sont des sous-ensembles de la forme $T=\Omega\cap\mathrm{P}(W)$ où W est un hyperplan de V, et T est un triangle dans $\mathrm{P}(W)$ dont le bord $\partial_{\mathrm{rel}}T$ dans $\mathrm{P}(W)$ est contenu dans $\partial\Omega$. Un point de $\partial\Omega$ est dit $\mathcal{C}^1$ ou *lisse* si Ω admet un unique hyperplan de support en ce point. Donnons quelques éléments du théorème de structure sur les orbivariétés de Benoist :

- Ω contient un nombre fini (non nul) de Γ-orbites de triangles proprement plongés ;
- deux triangles proprement plongés distincts sont d'adhérences disjointes ;
- les points non extrémaux de $\partial\Omega$ sont exactement les points des côtés des triangles proprement plongés ;
- les points non lisses de $\partial\Omega$ sont exactement les sommets des triangles proprement plongés ;

- le stabilisateur dans Γ d'un triangle proprement plongé est virtuellement isomorphe à $\mathbb{Z}^2$ (i.e. contient un sous-groupe d'indice fini isomorphe à $\mathbb{Z}^2$), et tout sous-groupe de Γ virtuellement isomorphe à $\mathbb{Z}^2$ stabilise un triangle proprement plongé.

Tout triangle proprement plongé $T \subset \Omega$ étant isométrique à $\mathbb{R}^2$ (muni d'une norme hexagonale, cf. la section 0.1.3), il peut être vu comme un plat de Ω, par analogie avec les variétés riemanniennes compactes de courbure négative ou nulle.

Notons que l'énoncé originel du théorème de structure de Benoist ne concerne que les variétés projectives convexes, mais il se généralise facilement aux orbivariétés grâce au lemme de Selberg ; rappelons que tout sous-groupe de $\mathrm{PGL}(\mathbf{V})$ qui divise un ouvert proprement convexe est de type fini, c'est un cas particulier du lemme de Milnor–Švarc (voir par exemple [BH99, Th. I.8.10]).

On a vu en section 0.1.2 que le flot géodésique $(\phi_t)_t$ sur T^1M n'est pas d'Anosov. Bray a étudié pendant sa thèse la dynamique de $(\phi_t)_t$. Les résultats qu'il a obtenus font notamment intervenir les notions de densité conforme et de mesure de Sullivan, définies en section 1.4.2, dont on a déjà parlé plus haut. Bray a démontré que :

- les orbites périodiques de $(\phi_t)_t$ sont denses dans T^1M ;
- le flot géodésique $(\phi_t)_t$ est *topologiquement mélangeant* sur le fibré unitaire tangent T^1M, c'est-à-dire que pour toute paire d'ouverts U et V non vides, $\psi_t(U)$ rencontre V pour tout t assez grand ;
- la notion de densité conforme peut être adaptée aux orbivariétés de Benoist (ce sont des familles de mesures sur le bord projectif $\partial\Omega$, et l'on leur associe naturellement une dimension), tout comme la notion associée de mesure de Sullivan sur T^1M (ce sont des mesures $(\phi_t)_t$-invariantes) ;
- il existe une unique densité conforme μ dont la dimension est *l'exposant critique* de Γ,
$$\delta_\Gamma := \limsup_{R\to\infty} \frac{1}{R} \log \# \left(\Gamma \cdot o \cap B_\Omega(o, R)\right) \; ; \tag{0.1.1}$$
- la mesure de Sullivan m sur T^1M associée à μ est ergodique, c'est-à-dire que tout sous-ensemble mesurable $(\phi_t)_t$-invariant de T^1M est de mesure pleine ou nulle ;
- on a l'estimée suivante sur le volume des boules dynamiques de rayon $R > 0$:
$$m\left(\left\{w \in T^1M : d_{T^1M}(\phi_t v, \phi_t w) \leq R \quad \forall\, 0 \leq t \leq T\right\}\right) \leq Ce^{-\delta_\Gamma T},$$
où C est une constante indépendante de $T > 0$ et $v \in T^1M$ (c'est la moitié d'une estimée classique en théorie des systèmes dynamiques, appelée propriété de Gibbs), et m est d'entropie maximale (voir la section 1.3.1 pour la définition précise) ;
- il existe une constante $C > 0$ telle que pour tout R assez grand,
$$C^{-1}e^{\delta_\Gamma R} \leq \# \left(\Gamma \cdot o \cap B_\Omega(o, R)\right) \leq Ce^{\delta_\Gamma R}.$$

Pour conclure cette section, notons que le théorème de structure des orbivariétés de Benoist est conjectural en dimension supérieure à 4, et fait l'objet de recherches actuelles très intéressantes : voir [BDL18, CLM20] pour des exemples et [IZ, Bob] pour des résultats partiels. Les résultats que nous allons présenter ici généralisent en dimension quelconque et à des situations non compactes les travaux de Bray, et donnent de plus une description plus fine de la dynamique du flot. Leur démonstration ne fait pas appel à une description similaire au théorème de Benoist.

0.1.5 Finitude géométrique et convexe cocompacité

Après les travaux de Benoist sur le flot géodésique, et avant ceux de Bray, Crampon–Marquis se sont intéressés [CM14b] au flot géodésique des orbivariétés projectives convexes non compactes de type "courbure négative" $M = \Omega/\Gamma$. Ils ont défini [CM14a, Déf. 5.14] les orbivariétés projectives convexes de type "courbure négative" *géométriquement finies* (voir aussi la section 9.1), qui généralisent les orbivariétés hyperboliques réelles du même nom. Rappelons que *l'ensemble limite* Λ_Γ de Γ est constitué des points d'accumulation dans $\partial\Omega$ de toute Γ-orbite dans Ω ; on appelle *cœur convexe* de M la projection dans M de l'enveloppe convexe dans Ω de Λ_Γ. L'orbivariété M est géométriquement finie si son cœur convexe se décompose en une partie compacte et un nombre fini de parties non compactes, dites *cuspidales*, dont la géométrie est élémentaire (plus précisément ce sont des quotients d'un convexe de Ω par un sous-groupe virtuellement nilpotent de Γ qui fixe un point du bord $\partial\Omega$). Le groupe Γ agit convexe cocompactement sur Ω si le cœur convexe de M est compact, c'est-à-dire si M est géométriquement finie avec une partie cuspidale vide.

Crampon–Marquis ont démontré des propriétés de récurrence topologiques de $(\phi_t)_t$ sur les orbivariétés projectives convexes de type "courbure négative" non élémentaires (l'hypothèse non-élémentaire est peu restrictive, et signifie que le groupe fondamental ne contient pas de sous-groupe nilpotent d'indice fini) :

- le cœur convexe contient toute la dynamique de $(\phi_t)_t$; plus précisément *l'ensemble non errant* $\mathrm{NW}(T^1M)$, c'est-à-dire l'ensemble des vecteurs $v \in T^1M$ dont les voisinages U vérifient $\phi_t U \cap U \neq \emptyset$ pour des temps t arbitrairement grands, est exactement l'ensemble des vecteurs tangents aux géodésiques contenues dans le cœur convexe ;
- les géodésiques périodiques sont denses dans l'ensemble non errant ;
- le flot géodésique est topologiquement mélangeant sur l'ensemble non errant.

Lorsque M est de plus géométriquement finie, Crampon et Marquis ont établi des propriétés dynamiques de $(\phi_t)_t$ plus fines comme l'uniforme hyperbolicité et des propriétés liées aux *exposants de Liapounov*, dont on ne parlera pas ici.

Crampon s'est aussi intéressé, dans sa thèse [Cra11], aux densités conformes et aux mesures d'entropie maximale. Il a remarqué que les notions de densités conformes et de mesures de Sullivan, et plusieurs résultats associés pour les orbivariétés hyperboliques, se généralisent facilement aux orbivariétés projectives convexes de type "courbure négative". Nous allons voir que la généralisation de ces notions aux orbivariétés projectives convexes qui ne sont pas nécessairement de type "courbure négative" est plus compliquée à mettre en œuvre (rappelons que cela a été fait par Bray dans le cas compact de dimension 3). Crampon a démontré que la mesure de Sullivan de dimension l'exposant critique sur les orbivariétés projectives convexes de type "courbure négative" géométriquement finie est l'unique mesure d'entropie maximale (*mesure de Bowen–Margulis*).

F. Zhu [Zhua] a récemment démontré dans le cadre des orbivariétés projectives convexes de type "courbure négative" que, si elle est finie, alors la mesure de Sullivan m de dimension δ_Γ est *mélangeante*, au sens où $m(\phi_t(A) \cap B)\, m(T^1M)$ tend vers $m(A)m(B)$ quand t tend vers l'infini, pour tous $A, B \subset T^1M$. Zhu s'est de plus servi du mélange pour obtenir des résultats d'équidistribution pour les Γ-orbites de Ω et pour les $(\phi_t)_t$-orbites périodiques de M, en s'inspirant des travaux de Roblin [Rob03].

La notion de finitude géométrique pour les orbivariétés projectives convexes générales, sans hypothèse de régularité sur le revêtement universel Ω, n'est pas encore fixée, bien que des avancées intéressantes aient été effectuées par A. Wolf dans sa thèse [Wol20].

Par contre il existe depuis peu une notion satisfaisante de convexe cocompacité, due à Danciger–Guéritaud–Kassel [DGKa], que nous utiliserons comme cadre naturel pour certains résultats présentés dans ce texte. Soit $M = \Omega/\Gamma$ une orbivariété projective convexe qui n'est pas nécessairement de type “courbure négative”. Comme Ω n'est pas forcément strictement convexe, les points d'accumulation dans $\partial\Omega$ d'une Γ-orbite dans Ω peuvent dépendre de l'orbite choisie. Danciger–Guéritaud–Kassel ont contourné ce problème en définissant *l'ensemble limite orbital total* comme

$$\Lambda_\Gamma^{\mathrm{orb}} = \bigcup_{x\in\Omega} \overline{\Gamma \cdot x} \smallsetminus \Gamma \cdot x. \tag{0.1.2}$$

Le *cœur convexe* de M est la projection de l'enveloppe convexe dans Ω de l'ensemble limite orbital total, et l'action de Γ sur Ω est dite *convexe cocompacte* si le cœur convexe de M est compact et non vide. Nous renvoyons le lecteur désireux de comprendre pourquoi ceci est la bonne notion de convexe cocompacité à l'article de Danciger–Guéritaud–Kassel, qui contient quantité de résultats et d'exemples, ainsi qu'à leur récente collaboration avec Lee et Marquis [DGKLM], qu'on a évoquée plus haut : ils classifient les actions convexes cocompactes obtenues par la théorie de Vinberg.

Une des motivations de Danciger–Guéritaud–Kassel pour introduire cette notion de convexe cocompacité était d'étendre la classe les représentations anosoviennes à des représentations de groupes non nécessairement Gromov-hyperboliques. Un sous-groupe de $\mathrm{PGL}(\mathbf{V})$ est dit *convexe cocompact dans* $\mathrm{P}(V)$ s'il existe un ouvert proprement convexe de $\mathrm{P}(V)$ sur lequel il agit convexe cocompactement. Remarquons que Danciger–Guéritaud–Kassel avaient d'abord défini dans [DGK18] une notion de $\mathbb{H}^{p,q-1}$*-convexe cocompacité* pour des sous-groupes de $\mathrm{PO}(p,q)$ (avec p,q quelconques), qui est un cas particulier de la notion de convexe cocompacité dans $\mathrm{P}(\mathbf{V})$ et qui généralise la notion de groupe *AdS-quasi-fuchsien* étudiée par Mess [Mes07] (pour $q = 2$ et $p = 2$) puis Barbot–Mérigot [BM12] (pour $q = 2$ et p quelconque).

Le lien avec les représentations anosoviennes est le suivant : un sous-groupe discret Gromov-hyperbolique $\Gamma \subset \mathrm{PGL}(\mathbf{V})$ qui préserve un ouvert proprement convexe est convexe cocompact dans $\mathrm{P}(V)$ si et seulement si l'inclusion de Γ dans $\mathrm{PGL}(\mathbf{V})$ est projectivement anosovienne, et dans ce cas il agit convexe cocompactement sur un ouvert proprement et strictement convexe à bord $\mathcal{C}^1$. Danciger–Guéritaud–Kassel [DGK18] ont d'abord démontré ceci pour des sous-groupes de $\mathrm{PO}(p,q)$, généralisant les travaux de Mess [Mes07] et Barbot–Mérigot [BM12], puis dans le cas général dans [DGKa]. Indépendamment, Zimmer [Zim20] a montré que si $\Gamma \subset \mathrm{PGL}(\mathbf{V})$ est un sous-groupe irréductible qui préserve un ouvert proprement convexe de $\mathrm{P}(\mathbf{V})$, alors l'inclusion $\Gamma \hookrightarrow \mathrm{PGL}(V)$ est projectivement anosovienne si et seulement s'il existe un ouvert proprement convexe Γ-invariant Ω et un convexe fermé Γ-invariant $C \subset \Omega$ tels que les points de $\overline{C} \cap \partial\Omega$ sont extrémaux et $\mathcal{C}^1$, et C/Γ est compact.

Remarquons que toute représentation projectivement anosovienne ρ dans $\mathrm{PGL}(\mathbf{V})$ ne préserve pas un ouvert proprement convexe, cependant on peut toujours trouver un espace plus grand V' et un plongement $\iota : \mathrm{PGL}(\mathbf{V}) \hookrightarrow \mathrm{PGL}(V')$ tel que $\iota \circ \rho$ préserve un ouvert proprement convexe, et soit encore anosovienne (cf. [DGKa] et [Zim20]).

Notons également que Kapovich–Leeb [KL18] et Zhu [Zhub] ont défini des notions de représentations *relativement anosoviennes*, qui généralisent aussi les représentations anosoviennes, mais dans une direction différente. Pour faire simple, les représentations projectivement convexe cocompactes ajoutent aux variétés hyperboliques fermées les variétés de Benoist (entre autres), tandis que les représentations relativement anosoviennes leur ajoutent les variétés hyperboliques non compactes de volume fini (entre autres). Plus généralement, Zhu montre que si Ω est strictement convexe de bord $\mathcal{C}^1$, et si l'action de Γ sur Ω est géométriquement finie au sens de Crampon–Marquis, alors l'inclusion de Γ dans $\mathrm{PGL}(\mathbf{V})$ est relativement anosovienne.

0.1.6 Orbivariétés convexes projectives de rang un

On a vu à travers les orbivariétés de Benoist (section 0.1.3) qu'il existe une analogie entre les orbivariétés projectives convexes et les variétés riemanniennes à courbure négative ou nulle. Cette analogie s'est récemment trouvée renforcée par les travaux de M. Islam [Isl] et A. Zimmer [Zim], qui ont défini les orbivariétés projectives convexes *de rang un*.

Rappelons qu'une variété riemannienne de courbure négative ou nulle est dite *de rang un* si elle possède une géodésique fermée de rang un, c'est-à-dire une géodésique le long de laquelle les champs de Jacobi parallèlement transportés par le flot géodésique sont tangents à la géodésique (voir par exemple [Kni02, Def. 5.1.1]) ; grosso modo cela signifie que la géodésique n'admet pas de géodésique qui lui soit parallèle, et cela implique que le flot géodésique a un comportement de type anosovien à proximité de la géodésique. Il existe une littérature très riche concernant la dynamique du flot géodésique sur les variétés riemanniennes de courbure négative ou nulle et de rang un (ou plus généralement sur les espaces localement CAT(0) de rang un), à laquelle ce texte doit beaucoup, voir la section 0.1.1. Ainsi, on considèrera principalement dans ce texte des orbivariétés projectives convexes de rang un, dont on donne à présent la définition.

Soit $M = \Omega/\Gamma$ une orbivariété projective convexe. On dira qu'un point de $\partial\Omega$ est *fortement extrémal* s'il n'appartient à aucun segment non trivial du bord $\partial\Omega$. Une géodésique de Ω — ainsi que sa projection dans M — est dite *de rang un* si ses extrémités dans $\partial\Omega$ sont $\mathcal{C}^1$ et fortement extrémales. L'orbivariété M est dite *de rang un* si elle possède une géodésique fermée de rang un. Les géodésiques d'une orbivariété projective convexe de type "courbure négative" sont toutes de rang un. Un élément g de $\mathrm{Aut}(\Omega)$ est dit *de rang un* s'il est d'ordre infini et s'il préserve une géodésique de rang un.

On peut démontrer facilement que si M est de rang un, alors Ω n'est pas un joint, et de plus Ω n'est pas symétrique de rang supérieur. Sous la condition que M est compacte, il existe une réciproque à ce fait, appelée théorème de rigidité en rang supérieur (analogue du théorème du même nom en géométrie riemannienne), et due à Zimmer [Zim, Th. 1.4]. En particulier, les orbivariétés de Benoist sont de rang un.

0.2 Présentation des résultats de la thèse

La principale problématique de la thèse est l'étude de la dynamique du flot géodésique sur les orbivariétés projectives convexes. Les résultats obtenus en ce sens, réunis ici dans la partie III, ont donné lieu à trois articles, dont un en collaboration avec Feng Zhu. Le premier [Bla] concerne des propriétés de récurrence topologiques du flot géodésique, et fait l'objet du chapitre 5. Le deuxième [Bla21b] concerne des aspects de récurrence probabilistes, par rapport à une famille de mesures dites de Sullivan, qu'on a déjà évoquées : cet article fait l'objet des chapitres 6 et 7. Le troisième article [BZ21], en collaboration avec F. Zhu, traite de questions d'équidistribution de Γ-orbites et de $(\phi_t)_t$-orbites périodiques, vis-à-vis aussi des mesures de Sullivan : cet article fait l'objet des chapitres 8 et 9.

Les recherches qui ont abouti aux résultats de [Bla21b], avant d'atteindre le niveau de généralité qui est à présent le leur, ne concernaient dans un premier temps que le flot géodésique des orbivariétés projectives convexes compactes. Les premières preuves qui ont été élaborées dans ce cadre étaient étroitement liées à une question de Benoist [Ben12, Prob. 5] à propos du bord des convexes divisibles, à laquelle nous avons pu apporter une réponse ; cette dernière fait l'objet du chapitre 4 et de l'article [Bla21a]. Le lien entre ce chapitre et la partie III est à présent plus ténu : celui-là permet de simplifier une preuve de celle-ci dans le cas particulier des orbivariétés projectives convexes compactes. Le chapitre 4 est contenu dans la partie II, qui comporte d'autres résultats préliminaires de géométrie projective convexe. On introduit dans le chapitre 3 les orbivariétés projectives convexes de rang un, qui constitueront le cadre principal de la partie III ; on y démontre de plus quelques résultats utiles. La notion "de rang un" en géométrie projective convexe est dûe à Islam [Isl], et est analogue à la notion du même nom en géométrie riemannienne.

Au cours de la thèse, un autre sujet a été abordé, qui concerne aussi les orbivariétés projectives convexes, mais pas directement leur flot géodésique. Partiellement en collaboration avec Harrison Bray, nous étudions certains espaces de modules de structures projectives convexes, dans le but de comprendre le comportement de l'exposant critique, ou au moins de mettre en lumière certains phénomènes intéressants. Ce projet fait l'objet de la partie IV, et de deux articles en cours de rédaction (le second avec H. Bray).

La partie I introduit les notions et résultats classiques dont nous aurons besoin par la suite : le chapitre 1 concerne la théorie des systèmes dynamiques, et le chapitre 2 la géométrie projective convexe.

0.2.1 L'ensemble limite proximal des groupes cocompacts

Le chapitre 4, très court et indépendant du reste de la thèse, répond à une question de Benoist concernant l'ensemble limite proximal (défini ci-dessous) des sous-groupes de $\mathrm{PGL}(\mathbf{V})$ qui divisent un ouvert proprement convexe.

Rappelons qu'un élément $g \in \mathrm{PGL}(\mathbf{V})$ est dit *proximal* si son action sur $\mathrm{P}(\mathbf{V})$ admet un point fixe attractif x_g^+, et *biproximal* si g et g^{-1} sont proximaux (on pose alors $x_g^- := x_{g^{-1}}^+$). *L'ensemble limite proximal* $\Lambda_\Gamma^{\mathrm{prox}}$ d'un sous-groupe $\Gamma \subset \mathrm{PGL}(\mathbf{V})$ est l'adhérence dans $\mathrm{P}(\mathbf{V})$ de l'ensemble des points fixes attractifs des éléments proximaux de Γ. Si Γ préserve un ouvert proprement convexe Ω, alors $\Lambda_\Gamma^{\mathrm{prox}}$ est contenu dans l'ensemble limite orbital total $\Lambda_\Gamma^{\mathrm{orb}} \subset \partial\Omega$ de (0.1.2).

Théorème 0.2.1. *Soient $\Omega \subset \mathrm{P}(\mathbf{V})$ un ouvert proprement convexe et $\Gamma \subset \mathrm{Aut}(\Omega)$ un sous-groupe discret et Zariski-dense dans $\mathrm{PGL}(\mathbf{V})$ qui divise Ω. Alors $\Lambda_\Gamma^{\mathrm{prox}} = \partial\Omega$.*

On utilise pour démontrer ce théorème un résultat de rigidité en rang supérieur de Zimmer [Zim, Th. 1.4] qui nous dit que Ω/Γ est de rang un sous les hypothèses de la proposition. De plus, on fait appel à une version faible du lemme de l'ombre de Sullivan, qui ne fait pas intervenir les densités conformes : on démontre que toute ombre de rayon suffisamment grand contient un point de $\Lambda_\Gamma^{\mathrm{prox}}$ (une ombre de rayon R étant une projection stéréographique dans $\partial\Omega$ d'une boule de rayon R de Ω pour la distance de Hilbert). À cela s'ajoute un lemme de topologie algébrique, qui permet de démontrer que s'il est non vide, l'intérieur (par rapport à $\partial\Omega$) de l'ensemble des points non extrémaux de $\partial\Omega$ contient des ombres de rayons arbitrairement grands.

0.2.2 Propriétés de récurrence topologiques

Soit $M = \Omega/\Gamma$ une orbivariété projective convexe. On s'intéresse dans le chapitre 5 aux propriétés de récurrence topologiques de $(\phi_t)_t$, plus précisément à l'adhérence des orbites périodiques, à l'ensemble non errant, et au mélange topologique. Dans le cas où M est de type "courbure négative" et non élémentaire, on a vu à la section 0.1.5 que les orbites périodiques sont denses dans l'ensemble non errant, qui coïncide avec l'ensemble, noté T^1M_{cor}

en général, des vecteurs tangents aux géodésiques contenues dans le cœur convexe, et le flot géodésique est topologiquement mélangeant dessus. Dans le cas général, la situation est plus variée : les divers sous-ensembles $(\phi_t)_t$-invariants de T^1M qui sont dynamiquement ou géométriquement intéressants peuvent différer ; cependant il existe toujours un fermé invariant naturel sur lequel $(\phi_t)_t$ est topologiquement mélangeant, que nous introduisons maintenant.

Par définition, *le fibré unitaire tangent biproximal* de M, noté T^1M_{bip}, est la projection dans T^1M de l'ensemble des vecteurs $v \in T^1\Omega$ tels que les extrémités de la géodesique rectiligne tangente à v, notées $\phi_\infty v := \lim_{t\to\infty} \pi\phi_t v$ et $\phi_{-\infty}v$, sont dans l'ensemble limite proximal $\Lambda_\Gamma^{\mathrm{prox}}$ (défini à la section 0.1.6).

Notons $\mathrm{Per} \subset T^1M$ l'ensemble des vecteurs périodiques, et $\mathrm{Per}_{\mathrm{bip}} \subset \mathrm{Per}$ l'ensemble des vecteurs périodiques dont les relevés dans $T^1\Omega$ sont tangents à une géodésique de la forme $]x_\gamma^-, x_\gamma^+[\subset \Omega$ avec $\gamma \in \Gamma$ biproximal. En général on a

$$\mathrm{Per}_{\mathrm{bip}} \subset T^1M_{\mathrm{bip}} \subset T^1M_{\mathrm{cor}}.$$

Si de plus Γ est fortement irréductible (i.e. ne préserve pas d'union finie de sous-espaces projectifs propres de $\mathrm{P}(\mathbf{V})$), alors

$$\emptyset \subset \overline{\mathrm{Per}}_{\mathrm{bip}} = T^1M_{\mathrm{bip}} \subset \overline{\mathrm{Per}} \subset \mathrm{NW}(T^1M) \subset T^1M_{\mathrm{cor}} \subset T^1M, \tag{0.2.1}$$

et il existe des exemples où aucune des inclusions, à part $\overline{\mathrm{Per}} \subset \mathrm{NW}(T^1M)$, n'est une égalité (et tels que Γ est Zariski-dense dans $\mathrm{PGL}(\mathbf{V})$ et agit convexe cocompactement sur Ω) ; de tels exemples peuvent être construits en utilisant l'article en préparation de Danciger–Guéritaud–Kassel [DGKb] (dont l'un des principaux outils est une technique de “ping-pong”), cf. [DGKa, Prop. 12.5]. En revanche, je ne connais pas d'exemple fortement irréductible où $\overline{\mathrm{Per}}$ est différent de $\mathrm{NW}(T^1M)$ (il existe des exemples compacts Ω/Γ où Ω est un triangle et le flot géodésique sur T^1M n'admet pas d'orbite périodique).

Le principal résultat du chapitre 5 est le suivant.

Théorème 0.2.2. *Soient $\Omega \subset \mathrm{P}(\mathbf{V})$ un ouvert proprement convexe et $\Gamma \subset \mathrm{Aut}(\Omega)$ un sous-groupe discret fortement irréductible. Alors $(\phi_t)_t$ est topologiquement mélangeant sur $T^1(\Omega/\Gamma)_{\mathrm{bip}}$.*

La preuve utilise de façon cruciale les travaux de Benoist [Ben96, Ben97, Ben00a]. Elle repose sur une petite étude des variétés stables des ouverts proprement convexes, sur la construction de sous-semi-groupes Schottky de Γ constitués d'éléments biproximaux, et enfin sur l'utilisation des travaux de Benoist sur les propriétés asymptotiques des projections de Jordan et de Cartan des sous-semi-groupes Zariski-denses des groupes de Lie réels semi-simples. Ces propriétés asymptotiques permettent notamment d'établir la *non arithméticité du spectre des longueurs*, c'est-à-dire le fait que les longueurs des $(\phi_t)_t$-orbites périodiques engendrent un sous-groupe (additif) dense dans $\mathbb{R}$; l'équivalence entre mélange topologique et non arithméticité du spectre des longueurs a été établie dans le cadre riemannien de courbure négative par Dal'bo [Dal00].

La collaboration avec F. Zhu a révélé que pour les orbivariétés de rang un, l'hypothèse d'irréductibilité forte peut être remplacée par l'hypothèse plus faible “non élémentaire”. Pour tout ouvert proprement convexe $\Omega \subset \mathrm{Aut}(\Omega)$, un sous-groupe de rang un discret $\Gamma \subset \mathrm{Aut}(\Omega)$ est dit *non élémentaire* s'il n'est pas virtuellement nilpotent, ce qui revient en fait à dire qu'il n'est pas virtuellement isomorphe à $\mathbb{Z}$. Le résultat suivant est contenu dans [BZ21].

Théorème 0.2.3. *Soient* $\Omega \subset \mathrm{P}(\mathbf{V})$ *un ouvert proprement convexe et* $\Gamma \subset \mathrm{Aut}(\Omega)$ *un sous-groupe discret. Supposons que* $M = \Omega/\Gamma$ *est de rang un et est non élémentaire. Alors le flot géodésique est topologiquement mélangeant sur* $T^1 M_{\mathrm{bip}}$.

Je pense que le théorème 0.2.3 implique le théorème 0.2.2, mais je ne sais pas le démontrer pour l'instant.

Conjecture 0.2.4. *Tout sous-groupe discret fortement irréductible de* $\mathrm{PGL}(\mathbf{V})$ *qui préserve un ouvert proprement convexe* Ω *est de rang un.*

Les théorèmes 0.2.2 et 0.2.3 sont complétés par le résultat de minimalité suivant, qui repose sur une étude des vecteurs de T^1M qui sont récurrents sous l'action de $(\phi_t)_t$. Rappelons qu'un flot continu $(\psi_t)_t$ sur un espace X est dit *topologiquement transitif* si tout ouvert non vide $(\phi_t)_t$-invariant est dense (voir la section 1.1.3), et $(\psi_t)_t$ est dit *non errant* sur X si son ensemble non errant est X.

Proposition 0.2.5. *Soient* $\Omega \subset \mathrm{P}(\mathbf{V})$ *un ouvert proprement convexe et* $\Gamma \subset \mathrm{Aut}(\Omega)$ *un sous-groupe discret. Soit* $X \subset T^1M$ *un fermé invariant non vide sur lequel* $(\phi_t)_t$ *est topologiquement transitif et non-errant. Si* $\emptyset \neq T^1M_{\mathrm{bip}} \subset X$*, alors* $X = T^1M_{\mathrm{bip}}$.

On trouvera dans la section 5.1.2 une comparaison des résultats énoncés ci-dessus avec un résultat sur le flot géodésique de variétés riemanniennes à courbure négative ou nulle et de rang un, dû à Coudène–Schapira [CS10].

Pour clôre cette section, mentionnons un dernier résultat du chapitre 5, qui concerne des orbivariétés projectives convexes pour lesquelles T^1M_{bip} est vide.

Proposition 0.2.6. *Soit* M *une orbivariété projective convexe compacte irréductible et de rang supérieur (i.e. pas de rang un). Alors* $\mathrm{NW}(T^1M)$ *a plusieurs composantes connexes, sur lesquelles le flot géodésique est topologiquement mélangeant.*

0.2.3 La dichotomie de Hopf–Tsuji–Sullivan–Roblin

On a vu que les densités conformes de Patterson et Sullivan sont des outils précieux pour étudier les variétés riemanniennes de courbure négative ou nulle, les orbivariétés projectives convexes de type “courbure négative”, et les orbivariétés de Benoist. Il semble donc naturel de vouloir les adapter à l'étude probabiliste du flot géodésique sur les orbivariétés projectives convexes de rang un $M = \Omega/\Gamma$.

Au chapitre 6 nous introduisons les densités conformes sur le bord projectif $\partial\Omega$, et les mesures de Sullivan associées sur $T^1\Omega$, $T^1M = T^1\Omega/\Gamma$ et $\mathrm{Geod}(\Omega) = T^1\Omega/(\phi_t)_t$, puis établissons la dichotomie de Hopf–Tsuji–Sullivan–Roblin (HTSR). Ce dernier résultat est analogue au théorème du même nom en géométrie riemannienne et plus généralement en géométrie CAT(0), où il tient un rôle central : il fait le lien entre la dynamique de $(\phi_t)_t$ — plus précisément les aspects probabilistes de cette dynamique vus par les mesures de Sullivan — et la géométrie de Γ en tant que sous-groupe de $\mathrm{PGL}(\mathbf{V})$.

Rappelons rapidement la définition de quelques propriétés de récurrence probabilistes qui apparaissent dans la dichotomie HTSR. Soit H un groupe topologique localement compact à base dénombrable et unimodulaire (dans notre cas H est $\mathbb{R}$ ou un groupe dénombrable ou le produit de $\mathbb{R}$ avec un groupe dénombrable), qui agit sur un espace mesurable X muni d'une mesure σ-finie G-invariante m. Un sous-ensemble mesurable $A \subset X$ est errant par rapport à m si $\{g \in G : gx \in A\}$ est relativement compact pour m-presque tout $x \in A$. L'action de G est *conservative* si les ensembles errants sont tous de mesure nulle, et *dissipative* si X est, à un ensemble de mesure nulle près, une union dénombrable d'ensembles errants.

L'*ensemble limite conique* $\Lambda_\Gamma^{\mathrm{con}}$ est l'ensemble des points $\xi \in \partial\Omega$ pour lesquels il existe $x \in \Omega$ et une suite divergente $(\gamma_n)_n \subset \Gamma$ tels que $(\gamma_n x)_n$ reste à distance bornée de $[x, \xi[\subset \Omega$. On note $\partial_{\mathrm{sse}}\Omega$ l'ensemble des points $\mathcal{C}^1$ et fortement extrémaux de $\partial\Omega$.

Théorème 0.2.7. *Soient $\Omega \subset \mathrm{P}(V)$ un ouvert proprement convexe et $\Gamma \subset \mathrm{Aut}(\Omega)$ un sous-groupe discret. Supposons que $M = \Omega/\Gamma$ est de rang un et non élémentaire. Soient $\delta \geq 0$ et $(\mu_x)_{x\in\Omega}$ une densité conforme Γ-équivariante de dimension δ sur $\partial\Omega$. On note respectivement m, m_Γ et $m_{\mathbb{R}}$ les mesures de Sullivan sur $T^1\Omega$, T^1M et $\mathrm{Geod}(\Omega)$. Fixons $o \in \Omega$. Il y a alors deux possibilités :*

1. *ou bien $\sum_{\gamma\in\Gamma} e^{-\delta d_\Omega(o,\gamma o)} < \infty$, auquel cas $\mu_o(\Lambda_\Gamma^{\mathrm{con}}) = 0$, et les systèmes $(T^1\Omega, \mathbb{R} \times \Gamma, m)$, $(T^1M, (\phi_t)_t, m_\Gamma)$ et $(\mathrm{Geod}(\Omega), \Gamma, m_{\mathbb{R}})$ sont dissipatifs et non ergodiques.*

2. *ou bien $\sum_{\gamma\in\Gamma} e^{-\delta d_\Omega(o,\gamma o)} = \infty$, auquel cas $\delta = \delta_\Gamma$, et*

   - *$(\mu_x)_x$ est l'unique densité conforme Γ-équivariante de dimension δ (à multiplication par un scalaire près);*
   - *$\mu_o(\partial_{\mathrm{sse}}\Omega \cap \Lambda_\Gamma^{\mathrm{prox}} \cap \Lambda_\Gamma^{\mathrm{con}}) = \mu_o(\partial\Omega)$ et μ_o n'a pas d'atome ; en particulier le support de m_Γ est T^1M_{bip};*
   - *les systèmes dynamiques $(T^1\Omega, \mathbb{R}\times\Gamma, m)$, $(T^1M, (\phi_t)_t, m_\Gamma)$ et $(\mathrm{Geod}(\Omega), \Gamma, m_{\mathbb{R}})$ sont conservatifs et ergodiques;*
   - *si m_Γ est finie, alors elle est mélangeante sous l'action de $(\phi_t)_t$.*

Par définition, δ_Γ est le supremum des nombres $\delta \geq 0$ tels que la *série de Poincaré* $\sum_{\gamma\in\Gamma} e^{-\delta d_\Omega(o,\gamma o)}$ diverge. D'après Danciger–Guéritaud–Kassel (voir le fait 2.2.9), le fait que la série de Poincaré diverge ne dépend pas de o ni de Ω, mais seulement de Γ en tant que sous-groupe de $\mathrm{PGL}(\mathbf{V})$.

Le théorème 0.2.7 donne beaucoup plus d'informations dans le cas (2), dit divergent, qui est d'ailleurs souvent considéré comme le cas le plus intéressant de la dichotomie.

La démonstration de la dichotomie HTSR que nous donnons dans le chapitre 6 est calquée sur celle de Roblin [Rob03] dans le cadre de la géométrie CAT(-1) ; en particulier, elle fait intervenir une version projective convexe du lemme de l'ombre de Sullivan, qui estime la mesure des ombres (voir la section 0.2.1). Un passage important de la démonstration, qui n'existe pas chez Roblin, est celui où l'on démontre que $\partial_{\mathrm{sse}}\Omega$ est de mesure pleine, qui est utilisé ensuite dans la preuve de l'ergodicité et du mélange. Pour démontrer ceci, nous établissons d'abord la conservativité (grâce aux idées de Roblin), puis utilisons une idée de Knieper (en géométrie riemannienne de courbure négative ou nulle), qui est que par le théorème de récurrence de Poincaré, presque tous les vecteurs sont récurrents, enfin on étudie la limite à l'infini $\phi_\infty v$ des vecteurs $v \in T^1\Omega$ qui se projettent sur un vecteur récurrent de T^1M.

Le fait que, dans le cas divergent, presque tous les points du bord sont $\mathcal{C}^1$ et fortement extrémaux s'avère utile lorsque l'on étudie des propriétés plus fines du flot géodésique, comme dans les chapitres 7 et 8.

0.2.4 La mesure d'entropie maximale

Le chapitre 7 est consacré aux orbivariétés projectives convexes de rang un $M = \Omega/\Gamma$ qui ont un cœur convexe compact non vide (l'action de Γ sur Ω est donc convexe cocompact au sens de la section 0.1.5). On y démontre, en s'inspirant des travaux de Knieper [Kni98] en géométrie riemannienne de courbure négative ou nulle, que la mesure de Sullivan associée à toute densité conforme de dimension δ_Γ, une fois renormalisée, est l'unique probabilité d'entropie maximale ; on l'appellera *probabilité de Bowen–Margulis*.

La définition d'entropie est un peu fastidieuse à énoncer, c'est pourquoi nous renvoyons à la section 1.3.1 ; on précise seulement qu'étant donné un flot continu sur un espace compact, on peut associer à chaque probabilité invariante un nombre positif appelé l'entropie, qu'on peut décrire comme le taux de croissance exponentiel quand t grandit de la quantité d'information qu'il faut avoir sur un point pour connaître sa trajectoire jusqu'au temps t ; l'entropie topologique est le supremum des entropies des probabilités invariantes.

Théorème 0.2.8. *Soient $\Omega \subset \mathrm{P}(\mathbf{V})$ un ouvert proprement convexe et $\Gamma \subset \mathrm{Aut}(\Omega)$ un sous-groupe discret. Supposons que $M = \Omega/\Gamma$ est non élémentaire de rang un, et que son cœur convexe est compact. Alors la mesure de Sullivan associée à une densité conforme de dimension δ_Γ est finie, et, après renormalisation, est l'unique probabilité d'entropie maximale sur T^1M_{cor}, d'entropie δ_Γ.*

Pour démontrer que la mesure de Sullivan est finie, il suffit de voir qu'elle est à support dans T^1M_{cor} qui est compact, puis utiliser le fait qu'elle est de Radon. Cela implique d'après le théorème de récurrence de Poincaré que l'on est dans le cas divergent de la dichotomie HTSR (théorème 0.2.7). Grâce à un argument dû à Manning [Man79], on démontre que l'entropie topologique est plus petite que δ_Γ. Il faut ensuite établir la propriété de Gibbs pour m_Γ sur T^1M, c'est-à-dire une estimation du volume des boules dynamiques, de la forme

$$C^{-1}e^{-\delta_\Gamma T} \leq m_\Gamma\left(\left\{w \in T^1M_{\mathrm{cor}} : d_{T^1M}(\phi_t v, \phi_t w) \leq R \quad \forall 0 \leq t \leq T\right\}\right) \leq Ce^{-\delta_\Gamma T},$$

pour tous $v \in T^1M_{\mathrm{cor}}$ et $T > 0$ et $R > 0$, où C ne dépend que de R. La majoration sert à démontrer que l'entropie de m_Γ est plus grande que δ_Γ, donc égale à δ_Γ et à l'entropie topologique. La minoration, combinée à l'ergodicité de m_Γ (résultant de la dichotomie HTSR, i.e. du théorème 0.2.7.2), et à un résultat général dû à Bowen sur les flots expansifs pour l'entropie, sert à démontrer l'unicité de la mesure d'entropie maximale.

En s'inspirant à nouveau des travaux de Knieper, on peut utiliser le théorème 0.2.8 et le mélange de m_Γ (théorème 0.2.7.2) pour établir des résultats d'équidistribution et de comptage sur l'ensemble des orbites périodiques. Rappelons que pour toute variété hyperbolique fermée, il existe une correspondance entre les $(\phi_t)_t$-orbites fermées et les classes de conjugaison du groupe fondamental. Cette correspondance ne tient plus dans le cas général des orbivariétés projectives convexes de rang un, mais il existe toujours un lien fort entre les $(\phi_t)_t$-orbites périodiques et les classes de conjugaison de Γ (voir la section 3.3). Notons $[\Gamma] = \{[\gamma] : \gamma \in \Gamma\}$ l'ensemble des classes de conjugaison de Γ, et $[\Gamma]^{\mathrm{r1}}$ (resp. $[\Gamma]^{\mathrm{sing}}$) le sous-ensemble de $[\Gamma]$ constitué des classes d'éléments de rang un (resp. qui ne sont pas de rang un) — cf. la section 0.1.6. Pour tout élément $c \in [\Gamma]^{\mathrm{r1}}$, on note $\mathcal{L}c$ l'unique probabilité $(\phi_t)_t$-invariante sur T^1M dont le support est la géodésique de rang un associée à c. Pour tout élément $g \in \mathrm{PGL}(\mathbf{V})$, on pose

$$\ell(g) = \frac{1}{2}\log\frac{\lambda_1(\tilde{g})}{\lambda_{\mathtt{d}+1}(\tilde{g})},$$

où $\tilde{g} \in \mathrm{GL}(\mathbf{V})$ est un relevé de g dont les modules des valeurs propres sont $\lambda_1(\tilde{g}) \geq \cdots \geq \lambda_{\mathtt{d}+1}(\tilde{g})$; cette quantité ne dépend que de la classe de conjugaison de g dans $\mathrm{PGL}(\mathbf{V})$. Pour tout sous-ensemble $A \subset [\Gamma]$ et tout $T \geq 0$, on pose $A_T := \{c \in A : \ell(c) \leq T\}$.

Théorème 0.2.9. *Sous les hypothèses du théorème 0.2.8, il existe $\delta < \delta_\Gamma$ et $C > 1$ telles que*

1. *$C^{-1}e^{\delta_\Gamma T} \leq \#[\Gamma]_T^{\mathrm{rl}} \leq Ce^{\delta_\Gamma T}$ pour tout $T \geq 0$ assez grand;*

2. *$\#[\Gamma]_T^{\mathrm{sing}} \leq Ce^{\delta T}$ pour tout $T \geq 0$;*

3. *pour tout $A \subset [\Gamma]$ tel que $\lim_{T\to\infty} T^{-1}\log(\#A_T) = \delta_\Gamma$, la famille $(\frac{1}{\#A_T}\sum_{c\in A_T}\mathcal{L}c)_{T\geq 0}$ converge, pour la topologie faible-étoile, vers la probabilité d'entropie maximale sur T^1M_{cor}, lorsque T tend vers l'infini.*

Rappelons que la topologie faible-étoile sur l'ensemble des mesure de Radon sur un espace topologique X est la topologie qui rend continues les évaluations sur les fonctions continues à support compact.

Islam [Isl, Th. 1.12] a obtenu l'estimée (1) dans le cas où M est compacte avec des méthodes différentes. Avec encore d'autres méthodes, cette estimée est améliorée dans le chapitre 8 (voir le théorème 0.2.11), issu de la collaboration avec F. Zhu, où l'on redémontre avec des méthodes différentes l'énoncé d'équidistribution 0.2.9.(3) pour $A = [\Gamma]^{\mathrm{rl}}$. On pourra rapprocher l'estimée (2) d'un autre résultat de Islam [Isl, Th. 1.11] concernant (dans le cas compact) les marches aléatoires sur Γ.

La démonstration du point (1) repose sur le mélange du flot géodésique et sur un lemme de fermeture des orbites qui reviennent très près de leur point de départ (c'est le lemme 5.4.13). Le point (2) est démontré à l'aide de l'unicité de la mesure d'entropie maximale : les classes de conjugaison d'éléments qui ne sont pas de rang un correspondent à des géodésiques fermées (pas forcément rectilignes) dans un fermé $(\phi_t)_t$-invariant de T^1M_{cor} qui est de mesure nulle pour la mesure d'entropie maximale, si bien que l'entropie topologique du flot géodésique sur ce fermé est strictement inférieure à δ_Γ. Enfin, pour établir le point (3), il s'agit de démontrer que tout point d'accumulation de la famille de probabilités en question a une entropie supérieure à δ_Γ.

0.2.5 Équidistribution des Γ-orbites et des géodésiques fermées

Dans le chapitre 8, qui résulte d'une collaboration avec Feng Zhu, on étend des résultats d'équidistribution de F. Zhu pour les orbivariétés projectives convexes de type "courbure négative" à la classe plus générale des orbivariétés projectives convexes de rang un. Les preuves sont fortement inspirées des travaux de Roblin [Rob03], et ont pour principaux outils le mélange du flot géodésique et le lemme de l'ombre. Soit $M = \Omega/\Gamma$ une orbivariété projective convexe. Le premier résultat décrit la façon dont les points d'une Γ-orbite se répartissent sur le bord $\partial\Omega$.

On note $\|\mu\|$ la masse totale d'une mesure finie μ ; étant donnés des espaces mesurés (X,μ) et (Y,ν), on note $\mu\otimes\nu$ la mesure produit $X\times Y$; étant donnés X et $x\in X$, on note $\mathcal{D}_x$ la mesure de Dirac de masse 1 en x.

Théorème 0.2.10. *Soient $\Omega \subset \mathrm{P}(\mathbf{V})$ un ouvert proprement convexe et $\Gamma \subset \mathrm{Aut}(\Omega)$ un sous-groupe discret tels que $M = \Omega/\Gamma$ est non élémentaire de rang un. Soient $(\mu_x)_{x\in\Omega}$ une densité conforme de dimension δ_Γ sur $\partial\Omega$, et m_Γ la mesure de Sullivan associée sur T^1M. Supposons que m_Γ est finie. Alors pour tous $x, y \in \Omega$,*

$$\delta_\Gamma\|m_\Gamma\|e^{-\delta_\Gamma T}\sum_{\substack{\gamma\in\Gamma\\ d_\Omega(x,\gamma y)\leq T}}\mathcal{D}_{\gamma y}\otimes\mathcal{D}_{\gamma^{-1}x} \underset{T\to\infty}{\longrightarrow} \mu_x\otimes\mu_y$$

pour la topologie faible-étoile. En particulier,

$$\#\{\gamma\in\Gamma : d_\Omega(x,\gamma y)\leq T\} \underset{T\to\infty}{\sim} \frac{\|\mu_x\|\cdot\|\mu_y\|}{\delta_\Gamma\|m_\Gamma\|}e^{\delta_\Gamma T}.$$

L'ingrédient le plus important dans la démonstration de ce théorème est le mélange de m_Γ (en fait, d'après Roblin, le théorème ci-dessus est équivalent au mélange de m_Γ). À cela il faut ajouter des estimées dans le style du lemme de l'ombre de Sullivan et de la propriété de Gibbs, mais en faisant attention à contrôler les constantes multiplicatives qui apparaissent.

Le deuxième résultat, dont la preuve fait appel au théorème 0.2.10, décrit la façon dont les géodésiques fermées de rang un (définies à la section 0.1.6) se répartissent dans T^1M. Il est similaire au théorème 0.2.9. On note $\mathcal{G}^{\mathrm{r1}}$ l'ensemble des $(\phi_t)_t$-orbites périodiques de rang un de T^1M. Pour chaque $c \in \mathcal{G}^{\mathrm{r1}}$ on note $\ell(c)$ sa période, et l'on note $\mathcal{L}c$ l'unique probabilité $(\phi_t)_t$-invariante sur c. Enfin pour tout $T \geq 0$ on pose $\mathcal{G}^{\mathrm{r1}}_T = \{c \in \mathcal{G}^{\mathrm{r1}} : \ell(c) \leq T\}$.

Théorème 0.2.11. *Sous les hypothèses du théorème 0.2.10, pour la topologie faible-étoile on a*

$$\delta_\Gamma T \|m_\Gamma\| e^{-\delta_\Gamma T} \sum_{c \in \mathcal{G}^{\mathrm{r1}}_T} \mathcal{L}c \underset{T\to\infty}{\longrightarrow} m_\Gamma.$$

En particulier, si Γ agit convexe cocompactement sur Ω, alors $\#\mathcal{G}^{\mathrm{r1}}_T \underset{T\to\infty}{\sim} e^{\delta_\Gamma T}/(\delta_\Gamma T)$.

Contrairement au théorème 0.2.9, la convergence des mesures dans le théorème ci-dessus ne nécessite pas d'hypothèse de convexe cocompacité ; de plus, le résultat de comptage obtenu dans le cas convexe cocompact est plus précis. Le théorème 0.2.9 avait en revanche ceci d'intéressant que l'équidistribution s'appliquait à des sous-ensembles de $[\Gamma]^{\mathrm{r1}}$, et que l'on avait des estimée sur le nombre de classes de conjugaison qui ne sont pas de rang un. Il est possible d'affiner le théorème 0.2.11 en y ajoutant un résultat sur les classes de conjugaison qui ne sont pas de rang un, mais ce résultat est plus délicat à formuler car il faut expliquer quand et comment on peut réaliser ces classes en tant que géodésiques fermées dans M ; mentionnons simplement le fait que pour tout $x \in \Omega$, le nombre d'éléments $\gamma \in \Gamma$ qui ne sont pas de rang un et tels que $d_\Omega(x, \gamma x) \leq T$ est négligeable devant $e^{\delta_\Gamma T}$ quand T tend vers l'infini.

La démonstration du théorème 0.2.11 a ceci en commun avec celle du théorème 0.2.9 qu'elle utilise un lemme de fermeture (lemme 5.4.11), qui implique que, ayant fixé $x \in \Omega$, la distance dans $\mathrm{P}(\mathbf{V})^2$ entre $(\gamma^{-1}x, \gamma x)$ et (x_γ^-, x_γ^+) (les points fixes répulsif et attractif de γ) tend vers zéro quand T tend vers l'infini pour presque tous les éléments de rang un $\gamma \in \Gamma$ tels que $d_\Omega(x, \gamma x) \leq T$. Il s'agit alors de combiner cette propriété avec le théorème 0.2.10.

0.2.6 Équidistribution dans les orbivariétés projectives convexes de type “courbure négative” géométriquement finies

Inspiré une fois encore par les travaux de Roblin [Rob03], ainsi que ceux de Crampon [Cra11], Feng Zhu [Zhua] a considéré des problèmes d'équidistribution dans les orbivariétés projectives convexes de type “courbure négative” géométriquement finies $M = \Omega/\Gamma$. Il a démontré que pour de telles orbivariétés, la mesure de Sullivan de dimension δ_Γ est finie, et il a établi une version plus forte de l'équidistribution du théorème 0.2.11, plus précisément la convergence a lieu pour une topologie plus fine, celle qui rend continues les évaluations sur les fonctions continues bornées (non nécessairement à support compact).

Il est pour le moment difficile de généraliser ce résultat au cadre des orbivariétés de rang un non nécessairement de type “courbure négative”, dans la mesure où la notion de finitude géométrique n'y est pas encore suffisamment bien développée. Toutefois nous avons pu améliorer légèrement ce résultat en remarquant qu'il s'étend à une classe plus large d'orbivariétés projectives convexes $M = \Omega/\Gamma$ de type “courbure négative”, dites *géométriquement finies au bord* (voir la section 9.1), qui furent définies par Crampon–Marquis [CM14a] en même temps que celles géométriquement finies. Cette généralisation fait l'objet du chapitre 9. Crampon–Marquis ont donné un exemple d'orbivariété géométriquement finie au bord mais pas géométriquement finie au sens fort, que Feng et moi avons trouvé très intéressant : nous voulions comprendre la dynamique de son flot géodésique. Crampon–Marquis expliquent que l'une des difficultés pour comprendre de telles orbivariétés provient du fait que le cœur convexe de M privé de ses pointes n'est pas compact ; l'observation faite au chapitre 9 qui permet de contourner cette difficulté est que T^1M_{cor} privé des pointes est lui compact (voir la proposition 9.3.4).

Comme M est de type "courbure négative", les géodésiques sont toutes de rang un, et les éléments $\gamma \in \Gamma$ qui sont de rang un sont exactement ceux pour lesquels $\ell(\gamma) > 0$ (Crampon–Marquis les qualifient d'*hyperboliques*, qui est la terminologie usuelle dans le cadre des variétés hyperboliques).

Théorème 0.2.12. *Soient $\Omega \subset \mathrm{P}(\mathbf{V})$ un ouvert proprement convexe et $\Gamma \subset \mathrm{Aut}(\Omega)$ un sous-groupe discret tels que $M = \Omega/\Gamma$ est non élémentaire de rang un. Soient $(\mu_x)_{x\in\Omega}$ une densité conforme de dimension δ_Γ sur $\partial\Omega$, et m_Γ la mesure de Sullivan associée sur T^1M. Supposons que M est géométriquement finie au bord. Alors m_Γ est finie, et pour toute fonction continue bornée f sur T^1M, on a*

$$\delta_\Gamma T \|m_\Gamma\| e^{-\delta_\Gamma T} \sum_{c\in[\Gamma]_T^{\mathrm{r1}}} \int_{T^1M} f \,\mathrm{d}\mathcal{L}c \underset{T\to\infty}{\longrightarrow} \int_{T^1M} f \,\mathrm{d}m_\Gamma.$$

En particulier, $\#[\Gamma]_T^{\mathrm{r1}} \underset{T\to\infty}{\sim} e^{\delta_\Gamma T}/(\delta_\Gamma T)$.

Pour démontrer que m_Γ est finie, nous nous inspirons de la thèse de Crampon [Cra11] sur les surfaces projectives convexes géométriquement finies. Un passage délicat est de démontrer que les sous-groupes paraboliques maximaux de Γ associés aux pointes de M sont divergents : lorsque M est géométriquement finie au sens fort, ces sous-groupes préservent un ellipsoïde de $\mathrm{P}(\mathbf{V})$ et cela permet d'utiliser des résultats classiques sur les variétés hyperboliques. Dans le cas général ce n'est plus vrai, mais on sait quand même d'après Crampon–Marquis que ces sous-groupes sont virtuellement des réseaux cocompacts de groupes unipotents, ainsi il suffit de montrer que les groupes unipotents sont divergents (au sens de la section 2.3.8), or ceci est une conséquence d'un lemme de géométrie algébrique de Benoist–Oh [BO12, Prop. 7.2].

Pour établir l'équidistribution, suivant l'exemple de Roblin, on utilise le théorème 0.2.11 pour se ramener au problème suivant. Pour chaque pointe $p \subset T^1M_{\mathrm{cor}}$, trouver une suite décroissante de sous-pointes $p = p_1 \supset p_2 \supset p_3 \supset \dots$ telles que $p \smallsetminus p_n$ est relativement compact pour tout n, et $(p_n)_n$ sort de tout compact du cœur convexe, puis montrer que $\limsup_{T\to\infty} Te^{-\delta_\Gamma T} \sum_{c\in[\Gamma]_T^{\mathrm{r1}}} \mathcal{L}c(p_n)$ tend vers zéro quand n tend vers l'infini.

0.2.7 L'exposant critique des orbivariétés projectives convexes

Dans la dernière partie de la thèse, il est encore question d'orbivariétés projectives convexes, mais on en examine un aspect différent : on fixe une orbivariété topologique et l'on s'intéresse au comportement asymptotique de l'exposant critique défini en (0.1.1) pour certaines familles de structures projectives convexes sur l'orbivariété.

De manière générale, à tout groupe Γ agissant par isométries sur un espace métrique propre (X, d) on peut associer l'exposant critique

$$\delta(\Gamma, X, d) := \limsup_{r\to\infty} \frac{1}{r} \log(\#\{\gamma \in \Gamma : d(x, \gamma x) \leq r\}),$$

indépendant du choix de $x \in X$. Cette quantité a fait l'objet de nombreux travaux, dans des cadres géométriques variés (par exemple riemanniens, CAT(-1), CAT(0), Gromov-hyperboliques, projectifs convexes). L'une des raisons est qu'elle coïncide souvent avec d'autres quantités géométriques ou dynamiques importantes, comme l'entropie du flot géodésique, le taux de croissance exponentiel du nombre de géodésiques fermées, ou la dimension de Hausdorff de l'ensemble limite ; parmi les travaux les plus anciens qui font état du lien entre ces diverses quantités, citons [Mar69,Pat76,Sul79,Man79] ; notons que les théorèmes 0.2.8, 0.2.9, 0.2.11 et 0.2.12 donnent aussi des exemples de ce lien, en géométrie projective convexe.

L'exposant critique intervient dans de nombreux résultats de rigidité : dans un article fondateur [Bow79], Bowen a montré que tout sous-groupe discret convexe cocompact de $\mathrm{PSL}_2(\mathbb{C})$ obtenu en déformant continûment un réseau cocompact de $\mathrm{PSL}_2(\mathbb{R})$ (*groupe quasi-fuchsien*) a un exposant critique supérieur à 1, avec égalité si et seulement s'il est conjugué à un sous-groupe de $\mathrm{PSL}_2(\mathbb{R})$. Des théorèmes de rigidité analogues à celui de Bowen ont été établis dans des cadres géométriques divers (mentionnons par exemple le célèbre article [BCG95] de Besson–Courtois–Gallot). En géométrie projective convexe, Crampon [Cra09] a montré que pour toute orbivariété projective convexe fermée $M = \Omega/\Gamma$ de type "courbure négative", l'exposant critique de Γ est inférieur à $\dim(M) - 1$, avec égalité si et seulement si M est hyperbolique. Ce résultat a été généralisé aux orbivariétés projectives convexes de volume fini par Barthelmé–Marquis–Zimmer [BMZ17] et aux orbivariétés projectives convexes de type "courbure négative" dont le cœur convexe est compact par Zimmer [Zim20] ; des résultats de rigidité différents, plus proches de [BCG95], ont récemment été obtenus dans [ABC19, BC]. Précisons enfin qu'un résultat de Tholozan [Tho17, Th. 2] (voir le Fait 2.3.17) implique que l'exposant critique de n'importe quelle orbivariété projective convexe M est inférieur à $\dim(M) - 1$.

Borne inférieure pour les exposants critique

Crampon a posé dans [Cra09, p. 3] la question suivante :

Question 0.2.13 ([Cra09]). *Quelle est la borne inférieure de l'ensemble des exposants critiques des orbivariétés projectives convexes compactes de type "courbure négative" de dimension* $\mathtt{d}$ *?*

X. Nie [Nie15b] y a répondu en dimensions deux, trois et quatre, en construisant des suites de telles orbivariétés dont l'exposant critique tend vers zéro. Les résultats de Nie sont en fait plus précis que cela. Il a considéré certaines orbivariétés topologiques compactes M de groupe fondamental Gromov-hyperbolique, puis il a considéré l'espace, noté X, de toutes les structures projectives convexes sur M, et enfin il a étudié l'exposant critique en tant que fonction sur X. Le choix de M fait par Nie est tel qu'on peut lui appliquer les travaux de Vinberg [Vin71] (le groupe fondamental de M est un groupe de Coxeter), et on peut utiliser ces travaux pour montrer que, dans les cas considérés par Nie, X est ou bien réduit à un point, qui correspond à une structure hyperbolique, ou bien homéomorphe à $[0, \infty[$, auquel cas $0 \in [0, \infty[$ est l'unique point correspondant à une structure hyperbolique. Supposons qu'on est dans le cas où X s'identifie à $[0, \infty[$. D'après Crampon [Cra09], l'exposant critique associé à un point $x \in]0, \infty[$ est strictement inférieur à $\dim(M) - 1$; Nie montre que cet exposant critique tend vers zéro quand x tend vers l'infini, autrement dit lorsque "$x \in X$ s'éloigne des orbivariétés hyperboliques".

Par la suite, des résultats analogues ont été obtenus dans le cadre des surfaces projectives convexes fermées (sans singularités, i.e. dont le groupe fondamental est sans torsion).

Fixons une surface topologique fermée et orientée Σ de genre supérieur à deux, et notons X l'espace des structures projectives convexes sur Σ. T. Zhang [Zha15b] a utilisé la paramétrisation de Goldman [Gol90] (qui généralise les coordonnées de Fenchel–Nielsen) de X pour donner un sens à "$x \in X$ s'éloigne des structures hyperboliques", et il a montré que cela impliquait que l'exposant critique associé à x tendait vers zéro. Indépendamment, Nie [Nie15a] a fait de même en utilisant une autre paramétrisation, due à Labourie [Lab07] et Loftin [Lof01].

Généralisation aux représentations de Hitchin

Zhang a généralisé ses résultats au cadre suivant. Fixons une représentation injective d'image discrète $\rho_0 : \pi_1(\Sigma) \to \mathrm{PSL}_2(\mathbb{R})$, et une représentation irréductible $\iota : \mathrm{PSL}_2(\mathbb{R}) \hookrightarrow \mathrm{PSL}(\mathbf{V})$ (où $\mathbf{V}$ est un espace vectoriel réel de dimension $\mathtt{d}+1 < \infty$). Les représentations de Hitchin de $\pi_1(\Sigma)$ dans $\mathrm{PSL}(\mathbf{V})$ sont celles obtenues en déformant continûment $\iota \circ \rho_0$. Une représentation de Hitchin préserve un ouvert proprement convexe de $\mathrm{P}(\mathbf{V})$ si et seulement si $\mathtt{d}$ est pair.

Il existe une généralisation naturelle de l'exposant critique défini en (0.1.1) qui s'applique à toutes les représentations dans $\mathrm{PGL}(\mathbf{V})$ (voir la section 2.3.8), et donc en particulier aux représentations de Hitchin. Notons que cet exposant critique "projectif" ne coïncide pas en général avec l'exposant critique classique obtenu en considérant l'action par isométries sur l'espace symétrique riemannien de $\mathrm{PGL}(\mathbf{V})$ (le premier est supérieur ou égal au second).

Zhang [Zha15a] a utilisé une paramétrisation de l'espace des représentations de Hitchin due à Bonahon–Dreyer [BD14] pour en construire une autre, plus proche des coordonnées de Fenchel–Nielsen. Il a alors donné un sens "s'éloigner du lieu fuchsien dans cette paramétrisation", où le lieu fuchsien désigne l'ensemble des représentations de la forme $\iota \circ \rho$, où $\rho : \pi_1(\Sigma) \to \mathrm{PSL}_2(\mathbb{R})$ est injective d'image discrète. Puis Zhang a montré que l'exposant critique projectif tend vers zéro lorsque la représentation s'éloigne du lieu fuchsien. Ces résultats sont à comparer au résultat de rigidité plus récent de Potrie–Sambarino [PS17, Th. A], qui concerne aussi les représentations de Hitchin.

Les surfaces projectives convexes de volume fini

Crampon [Cra11, Prop. 4.3.4] a montré que l'exposant critique d'une surface projective convexe non compacte de volume fini de type "courbure négative" est strictement supérieur à $1/2$, qui est l'exposant critique des pointes de la surface. Ainsi la question 0.2.13, transposée à ce type de surface, doit admettre une réponse différente, et nous montrons que cette réponse est simplement $1/2$. Ceci est la conséquence d'un résultat plus précis, similaire à celui de Nie [Nie15a], résultat qui utilise une paramétrisation de ces surfaces due à Benoist–Hulin [BH13], généralisant la paramétrisation de Labourie et Loftin.

Soit Σ une surface (de caractéristique d'Euler strictement négative) obtenue en enlevant un ensemble fini $\{\overline{p}_1, \dots, \overline{p}_k\}$ de points à une surface orientée fermée $\overline{\Sigma}$. Benoist–Hulin [BH13, Th. 1.1] ont construit un homéomorphisme entre l'espace des structures projectives convexes marquées de volume fini sur Σ et un certain fibré vectoriel (de dimension finie) au-dessus de l'espace des structures hyperboliques marquées de volume fini sur Σ. (Voir le chapitre 10 pour plus de détails.) Nous démontrons le résultat suivant.

Théorème 0.2.14. *Soit S une surface hyperbolique non compacte de volume fini. Notons $\mathcal{V}$ l'ensemble des structures projective convexes marquées de volume fini au-dessus de S dans la paramétrisation de Benoist–Hulin, et fixons une norme $\|\cdot\|$ dessus. Alors il existe* *une constante $C > 0$ telle que pour tout $v \in \mathcal{V}$, si δ est l'exposant critique de v, alors*

$$\frac{1}{2} < \delta \leq \frac{1}{2} + Ce^{-\frac{\|v\|}{C}}.$$

Rappelons l'idée de Nie [Nie15a] pour traiter le cas d'une surface hyperbolique compacte $S' = \mathbb{H}^2/\Gamma'$ (où $\mathbb{H}^2$ est le disque de Poincaré), dont la fibre dans la paramétrisation de Labourie et Loftin, notée $\mathcal{V}'$, est munie d'une norme $\|\cdot\|$. Chaque structure projective convexe marquée $v \in \mathcal{V}'$ induit sur $\mathbb{H}^2$ une distance Γ'-invariante d_v (la distance de Hilbert), de sorte que l'exposant critique de v est l'exposant critique de l'action de Γ' sur $(\mathbb{H}^2, d_v)$. La distance d_0 est la distance usuelle de $\mathbb{H}^2$. On peut alors reformuler la stratégie de Nie comme suit : il existe une constante $C > 0$ telle que pour tout $v \in \mathcal{V}'$ non nul, on a $d_v \geq C^{-1}\|v\|^{1/3}(d_0 - C)$. Ainsi, l'exposant critique de v est plus petit que $C\|v\|^{-1/3}$ (rappelons que l'exposant critique de S' est égal à 1).

Autrement dit, la surface S' équippée de la structure projective convexe $v \in \mathcal{V}'$ est de plus en plus “grosse” à mesure que v grandit en norme, et plus elle est “grosse”, plus son exposant critique est petit.

L'idée de la preuve du théorème 0.2.14 est la suivante. On décompose S en une partie compacte et un nombre fini de pointes disjointes ; on applique alors sur la partie compacte la stratégie de Nie [Nie15a] ; puis on étudie les pointes, plus précisément on démontre que pour tout $v \in \mathcal{V}$ de norme suffisamment grande, chaque pointe A de S contient une sous-pointe $A' \subset A$ comparable à une pointe hyperbolique standard (et A' est de plus en plus loin dans A à mesure que v grandit en norme).

Notons que l'exposant critique est une fonction continue sur l'espace tout entier des structures projectives convexes marquées de volume fini sur Σ, d'après Crampon [Cra11, Prop. 5.4.1]. Ce résultat, combiné au théorème 0.2.14, au théorème de Tholozan [Tho17], et au fait que l'exposant critique d'une surface hyperbolique de volume fini est égal à 1, a pour corollaire le résultat suivant.

Corollaire 0.2.15. *L'ensemble des exposants critiques des surfaces projectives convexes non compactes de volume fini est égal à* $]1/2, 1]$.

0.2.8 L'exposant critique de certains groupes de réflexions projectives

Le dernier chapitre de la thèse est issu d'une collaboration avec Harrison Bray. Notre objectif est de prolonger les travaux de Nie [Nie15b] présentés à la section 0.2.7, en répondant à la question 0.2.13 pour d'autres types d'orbivariétés projectives convexes issues de la théorie de Vinberg [Vin71], comme des orbivariétés de Benoist, ou des orbivariétés projectives convexes non compactes de volume fini. Précisons que le théorème 0.2.14 ne nous était pas connu lorsque nous avons commencé ce projet.

Les travaux de Marquis [Mar10] fournissent beaucoup d'exemples d'orbivariétés projectives convexes de dimension 3 “à la Vinberg”. Fixons une telle orbivariété M. Le critère d'hyperbolicité de Moussong [Mou88, Th. B], allié au théorème de Benoist discuté en section 0.1.2, permet de savoir si M est une orbivariété de Benoist. Par ailleurs, d'autres travaux de Marquis [Mar17, Th. A] permettent de savoir si M est de volume fini.

Il reste donc à trouver une méthode pour estimer l'exposant critique des orbivariétés projectives convexes “à la Vinberg” $M = \Omega/\Gamma$. Il existe un moyen simple de le minorer, qui n'utilise d'ailleurs pas les travaux de Vinberg : si Γ possède un sous-groupe libre non abélien Γ', alors $\delta_{\Gamma'} \leq \delta_\Gamma$, or l'exposant critique d'un groupe libre est plus facile à estimer (voir la section 11.2). Une application de ceci est que les exposants critiques de la famille d'orbivariétés construite par Benoist [Ben06a, §4.3] pour établir l'existence des orbivariétés de Benoist, sont minorés par une constante strictement positive (cf. la section 11.4.1).

Pour établir une majoration de l'exposant critique, nous reprenons la stratégie de Nie, et l'adaptons à des classes plus larges d'orbivariétés "à la Vinberg". Cela nous amène notamment à définir pour ces orbivariétés une quantité géométrique appelée *grosseur*, à laquelle s'applique la même idée que dans la section 0.2.7 précédente : plus l'orbivariété est "grosse", plus son exposant critique est petit.

Appelons *polytope de Coxeter hilbertien* tout couple (P, Ω) où $\Omega \subset \mathrm{P}(\mathbf{V})$ est un ouvert proprement convexe et $P \subset \Omega$ un polytope de Ω qui vérifie les conditions suivantes :

- le long de chaque face de P, il existe une réflexion projective qui préserve Ω ;
- soit S l'ensemble des réflexions le long des faces, et Γ le groupe engendré par S, alors P est un domaine fondamental pour l'action de Γ sur Ω.

Le couple (Γ, S) est appelé le *groupe de* (P, Ω), et on définit l'exposant critique de (P, Ω) comme celui de Γ (agissant sur Ω).

Les polytopes de Coxeter hilbertiens (P, Ω) considérés par Nie sont ceux pour lesquels Ω est strictement convexe et P est un simplexe compact de Ω.

On définit la *grosseur* d'un polytope de Coxeter hilbertien (P, Ω) comme l'infimum des *écarts* entre les facettes disjointes de P, où par écart entre deux sous-ensembles A et B de Ω on entend l'infimum des $d_\Omega(x, y)$ où $x \in A$ et $y \in B$.

Théorème 0.2.16. *Soit (Γ, S) un système de Coxeter. Il existe une constante $C > 0$ telle que pour tout polytope de Coxeter hilbertien de groupe isomorphe à (Γ, S), d'exposant critique δ et de grosseur R, on a*

$$\delta \leq \frac{C}{R}.$$

Pour construire des exemples de suites de polytopes de Coxeter hilbertiens dont la grosseur tend vers l'infini, de sorte l'exposant critique tende vers zéro, on utilise les travaux de Vinberg, et l'observation élémentaire suivante servant à minorer la grosseur.

Observation 0.2.17. *Soient (P, Ω) un polytope de Coxeter hilbertien de groupe (Γ, S) et F_1, F_2 deux faces de P. Soit γ_1 (resp. γ_2) un élément du stabilisateur de F_1 (resp. F_2) dans Γ. Alors l'écart entre F et F' est plus supérieur ou égal à $\frac{1}{2}\ell(\gamma_1\gamma_2)$.*

Pour appliquer l'observation précédente à des exemples concrets, i.e. pour obtenir des estimations des longueurs de translations, on peut faire appel au théorème de Puiseux. Plus précisément, étant donnée une famille de matrices $(\gamma_t)_{t \in \mathbb{R}} \subset \mathrm{GL}(\mathbf{V})$ dont les coefficients dépendent de t comme des fractions rationnelles, le théorème de Puiseux donne un algorithme pour estimer les coefficients de $\ell(\gamma_t)$ en tant que série de Puiseux en t.

Le dernier résultat de la thèse concerne l'exposant critique de certains polytopes de Coxeter hilbertiens dits *à pointes paraboliques de type* $\widetilde{A}_{\mathtt{d}-1}$, qui sont de volume fini d'après les travaux de Marquis [Mar17]. Un tel polytope P est l'intersection avec Ω d'un polytope de $\overline{\Omega}$ qui rencontre $\partial\Omega$ en un nombre fini de sommets appelés *sommets à l'infini* ; de plus, on demande que, pour chaque sommet à l'infini, le groupe engendré par les réflexions le long des faces adjacentes au sommet soit conjugué à un sous-groupe de $\mathrm{GL}(\mathbf{V})$ bien précis (celui donné par la représentation de Tits du groupe de Coxeter affine de type $\widetilde{A}_{\mathtt{d}-1}$).

La grosseur de ces polytopes est nulle car l'écart entre deux facettes disjointes adjacentes à un même sommet à l'infini est nul. C'est pourquoi nous définissons un autre type de grosseur. On montre dans la section 11.5.2 que, pour chaque sommet à l'infini de P, on peut choisir de façon canonique un voisinage dans Ω. La *grosseur non cuspidale* de (P, Ω) est définie comme l'infimum sur deux ensembles : d'une part l'ensemble des écarts entre deux facettes disjointes non adjacentes à un même sommet à l'infini ; d'autre part l'ensemble des écarts entre le voisinage canonique d'un sommet à l'infini et une facette non adjacente au sommet.

Théorème 0.2.18. *Soit* (Γ, S) *un groupe de Coxeter. Il existe une constante* $C > 1$ *telle que pour tout polytope de Coxeter hilbertien* (P, Ω) *à pointes paraboliques de type* $\widetilde{A}_{\mathtt{d}-1}$*, de groupe* (Γ, S)*, d'exposant critique* δ *et de grosseur non cuspidale* R*, on a*

$$\frac{\mathtt{d}-1}{2} < \delta \leq \frac{\mathtt{d}-1}{2} + Ce^{-\frac{R}{C}}.$$

Part I

Reminders

Chapter 1

Reminders on dynamical systems

In this chapter we recall the definitions of several classical dynamical properties for a general action by a topological group. Historically, the first recurrence properties to be considered were of probabilistic nature, and they came to mathematics through statistical physics. The first mathematical theorem on this topic is probably the celebrated Poincaré recurrence theorem.

By *dynamical system* we mean here a group acting on a set, preserving some structure, like a topology or a measure. The three kinds of dynamical systems we shall keep in mind are the following. Let $M = \Omega/\Gamma$ be a convex projective orbifold. The group $\mathbb{R}$ acts on $T^1M := T^1\Omega/\Gamma$ via the geodesic flow $(\phi_t)_t$; the group Γ acts on $\mathrm{Geod}(\Omega) := T^1\Omega/(\phi_t)_t$, which identifies with $\{(\xi,\eta) \in \partial\Omega^2 : [\xi,\eta] \cap \Omega \neq \emptyset\} \subset \partial\Omega^2$; finally, it will be natural to consider the combination of the two previous actions (which commute), in other words that of $\Gamma \times \mathbb{R}$ on $T^1\Omega$.

To make the interaction between these three dynamical systems formal, we consider the following general setting. We fix a group G acting on a set X, and we fix a normal subgroup $H \subset G$, so that G/H acts naturally on $H\backslash X$, and we have a G-equivariant projection $\pi_H : X \to H\backslash X$. In this generality, the idea that the dynamics of G on X and that of G/H on $H\backslash X$ have the same features works well for the most basic recurrence properties, such as non-wandering and transitivity properties (whose definitions are recalled in the present chapter). However, this idea fails when we look at more sophisticated dynamical notions such as mixing or entropy (which are notions for $G = \mathbb{Z}$ or $\mathbb{R}$).

Indeed, suppose we have two simply connected spaces Y and Y' both equipped with free and proper commuting actions by $\mathbb{R}$ (we denote the induced flows by $(\phi_t)_t$) and a discrete group Γ, and suppose we have a homeomorphism $f : Y/\Gamma \to Y'/\Gamma$ that sends every $(\phi_t)_t$-orbit on a $(\phi_t)_t$-orbit. Then f lifts to a Γ-equivariant homeomorphism $Y \to Y'$, that descends to a Γ-equivariant homeomorphism $Y/(\phi_t)_t \to Y'/(\phi_t)_t$. Thus, the two dynamical systems $(Y/(\phi_t)_t, \Gamma)$ and $(Y'/(\phi_t)_t, \Gamma)$ are the same, although the homeomorphism $f : Y/\Gamma \to Y'/\Gamma$ we started with might not be $(\phi_t)_t$-equivariant. To be able to distinguish the two Γ-actions on $Y/(\phi_t)_t$ and $Y'/(\phi_t)_t$, and retrieve the dynamical systems $(Y/\Gamma, (\phi_t)_t)$ and $(Y'/\Gamma, (\phi_t)_t)$, one needs extra data: for instance, in the setting of convex projective orbifolds $M = \Omega/\Gamma$ (or Riemannian manifolds), one needs to know the periods of the elements of Γ, i.e. the lengths of the associated periodic $(\phi_t)_t$-orbits; this extra data is usually refered to as the length spectrum. Ledrappier wrote a very interesting and complete paper [Led95] on this subject, in the setting of negatively curved Riemannian manifolds.

The material presented in this chapter is classical. Let us give a (very) small list of references on the theory of dynamical systems: [HK95, FH19] for $G = \mathbb{N}, \mathbb{Z}, \mathbb{R}$, and X compact; [Kre85] for $G = \mathbb{Z}$ and ergodic theory; [Zim84] for general G and ergodic theory;

[GH55] for general G and topological dynamics; [CKN07] for discrete G and topological dynamics.

When $G = \mathbb{R}$, we shall denote its action by $t \cdot x = \phi_t(x)$.

1.1 Topological recurrence

Let us fix for the whole section a secound countable locally compact topological group G acting continuously on a secound countable locally compact topological space X, and a normal closed subgroup $H \subset G$. The quotient space $H\backslash X$ is endowed with the quotient topology, so that the projection $\pi_H : X \to H\backslash X$ is continuous and open, and induces a bijection between the G-invariant closed subsets of X and the G/H-invariant closed subsets of $H\backslash X$.

1.1.1 Proper actions

The action of G on X is said to be *proper* (properly discontinuous if G is discrete) if for every compact subset $K \subset X$, the set of elements $g \in G$ such that $gK \cap K \neq \emptyset$ is compact. In this case, the quotient space $G\backslash X$ equipped with the quotient topology is second countable and locally compact.

Let d be a metric on X that induces its topology. The metric is said to be *proper* if every closed ball is compact. In this case, the group of isometries $\mathrm{Isom}(X, d)$, endowed with the compact-open topology, acts properly on X. If G is a closed subgroup of $\mathrm{Isom}(X, d)$, and $\pi_G : X \to G\backslash X$ denotes the natural projection, then the formula $d_{G\backslash X}(\pi_G(x), \pi_G(y)) = \inf\{d(x, gy) : g \in G\}$ yields a proper metric on $G\backslash X$.

We assume for the rest of this section that H acts properly on X.

1.1.2 Non-wandering, recurrent and attracting points

Let $x \in X$. If x is fixed by some element $g \in G$, then x is said to be *attracting* for g if it admits a neighbourhood U such that $(g^n y)_n$ converges to x for any $y \in U$. If x is attracting for g^{-1}, then it is said to be *repelling* for g. The point x is said to be *recurrent* if for any neighbourhood U of x, the set $\{g \in G : gx \in U\}$ is not relatively compact. The point x is said to be *non-wandering* if for any neighbourhood U of x, the set $\{g \in G : gU \cap U \neq \emptyset\}$ is not relatively compact.

That x is an attracting fixed point of some $g \in G$ (and is not isolated, i.e. $\{x\}$ is not open) implies that x is a fixed point of some $g \in G$ with $\{g^n\}_{n\geq 0}$ not relatively compact, which implies that x is recurrent, which implies that x is non-wandering.

If $G = \mathbb{R}$, then x is called *periodic* when $\{t > 0 : \phi_t x = x\}$ is non-empty, in which case the smallest element of this set is called the *period* of x. The point x is called *forward* (resp. *backward*) *recurrent* when $\{t > 0 : \phi_t x \in U \neq \emptyset\}$ (resp. $\{t < 0 : \phi_t x \in U \neq \emptyset\}$) is not relatively compact for any neighbourhood U of x. Observe that any recurrent point is forward recurrent or backward recurrent. Note also that given a non-wandering point $x \in X$ and a neighbourhood U of x, one can find arbitrarily large *positive* times t such that $\phi_t U \cap U \neq \emptyset$; indeed, if $\phi_t U \cap U \neq \emptyset$, then $\phi_{-t} U \cap U \neq \emptyset$.

The set of non-wandering points is called the *non-wandering set* and is denoted by $\mathrm{NW}(X, G)$; it is closed and G-invariant. The action of G on X is called *non-wandering* if $\mathrm{NW}(X, G) = X$. Note that if X is compact but G is not, then the set of recurrent points, and hence also the non-wandering set, are non-empty.

Recall that a subset of X is G_δ-dense if it is the intersection of countably many open and dense subsets; it can be seen as a topological analogue of measurable subsets with full measure. The following two facts are classical and elementary.

Fact 1.1.1. *If the action of G is non-wandering, then the set of recurrent points contains a G_δ-dense set.*

If moreover $G = \mathbb{R}$, then the set of forward recurrent points also contains a G_δ-dense set.

Fact 1.1.2. *Any point $x \in X$ is non-wandering (resp. recurrent) with respect to G if and only if $\pi_H(x)$ is non-wandering (resp. recurrent) with respect to G/H.*

1.1.3 Topological transitivity and minimality

The action of G on X is called *topologically transitive* if any G-invariant non-empty open subset of X is dense, i.e. if any G-invariant, proper and closed subset has empty interior, i.e. if for any non-empty open subsets $U, V \subset X$, there exists $g \in G$ such that gU intersects V. The action of G is said to be *minimal* if X does not admit any G-invariant, non-empty, proper and closed subset; this implies topological transitivity.

If $G = \mathbb{R}$, then the action is *forward topologically transitive* if for any two non-empty open subsets $U, V \subset X$, there exists $t > 0$ such that $\phi_t U$ intersects V. Note that forward topological transitive is equivalent to the following: for any two non-empty open subsets $U, V \subset X$, there exists $t < 0$ such that $\phi_t U$ intersects V.

The following fact is classical and elementary.

Fact 1.1.3. *If the action of G on X is topologically transitive, then the set of points with a dense orbit contains a G_δ-dense set.*

If moreover $G = \mathbb{R}$ and its action is non-wandering, then the set of points with a dense forward orbit and a dense backward orbit contains a G_δ-dense set; in particular the action is forward topologically transitive.

In general, topological transitivity does not imply non-wandering. However it does under some local topological condition on X and G, which can roughly be formulated by saying that X is locally "much bigger" than G, so that a G-orbit has to come back often in order to fill an open set. For instance, if G is discrete, X has no isolated point and the action is topologically transitive, then it is non-wandering.

The following fact is classical and elementary.

Fact 1.1.4. *The action of G on X is topologically transitive (resp. minimal) if and only if that of G/H on $H\backslash X$ is topologically transitive (resp. minimal).*

1.1.4 Topological mixing

If $G = \mathbb{R}$, then its action on X is called *topologically mixing* if for any two non-empty open subsets $U, V \subset X$, there exists $T > 0$ such that $\phi_t U$ intersects V for any $t \geq T$. In this case, the reversed flow $(\phi_{-t})_t$ is also topologically mixing. Observe that topological mixing implies forward topological transitivity and non-wandering.

One could extend the previous definition to other groups by asking that for any two non-empty open subsets $U, V \subset X$, there exists $K \subset G$ compact such that $g \cdot U$ intersects V for any $g \in G \smallsetminus K$. However this definition is not relevant for us, because there is no analogue of Facts 1.1.2 and 1.1.4. For instance, the geodesic flow on a compact hyperbolic manifold $M = \Omega/\Gamma$ is topologically mixing, while the action of Γ on the set of geodesics $\mathrm{Geod}(\Omega)$ is not topologically mixing in the above sense.

Let us recall roughly how the topological mixing of the geodesic flow on T^1M can be interpreted in terms of the action of Γ on $\mathrm{Geod}(\Omega)$ and its length spectrum. The proof of the topological mixing relies on the two following properties:

1. the local non-arithmeticity of the length spectrum, i.e. the set of length of periodic $(\phi_t)_t$-orbits through any given open set generates a dense subgroup of $\mathbb{R}$ (in particular periodic orbits are dense);

2. the existence for each periodic $(\phi_t)_t$-orbit x of period τ of a sufficiently large "attracting (resp. repelling) manifold" — usually called strong stable (resp. unstable) manifold —, consisting of points y such that $(\phi_{n\tau}y)_n$ converges to x as n goes to ∞ (resp. $-\infty$).

A periodic orbit of the geodesic flow naturally corresponds to a pair $(\xi,\eta) \in \mathrm{Geod}(\Omega)$ fixed by an element of $\gamma \in \Gamma$, the period may be expressed as an algebraic quantity of γ, and the existence of attracting and repelling manifolds is interpreted in terms of ξ and η being attracting or repelling points of γ in $\partial\Omega$.

1.2 Measure-theoretic recurrence

Let us fix for the whole section a locally compact, secound countable and unimodular group G acting measurably on a standard Borel space X (i.e. X is measurably isomorphic to $\mathbb{R}$), and a G-invariant and σ-finite measure m on X. We also fix a Haar measure on G, denoted by $\mathrm{d}g$, and an integrable positive function σ on X.

1.2.1 The Hopf decomposition and quotients of measures

For any non-negative measurable function f on X, we denote by $\int_G f$ the G-invariant measurable function defined by $\int_G f(x) = \int_G f(gx)\,\mathrm{d}g$; this also denotes the induced function on $G\backslash X$.

A measurable subset $A \subset X$ is said to *wandering* (with respect to m) under the action of G if for m-almost any $x \in A$, the *transporter* $T(x,A) := \{g \in G : gx \in A\}$ is relatively compact. The following fact is classical, and serves as a definition of the Hopf decomposition.

Fact 1.2.1. *Let $\mathcal{C} := \{\int_G \sigma = \infty\}$ and $\mathcal{D} := \{\int_G \sigma < \infty\} \subset X$. The decomposition $X = \mathcal{C} \sqcup \mathcal{D}$ is a* Hopf decomposition*, in the sense that every wandering subset of $\mathcal{C}$ has m-measure zero, and $\mathcal{D}$ is a countable union of wandering subsets of X. Any two Hopf decompositions agree on some m-full subset of X. The dynamical system (X,G,m) is said to be* conservative *(resp.* dissipative*) if $m(\mathcal{D}) = 0$ (resp. $m(\mathcal{C}) = 0$).*

Proof. Let $A \subset \mathcal{C}$ be a measurable subset, and let us prove that $\int_G 1_A$ is infinite on m-almost every point of A. On the one hand, if, for $R > 0$, we denote $A_R := \{\int_G 1_A \leq R\} \cap A$, then

$$\int_{X\times G} \sigma(gx) 1_{A_R}(x)\,\mathrm{d}g\,\mathrm{d}m(x) = \infty \cdot m(A_R),$$

while on the other hand, since G is unimodular and m is G-invariant,

$$\begin{aligned}\int_{X\times G} \sigma(gx)1_{A_R}(x)\,\mathrm{d}g\,\mathrm{d}m(x) &= \int_{X\times G} \sigma(x)1_{A_R}(gx)\,\mathrm{d}g\,\mathrm{d}m(x)\\ &= \int_X \sigma\left(\int_G 1_{A_R}\right)\mathrm{d}m\\ &\leq R\int_X \sigma\,\mathrm{d}m < \infty.\end{aligned}$$

Therefore $m(A_R) = 0$ for any $R > 0$, hence $m(\cup_R A_R) = 0$. For any compact subset $K \subset G$, we consider $B_K := \{\int_K \sigma > 1/2 \cdot \int_G \sigma\} \subset \mathcal{D}$, and observe that it is wandering. Indeed if $x \in B_K$ and $g \in G$ are such that $gx \in B_K$, then $\int_K \sigma(x) + \int_{Kg} \sigma(x) > \int_G \sigma(x)$, thus $K \cap Kg \neq \emptyset$ and $g \in K^{-1} \cdot K$ which is compact. Furthermore $\mathcal{D} = \cup_n B_{K_n}$ if $G = \cup_n K_n$. □

Fact 1.2.2. *Assume that X is a locally compact topological space with countable basis, and that the action of G is continuous. If m is conservative, then m-almost all points are recurrent. In particular the action of G is non-wandering on the support of m.*

If moreover $G = \mathbb{R}$, then m-almost all points are forward and backward recurrent.

Proof. Without loss of generality we may assume that $X = \mathrm{supp}(m)$. Let us prove that for any measurable subset $A \subset X$, the function $\int_G 1_A$ is infinite on m-almost every point of A. Suppose by contradiction that there exists $R \geq 0$ and a measurable subset $A \subset X$ with finite and positive measure such that $1_A \int_G 1_A \leq R$. Since m is G-invariant,

$$\int_A \int_G 1_A(ga)\,\mathrm{d}g\,\mathrm{d}m(a) \leq Rm(A) < \infty.$$

This contradicts the conservativity of m.

Let $\mathcal{V}$ be a countable basis of open sets of X which have finite m-measure. The set of non-recurrent points is contained in

$$\bigcup_{V\in\mathcal{V}} \left\{y \in V : \int_G 1_V(gy)\,\mathrm{d}g < \infty\right\},$$

which has zero measure.

When $G = \mathbb{R}$, the rest of the lemma is proved by using the following observation.

$$\int_A \int_{\mathbb{R}} 1_A(\phi_t a)\,\mathrm{d}t\,\mathrm{d}m(a) = 2\int_A \int_0^\infty 1_A(\phi_t a)\,\mathrm{d}t\,\mathrm{d}m(a).$$ □

Note that if the action of G on X is *smooth* (namely $G\backslash X$ is a standard Borel space) and has compact stabilisers, then (X, G, m) is dissipative. In particular, this observation applies when X is a locally compact second countable topological space and the action of G is continuous and proper.

Definition 1.2.3. The *quotient* of m on $G\backslash X$ is the σ-finite measure defined as

$$m_G := \left(\int_G \sigma\right)^{-1} \pi_{G*}(\sigma m),$$

where π_G denotes the projection $X \to G\backslash X$, and we use the convention $1/\infty = 0$. This definition is independent of the choice of σ.

Remark 1.2.4. Assume that m is dissipative. Then for any non-negative (or integrable) function f on X,

$$\int_X f \,\mathrm{d}m = \int_{G\backslash X} \left(\int_G f \right) \mathrm{d}m_G.$$

As a consequence, if $\pi_{G*}m' = m_G$ for some measure m', then $m = \int_G g_* m' \,\mathrm{d}g$, in the sense that $\int_X f \,\mathrm{d}m = \int_X \left(\int_G f \right) \mathrm{d}m'$ for any non-negative measurable function f.

We fix for the whole section a unimodular, normal and closed subgroup $H \subset G$ whose action on X is smooth and with compact stabilisers, so that m is dissipative with respect to H, and we denote by m_H its quotient on $H\backslash X$ (which is G/H-invariant). The projection $\pi_H : X \to H\backslash X$ induces a correspondence between G-invariant measurable subsets of X and G/H-invariant measurable subsets of $H\backslash X$, where sets of zero m-measure correspond to sets of zero m_H-measure. The projection π_H also induces a correspondence between G-invariant σ-finite measures on X and G/H-invariant σ-finite measures on $H\backslash X$.

Fact 1.2.5. *The Hopf decomposition of X projects under π_H onto the Hopf decomposition of $H\backslash X$. In particular, (X, G, m) is conservative (resp. dissipative) if and only if $(H\backslash X, G/H, m_H)$ is conservative (resp. dissipative).*

Proof. Observe that the Haar measure on G/H is the quotient of the Haar measure on G by the action of H. Therefore, if σ is a positive integrable function on X, then $\int_H \sigma$ is a positive integrable function on $H\backslash X$, and $\int_G \sigma = \int_{G/H} \int_H \sigma$. □

1.2.2 Ergodicity

The action of G on (X, m) is said to be *ergodic* if any G-invariant measurable subset of X has null or full measure. The idea is that a dynamical system is ergodic when almost any orbit "equidistributes". This statement is made formal in the case where $G = \mathbb{R}$ and m is finite by the celebrated Birkhoff ergodic theorem, which we do not need nor state here. One can weaken this equidistribution property to obtain a topological statement, whose proof is elementary.

Fact 1.2.6. *Assume that X is a locally compact topological space with countable basis, and that the action of G is continuous. If m is ergodic, then m-almost all points have a dense orbit in* $\mathrm{supp}(m)$*; in particular G acts topologically transitively on* $\mathrm{supp}(m)$.

If moreover $G = \mathbb{R}$ and m is conservative, then m-almost all points have a dense forward orbit and a dense backward orbit in $\mathrm{supp}(m)$*, and the action of* $\mathbb{R}$ *on* $\mathrm{supp}(m)$ *is forward topologically transitive.*

Remark 1.2.7. If X is a locally compact second countable space and G acts continuously and ergodically, and if every G-orbit has measure zero, then the action of G is conservative.

The following fact is classical and elementary.

Fact 1.2.8. *(X, G, m) is ergodic if and only if $(H\backslash X, G/H, m_H)$ is ergodic.*

1.2.3 Mixing

Suppose that $G = \mathbb{R}$ and that m is finite. The action of $\mathbb{R}$ on (X, m) is said to be (measure-theoretically) *mixing* if for any two measurable subsets $U, V \subset X$,

$$m(U \cap \phi_t V) \underset{t\to\infty}{\longrightarrow} \frac{m(U)m(V)}{m(X)}.$$

Note that in this case the reversed flow is also mixing.

Fact 1.2.9. *Assume that X is a second countable locally compact topological space, and that the action of $G = \mathbb{R}$ is continuous (and that m is finite). If the action of $\mathbb{R}$ is mixing, then its restriction to the support of m is topologically mixing.*

Given a convex projective orbifold $M = \Omega/\Gamma$, we will need to interpret what it means in the universal cover Ω that the geodesic flow on T^1M is mixing.

Remark 1.2.10. Assume that the action of G on X is smooth with compact stabilisers, that the quotient m_G of m on $G\backslash X$ is finite, and that we have a measurable flow $(\phi_t)_t$ on X which preserves m, commutes with the action of G and descends to a mixing flow on $(G\backslash X, m_G)$.

If α, β are integrable functions on X such that $|\alpha| \int_G |\alpha|$ and $|\beta| \int_G |\beta|$ are integrable on X (e.g. if α and β are bounded and zero outside of a measurable set of finite measure A such that $T(x, A) \subset K$ for any $x \in A$, where $K \subset G$ is a fixed compact subset), then

$$\left(\int_G \alpha\right)\left(\int_G \beta\right) = \int_{g\in G}\int_G (\alpha \cdot (\beta \circ g))\, \mathrm{d}g.$$

Therefore the mixing property can be reformulated as

$$\int_{g\in G}\int_X \alpha \cdot (\beta \circ g \circ \phi_t)\, \mathrm{d}m\, \mathrm{d}g \underset{t\to\infty}{\longrightarrow} \frac{1}{\|m\|}\int_X \alpha\, \mathrm{d}m \cdot \int_X \beta\, \mathrm{d}m.$$

Also, if $\tilde{\beta}$ is a G-invariant function on X that lifts a square-integrable function β on $G\backslash X$ and α is an integrable function on X such that $|\alpha| \int_G |\alpha|$ is integrable, then

$$\int_X \alpha \cdot (\tilde{\beta} \circ \phi_t)\, \mathrm{d}m \underset{t\to\infty}{\longrightarrow} \frac{1}{\|m_G\|}\int_X \alpha\, \mathrm{d}m \cdot \int_{G\backslash X} \beta\, \mathrm{d}m_G.$$

Assume further that X is a locally compact second countable space and that the action of G is continuous and proper. Then for any two relatively compact Borel subsets $A, B \subset X$,

$$\int_{g\in G} m(A \cap \phi_t g B) \underset{t\to\infty}{\longrightarrow} \frac{m(A)m(B)}{\|m_G\|}.$$

As for topological mixing, with the notations of Remark 1.2.10, the mixing property of $(\phi_t)_t$ on $G\backslash X$ cannot be interpreted purely in terms of the action of G on $X/(\phi_t)_t$; for instance in the setting of convex projective orbifolds (or negatively curved Riemannian manifolds), one needs to take into account the length spectrum of G.

1.2.4 Criterions for ergodicity and for mixing

Coudène stated and proved a convenient criterion for the ergodic property, based on the ergodic theorem and the celebrated Hopf argument, and another for the mixing property, based on an idea of Babillot [Bab02]. We will use these criteria in Chapter 6.

Consider a Borel flow $(\phi_t)_{t\in\mathbb{R}}$ on a metric space (X, d). The *strong stable manifold* of a point $x \in X$ is

$$W^{ss}(x) = \{y \in X :\ d(\phi_t x, \phi_t y) \underset{t\to\infty}{\longrightarrow} 0\}.$$

The *strong unstable manifold* $W^{us}(x)$ is the strong stable manifold of the time-reversed flow. Consider a $(\phi_t)_{t\in\mathbb{R}}$-invariant σ-finite measure on X. A measurable function $f : X \to \mathbb{R}$ is said to be W^{ss}*-invariant* when there exists a measurable subset $E \subset X$ with full measure such that for all x and y in E, if they are on the same strong stable manifold then $f(x) = f(y)$. The notion of W^{su}*-invariance* is similarly defined.

Coudène's criteria are the following.

Fact 1.2.11 ([Cou07a]). *Let (X,d) be a metric space, $(\phi_t)_t$ a measurable flow on it, m a conservative $(\phi_t)_t$-invariant measure, and assume that some m-full subset of X is covered by a countable family of open sets with finite m-measure. If every W^{ss}, W^{su} and $(\phi_t)_t$-invariant measurable function is essentially constant then the flow is ergodic.*

Fact 1.2.12 ([Cou07b]). *Consider a Borel flow preserving a finite measure on a metric space. If every W^{ss}- and W^{su}-invariant Borel function is essentially constant then the flow is mixing.*

1.3 Entropy

As for topological and measure-theoretic mixing, the notion of entropy has only been developped (to my knowledge) for dynamical systems (X,G) with $G=\mathbb{R}$ or $\mathbb{Z}$ (or $\mathbb{R}_{\geq 0}$ or $\mathbb{N}$ with their semi-group structure). As mentioned in the introduction, one can think of the entropy as the exponential growth, as t tends to infinity, of the minimum quantity of information that we need to know for each point in order to know its trajectory up to time t. The nature of the information can be topological, metrical or measure-theoretic.

1.3.1 Topological entropy

Let (X,d) be a proper metric space, and $(\phi_t)_{t\in\mathbb{R}}$ be a continuous flow on it, such that ϕ_t is uniformly continuous for any $t\in\mathbb{R}$.

- Let $\epsilon>0$. A subset $S\subset X$ is (d,ϵ)-*separated* if $d(s,s')\geq\epsilon$ for all $s\neq s'$ in S. For any $A\subset X$, we denote by $N(A,d,\epsilon)$ the maximal cardinality of a (d,ϵ)-separated subset of A, and we set $N(d,\epsilon)=N(X,d,\epsilon)$.

- Let $\epsilon>0$ and $A\subset X$. A subset $S\subset X$ (d,ϵ)-*spans* A if for any $a\in A$, there exists $s\in S$ with $d(a,s)<\epsilon$. We denote by $S(A,d,\epsilon)$ the minimal cardinality of such a set S, and $S(d,\epsilon)=S(X,d,\epsilon)$.

- We take the classical notation $d^{(t)}(x,y):=\max_{0\leq s\leq t}d(\phi_s x,\phi_s y)$ for $t\geq 0$ and $x,y\in X$; this defines a family of metrics on X.

- The *topological entropy* of ϕ on a compact subset $K\subset X$ is

$$h_{\mathrm{top}}(\phi,K):=\lim_{\epsilon\to 0}\limsup_{t\to\infty}\frac{1}{t}\log N(K,d^{(t)},\epsilon)=\lim_{\epsilon\to 0}\limsup_{t\to\infty}\frac{1}{t}\log S(K,d^{(t)},\epsilon).$$

  The topological entropy on the whole space X is denoted by $h_{\mathrm{top}}(\phi)$ and defined as the supremum over all topological entropies on compact subsets of X.

One defines similarly the topological entropy of a uniformly continuous homeomorphism by replacing $t\in\mathbb{R}_{\geq 0}$ by $n\in\mathbb{Z}_{\geq 0}$. Note that the topological entropies of the flow and of the underlying time-one map are the same. Also note that the the reparametrised flow $(\phi_{kt})_{t\in\mathbb{R}}$, where $k\in\mathbb{R}_{\neq 0}$, has topological entropy equal to $|k|$ times the topological entropy of $(\phi_t)_{t\in\mathbb{R}}$.

In practice, we will only talk about entropy on compact spaces. For these, the topological entropy is independent of the metric d.

1.3.2 Measure-theoretic entropy

Let f be an inversible measurable map of a measurable space X, which preserves a probability measure μ. Let $\mathcal{P}$ be a finite measurable partition of X.

- The *entropy of* $\mathcal{P}$ is
$$H_\mu(\mathcal{P}) := -\sum_{P\in\mathcal{P}} \mu(P)\log\mu(P).$$

- For each integer $n \geq 1$, we set
$$\mathcal{P}^{(n)} := \{P_0 \cap f^{-1}P_1 \cap \cdots \cap f^{-n+1}P_{n-1} : P_k \in \mathcal{P},\ 0 \leq k \leq n-1\}.$$

- The *entropy of* f *with respect to* μ *and* $\mathcal{P}$ is
$$H_\mu(f,\mathcal{P}) := \lim_{n\to\infty} \frac{H_\mu(\mathcal{P}^{(n)})}{n}.$$

- The *entropy of* f *with respect to* μ is
$$h_\mu(f) := \sup\{H_\mu(f,\mathcal{Q}) : \mathcal{Q} \text{ measurable partition}\}.$$

Let us now consider a measurable flow $(\phi_t)_{t\in\mathbb{R}}$ on a measurable space X preserving a probability measure μ. The *entropy of the flow with respect to* μ is defined to be the entropy of the time-one map ϕ_1.

Remark 1.3.1. The reparametrised flow $(\phi_{kt})_{t\in\mathbb{R}}$ has measure-theoretic entropy equal to $|k|$ times the measure-theoretic entropy of $(\phi_t)_{t\in\mathbb{R}}$.

The relation between topological and measure-theoretic entropies is given by the following famous principle, for more details see [HK95, Th. 4.5.3].

Fact 1.3.2 (Variational Principle)**.** *Let* $\phi = (\phi_t)_{t\in\mathbb{R}}$ *be a continuous flow on a compact metric space* (X,d)*. Denote by* $\mathcal{P}^\phi(X)$ *the set of* ϕ*-invariant probability measures on* X*. Then*
$$h_{\mathrm{top}}(\phi) = \sup_{\mu\in\mathcal{P}^\phi(X)} h_\mu(\phi).$$

1.3.3 Entropy-expansive maps

We recall the definition of entropy-expansive maps from Bowen [Bow72]. Consider a uniformly continuous homeomorphism f of a proper metric space (X,d) and $\epsilon > 0$. The map f is said to be (d,ϵ)*-entropy-expansive* if for each $x \in X$, the action of f on the (compact) *Bowen ball*
$$\{y \in X : \forall n \in \mathbb{Z},\ d(f^n x, f^n y) \leq \epsilon\}$$
has zero entropy. If X is compact, then we say that f is entropy-expansive if there exists ϵ such that f is (d,ϵ)-entropy expansive, and this does not depend on d by compactness.

The notion of entropy-expansivity generalises the notion of expansivity (f is expansive if there exists $\epsilon > 0$ such that any Bowen ball of radius ϵ is reduced to a singleton).

Remark 1.3.3. If f is (d,ϵ)-entropy-expansive then f^n is $(d^{(n)},\epsilon)$-entropy-expansive.

The following is one of the main result of Bowen on entropy-expansive maps.

Fact 1.3.4 ([Bow72, Th. 3.5]). *Let ϵ be a positive number; let f be a (d, ϵ)-entropy-expansive homeomorphism of a compact metric space (X, d); let μ be a f-invariant probability measure on X; let $\mathcal{P}$ be a finite measurable partition all of whose elements have diameter less than ϵ. Then*

$$h_\mu(f) = H_\mu(f, \mathcal{P}).$$

Remark 1.3.5. Consider an entropy-expansive homeomorphism of a compact metric space. By Fact 1.3.4, the measure-theoretic entropy depends upper semi-continuously on the measure (with respect to the weak* topology), and there exists a measure of maximal entropy.

Unit tangent bundles of closed hyperbolic surfaces $\mathbb{H}^2/\Gamma$ equipped with their geodesic flow are examples of entropy-expansive systems, if Γ is torsion-free. However, if $\Gamma \subset \mathrm{PGL}_2(\mathbb{R})$ is a discrete cocompact triangle group generated by the orthogonal reflections along the sides of a triangle of $\mathbb{H}^2$, then the geodesic flow on $T^1\mathbb{H}^2/\Gamma$ is not entropy-expansive. We want to include such examples in the present work, and we need in Chapter 7 to be able to apply the work of Bowen.

This is possible thanks Selberg's lemma [Sel60], which ensures the existence of a torsion-free normal subgroup $\Gamma' \subset \Gamma$, as soon as Γ is finitely generated (this is true if Γ is a triangle group, and this condition will also be satisfied in Chapter 7). Then the group Γ/Γ' is finite and acts on $\Gamma'\backslash T^1\mathbb{H}^2$, and the quotient by Γ/Γ' is $\Gamma\backslash T^1\mathbb{H}^2$. We can then apply the following elementary observation.

Observation 1.3.6. *Let $(\phi_t)_t$ be a measurable flow on a measurable space X that preserves a probability measure μ, and G a finite group that acts measurably on X and commutes with $(\phi_t)_t$. We denote by $\pi : X \to X/G$ the natural projection, and $(\phi_t)_t$ the induced flow on the quotient. Then $h_\mu(X, (\phi_t)_t) = h_{\pi_*\mu}(X/G, (\phi_t)_t)$. In particular, if X is a compact topological space and the actions of $(\phi_t)_t$ and G are continuous, then $h_{\mathrm{top}}(X, (\phi_t)_t) = h_{\mathrm{top}}(X/G, (\phi_t)_t)$.*

I do not know of a way to prove that $h_{\mathrm{top}}(X, (\phi_t)_t) = h_{\mathrm{top}}(X/G, (\phi_t)_t)$ (with the notations of the previous observation) without using the variational principle.

1.4 Horoboundary, Patterson–Sullivan densities and Sullivan measures

Fix a proper metric space (X, d).

1.4.1 The horocompactification

The *horofunction* at points $x, y, z \in X$ is defined as follows:

$$\mathtt{b}_z(x, y) = d(x, z) - d(y, z).$$

The set of points $y' \in X$ such that $\mathtt{b}_z(x, y') = 0$ is the sphere centred at z and passing through x. Note also that horofunctions satisfy the cocycle relation $\mathtt{b}_z(x, y') = \mathtt{b}_z(x, y) + \mathtt{b}_z(y, y')$.

We now recall the definition of the horocompactification, due to Gromov [Gro81, §1.2]. The idea is that it is the smallest compactification where balls and spheres extends continuously to generalised balls and spheres, respectively called horoballs and horospheres. By a *compactification* of X we mean a compact topological space Y together with an embedding

$X \hookrightarrow Y$ with open and dense image; then the subset $Y \setminus X = \partial_Y X$ is called the boundary of the compactification. Another compactification Z *dominates* Y if there is a continuous map from Z to Y which is compatible with the embeddings of X. Using the Arzelà–Ascoli theorem, one can show that the following is well defined.

The *horocompactification* of (X, d) is denoted by $\overline{X}^{\mathtt{h}}$ and is the smallest compactification of X such that $z \mapsto \mathtt{b}_z(x, y)$ extends continuously to $\overline{X}^{\mathtt{h}}$ for every $x, y \in X$. The *horoboundary* $\partial_{\mathtt{h}} X$ of (X, d) is the boundary of $\overline{X}^{\mathtt{h}}$.

For any $\xi \in \overline{X}^{\mathtt{h}}$ and $x \in X$, the (open) *horoball* (resp. closed horoball, resp. *horosphere*) centred at ξ and passing through x, denoted by $\mathcal{H}_\xi(x)$ (resp. $\overline{\mathcal{H}}_\xi(x)$, resp. $\partial\mathcal{H}_\xi(x)$), consists of the points $y \in X$ such that $\mathtt{b}_\xi(x, y) > 0$ (resp. $\mathtt{b}_\xi(x, y) \geq 0$, resp. $\mathtt{b}_\xi(x, y) = 0$).

Under some additional conditions, for instance if X is a geodesic space (i.e. any two points are connected by a geodesic segment), a sequence $(\xi_n)_n$ in $\overline{X}^{\mathtt{h}}$ converges to ξ if and only if $(\overline{\mathcal{H}}_{\xi_n}(x))_n$ converges to $\overline{\mathcal{H}}_\xi(x)$ for any $x \in X$, in the sense that the limit of any converging sequence of $\prod_n \overline{\mathcal{H}}_{\xi_n}(x)$ is in $\overline{\mathcal{H}}_\xi(x)$, and conversely any point of $\overline{\mathcal{H}}_\xi(x)$ is the limit of such a sequence.

Note that $\overline{X}^{\mathtt{h}}$ is metrisable and the function $(\xi, x, y) \in \overline{X}^{\mathtt{h}} \times X \times X \mapsto \mathtt{b}_\xi(x, y)$ is continuous. Any isometry of X extends continuously to a bi-Lipschitz homeomorphism of $\overline{X}^{\mathtt{h}}$.

1.4.2 The Patterson–Sullivan densities

Let us recall the definition of Patterson–Sullivan measures on the horoboundary of a proper metric space.

Let $\Gamma \subset \mathrm{Isom}(X, d)$ be a closed subgroup. Given $\delta \in \mathbb{R}$, a (Γ-equivariant) δ-*conformal density* on $\partial_{\mathtt{h}} X$ is a family of finite measures $(\mu_x)_{x \in X}$ on $\partial_{\mathtt{h}} X$ such that

- μ_y is absolutely continuous with respect to μ_x for all $x, y \in X$, and the Radon–Nikodym derivative is :
$$\frac{d\mu_y}{d\mu_x}(\xi) = e^{-\delta\, \mathtt{b}_\xi(y,x)};$$
(This implies that the family is entirely determined by μ_o for any $o \in X$.)

- for every $\gamma \in \Gamma$ and $x \in X$ the push-forward by γ of μ_x is :
$$\gamma_* \mu_x = \mu_{\gamma x}.$$

Let us recall the classical example of a conformal density, which we will need in this paper. For any measured metric space (X, d, μ) with infinite mass, the *volume entropy* is, for any $o \in X$,
$$\delta_\mu := \limsup_{r \to \infty} \frac{1}{r} \log \mu(B_X(o, r)) \in \mathbb{R}_{\geq 0} \cup \{\infty\}. \tag{1.4.1}$$

Fact 1.4.1 ([Pat76, §3]). *Let (X, d) be a proper metric space, let $o \in X$ be a basepoint, let Γ be a non-compact closed subgroup of* $\mathrm{Isom}(X, d)$ *and let* Vol *be a Γ-invariant Radon measure on X. We assume that the volume entropy δ_{Vol} is finite. Then there exists a continuous non-decreasing function $h : \mathbb{R}_+ \to \mathbb{R}_{>0}$ such that*

- $\int_{x \in X} h(d(o, x)) e^{-\delta_{\mathrm{Vol}} d(o,x)} \, \mathrm{d}\,\mathrm{Vol}(x) = \infty$;
- *for every $\epsilon > 0$, there exists $R > 0$ such that $h(r + t) \leq e^{\epsilon t} h(r)$ for all $r \geq R$ and $t \geq 0$.*

Furthermore, if we define, for $s > \delta_{\mathrm{Vol}}$, the probability measure $\mu_{o,s}$ on X such that

$$\mu_{o,s}(A) = \frac{\int_{x\in A} h(d(o,x))e^{-sd(o,x)}\,\mathrm{d}\,\mathrm{Vol}(x)}{\int_{x\in X} h(d(o,x))e^{-sd(o,x)}\,\mathrm{d}\,\mathrm{Vol}(x)},$$

for any Borel subset $A \subset X$, then any accumulation point of $(\mu_{o,s})_{s\to\delta_{\mathrm{Vol}}}$ in the space $\mathcal{P}(\overline{X}^{\mathtt{h}})$ of probability measures on $\overline{X}^{\mathtt{h}}$ is supported on $\partial_{\mathtt{h}} X$ and is a δ_{Vol}-conformal density.

A typical example of Γ-invariant Radon measure on X is the push-forward by an orbital map of the Haar measure Haar_Γ on Γ (i.e. the counting measure if Γ is discrete). In this case, the volume entropy is called the *critical exponent* of Γ, and is given by

$$\delta_\Gamma := \limsup_{r\to\infty} \frac{1}{r} \log\left(\mathrm{Haar}_\Gamma\{\gamma\in\Gamma : d(x,\gamma x) \le r\}\right), \tag{1.4.2}$$

which does not depend on the choice of $x \in X$.

1.4.3 The Gromov product and the Sullivan measures

The Gromov product of three points $x, \xi, \eta \in X$ is defined by

$$\langle \xi,\eta\rangle_x := \frac{1}{2}\left(d(x,\xi) + d(x,\eta) - d(\xi,\eta)\right) \ge 0. \tag{1.4.3}$$

For any $y \in X$, we have

$$\langle \xi,\eta\rangle_x = 0 \Leftrightarrow d(\xi,\eta) = d(\xi,x) + d(x,\eta); \tag{1.4.4}$$

$$\langle \xi,\eta\rangle_x = \langle \xi,\eta\rangle_y + \frac{1}{2}\mathtt{b}_\xi(x,y) + \frac{1}{2}\mathtt{b}_\eta(x,y); \tag{1.4.5}$$

$$|\langle \xi,\eta\rangle_x - \langle \xi,\eta\rangle_y| \le d_\Omega(x,y). \tag{1.4.6}$$

Note that $d(\xi,\eta) = d(\xi,x) + d(x,\eta)$ holds for example if x lies on a geodesic segment from ξ to η.

Suppose we are given an open subset $\mathcal{G} \subset (\overline{X}^{\mathtt{h}})^2$ that contains X^2 and such that $(\xi,\eta,x) \mapsto \langle \xi,\eta\rangle_x$ extends continuously to $\mathcal{G}\times X$. Then (1.4.5) and (1.4.6) also extend to $(\xi,\eta,x,y) \in \mathcal{G}\times X^2$. Set $\mathcal{G}^\infty = \mathcal{G}\cap \partial_{\mathtt{h}} X^2$.

Suppose further that we are given a closed subgroup $\Gamma \subset \mathrm{Isom}(X,d)$ and a δ-conformal density $(\mu_x)_x$ for some $\delta \ge 0$. Then the *Sullivan measure* induced by $(\mu_x)_x$ on $\mathcal{G}^\infty$ is defined by the following formula due to Sullivan [Sul79, Prop. 11].

$$\mathrm{d}m_{\mathbb{R}}(\xi,\eta) = e^{2\delta\langle\xi,\eta\rangle_o} d\mu_o(\xi) d\mu_o(\eta),$$

for any $o \in X$. The measure $m_{\mathbb{R}}$ does not depend on the choice of o, and is Γ-invariant, and invariant under the flip action $\iota(\xi,\eta) = (\eta,\xi)$. Moreover it is Radon (i.e. gives finite measure to compact sets) since the Gromov product is continuous and μ_o is finite; however it may be zero.

The Sullivan measure on $\mathcal{G}^\infty\times\mathbb{R}$ is simply defined by

$$\mathrm{d}m(\xi,\eta,t) = \mathrm{d}m_{\mathbb{R}}(\xi,\eta)dt.$$

It is Radon and invariant under the natural flow $\phi_s(\xi,\eta,t) = (\xi,\eta,t+s)$, and under the actions by Γ of the form $\gamma\cdot(\xi,\eta,t) = (\gamma\xi,\gamma\eta,t+\mathtt{b}_\eta(\gamma^{-1}o,o))$, where $o \in \Omega$. These actions by Γ are interesting for the following reasons. First because they are proper, hence the action of $(\phi_t)_t$ on $\mathcal{G}^\infty\times\mathbb{R}/\Gamma$ is relevant and can give information on the action of Γ on $\mathcal{G}^\infty$ and

$\partial_{\mathtt{h}} X$. Second because in the case where X is the hyperbolic plane $\mathbb{H}^2$ and $\Gamma \subset \mathrm{Isom}(\mathbb{H}^2)$, the horoboundary is the usual visual boundary ∂X, we can take $\mathcal{G}^\infty$ to be the set of pairs of distinct points of ∂X, and the $(\Gamma, (\phi_t)_t)$-space $\mathcal{G}^\infty \times \mathbb{R}$ is isomorphic to $T^1\mathbb{H}^2$, via the celebrated Hopf parametrisation. We will check that this fact also holds, to some extent, in the setting of convex projective orbifolds.

Chapter 2

Reminders on convex projective geometry

2.1 Properly convex open subsets of real projectives spaces and their geodesic flow

In the whole thesis we fix a finite-dimensional real vector space $\mathbf{V} = \mathbb{R}^{\mathtt{d}+1}$. Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set. Recall that Ω admits a proper metric called the *Hilbert metric* and defined by the following formula: for $(a,x,y,b) \in \partial\Omega \times \Omega \times \Omega \times \partial\Omega$ aligned in this order (see Figure 2.1),

$$d_\Omega(x,y) = \frac{1}{2}\log([a,x,y,b]),$$

where $[a,x,y,b]$ is the cross-ratio of the four points, normalised so that $[0,1,t,\infty] = t$ in $\mathrm{P}(\mathbb{R}^2)$ identified with $\mathbb{R} \cup \{\infty\}$. In other words, if $\mathbb{A} \subset \mathrm{P}(\mathbf{V})$ is an affine chart which contains $\overline{\Omega}$ and is equipped with some norm $\|\cdot\|$, then

$$[a,x,y,b] = \frac{\|b-x\| \cdot \|a-y\|}{\|a-x\| \cdot \|b-y\|}. \tag{2.1.1}$$

By definition, this metric is invariant under projective transformations: $g_{|\Omega} : (\Omega, d_\Omega) \to (g\Omega, d_{g\Omega})$ is an isometry for any $g \in \mathrm{PGL}(\mathbf{V})$. In particular, any element of $\mathrm{Aut}(\Omega) = \{g \in \mathrm{PGL}(\mathbf{V}) : g\Omega = \Omega\}$ yields an isometry of Ω. Most results on the geometry of the Hilbert metric that we present in this section, as well as many others, can be found in the Handbook of Hilbert geometry [PT14].

If Ω is an ellipsoid, then (Ω, d_Ω) is the Klein model of the real hyperbolic space of dimension $\mathtt{d}$, and if Ω is a $\mathtt{d}$-simplex, then (Ω, d_Ω) is isometric to $\mathbb{R}^{\mathtt{d}}$ endowed with a hexagonal norm. In general, the Hilbert metric is a Finsler metric, and (Ω, d_Ω) is Riemannian if and only if it is CAT(0), if and only if Ω is an ellipsoid (this is due to Kelly–Strauss [KS58], see also [PT14, §6.4-5]). The formula to compute the Finsler metric will not be needed in this thesis.

Any discrete subgroup $\Gamma \subset \mathrm{Aut}(\Omega)$ must act properly discontinuously on Ω since it preserves the proper metric d_Ω (see Section 1.1.1), and therefore the quotient $M = \Omega/\Gamma$ is an orbifold. Furthermore, M is a manifold if the action is free. If Γ is torsion-free, then it clearly acts freely on Ω; the converse holds by Brouwer's fixed point theorem, applied to the convex hull of a finite orbit of a torsion element. Note that by Selberg's lemma [Sel60], if Γ is finitely generated, then it has a torsion-free finite-index subgroup. We will work in general with Γ not necessarily torsion-free, so we set the notation $T^1M = T^1\Omega/\Gamma$.

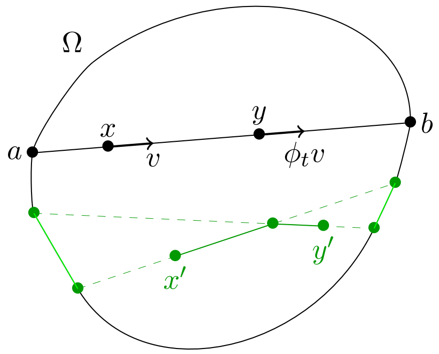

Figure 2.1: The Hilbert metric and the geodesic flow ($t = d_\Omega(x, y)$).

The intersections of Ω with projective lines can be parametrised to be geodesics (for d_Ω), which are said to be *straight*. However, an interesting feature in the non-strictly convex case is that when there are two coplanar non-trivial segments in the boundary $\partial\Omega$, one can construct geodesics which are not straight, see for instance the green broken segment from x' to y' in Figure 2.1. In order to define the geodesic flow we only take into account straight geodesics: for v in $T^1\Omega$, let $t \mapsto c(t)$ be the parametrisation of the projective line tangent to v such that c is an isometric embedding from $\mathbb{R}$ to Ω and $c'(0) = v$. For $t \in \mathbb{R}$ we set $\phi_t(v) = c'(t) \in T^1\Omega$. See Figure 2.1.

The geodesic flow on T^1M is well defined because the two actions of $\mathrm{Aut}(\Omega)$ and $(\phi_t)_{t\in\mathbb{R}}$ on $T^1\Omega$ commute.

Notation 2.1.1. We denote by $\pi : T^1M \to M$ and $\pi : T^1\Omega \to \Omega$ the foot-point projections. For any $v \in T^1\Omega$, we denote by $\phi_\infty v$ (resp. $\phi_{-\infty}v$) the limit in $\partial\Omega$ (for the topology of $\mathrm{P}(\mathbf{V})$) of $\pi\phi_t v$ as t tends to ∞ (resp. $-\infty$). We denote by ι the flip involution of $T^1\Omega$, that satisfies $\phi_{-\infty}\iota v = \phi_\infty v$, $\phi_\infty \iota v = \phi_{-\infty}v$ and $\pi\iota v = \pi v$.

2.1.1 The metrics we use

We consider the following metrics for $T \geq 0$ (if $T = 0$ then we omit it):

$$\begin{aligned}
\forall x, y \in M, \qquad & d_M(x, y) = \min\{d_\Omega(\tilde{x}, \tilde{y}) : \tilde{x}, \tilde{y} \in \Omega \text{ lifts of } x, y\},\\
\forall v, w \in T^1\Omega, \qquad & d^{(T)}_{T^1\Omega}(v, w) = \max_{0\leq t\leq T+1} d_\Omega(\pi\phi_t v, \pi\phi_t w),\\
\forall v, w \in T^1M, \qquad & \tilde{d}^{(T)}_{T^1M}(v, w) = \min\{d^{(T)}_{T^1\Omega}(\tilde{v}, \tilde{w}) : \tilde{v}, \tilde{w} \in T^1\Omega \text{ lifts of } v, w\}.
\end{aligned}$$

Notation 2.1.2. For any space X equipped with a metric of the form d^α_X (resp. $\tilde{d}^\alpha_X$), we denote by $B^\alpha_X(x, r)$ (resp. $\widetilde{B}^\alpha_X(x, r)$) the open ball of radius $r > 0$ centred at $x \in X$, and $\overline{B}^\alpha_X(x, r)$ (resp. $\widetilde{\overline{B}}^\alpha_X(x, r)$) the corresponding closed ball.

2.1.2 Comparison between the Hilbert metric and the projective metric

Fix an affine chart of $\mathrm{P}(\mathbf{V})$ containing $\overline{\Omega}$. Then an elementary computation yields

$$\overline{B}_\Omega(x, r) \subset (1 - e^{-2r})(\overline{\Omega} - x) + x$$

for all $x \in \Omega$ and $r > 0$, where $(1 - e^{-2r})(\overline{\Omega} - x) + x$ is the image of $\overline{\Omega}$ under the homothety (of the affine chart) centred at x and with ratio $1 - e^{-2r}$.

A consequence of the above formula is that for any Riemannian metric $d_{\mathrm{P}(\mathbf{V})}$ on $\mathrm{P}(\mathbf{V})$, there exists a constant $C > 0$ (depending on Ω) such that $d_\Omega \geq C d_{\mathrm{P}(\mathbf{V})}$.

2.1.3 Comparison between two Hilbert metrics

Fact 2.1.3 ([Bir57]). *Let $\Omega, \Omega' \subset \mathrm{P}(\mathbf{V})$ be two properly convex open sets. If $\Omega' \subset \Omega$, then $d_\Omega(x,y) \leq d_{\Omega'}(x,y)$ for all $x,y \in \Omega'$. If $\overline{\Omega}' \subset \Omega$, then there exists $C < 1$ such that $d_\Omega(x,y) \leq C d_{\Omega'}(x,y)$ for all $x,y \in \Omega'$.*

Proof. Fix an affine chart that contains $\overline{\Omega}$, and equip it with a norm, denoted by $\|\cdot\|$. Consider $a \in \partial\Omega$, $a' \in \partial\Omega'$, $x,y \in \Omega'$, $b' \in \partial\Omega'$, and $b \in \partial\Omega$ aligned in this order. Set $\alpha = \|a'-x\|$, $\beta = \|a'-y\|$, $t = \|a-a'\|$, $\alpha' = \|b'-y\|$, $\beta' = \|b'-x\|$ and $t' = \|b-b'\|$. By definition of the Hilbert metric,

$$d_\Omega(x,y) = \log\left(\frac{t+\alpha}{t+\beta}\cdot\frac{t'+\alpha'}{t'+\beta'}\right) \leq d_{\Omega'}(x,y) = \log\left(\frac{\alpha}{\beta}\cdot\frac{\alpha'}{\beta'}\right).$$

Let us now assume that $\overline{\Omega}' \subset \Omega$. Set $r = \min\{\|p-q\| : p \in \partial\Omega,\ q \in \partial\Omega'\}$ and $R = \max\{\|p-q\| : p,q \in \overline{\Omega}\}$. To conclude the proof of Fact 2.1.3, it is enough to show that

$$\log\left(\frac{t+\alpha}{t+\beta}\right) \leq e^{-\frac{r}{R}}\log\left(\frac{\alpha}{\beta}\right), \quad \text{and} \quad \log\left(\frac{t'+\alpha'}{t'+\beta'}\right) \leq e^{-\frac{r}{R}}\log\left(\frac{\alpha'}{\beta'}\right).$$

The two inequalities are proved the same way. Set $f(s) = \log\log\frac{s+\alpha}{s+\beta}$ for $s \in [0,t]$.

$$f(t) - f(0) = \int_0^t f'(s)\,\mathrm{d}s \leq t\max\{f'(s) : s \in [0,t]\} \leq r\max\{f'(s) : s \in [0,t]\}.$$

Thus it is enough to prove that $f'(s) \leq \frac{-1}{R}$ for any $s \in [0,t]$. An elementary computation yields:

$$f'(s) = \frac{1 - \frac{s+\beta}{s+\alpha}}{(s+\beta)\log\left(\frac{s+\beta}{s+\alpha}\right)}$$

for $s \in [0,t]$. This concludes the proof since $\log u \leq u-1$ for any $u > 0$, and $t+\beta = \|a-y\| \leq R$. ☐

2.1.4 The Hilbert balls are convex

Busemann proved [Bus55, 18.6] that the Hilbert balls of any properly convex open set $\Omega \subset \mathrm{P}(\mathbf{V})$ are convex. In fact, he even proved that any uniform neighbourhood of a convex subset of Ω is convex. If Ω is strictly convex then Hilbert balls are also strictly convex. Finally, for any $r \in \mathbb{N}$, the boundary $\partial\Omega$ is $\mathcal{C}^r$ if and only if the Hilbert spheres are so.

2.1.5 Crampon's lemma

The Hilbert balls are convex, but the Hilbert metric does not satisfy the stronger convexity property that the functions $t \mapsto d_\Omega(c_1(t), c_2(t))$ are convex for c_1 and c_2 straight Hilbert geodesics (see [SM02]). However, a weaker property holds, which was observed by Crampon, and of which we make extensive use.

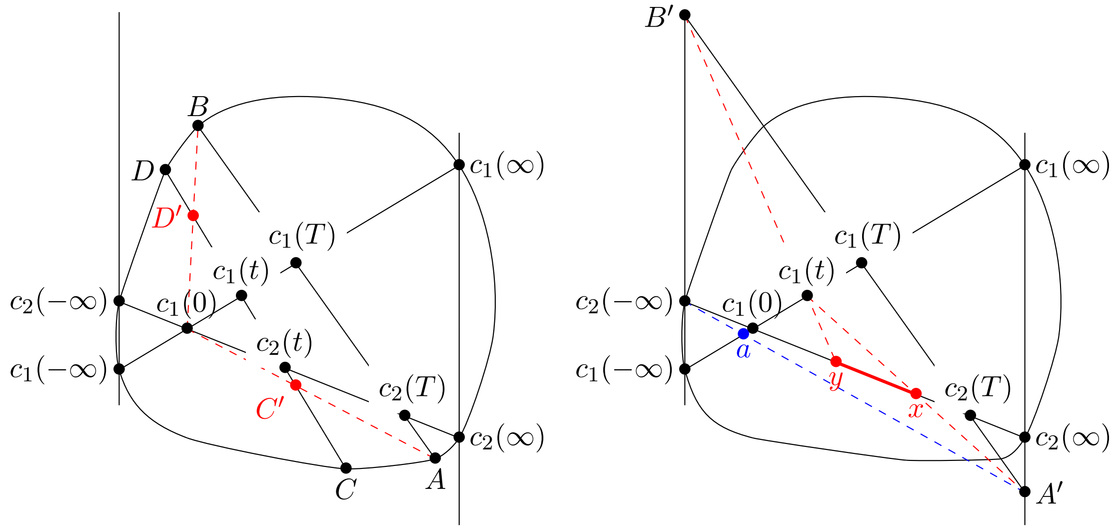

Figure 2.2: Proof of Crampon's Lemma 2.1.4

Lemma 2.1.4 ([Cra09, Lem. 8.3])**.** *Let Ω be a properly convex open subset of* $\mathrm{P}(\mathbf{V})$*. Let c_1 and c_2 be two straight geodesics parametrised with constant speed, but not necessarily with the same speed. Then for all $0 \leq t \leq T$,*

$$d_\Omega(c_1(t), c_2(t)) \leq d_\Omega(c_1(0), c_2(0)) + d_\Omega(c_1(T), c_2(T)).$$

Proof. It is enough to establish Lemma 2.1.4 when $c_1(0) = c_2(0)$. Indeed, suppose the lemma true in this case. Consider two straight geodesics c_1 and c_2, each parametrised with constant speed. Let c_3 be the straight geodesic, parametrised with constant speed, such that $c_3(0) = c_1(0)$ and $c_3(T) = c_2(T)$. For $t \leq T$ we have

$$\begin{aligned} d_\Omega(c_1(t), c_2(t)) &\leq d_\Omega(c_1(t), c_3(t)) + d_\Omega(c_3(t), c_2(t)) \\ &\leq d_\Omega(c_1(T), c_3(T)) + d_\Omega(c_3(0), c_2(0)) \\ &\leq d_\Omega(c_1(T), c_2(T)) + d_\Omega(c_1(0), c_2(0)). \end{aligned}$$

We now assume $c_1(0) = c_2(0)$ (and that c_1 and c_2 are not constant, otherwise the proof is trivial). We can then assume that Ω has dimension 2, and we can consider an affine chart in which both projective lines $(c_1(-\infty) \oplus c_2(-\infty))$ and $(c_1(\infty) \oplus c_2(\infty))$ are vertical. Fix $0 < t < T$. We draw Figure 2.2 (left-hand side) which contains the following points:

- A and B are the intersection points of the line $(c_2(T) \oplus c_1(T))$ with $\partial\Omega$;
- C and D are the intersection points of the line $(c_2(t) \oplus c_1(t))$ with $\partial\Omega$;
- C' and D' are the intersection points of the line $(c_2(t) \oplus c_1(t))$ with the lines $(c_1(0) \oplus A)$ and $(c_1(0) \oplus B)$.

If we are in the case, as in Figure 2.2, where the lines $(c_1(t) \oplus c_2(t))$ and $(c_1(T) \oplus c_2(T))$ do not intersect inside Ω, then by convexity of Ω the point C' lies between C and $c_2(t)$ and D' lies between D and $c_1(t)$. Therefore by definition of the cross-ratio we deduce that

$$\begin{aligned} d_\Omega(c_1(T), c_2(T)) &= d_{(C',D')}(c_1(t), c_2(t)) \\ &\geq d_{(C,D)}(c_1(t), c_1(t)) \\ &\geq d_\Omega(c_1(t), c_2(t)). \end{aligned}$$

It remains to prove that the lines $(c_1(t) \oplus c_2(t))$ and $(c_1(T) \oplus c_2(T))$ do not cross inside Ω (this is the missing explanation in Crampon's original proof). We draw Figure 2.2 (right-hand side) which contains the points:

- A' and B' are the intersection points of the line $(c_2(T)\oplus c_1(T))$ with the lines $(c_1(\infty)\oplus c_2(\infty))$ and $(c_1(-\infty) \oplus c_2(-\infty))$.
- x and y are the intersection points of the line $(c_2(-\infty) \oplus c_2(\infty))$ with the lines $(c_1(t) \oplus A')$ and $(c_1(t) \oplus B')$.
- a is the intersection point of the line $(c_1(-\infty)\oplus c_1(\infty))$ with the line $(c_2(-\infty)\oplus A')$.

And we observe that it is enough to prove that $c_2(t)$ is on the segment $[x, y]$. In other words we want to establish:

$$\frac{d_\Omega(c_2(0), y)}{d_\Omega(c_2(0), c_2(T))} \leq \frac{d_\Omega(c_2(0), c_2(t))}{d_\Omega(c_2(0), c_2(T))} = \frac{t}{T} = \frac{d_\Omega(c_1(0), c_1(t))}{d_\Omega(c_1(0), c_1(T))} \leq \frac{d_\Omega(c_2(0), x)}{d_\Omega(c_2(0), c_2(T))}.$$

For example, if we want to establish the inequality on the right, we see by definition of the cross-ratio that it is enough to prove:

$$\frac{d_\Omega(c_1(0), c_1(t))}{d_\Omega(c_1(0), c_1(T))} \leq \frac{d_{(a,c_1(\infty))}(c_1(0), c_1(t))}{d_{(a,c_1(\infty))}(c_1(0), c_1(T))}.$$

It is a consequence of the following lemma. This, and a similar argument for the inequality on the left, conclude the proof of Lemma 2.1.4. □

Observation 2.1.5. $\frac{d_{(a,b)}(x,y)}{d_{(a,b)}(x,z)} \leq \frac{d_{(a',b)}(x,y)}{d_{(a',b)}(x,z)}$ *for all* $a < a' < x < y < z < b \in \mathbb{R}$.

Proof. Up to acting by a projective transformation we can assume that $x = 0$, $y = 1$ and $b = \infty$. For $z > 1$ we consider the function: $a \mapsto f_z(a) = \frac{d_{(a,\infty)}(0,1)}{d_{(a,\infty)}(0,z)}$ on $(-\infty, 0)$. We have to check that this function f_z is non-decreasing. This follows immediatly from the fact that $f_z(a) = \frac{\log(1+\frac{-1}{a})}{\log(1+\frac{-z}{a})}$ for every $a < 0$, and from the computation of the derivative. □

2.1.6 Weak stable manifolds

The metrics we have chosen on $T^1\Omega$ (Section 2.1.1) combines well with Lemma 2.1.4:

Lemma 2.1.6. *Let* $\Omega \subset \mathrm{P}(\mathbf{V})$ *be a properly convex open set and* $v, w \in T^1\Omega$ *such that* $\phi_\infty w = \phi_\infty v$. *Then the function* $t \mapsto d_{T^1\Omega}(\phi_t v, \phi_t w)$ *is non-increasing.*

Proof. In order to prove that $t \mapsto d_{T^1\Omega}(\phi_t v, \phi_t w)$ is non-increasing, it is enough to prove that $t \mapsto d_\Omega(\pi\phi_t v, \pi\phi_t w)$ is non-increasing. Observe that it will also have as a consequence that $d_{T^1\Omega}(v, w) \leq d_\Omega(\pi v, \pi w)$. We fix $t \geq 0$. Consider a sequence $(x_n)_{n\in\mathbb{N}} \in \Omega^{\mathbb{N}}$ converging to ξ, and for each $n \in \mathbb{N}$, take $v_n \in T^1_{\pi v}\Omega$ and $w_n \in T^1_{\pi w}\Omega$ which define geodesic rays containing x_n. Then $\phi_t(v) = \lim_{n\to\infty} \phi_t(v_n)$ and $\phi_t(w) = \lim_{n\to\infty} \phi_t(w_n)$. By Lemma 2.1.4,

$$d_\Omega(\pi\phi_t v_n, \pi\phi_{t\frac{d_\Omega(\pi w, x_n)}{d_\Omega(\pi v, x_n)}} w_n) \leq d_\Omega(\pi v, \pi w)$$

for any n, and we get the desired inequality by taking the limit, since $(\frac{d_\Omega(\pi w, x_n)}{d_\Omega(\pi v, x_n)})_n$ tends to 1. □

2.1.7 Shadows

Let $x \in \overline{\Omega}$ and $y \in \Omega$ and $R > 0$. The shadow cast on $\partial\Omega$ by the balls of radius R around y with a light source at x are:

$$\begin{aligned} \mathcal{O}_R(x,y) &= \{\xi \in \partial\Omega : [x,\xi] \cap B_\Omega(y,R) \neq \emptyset\}, \\ \overline{\mathcal{O}}_R(x,y) &= \{\xi \in \partial\Omega : [x,\xi] \cap \overline{B}_\Omega(y,R) \neq \emptyset\}. \end{aligned}$$

By convexity of Hilbert balls (Section 2.1.4), these two shadows are homeomorphic to respectively $\mathbb{R}^{d-1}$ and its closed unit ball, as soon as $x \notin \overline{B}_\Omega(y,R)$. By Lemma 2.1.4, for any $x' \in [x,y]$ we have $\mathcal{O}_R(x,y) \subset \mathcal{O}_R(x',y)$.

2.1.8 Terminology for convex sets

We recall here some terminology on convex sets.

Notation 2.1.7. For any subset X of the projective space $\mathrm{P}(\mathbf{V})$, the closure (resp. interior resp. boundary) of X, denoted by $\overline{X}$ (resp. $\mathrm{int}(X)$ resp. ∂X), will always be considered *with respect to* $\mathrm{P}(\mathbf{V})$.

Let $K \subset \mathrm{P}(\mathbf{V})$ be properly convex, i.e. convex and bounded in some affine chart.

- The *relative interior* (resp. *relative boundary*) of K, denoted by $\mathrm{int}_{\mathrm{rel}}(K)$ (resp. $\partial_{\mathrm{rel}} K$) is its topological interior (resp. boundary) with respect to the projective subspace it spans.

- For $x \in \overline{K}$, the *open face* of x in $\overline{K}$, denoted by $F_K(x)$, consists of the points $y \in \overline{K}$ such that $[x,y]$ is contained in the relative interior of a (possibly trivial) segment contained in $\overline{K}$. The *closed face* of x is $\overline{F}_K(x) = \overline{F_K(x)}$.

- A point $x \in \overline{K}$ is said to be *extremal* (resp. *strongly extremal*) if $F_K(x) = \{x\}$ (resp. $F_K(x) = \{x\}$ and $[x,y] \cap \mathrm{int}_{\mathrm{rel}} K \neq \emptyset$ for $y \in \partial_{\mathrm{rel}} K \smallsetminus \{x\}$); one says that K is *strictly convex* if all the points in the relative boundary are extremal (and hence strongly extremal).

- Assume that K spans $\mathrm{P}(\mathbf{V})$ and let $\xi \in \partial K$. A *supporting hyperplane* of K at ξ is a hyperplane which contains ξ but does not intersect $\mathrm{int}(K)$. Note that there always exists such a hyperplane. The point ξ is said to be a *smooth* or $\mathcal{C}^1$ point of ∂K if there is only one supporting hyperplane of K at ξ, denoted by $T_\xi \partial K$.

- Assume that K spans $\mathrm{P}(\mathbf{V})$. The set of smooth (resp. non-smooth, resp. smooth and strongly extremal) points of ∂K is denoted by $\partial_{\mathrm{smooth}} K$ (resp. $\partial_{\mathrm{sing}} K$, resp. $\partial_{\mathrm{sse}} K$).

2.1.9 Duality

Let us recall the notion of duality for properly convex open sets, and set some notations and identifications.

Notation 2.1.8.

- We denote by $\mathbf{V}^*$ the dual of $\mathbf{V}$, i.e. the vector space of (real) linear forms on $\mathbf{V}$, and we identify $\mathbf{V}^{**}$ with $\mathbf{V}$;

- for each element $g \in \mathrm{End}(\mathbf{V})$, we denote by ${}^t g \in \mathrm{End}(\mathbf{V}^*)$ its transpose, defined by ${}^t g \cdot \alpha = \alpha \circ g$ for any $\alpha \in \mathbf{V}^*$, and if g is inversible, then we set $g^* = {}^t g^{-1}$;

- the dual projective space $\mathrm{P}(\mathbf{V}^*)$ identifies with the set of projective hyperplanes of $\mathrm{P}(\mathbf{V})$; note that for any $g \in \mathrm{PGL}(\mathbf{V})$ and any $H \in \mathrm{P}(\mathbf{V}^*)$ seen as a hyperplane of $\mathrm{P}(\mathbf{V}^*)$, the hyperplane $gH = \{gx : x \in H\} \subset \mathrm{P}(V)$ identifies with $g^*H \in \mathrm{P}(\mathbf{V}^*)$;
- the dual of Ω, denoted by Ω^*, is the properly convex open subset of $\mathrm{P}(\mathbf{V}^*)$ defined as the set of projective hyperplanes which do not intersect $\overline{\Omega}$;
- Ω identifies with Ω^{**};
- $\partial\Omega^*$ identifies with the set of supporting hyperplanes of Ω, and if we let $x \in \partial\Omega$ and $H \in \partial\Omega^*$, then $x \in H$ (where H is seen as a supporting hyperplane of Ω) if and only if $H \in x$ (where $x \in \partial\Omega^{**}$ is seen as a supporting hyperplane of Ω^*);
- $g \in \mathrm{PGL}(\mathbf{V}) \mapsto g^* \in \mathrm{PGL}(\mathbf{V}^*)$ induces an isomorphism between $\mathrm{Aut}(\Omega)$ and $\mathrm{Aut}(\Omega^*)$.

Observation 2.1.9. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set.*

(i) $H \in \partial\Omega^*$ is smooth if and only if $H \cap \partial\Omega$ is a singleton.

(ii) A smooth point $x \in \partial\Omega$ is strongly extremal if and only if its tangent space $T_x\partial\Omega$ is a smooth point of $\partial\Omega^*$; in this case $T_x\partial\Omega$ is strongly extremal.

(iii) For any $H, H' \in \partial\Omega^*$, the segment $[H, H'] \subset \overline{\Omega}^*$ is contained in $\partial\Omega^*$ if and only if $H \cap H' \cap \partial\Omega$ is non-empty.

2.1.10 More metrics

In this section we define two metrics : one on the projective boundary $\partial\Omega$, and one on $\overline{\Omega}$.

- The *simplicial distance* between two points ξ and η of $\partial\Omega$ is defined as follows (and it is possibly infinite).

  $$d_{\mathrm{spl}}(\xi, \eta) := \inf\{k : \exists a_0, \dots, a_k \in \partial\Omega : a_0 = \xi, a_k = \eta, \forall 0 \le i < k, [a_i, a_{i+1}] \subset \partial\Omega\}.$$

  We set $B_{\mathrm{spl}}(\xi, R) = \{\xi' \in \partial\Omega : d_{\mathrm{spl}}(\xi, \xi') < R\}$ and $\overline{B}_{\mathrm{spl}}(\xi, R) = \{\xi' : d_{\mathrm{spl}}(\xi, \xi') \le R\}$ for any $R \ge 0$.

- Consider $\xi, \eta \in \overline{\Omega}$. When ξ and η are on the same open face F of $\overline{\Omega}$, we set $d_{\overline{\Omega}}(\xi, \eta) := d_F(\xi, \eta)$, and otherwise we set $d_{\overline{\Omega}}(\xi, \eta)$ to be infinite. We set $B_{\overline{\Omega}}(\xi, R) = \{\xi' \in \overline{\Omega} : d_{\overline{\Omega}}(\xi, \xi') < R\}$ and $\overline{B}_{\overline{\Omega}}(\xi, R) = \{\xi' : d_{\overline{\Omega}}(\xi, \xi') \le R\}$, and $\partial B_{\overline{\Omega}}(\xi, R) = \{\xi' : d_{\overline{\Omega}}(\xi, \xi') = R\}$ for any $R \ge 0$.

If Ω is strictly convex, then $d_{\overline{\Omega}}(\xi, \eta) = d_{\mathrm{spl}}(\xi, \eta) = \infty$ for all $\xi, \eta \in \partial\Omega$.

The following elementary observation is quite useful, we use it several times in this thesis.

Fact 2.1.10. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set. The functions $d_{\mathrm{spl}} : \partial\Omega^2 \to \mathbb{R} \cup \{\infty\}$ and $d_{\overline{\Omega}} : \overline{\Omega}^2 \to \mathbb{R} \cup \{\infty\}$ are lower semi-continuous. As a consequence, for any $R > 0$, the map*

$$\begin{array}{rcccl} \overline{B}_{\overline{\Omega}}(\cdot, R) & : & \overline{\Omega} & \longrightarrow & \{\textit{compact subsets of } \overline{\Omega}\} \\ & & \xi & \longmapsto & \overline{B}_{\overline{\Omega}}(\xi, R) \end{array}$$

is upper semi-continuous in the following sense: all accumulation points of $\overline{B}_{\overline{\Omega}}(\eta, R)$ when $\eta \to \xi$ for the Hausdorff topology must be contained in $\overline{B}_{\overline{\Omega}}(\xi, R)$.

Proof. Let us check that $d_{\overline{\Omega}}$ is lower semi-continuous. Let $(x_n, y_n)_n$ converge to (x, y) in $\overline{\Omega}^2$ and be such that $(d_{\overline{\Omega}}(x_n, y_n))_n$ converges; let us show that the limit is at least $d_{\overline{\Omega}}(x, y)$. We may assume that $x \neq y$ and $x_n \neq y_n$ for all n. For each n, let $a_n, b_n \in \partial\Omega$ (resp. $a, b \in \partial\Omega$) be such that a_n, x_n, y_n, b_n (resp. a, x, y, b) are aligned in this order and $[a_n, b_n]$ (resp. $[a, b]$) is maximal for inclusion among segments of $\overline{\Omega}$; by definition $d_{\overline{\Omega}}(x_n, y_n) = \log[a_n, x_n, y_n, b_n]/2$ and $d_{\overline{\Omega}}(x, y) = \log[a, x, y, b]/2$, where we set $[a, x, y, b] = \infty$ if $a = x$ or $b = y$. Up to extracting, we may assume that $(a_n, b_n)_n$ converges to some $(a', b') \in \partial\Omega^2$. Since $[a, b]$ is maximal in $\overline{\Omega}$, it contains $[a', b']$, and a, a', x, y, b', b are aligned in this order. The following concludes the proof:

$$[a_n, x_n, y_n, b_n] \underset{n\to\infty}{\longrightarrow} [a', x, y, b'] \geq [a, x, y, b]. \qquad \square$$

2.1.11 Benzécri's compactness theorem and proper densities

In this section we recall Benzécri's famous compactness theorem, and we see a first consequence for $\mathrm{PGL}(\mathbf{V})$-equivariant volume form on properly convex open sets. We denote by $\mathcal{E}_{\mathbf{V}}$ (resp. $\mathcal{E}^{\bullet}_{\mathbf{V}}$) the set of properly convex open sets (resp. pointed properly convex open sets) of $\mathrm{P}(\mathbf{V})$. We equip $\mathcal{E}^{\bullet}_{\mathbf{V}}$ with the pointed Hausdorff topology (i.e. the metrisable topology such that any sequence (x_n, Ω_n) in $\mathcal{E}^{\bullet}_{\mathbf{V}}$ converges to $(x, \Omega) \in \mathcal{E}^{\bullet}_{\mathbf{V}}$ if and only if $(x_n)_n$ converges to x, any sequence in $\prod_n \overline{\Omega}_n$ that converges in $\mathrm{P}(\mathbf{V})$ has its limit in $\overline{\Omega}$, and any point of $\overline{\Omega}$ is the limit of such a sequence).

Fact 2.1.11 ([Ben60, Ch. 5,§2,Th. 2])**.** *The action of* $\mathrm{PGL}(\mathbf{V})$ *on* $\mathcal{E}^{\bullet}_{\mathbf{V}}$ *is continuous, proper and cocompact.*

We recall the notion of a proper density on the set of properly convex open sets. They prescribe the way to choose a volume form on each properly convex open set in a $\mathrm{PGL}(\mathbf{V})$-equivariant manner. For the whole paper we fix a density $\mathrm{Vol}_{\mathrm{P}(\mathbf{V})}$ on $\mathrm{P}(\mathbf{V})$, seen as a measure.

Definition 2.1.12. A *proper density* on $\mathcal{E}_{\mathbf{V}}$ is a map of the form $\Omega \mapsto \mathrm{Vol}_{\Omega}$, where $\Omega \subset \mathrm{P}(\mathbf{V})$ is a properly convex open set and Vol_{Ω} is a density on Ω with Radon–Nikodym derivative $f(x, \Omega) > 0$ with respect to $\mathrm{Vol}_{\mathrm{P}(\mathbf{V})}$, satisfying the following three conditions.

- (Continuity) The function $f : \mathcal{E}^{\bullet}_{\mathbf{V}} \to \mathbb{R}_{>0}$ is continuous.
- (Monotone decreasing) Let (x, Ω) and $(y, \Omega') \in \mathcal{E}^{\bullet}_{\mathbf{V}}$. If $x = y \in \Omega \subset \Omega'$ then
$$f(x, \Omega') \leq f(x, \Omega).$$
- ($\mathrm{PGL}(\mathbf{V})$-equivariance) For any $T \in \mathrm{PGL}(\mathbf{V})$,
$$T_* \mathrm{Vol}_{\Omega} = \mathrm{Vol}_{T(\Omega)}\,.$$

See [Ver17] for more details and examples. We fix for the whole paper a proper density $\Omega \mapsto \mathrm{Vol}_{\Omega}$ on $\mathcal{E}_{\mathbf{V}}$. One of the key observations that we will need on proper densities is that for any $R > 0$, the following quantities are positive and finite (this is a direct consequence of Fact 2.1.11).

$$0 < \chi_-(R) := \min_{(x,\Omega)\in\mathcal{E}^{\bullet}_{\mathbf{V}}} \mathrm{Vol}_{\Omega}(\overline{B}_{\Omega}(x, R)) \leq \chi_+(R) := \max_{(x,\Omega)\in\mathcal{E}^{\bullet}_{\mathbf{V}}} \mathrm{Vol}_{\Omega}(\overline{B}_{\Omega}(x, R)) < \infty. \tag{2.1.2}$$

2.1.12 Non-straight geodesics

The following fact says that the only case where a non-straight geodesic can appear is the one shown in Figure 2.1.

Fact 2.1.13. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set. For $x, y \in \Omega$ distinct, consider (a_{xy}, b_{xy}) in $\partial\Omega^2$ such that a_{xy}, x, y, b_{xy} are aligned in this order. Let $x, y, z \in \Omega$ be pairwise distinct. Then $d_\Omega(x,z) = d_\Omega(x,y) + d_\Omega(y,z)$ if and only if $a_{xz} \in [a_{xy}, a_{yz}]$ and $b_{xz} \in [b_{xy}, b_{yz}]$.*

This property can be used to prove the following.

Fact 2.1.14 ([FK05])**.** *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set, $I \subset \mathbb{R}$ a non-trivial interval and $c : I \to \Omega$ an isometric embedding. For all $t < s \in I$, consider (a_{ts}, b_{ts}) in $\partial\Omega^2$ such that $a_{ts}, c(t), c(s), b_{ts}$ are aligned in this order. Let F_+ (resp. F_-) be the smallest closed face of Ω that contains $\{b_{ts} : t < s \in I\}$ (resp. $\{a_{ts} : t < s \in I\}$). Then F_+ and F_- are proper faces of Ω, whose dimension is the dimension of the convex hull of $c(I)$ minus 1.*

Moreover, if $\sup I = \infty$ (resp. $\inf I = -\infty$), then $(c(t))_t$ converges to a point of F_+ (resp. F_-) when t goes to $+\infty$ (resp. $-\infty$).

2.2 Automorphisms of properly convex open sets

2.2.1 Automorphisms with a spectral gap

Notation 2.2.1. If W_1 and W_2 are two subspaces of $\mathbf{V}$ such that $W_1 \cap W_2 = \{0\}$, we write $W_1 \oplus W_2 \subset \mathbf{V}$ for their direct sum and $\mathrm{P}(W_1) \oplus \mathrm{P}(W_2) = \mathrm{P}(W_1 \oplus W_2)$ for its projectivisation. In particular, if $x, y \in \mathrm{P}(\mathbf{V})$ are two distinct points, we write $x \oplus y$ for the projective line through x and y. We will often identify subspaces of $\mathbf{V}$ with subspaces of its projective space $\mathrm{P}(\mathbf{V})$.

Notation 2.2.2. Let $g \in \mathrm{End}(\mathbf{V})$ be non-zero.

1. $\lambda_1(g) \geq \cdots \geq \lambda_{\mathtt{d}+1}(g) \geq 0$ are the moduli of the complex eigenvalues of g;

2. $x_g^+ \subset \mathbf{V}$ (resp. x_g^-, resp. x_g^0) is sum of generalised eigenspaces associated to eigenvalues of modulus $\lambda_1(g)$ (resp. $\lambda_{\mathtt{d}+1}(g)$, resp. in the interval $(\lambda_{\mathtt{d}+1}(g), \lambda_1(g))$);

3. we set $y_g^+ = x_g^+ \oplus x_g^0$ and $y_g^- = x_g^- \oplus x_g^0$, and $f_g^+ = (x_g^+, y_g^+)$ and $f_g^- = (x_g^-, y_g^-)$;

4. $\ell(g) = \frac{1}{2} \log \frac{\lambda_1(g)}{\lambda_{\mathtt{d}+1}(g)}$ is set to be zero if $\lambda_1(g) = 0$ and infinite if $\lambda_1(g) > \lambda_{\mathtt{d}+1}(g) = 0$.

Note that $x_g^\pm$, $y_g^\pm$, x_g^0, $f_g^\pm$, and $\ell(g)$ only depends on the image of g in $\mathrm{P}(\mathrm{End}(\mathbf{V}))$.

Any non-zero element $g \in \mathrm{End}(\mathbf{V})$ defines a map $\mathrm{P}(\mathbf{V}) \smallsetminus \mathrm{Ker}(g) \to \mathrm{P}(\mathbf{V})$. If $\ell(g) > 0$ (i.e. g has a non-zero spectral gap), then we can iterate this map on the smaller subset $\mathrm{P}(\mathbf{V}) \smallsetminus x_g^-$, and its dynamics has the following elementary but interesting feature: the forward orbit of any compact subset of $\mathrm{P}(\mathbf{V}) \smallsetminus y_g^-$ accumulates on x_g^+, which can be seen as an attracting space for g. If moreover g is an automorphism of a properly convex open set, the attracting space x_g^+ must be located on the boundary of the convex set.

Fact 2.2.3. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set.*

1. *Let $g \in \mathrm{P}(\mathrm{End}(\mathbf{V}))$ be in the closure of $\mathrm{Aut}(\Omega)$, then its kernel and its image intersect $\overline{\Omega}$ but not Ω;*

2. *let $g \in \mathrm{Aut}(\Omega)$ be such that $\ell(g) > 0$, let $I \subset \mathbb{R}$ be a segment, and let X be the sum of the generalised eigenspaces associated to the eigenvalues of norm in I, then $X \cap \overline{\Omega}$ is a closed face of Ω, which is non-empty if I contains $\lambda_1(g)$ or $\lambda_{\mathtt{d}+1}(g)$.*

The second point applies in particular when $X = x_g^+$, y_g^+, x_g^- or y_g^-.

For any $g \in \mathrm{End}(\mathbf{V})$, its transpose ${}^t g$ (see Notation 2.1.8) has the same complex eigenvalues as g (with the same multiplicities) and, under the identification of $\mathrm{P}(\mathbf{V}^*)$ with the set of hyperplanes of $\mathrm{P}(\mathbf{V})$, for any subset $A \in \mathbb{R}$, the sum of generalised eigenspaces of ${}^t g$ associated to eigenvalues with real part in A is exactly the set of hyperplanes of $\mathrm{P}(\mathbf{V})$ that contain the sum of generalised eigenspaces of g associated with eigenvalues with real part in $\mathbb{R} \smallsetminus A$.

2.2.2 Proximal elements

Among the endomorphisms with a spectral gap, those which display the simplest (forward) dynamics on the projective space are the proximal ones.

Definition 2.2.4. Let $g \in \mathrm{End}(\mathbf{V})$ be non-zero.

1. g is *proximal* if $\lambda_1(g) > \lambda_2(g)$, i.e. if x_g^+ is a line of $\mathbf{V}$, which we identify with the associated point in $\mathrm{P}(\mathbf{V})$;

2. g is *biproximal* if it is proximal, invertible and if its inverse is proximal, in this case $x_g^+ \oplus x_g^-$ (see Notation 2.2.1) is the *axis* of g and is denoted by $\mathrm{axis}(g)$.

Note that proximality, biproximality and $\mathrm{axis}(g)$ are well defined for $g \in \mathrm{P}(\mathrm{End}(\mathbf{V}))$. Also note that g is proximal if and only if its transpose ${}^t g$ (Notation 2.1.8) is proximal, and $x_{{}^t g}^+$ identifies with y_g^-, which is a hyperplane of $\mathrm{P}(\mathbf{V})$.

Remark 2.2.5. The set of proximal linear transformations is open in $\mathrm{End}(\mathbf{V})$, and the map sending a proximal linear transformation g to the pair (x_g^+, associated real eigenvalue) is continuous.

The previous section implies that proximal elements have an attracting fixed point in $\mathrm{P}(\mathbf{V})$ (i.e. a fixed point $x \in \mathrm{P}(\mathbf{V})$, with a neighbourhood K such that $(g^n K)_n$ converges to x as n tends to infinity). The converse is also true, as attested by the following classical result.

Fact 2.2.6. *Let $g \in \mathrm{PGL}(\mathbf{V})$ and $x \in \mathrm{P}(\mathbf{V})$. Suppose that x is an attracting fixed point of g. Then g is proximal and $x_g^+ = x$.*

Proof. Set $W := x_g^+$. Since any orbit $(g^n y)_n$ with $y \notin y_g^-$ accumulates on x_g^+, the point x must lie in x_g^+, and x is an attracting fixed point of the restriction of g to $\mathrm{P}(W)$, denoted by h. Let us show that W has dimension 1. Consider lifts $v \in W$ of x and $\tilde{h} \in \mathrm{GL}(\mathbf{V})$ of h such that $\tilde{h}v = v$. All eigenvalues of $\tilde{h}$ are on the unit circle. If by contradiction $\tilde{h}$ has another eigenvector $w \in W$, then the restriction of $\tilde{h}^2$ to $\mathrm{Span}(v, w)$ is the identity, which contradicts the fact that x is attracting. If by contradiction $\tilde{h}$ has a complex eigenvalue, then there exists a plane $P \subset \mathbf{V}$ (of dimension 2), and a basis $v_1, v_2 \in P$ such that the restriction of $\tilde{h}$ to $\mathrm{Span}(v, P)$, written in the basis v, v_1, v_2, is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

for some $\theta \in \mathbb{R}$, which also contradicts the fact x is attracting. We have proved that the only eigenvalue of $\tilde{h}$ is 1 and that $\tilde{h}$ has only one eigenvector. If by contradiction $\dim(W) \geq 2$, then we can find a plane $P \subset W$ (of dimension 2) which is preserved by $\tilde{h}$ and contains v. In a suitable basis, the restriction of $\tilde{h}$ to P is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and this contradicts the fact that x is attracting. ☐

This, combined with Fact 2.1.3, yields the following generalisation of (a weak version of) the Perron–Frobénius theorem.

Fact 2.2.7 ([Bir57]). *Let $g \in \mathrm{PGL}(\mathbf{V})$. Suppose that g contracts a properly convex open set, in the sense that there exists a properly convex open set $\Omega \subset \mathrm{P}(\mathbf{V})$ such that $g(\overline{\Omega}) \subset \Omega$. Then the restriction of g to Ω is C-Lipschitz for some $0 < C < 1$, with respect to the Hilbert metric, and g is proximal with $x_g^+ \in \Omega$ (which is the only fixed point of g in $\overline{\Omega}$).*

2.2.3 The translation length of automorphisms of properly convex open sets

For any automorphism g of a properly convex open set Ω, the algebraic quantity $\ell(g)$ (Notation 2.2.2) may be interpreted thanks to the following fact as a geometrical quantity of the action of g on Ω: more precisely as its translation length.

Fact 2.2.8 ([CLT15, Prop. 2.1]). *For any properly convex open set $\Omega \subset \mathrm{P}(\mathbf{V})$ and $g \in \mathrm{Aut}(\Omega)$ and $o \in \Omega$,*

$$\ell(g) = \inf\{d_\Omega(x, gx) : x \in \Omega\} = \lim_{n\to\infty} \frac{1}{n} d_\Omega(o, g^n o).$$

If o, go and $g^2 o$ are aligned, then $d_\Omega(o, go) = \ell(g)$.

The question whether this infimum can be attained is interesting. Let Ω be a properly convex open set and $g \in \mathrm{Aut}(\Omega)$. Recall that if Ω is an ellipsoid, then there are three possibilities: either g fixes a point of Ω (g is elliptic), or $\ell(g) = 0$ and g fixes exactly one point of $\overline{\Omega}$, which is in $\partial\Omega$ (g is parabolic), or $\ell(g) > 0$ and g fixes exactly two points of $\partial\Omega$, and the geodesic between them is exactly the set of x such that $d_\Omega(x, gx) = \ell(g)$ (g is hyperbolic). This classification still hold when Ω is strictly convex case (see Section 9.1), but not in general.

2.2.4 Quantifying the properness of the action of automorphisms

For any element $g \in \mathrm{GL}(\mathbf{V})$, define

$$\kappa(g) = \frac{1}{2} \log\left(\|g\| \cdot \|g^{-1}\|\right), \tag{2.2.1}$$

where $\|\cdot\|$ is the operator norm on $\mathrm{End}(\mathbf{V})$; this quantity is also well defined if $g \in \mathrm{PGL}(\mathbf{V})$.

Fact 2.2.9 ([DGKa, Prop. 10.1]). *For any properly convex open set $\Omega \subset \mathrm{P}(\mathbf{V})$ and any point $x \in \Omega$, there exists $C > 0$ such that $\kappa(g) - C \leq d_\Omega(x, gx) \leq \kappa(g) + C$ for any $g \in \mathrm{Aut}(\Omega)$.*

The constant C is a function of $(x, \Omega) \in \mathcal{E}_{\mathbf{V}}^{\bullet}$ that can be taken continuous.

2.3 Groups of automorphisms of properly convex open sets

2.3.1 Irreducibility and strong irreducibility

A subgroup $\Gamma \subset \mathrm{PGL}(\mathbf{V})$ is *irreducible* if it does not preserve any proper subspace of $\mathrm{P}(\mathbf{V})$. It is moreover called *strongly irreducible* if every finite-index subgroup is irreducible.

Notation 2.3.1. For any subgroup $\Gamma \subset \mathrm{PGL}(\mathbf{V})$, we denote by Γ_0^{Z} the Zariski-connected component of the identity component. It is a finite-index subgroup, and it is **topologically** irreducible for the Zariski topology, in the sense that every Zariski-open non-empty subset is Zariski-dense.

If the action of Γ on $\mathrm{P}(\mathbf{V})$ is strongly irreducible, then the action of Γ_0^{Z} is irreducible. Conversely, assume that Γ_0^{Z} is irreducible. Let $\mathrm{P}(W_1)\cup\cdots\cup\mathrm{P}(W_n) \subset \mathrm{P}(\mathbf{V})$ be a finite union of proper subspaces, and $x \in \mathrm{P}(\mathbf{V})$. The sets $\{g \in \Gamma_0^{\mathrm{Z}} : gx \notin \mathrm{P}(W_i)\}_{1\leq i\leq n}$ are Zariski-open and non-empty since the action of Γ_0^{Z} irreducible; by topological Zariski-irreducibility, these sets are Zariski-dense, and hence have a non-empty intersection. We have proved that Γ_0^{Z}, and hence Γ, act strongly irreducibly on $\mathrm{P}(\mathbf{V})$.

Fact 2.3.2 ([Ben97, Lem. 3.6.ii]). *Let $\Gamma \subset \mathrm{PGL}(\mathbf{V})$ be a strongly irreducible subgroup that contains a proximal element. Then Γ contains a biproximal element.*

Proof. Let $\gamma \in \Gamma_0^{\mathrm{Z}}$ be proximal. Let $A \in \mathrm{P}(\mathrm{End}(\mathbf{V}))$ be an accumulation point of $(\gamma^{-n})_n$. By strong irreducibility, we can find g and $h \in \Gamma_0^{\mathrm{Z}}$ such that $gx_\gamma^+ \notin \mathrm{Ker}(A)$, and $hAgx_\gamma^+ \notin y_g^-$, and $g^{-1}x_\gamma^+ \notin \mathrm{Ker}(A)$ and $h^{-1}Ag^{-1}x_\gamma^+ \notin y_\gamma^-$. Then $g\gamma^n h\gamma^{-n}$ is biproximal for n large enough. □

2.3.2 The proximal limit set

Following the work of Guivarc'h and Benoist, we define as follows the proximal limit set of any subgroup of $\mathrm{PGL}(\mathbf{V})$ (which do not necessarily preserve a properly convex open set).

Definition 2.3.3. Let $\Gamma \subset \mathrm{PGL}(\mathbf{V})$ be a subgroup. The *proximal limit set* of Γ, denoted by $\Lambda_\Gamma^{\mathrm{prox}}$ or simply Λ_Γ, is the closure of the set of attracting fixed points of proximal elements of Γ.

Of course, the proximal limit set is sometimes empty, even for interesting groups. The following fact ensures that most of the groups we are concerned with in this thesis have a proximal element.

Fact 2.3.4 ([Ben00a, Prop. 1.1]). *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set and $\Gamma \subset \mathrm{Aut}(\Omega)$ an irreducible subgroup. Then Γ contains a proximal element.*

We will also see (Proposition 2.3.15) that if Γ acts cocompactly (or more generally convex cocompactly) on Ω, then Γ must contain a proximal element, even if it is not irreducible.

Remark 2.3.5. As observed by Benoist [Ben97, Lem. 3.6.ii], for any irreducible subgroup $\Gamma \subset \mathrm{PGL}(\mathbf{V})$ which contains a proximal element, the proximal limit set is the smallest closed Γ-invariant non-empty subset of $\mathrm{P}(\mathbf{V})$; in particular, the action of Γ on Λ_Γ is minimal (i.e. any orbit is dense). Indeed, consider any closed Γ-invariant non-empty subset $X \subset \mathrm{P}(\mathbf{V})$ and any proximal element $\gamma \in \Gamma$. By irreducibility, X contains a point x outside y_γ^-, and then x_γ^+, which is the limit of the sequence $(\gamma^n x)_{n\in\mathbb{N}}$, belongs to X.

If $\Gamma \subset \mathrm{PO}(\mathsf{d}, 1)$ is the fundamental group of a (non-elementary) real hyperbolic orbifold M of the form Ω/Γ (where Ω is an ellipsoid), then the proximal limit set of Γ in $\mathrm{P}(\mathbb{R}^{\mathsf{d}+1})$ coincides with the set of accumulation points in $\partial\Omega$ of any Γ-orbit of Ω, simply called limit set in this setting. Then there is a rich interaction between the action of Γ on pairs of points of the limit and the action of the geodesic flow on T^1M. This is the reason why we want to make the following definition in the more general setting of convex projective orbifolds.

Definition 2.3.6. Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set and $\Gamma \subset \mathrm{Aut}(\Omega)$ a discrete subgroup; set $M = \Omega/\Gamma$. The *biproximal unit tangent bundle* is denoted by T^1M_{bip}, and consists of the vectors whose lifts $v \in T^1\Omega$ have $\phi_\infty v$ and $\phi_{-\infty} v$ in Λ_Γ.

2.3.3 The biproximal limit set

Guivarc'h and Benoist actually defined much more general limit sets for subgroups of more general semi-simple Lie groups. More precisely these limit sets are subsets of a flag variety of the semi-simple Lie groups. We will not enter such a generality here, but we need to introduce the limit set in another flag variety of $\mathrm{PGL}(\mathbf{V})$:

Notation 2.3.7. Using the same notations as Benoist [Ben00a], we denote by $\mathrm{Q}(\mathbf{V})$ the set of pairs $(x, H) \in \mathrm{P}(\mathbf{V}) \times \mathrm{P}(\mathbf{V}^*)$ such that $x \in H$. Two pairs $(x, H), (x', H') \in \mathrm{P}(\mathbf{V}) \times \mathrm{P}(\mathbf{V}^*)$ are called *transverse* if $x \neq x'$, $H \neq H'$, $x \notin H'$ and $x' \notin H$, in which case we write $(x, H) \pitchfork (x', H')$. Observe that, for any biproximal element $g \in \mathrm{PGL}(\mathbf{V})$, the point f_g^+ is an attracting fixed point for the action of g on $\mathrm{P}(\mathbf{V}) \times \mathrm{P}(\mathbf{V}^*)$, i.e. $(g^n(x, H))_n$ converges to f_g^+ for any pair $(x, H) \in \mathrm{P}(\mathbf{V}) \times \mathrm{P}(\mathbf{V}^*)$ transverse to f_g^-.

Definition 2.3.8. Let $\Gamma \subset \mathrm{PGL}(\mathbf{V})$ be a subgroup. The *biproximal limit set*, denoted by $\Lambda_\Gamma^{\mathrm{bip}}$, is the closure in $\mathrm{Q}(\mathbf{V})$ of $\{f_g^+ : g \in \Gamma \text{ biproximal}\}$.

Benoist proved that the following results, analogous to Remark 2.3.5. Later on we will need a generalisation of them, that encompasses groups which are not necessarily strongly irreducible (see Proposition 3.2.3).

Fact 2.3.9 ([Ben97, Lem. 3.6.ii]). *Let $\Gamma \subset \mathrm{PGL}(\mathbf{V})$ be a strongly irreducible subgroup that contains a proximal element. Then the biproximal limit set is the smallest closed Γ-invariant non-empty subset of $\mathrm{P}(\mathbf{V}) \times \mathrm{P}(\mathbf{V}^*)$; in particular, the action of Γ on $\Lambda_\Gamma^{\mathrm{bip}}$ is minimal, and the projection of $\Lambda_\Gamma^{\mathrm{bip}}$ on the first coordinate in $\mathrm{P}(\mathbf{V})$ is $\Lambda_\Gamma^{\mathrm{prox}}$.*

More precisely, for any $(x, H) \in \mathrm{P}(\mathbf{V}) \times \mathrm{P}(\mathbf{V}^*)$, for any biproximal element $\gamma \in \Gamma$, for any neighbourhood $U \subset \mathrm{P}(\mathbf{V}) \times \mathrm{P}(\mathbf{V}^*)$ of f_γ^+, the subset $\{g \in \Gamma_0^{\mathsf{Z}} : g(x, H) \in U\} \subset \Gamma_0^{\mathsf{Z}}$ is Zariski-dense. In particular, the set of flags of $\Lambda_\Gamma^{\mathrm{bip}}$ which are transverse to (x, H) is dense.

2.3.4 Convex cocompact actions

Let $M = \Omega/\Gamma$ be a convex projective orbifold. The proximal limit set of Γ is contained in $\partial\Omega$, but if Ω is not strictly convex, then it can happen that the set of accumulation points in $\partial\Omega$ of $\Gamma \cdot x$ depends on $x \in \Omega$, and contains Λ_Γ as a proper subset. Based on this observation, Danciger–Guéritaud–Kassel defined another limit set.

Definition 2.3.10 ([DGKa]). Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set and $\Gamma \subset \mathrm{Aut}(\Omega)$ a subgroup, set $M = \Omega/\Gamma$. The *full orbital limit set* of Γ is

$$\Lambda_\Omega^{\mathrm{orb}}(\Gamma) = \partial\Omega \cap \bigcup_{x \in \Omega} \overline{\Gamma \cdot x},$$

which we denote by $\Lambda_\Gamma^{\mathrm{orb}}$ when the context is clear. Similarly to $T^1 M_{\mathrm{bip}}$, we consider

$$T^1 M_{\mathrm{cor}} := \{v \in T^1\Omega : \phi_{\pm\infty} v \in \Lambda_\Gamma^{\mathrm{orb}}\}/\Gamma \subset T^1 M.$$

Finally, the convex hull in Ω of the full orbital limit set is denoted by $\mathcal{C}_\Omega^{\mathrm{cor}}(\Gamma)$, and its projection in M is called the *convex core* of M. Observe that $\Lambda_\Gamma^{\mathrm{orb}}$ (resp. $T^1 M_{\mathrm{cor}}$) always contains $\Lambda_\Gamma = \Lambda_\Gamma^{\mathrm{prox}}$ (resp. $T^1 M_{\mathrm{bip}}$).

One can check that if Ω is strictly convex with $\mathcal{C}^1$ boundary and Γ is non-elementary rank-one, then $\Lambda_\Gamma = \Lambda_\Gamma^{\mathrm{orb}}$.

The full orbital limit set turns out to be the good way to define convex cocompact actions in general convex projective geometry. Recall that in real hyperbolic geometry, many properties of compact manifolds are actually also true for the broader class of manifolds with a compact convex core, in other words manifolds that are quotients of convex cocompact actions, and the proofs of these properties generally work verbatim. This observation remains valid in convex projective geometry.

Definition 2.3.11 ([DGKa, Def. 1.11])**.** Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set, and $\Gamma \subset \mathrm{Aut}(\Omega)$ a discrete subgroup. The action of Γ on Ω is said to be *convex cocompact* if the convex core of $M = \Omega/\Gamma$ is non-empty and compact.

If the action of Γ on Ω is convex cocompact, then the convex core of M and $\mathcal{C}_\Omega^{\mathrm{cor}}(\Gamma) \subset \Omega$ are closed subsets, and one can further prove the following.

Fact 2.3.12 ([DGKa, Cor. 4.9 & 4.11])**.** *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set, and $\Gamma \subset \mathrm{Aut}(\Omega)$ a discrete subgroup that acts convex cocompactly. Then $\Lambda^{\mathrm{orb}} = \overline{\mathcal{C}_\Omega^{\mathrm{cor}}(\Gamma)} \cap \partial\Omega$ (and hence is closed), and it contains the open face of all its points.*

This implies that $T^1 M_{\mathrm{cor}}$ consists exactly of the vectors whose entire $(\phi_t)_t$-orbit lies in the convex core of M.

Note also that if Γ divides a properly convex open set Ω, then the convex hull of any Γ-orbit in Ω is equal to Ω (this is due to Vey [Vey70, Prop. 3]), hence $\mathcal{C}_\Omega^{\mathrm{cor}}(\Gamma) = \Omega$ and Γ acts convex cocompactly on Ω. Compared to cocompact actions, one of the advantage of convex cocompact actions is that they are easier to construct: see for instance [DGKa, §1.4–1.6–1.7–4.1–10.7] and [DGKLM].

2.3.5 The conical limit set

Like in hyperbolic geometry, there is a characterisation of convex cocompactness in terms of a third limit set. This limit set also has a dynamical interpretation, which, as we saw in the introduction, plays an important role in the HTSR dichotomy (see Theorem 6.0.1).

Definition 2.3.13. Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set, and $\Gamma \subset \mathrm{Aut}(\Omega)$ a discrete subgroup. $\xi \in \Lambda_\Gamma^{\mathrm{orb}}$ is a *conical limit point* if there exists a sequence of elements $(\gamma_n)_n \subset \Gamma$ such that $(\gamma_n x)_n$ converges to ξ and $\sup_n d_\Omega(\gamma_n x, [x, \xi)) < \infty$ for some $x \in \Omega$. We denote by $\Lambda_\Gamma^{\mathrm{con}}$ the set of conical limit points.

One can check that if $\xi \in \Lambda_\Gamma^{\mathrm{con}}$ then the entire open face $F_\Omega(\xi)$ of ξ is contained in $\Lambda_\Gamma^{\mathrm{con}}$.

Fact 2.3.14 ([DGKa, Cor. 4.9 & Lem. 4.19])**.** *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set, and $\Gamma \subset \mathrm{Aut}(\Omega)$ a discrete subgroup. Then the action of Γ on Ω is convex cocompact if and only if $\Lambda_\Gamma^{\mathrm{orb}}$ is closed and every point in it is a conical limit point.*

2.3.6 Proximality of convex cocompact actions

Proposition 2.3.15. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set, and $\Gamma \subset \mathrm{Aut}(\Omega)$ a discrete subgroup that acts convex cocompactly on Ω. Then Γ contains a proximal element, and $\Lambda_\Gamma^{\mathrm{prox}}$ is the closure of the set of points $\Lambda_\Gamma^{\mathrm{orb}}$ that are extremal in Ω.*

Proof. Before giving the proof in full generality, let us explain it in a particular case. Suppose Γ is irreducible and acts cocompactly on Ω, and is a subgroup of $\mathrm{SL}(\mathbf{V})$ instead of $\mathrm{PGL}(\mathbf{V})$. Let $\xi \in \partial\Omega$ be extremal, and $(x_n)_n$ a sequence in Ω converging to ξ. By cocompactness, there exists $(\gamma_n)_n \subset \Gamma$ such that $(\gamma_n x)_n$ is at bounded distance from $(x_n)_n$ for any $x \in \Omega$. Moreover $F_\Omega(\xi) = \{\xi\}$ since ξ is extremal, so $(\overline{B}_\Omega(x_n, R))_n$ converges to $\{\xi\}$ for $R > 0$ by Fact 2.1.10. This implies that

$$(\gamma_n x)_n \underset{n\to\infty}{\longrightarrow} \xi \quad \forall x \in \Omega. \tag{2.3.1}$$

Up to extraction, we may assume that $(\|\gamma_n\|^{-1}\gamma_n)_n$ converges to some non-zero $\gamma \in \mathrm{End}(\mathbf{V})$, where $\|\cdot\|$ is any fixed norm on $\mathrm{End}(\mathbf{V})$. Then (2.3.1) implies that the image of γ is contained in the line ξ. Since γ is non-zero, its image is exactly ξ. Since Γ is irreducible, there is $g \in \Gamma$ such that the kernel of γg does not contain ξ, so that γg is proximal with attracting fixed point ξ. Proximality is an open condition, so $\|\gamma_n\|^{-1}\gamma_n g$, and hence also $\gamma_n g \in \Gamma$, is proximal for n large, with attracting fixed point near ξ.

We now give the proof in full generality. We denote by $\overline{\Gamma}$ the closure of Γ in $\mathrm{P}(\mathrm{End}(\mathbf{V}))$. By Fact 2.3.12, each extremal point of the properly convex open set $\mathcal{C}^{\mathrm{cor}} = \mathcal{C}_\Omega^{\mathrm{cor}}(\Gamma)$ is an extremal point of Ω. For each extremal point $\xi \in \Lambda^{\mathrm{orb}}$ of $\mathcal{C}^{\mathrm{cor}}$, we can choose a point $x \in \Omega$ and a sequence $(\gamma_n)_n$ in Γ such that $(\gamma_n x)_n$ converges to ξ, and $(\gamma_n)_n$ converges to an element A_ξ of $\overline{\Gamma}$. By Fact 2.2.3, the kernel of A_ξ does not intersect Ω, and $(\gamma_n y)_n$ converges to $A_\xi y$ for any $y \in \Omega$. By Fact 2.1.10, and since ξ is extremal for Ω, the sequence $(\gamma_n y)_n$ also converges to ξ for any $y \in \Omega$. As a consequence, the image of A_ξ is exactly ξ.

Let us fix $\xi_1 \in \Lambda^{\mathrm{orb}}$ extremal and prove that $\xi_1 \in \Lambda^{\mathrm{prox}}$. Since $\overline{\mathcal{C}^{\mathrm{cor}}}$ is the convex hull of its extremal points, we can find one outside $\mathrm{Ker}(A_{\xi_1})$, which we denote by ξ_2. By induction, we can find $\xi_3, \dots, \xi_{\mathtt{d}+1} \in \Lambda^{\mathrm{orb}}$ extremal such that $\xi_{i+1} \notin \mathrm{Ker}(A_{\xi_i})$ for any $1 \leq i \leq \mathtt{d}$, and this implies that $\xi_i = A_{\xi_i}\xi_{i+1} \in \overline{\Gamma \cdot \xi_{i+1}}$. By induction, $\xi_1 \in \overline{\Gamma \cdot \xi_i}$ for any $1 \leq i \leq \mathtt{d}+1$. For each i we set $A_i = A_{\xi_i}$.

Let us assume by contradiction that $\xi_1 \notin \Lambda^{\mathrm{prox}}$. Then $\xi_i \notin \Lambda^{\mathrm{prox}}$ for any $1 \leq i \leq \mathtt{d}+1$ as Λ^{prox} is closed and Γ-invariant. For all $1 \leq i \leq j \leq \mathtt{d}+1$, the image of $A_i \cdots A_j$ is ξ_i, therefore $A_i \cdots A_j$ is not proximal, so its kernel, i.e. $\mathrm{Ker}(A_j)$, must contain ξ_i. For any $1 \leq i \leq \mathtt{d}$, the point ξ_{i+1} is in $\mathrm{Ker}(A_{i+1}) \cap \dots \cap \mathrm{Ker}(A_{\mathtt{d}+1})$ but not in $\mathrm{Ker}(A_i)$. We have proved that $(\mathrm{Ker}(A_i) \cap \dots \cap \mathrm{Ker}(A_{\mathtt{d}+1}))_{i=1}^{\mathtt{d}+1}$ is an increasing sequence of non-empty subspaces of $\mathrm{P}(\mathbf{V})$ of length $\mathtt{d}+1$; this contradicts the fact that $\mathbf{V}$ has dimension $\mathtt{d}+1$. ☐

2.3.7 Duality for convex cocompact actions

Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set and $\Gamma \subset \mathrm{Aut}(\Omega)$ a discrete subgroup. If Γ acts cocompactly on Ω, then it also act cocompactly on the dual Ω^* (see [Ben04, Lem. 2.8]). However, it is not always true that if Γ acts convex cocompactly on Ω then it also acts convex cocompactly on Ω^*. Convex cocompactness is stable under duality in a weaker sense thanks to the following result.

Fact 2.3.16 ([DGKa, Cor. 4.18-5.4 & Prop. 5.10])**.** *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set and $\Gamma \subset \mathrm{Aut}(\Omega)$ a discrete subgroup that acts convex cocompactly on Ω. Then there exists a Γ-invariant open convex subset $\Omega_1 \subset \Omega$ such that Γ acts convex cocompactly on both Ω_1 and Ω_1^*.*

Moreover, $\Lambda^{\mathrm{orb}}_{\Omega}(\Gamma) = \Lambda^{\mathrm{orb}}_{\Omega_1}(\Gamma)$ *and* $\mathcal{C}^{\mathrm{cor}}_{\Omega}(\Gamma) = \mathcal{C}^{\mathrm{cor}}_{\Omega_1}(\Gamma)$*, hence* $(T^1\Omega/\Gamma)_{\mathrm{cor}} = (T^1\Omega_1/\Gamma)_{\mathrm{cor}}$*. Finally,* $\overline{\Omega}_1 \smallsetminus \Lambda^{\mathrm{orb}}_{\Omega_1}(\Gamma)$ *has compact quotient under* Γ*, and has* bisaturated *boundary, in the sense that* $[\xi,\eta] \cap \Omega_1$ *is non-empty for all* $\xi \in \Lambda^{\mathrm{orb}}$ *and* $\eta \in \overline{\Omega}_1 \smallsetminus \Lambda^{\mathrm{orb}}$*.*

Given a discrete subgroup $\Gamma \subset \mathrm{PGL}(\mathbf{V})$ that acts convex cocompactly on some properly convex open set, Danciger–Guéritaud–Kassel gave an interesting description of the set of Γ-invariant properly convex open sets [DGKa].

2.3.8 A convex projective critical exponent

Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set and $\Gamma \subset \mathrm{Aut}(\Omega)$ a closed subgroup equipped with a Haar measure Haar_Γ. We have seen in Section 1.4.2 the importance of a geometrical quantity associated to Γ and the Hilbert metric on Ω called the critical exponent, and defined, for any $x \in \Omega$, as

$$\delta_\Gamma = \limsup_{r\to\infty} \frac{1}{r} \log \mathrm{Haar}_\Gamma\{\gamma \in \Gamma : d_\Omega(x,\gamma x) \leq r\}.$$

In Section 2.1.11, we have fixed a $\mathrm{Aut}(\Omega)$-invariant measure on Ω, denoted by Vol_Ω, and in Section 1.4.2 we have also seen the importance of the volume entropy of Vol_Ω, denoted by $\delta_{\mathrm{Vol}_\Omega}$. One can check that δ_Γ is always bounded above by the volume entropy $\delta_{\mathrm{Vol}_\Omega}$. The following result will allow us to apply Fact 1.4.1 to our convex projective setting.

Fact 2.3.17 ([Tho17, Th. 2])**.** *Let* $\Omega \subset \mathrm{P}(\mathbf{V})$ *be a properly convex open set. Then* $\delta_{\mathrm{Vol}_\Omega} \leq \mathtt{d} - 1$*. In particular, for any closed subgroup* $\Gamma \subset \mathrm{Aut}(\Omega)$*, the numbers* δ_Γ *and* $\delta_{\mathrm{Vol}_\Omega}$ *are finite.*

Acoording to Fact 2.2.9, it turns out that the critical exponent δ_Γ is independent of Ω and its Hilbert metric, and the defition of critical exponent can naturally be extended to any closed subgroup of $\mathrm{PGL}(\mathbf{V})$, and more generally to any representation of a locally compact second countable group in $\mathrm{PGL}_{\mathtt{d}+1}(\mathbb{C})$. We will need this generality for several reasons in Sections 9.2 and 11.2.

Let Γ be a locally compact second countable group with a Haar measure Haar_Γ and $\rho : \Gamma \to \mathrm{PGL}_{\mathtt{d}+1}(\mathbb{C})$ a representation. The (convex projective) *critical exponent* of ρ is

$$\delta_\rho := \limsup_{r\to\infty} \frac{1}{r} \log \mathrm{Haar}_\Gamma\{\gamma \in \Gamma : \kappa(\rho(\gamma)) \leq r\} \in [0,\infty]. \tag{2.3.2}$$

In other words, if Γ is non-compact, then δ_ρ is the supremum of the numbers δ such that $\sum_{\gamma\in\Gamma} e^{-\delta\kappa(\rho(\gamma))}$ diverges. Moreover, ρ is said to be *divergent* if $\sum_{\gamma\in\Gamma} e^{-\delta_\rho\kappa(\rho(\gamma))}$ diverges, with the convention that $e^{-\infty} = 0$.

If $\Gamma \subset \mathrm{PGL}_{\mathtt{d}+1}(\mathbb{C})$ (resp. $\mathrm{GL}_{\mathtt{d}+1}(\mathbb{C})$), then we denote by δ_Γ the critical exponent of the inclusion (resp. projection) of Γ in $\mathrm{PGL}_{\mathtt{d}+1}(\mathbb{C})$, and Γ is said to be divergent if this inclusion (resp. projection) is.

The following classical fact is a consequence of the Tits alternative and of the sub-additivity of κ. It applies to all strongly irreducible discrete subgroups of $\mathrm{PGL}_{\mathtt{d}+1}(\mathbb{C})$, and non-elementary rank-one discrete groups of properly convex open sets. See also Lemma 11.2.1.

Fact 2.3.18. *Let* $\Gamma \subset \mathrm{PGL}_{\mathtt{d}+1}(\mathbb{C})$ *be a discrete subgroup that is not virtually solvable. Then* $\delta_\Gamma > 0$*.*

Part II

Preliminary results on rank-one convex projective geometry

Chapter 3

The rank-one condition

In this chapter, we recall the definition, due to M. Islam [Isl], of rank-one automorphisms and rank-one groups of automorphisms of a properly convex open set, and of rank-one convex projective orbifolds. We establish several useful properties of these.

3.1 Rank-one automorphisms

3.1.1 The definition

The dynamics of biproximal elements $g \in \mathrm{PGL}(\mathbf{V})$ on $\mathrm{P}(\mathbf{V})$ display a generalised (higher-rank) form of North-South dynamics with poles x_g^+ and x_g^-, in the sense that the forward (resp. backward) orbit of a point in $\mathrm{P}(\mathbf{V}) \smallsetminus y_g^-$ (resp. $\mathrm{P}(\mathbf{V}) \smallsetminus y_g^+$) tends to x_g^+ (resp. x_g^-). (Recall Notation 2.2.2 and Definition 2.2.4.) However, the forward orbit of a point in $y_g^- \smallsetminus \{x_g^-\}$ accumulates in x_g^0. For our purpose, it will be convenient to consider biproximal automorphisms of a properly convex open set Ω that satisfy additional properties: having a North-South dynamics on $\overline{\Omega}$ in the usual sense, i.e. the forward (resp. backward) orbit of any point of $\overline{\Omega}$ distinct from x_g^- (resp. x_g^+) tends to x_g^+ (resp. x_g^-), and also having an axis inside Ω; following the work of M. Islam [Isl] (and also A. Zimmer [Zim]), we shall call such automorphisms rank-one. Figure 3.1 illustrates three basic examples of rank-one and non-rank-one automorphisms. Note that Islam's definition is formulated differently; we check in the next section that his definition is equivalent.

Definition 3.1.1 ([Isl, Def. 6.2 & Prop. 6.3]). Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set and $g \in \mathrm{Aut}(\Omega)$. The element g is called *rank-one* if it is biproximal, if $y_g^+ \cup y_g^-$ intersects $\overline{\Omega}$ exactly at x_g^+ and x_g^-, and if the axis of g intersects Ω.

Definition 3.1.1 was inspired by a similar notion in Riemannian geometry, introduced by Ballmann–Brin–Eberlein [BBE85, Def. p. 1].

3.1.2 Characterisations

Given a biproximal automorphism g of a properly convex open set Ω, there is an interesting relation between the rank-one property of g, that of its dual $g^* \in \mathrm{Aut}(\Omega^*)$ (Notation 2.1.8), and several regularity properties (from Section 2.1.8) of x_g^+ and $x_g^- \in \partial\Omega$. The following results generalise characterisations of Islam [Isl, Prop. 6.3], with shorter proofs.

Lemma 3.1.2. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set and $g \in \mathrm{Aut}(\Omega)$ biproximal. Then*

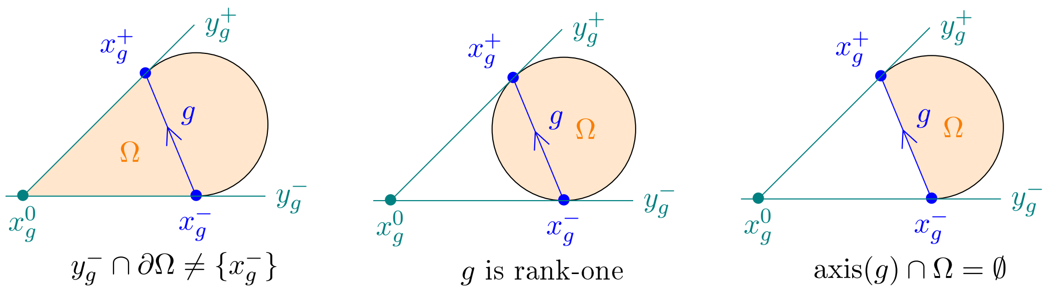

Figure 3.1: Rank-one and non-rank-one biproximal automorphisms. Note that the right picture is the dual of the left picture.

1. $\mathrm{axis}(g) \cap \Omega$ *is non-empty if and only if* x_g^+ *is smooth, in which case* $T_{x_g^+}\partial\Omega = y_g^+$*;*
2. g *is a rank-one automorphism of* Ω *if and only if its dual is a rank-one automorphism of* Ω^**, if and only if* x_g^+ *and* x_g^- *are smooth and strongly extremal points of* $\partial\Omega$*, if and only if* x_g^+ *and* x_g^- *are strongly extremal.*

Proof. The proof basically follows from Observation 2.1.9. Recall also that $x_g^+, x_g^- \in \partial\Omega$ and that y_g^- and y_g^+ are supporting hyperplanes of Ω by Fact 2.2.3. The fact that $y_g^+ \cap \partial\Omega = \{x_g^+\}$ exactly means that y_g^+ is a smooth point of the boundary of the dual Ω^* of Ω. Since the backward orbit of any point of $y_g^+ \smallsetminus \{x_g^+\}$ accumulates in x_g^0, it is also equivalent to asking that $x_g^0 \cap \overline{\Omega}$ is empty, or, in other words, that the axis of g^* intersects Ω^*. Therefore, if g is rank-one, then x_g^+ and x_g^- are smooth and strongly extremal points of $\partial\Omega$; the converse holds trivially. ☐

Let us give another characterisation of the rank-one property, which uses the simplicial metric d_{spl} introduced in Section 2.1.10.

Lemma 3.1.3. *Let* $\Omega \subset \mathrm{P}(\mathbf{V})$ *be a properly convex open set and* $g \in \mathrm{Aut}(\Omega)$ *such that* $(g^n)_n$ *is not relatively compact, and such that* g *fixes two points* $\xi, \eta \in \partial\Omega$ *with* $d_{\mathrm{spl}}(\xi, \eta) > 1$.

(1) If ξ *and* η *are extremal, then* g *is biproximal and* $\{\xi, \eta\} = \{x_g^+, x_g^-\}$.

(2) If $d_{\mathrm{spl}}(\xi, \eta) > 2$*, then* g *is rank-one and* $\{\xi, \eta\} = \{x_g^+, x_g^-\}$.

Proof. We fix $x \in (\xi \oplus \eta) \cap \Omega$; up to switching ξ and η, we can assume, since g has infinite order, that $(g^n x)_n$ converges to ξ (resp. η) when n goes to $+\infty$ (resp. $-\infty$). Then $\xi \in x_g^+$ and $\eta \in x_g^-$ by Fact 2.2.3.

Let us assume that ξ and η are extremal, and show that g is proximal. The natural projections of $(B_\Omega(g^n x, 1))_n$ in $\mathrm{P}(\mathbf{V}/\eta)$ yields a sequence of properly convex open sets $(\Omega_n)_n$, which is non-increasing for the inclusion by Lemma 2.1.6. Since ξ is extremal, $d_{\overline{\Omega}}$ is lower semi-continuous, and since $(g^n x)_{n\geq 0}$ converges to ξ, the sequence $(\overline{\Omega}_n)_{n\geq 0}$ converges to the projection of ξ in $\mathrm{P}(\mathbf{V}/\eta)$. The projection in $\mathrm{P}(\mathbf{V}/\eta)$ of ξ lies in Ω_0, so it is an attracting fixed point for the action of g. By Fact 2.2.6, the action of g on $\mathrm{P}(\mathbf{V}/\eta)$ is proximal, and so it is on $\mathrm{P}(\mathbf{V})$ (since $\eta \in x_g^-$). Since $\xi \in x_g^+$, we have $\xi = x_g^+$. Symmetrically, g^{-1} is proximal with $\eta = x_g^-$.

Assume that $d_{\mathrm{spl}}(\xi, \eta) > 2$. According to (1) and Lemma 3.1.2, it is enough to show that ξ (and similarly η) is strongly extremal. Assume the contrary: then we can find $\xi' \in \partial\Omega \smallsetminus \{\xi\}$ extremal with $[\xi, \xi'] \subset \partial\Omega$. By upper semi-continuity of $d_{\overline{\Omega}}$ and since $d_{\overline{\Omega}}(\xi, [\eta, \xi']) = \infty$, the Hilbert distance from $g^n x$ to $[\eta, \xi']$ goes to infinity with n. As a consequence, the Hilbert distance from x to $[\eta, g^{-n}\xi']$ also goes to infinity, which implies that $[\eta, \xi''] \subset \partial\Omega$ for any accumulation point ξ'' of $(g^{-n}\xi')_n$. But $[\xi, \xi''] \subset \partial\Omega$, which contradicts that $d_{\mathrm{spl}}(\xi, \eta) > 2$. □

Corollary 3.1.4. *For any properly convex open set $\Omega \subset \mathrm{P}(\mathbf{V})$, the set of rank-one elements of $\mathrm{Aut}(\Omega)$ is open.*

Proof. This is an immediate consequence of the fact that biproximality is an open condition, of the fact that d_{spl} is lower semi-continuous, and of Lemma 3.1.3. □

3.2 Rank-one groups

We shall use the following terminology.

Definition 3.2.1. Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set. A subgroup $\Gamma \subset \mathrm{Aut}(\Omega)$ is *rank-one* if it contains a rank-one automorphism of Ω.

In this case, we say that Γ is *non-elementary* if no finite-index subgroup fixes a point of $\overline{\Omega}$, or, equivalently, if Γ_0^{Z} (from Notation 2.3.1) fixes no point of $\overline{\Omega}$.

3.2.1 Non-elementary rank-one groups act minimally on their limit set

The goal of this section is to prove the following result, which is analogous to Remark 2.3.5 and Fact 2.3.9. Recall that transversality was defined in Notation 2.3.7, and the biproximal limit set in Definition 2.3.8.

Proposition 3.2.2. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set and $\Gamma \subset \mathrm{Aut}(\Omega)$ a non-elementary rank-one subgroup. Then $\Lambda_\Gamma^{\mathrm{bip}}$ projects onto $\Lambda_\Gamma^{\mathrm{prox}}$ and is the smallest Γ-invariant subset of $\overline{\Omega} \times \overline{\Omega}^*$; in particular Γ acts minimally on it.*

More precisely, for any $(x, H) \in \overline{\Omega} \times \overline{\Omega}^*$, for any biproximal element $\gamma \in \Gamma$, for any neighbourhood $U \subset \mathrm{P}(\mathbf{V}) \times \mathrm{P}(\mathbf{V}^*)$ of f_γ^+, the subset $\{g \in \Gamma_0^{\mathrm{Z}} : g(x, H) \in U\} \subset \Gamma_0^{\mathrm{Z}}$ is Zariski-dense. In particular, the set of flags of the biproximal limit set $\Lambda_\Gamma^{\mathrm{bip}}$ which are transverse to (x, H) is dense.

A more general result

In order to prove Proposition 3.2.2, we will use the following more general result, which can also be used to prove Fact 2.3.9.

Proposition 3.2.3. *Let $\Gamma \subset \mathrm{PGL}(\mathbf{V})$ be a subgroup that contains a biproximal element, and $F \subset \mathrm{P}(\mathbf{V}) \times \mathrm{P}(\mathbf{V}^*)$ a closed Γ-invariant subset that contains $\Lambda_\Gamma^{\mathrm{bip}}$. Assume that for all $f, f' \in F$, there exists $g \in \Gamma_0^{\mathrm{Z}}$ such that gf is tranverse to f'. Then for any $f \in F$, for any biproximal element $\gamma \in \Gamma$, for any neighbourhood $U \subset \mathrm{P}(\mathbf{V}) \times \mathrm{P}(\mathbf{V}^*)$ of f_γ^+, the subset $\{g \in \Gamma_0^{\mathrm{Z}} : gf \in U\} \subset \Gamma_0^{\mathrm{Z}}$ is Zariski-dense. In particular, $\Lambda_\Gamma^{\mathrm{bip}}$ is the smallest Γ-invariant non-empty closed subset of F, so Γ acts minimally on $\Lambda_\Gamma^{\mathrm{bip}}$, and finally the set of flags in $\Lambda_\Gamma^{\mathrm{bip}}$ which are transverse to f is dense.*

Proof. The subsets $\{g \in \Gamma_0^{\mathrm{Z}} : gf_\gamma^+ \pitchfork f_\gamma^-\}$ and $\{g : gf \pitchfork f_\gamma^+\}$ of Γ_0^{Z} are Zariski-open, and non-empty by assumption; by topological irreducibility, they are dense. Thus, their intersection A is Zariski-open and Zariski-dense in Γ_0^{Z}.

Let us show that the Zariski-closure B of $\{g \in \Gamma_0^{\mathrm{Z}} : gf \in U\}$ in Γ_0^{Z} contains A. Let $g \in A$. Let $K \subset U$ be a compact neighbourhood of f_γ^+ such that $K \cup gK$ is made of elements transverse to f_γ^-. The set $S = \{g \in \Gamma_0^{\mathrm{Z}} : g(K \cup \{f\} \subset K\}$ is a sub-semi-group of Γ_0^{Z} which is contained in B. Let $N \geq 0$ be large enough so that $\gamma^n(K \cup gK \cup \{f\} \subset K$ for any $n \geq N$. Then γ^n and $\gamma^n g \gamma^n$ belong to S for any $n \geq N$. Since the Zariski-closure of a semi-group is a group, we conclude that $g \in B$. □

Putting flags into general position

Let us now check that we can apply Proposition 3.2.3 to non-elementary rank-one groups. For this, let us first state an immediate consequence of Fact 3.1.3. Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set, $g \in \mathrm{Aut}(\Omega)$ a rank-one element, and $(x, H) \in \overline{\Omega} \times \overline{\Omega^*}$ any pair. If $x \neq x_g^+$ and $H \neq y_g^+$, then $(x, H) \pitchfork f_g^+$; if $(x, H) \in \mathrm{Q}(\mathbf{V})$ (i.e. if $x \in H$) and $(x, H) \neq f_g^+$, then $f_g^+ \pitchfork (x, H)$.

Lemma 3.2.4. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set and $\Gamma \subset \mathrm{Aut}(\Omega)$ a non-elementary rank-one subgroup. Then for all $(x, H), (x', H') \in \overline{\Omega} \times \overline{\Omega^*}$, there exists $g \in \Gamma_0^{\mathrm{Z}}$ such that $g(x, H)$ is transverse to (x', H').*

Proof. Let $\gamma \in \Gamma$ be rank-one. Let us show first that there exists $g \in \Gamma_0^{\mathrm{Z}}$ such that $g(x, H) \pitchfork f_\gamma^+$. Indeed let $g, h \in \Gamma_0^{\mathrm{Z}}$ be such that f_γ^+, gf_γ^+ and hf_γ^+ are pairwise distinct, and hence transverse. If (x, H) is not transverse to f_γ^+ nor gf_γ^+, then it must be transverse to hf_γ^+.

Let us conclude the proof. Let $g \in \Gamma_0^{\mathrm{Z}}$ be such that $g(x, H) \pitchfork f_\gamma^-$, so $(\gamma^n g(x, H))_n$ converges to f_γ^+. Let $h \in \Gamma_0^{\mathrm{Z}}$ be such that $hf_\gamma^+ \pitchfork (x', H')$, then $h\gamma^n g(x, H) \pitchfork (x', H')$ for n large enough. □

Proof of Proposition 3.2.2

Thanks to Proposition 3.2.3 and Lemma 3.2.4, we only have to prove that $\Lambda_\Gamma^{\mathrm{bip}}$ projects onto $\Lambda_\Gamma^{\mathrm{prox}}$. Consider $g \in \Gamma_0^{\mathrm{Z}}$ proximal. Pick $(x, H) \in \Lambda_\Gamma^{\mathrm{bip}}$. By Lemma 3.2.4, there exists $h \in \Gamma_0^{\mathrm{Z}}$ such that $h(x, H)$ is transverse to (x, y_g^-), where $y_g^- = x_g^- \oplus x_g^0$. Then $hx \notin y_g^-$, and $(g^n hx)_n$ converges to x_g^+.

3.2.2 How to make non-elementary rank-one groups strongly irreducible

In this section, which is part of a collaboration with F. Zhu [BZ21], we show that non-elementary rank-one groups behave as if they were strongly irreducible, although this is not always the case; consider for example a cocompact lattice Γ of $\mathrm{SO}(1,2) \subset \mathrm{SO}(1,3) \subset \mathrm{SL}(4, \mathbb{R})$, where Γ preserves the span of the proximal limit set $\Lambda_\Gamma^{\mathrm{prox}}$ (Definition 2.3.3), which is a proper subspace of $\mathbf{P}(\mathbb{R}^4)$.

A criterion for irreducibility

The following result asserts that the only other possible obstruction is that the span of the dual proximal limit set is a proper subspace of the dual projective space.

Proposition 3.2.5. *Suppose we have $\Omega \subset \mathbf{P}(\mathbf{V})$ a properly convex open set, and Γ a non-elementary rank-one subgroup of $\mathrm{Aut}(\Omega)$. Then the following are equivalent:*

(i) $\Lambda_\Gamma^{\mathrm{prox}}$ *spans* $\mathbf{P}(\mathbf{V})$ *and* $\Lambda_{\Gamma^*}^{\mathrm{prox}}$ *spans* $\mathbf{P}(\mathbf{V}^*)$*;*

(ii) Γ *is irreducible, i.e. the only* Γ*-invariant subspaces of* V *are trivial;*

(iii) Γ *is strongly irreducible, i.e. all finite-index subgroups of* Γ *are irreducible.*

Proof. Strong irreducibility implies *a fortiori* irreducibility, i.e. (iii) implies (ii).

(ii) implies (i) because $\Lambda_\Gamma^{\mathrm{prox}}$ (resp. $\Lambda_{\Gamma^*}^{\mathrm{prox}}$) is Γ-invariant (resp. Γ^*-invariant), and Γ is irreducible if and only if Γ^* is irreducible.

Any finite-index subgroup of Γ is non-elementary and rank-one, with limit set $\Lambda_\Gamma^{\mathrm{prox}}$, and dual limit set $\Lambda_{\Gamma^*}^{\mathrm{prox}}$. Thus, to establish that (i) implies (iii), it is enough to prove that (i) implies (ii).

Suppose we have a Γ-invariant subspace $W \subset \mathbf{V}$. If $\mathbf{P}(W)$ contains a point of $\Lambda_\Gamma^{\mathrm{prox}}$, then it contains them all by Proposition 3.2.2, and $W = \mathbf{V}$ since $\Lambda_\Gamma^{\mathrm{prox}}$ spans $\mathbf{V}$. Let us assume the contrary. Then for any proximal element $\gamma \in \Gamma$, we have $x_\gamma^+ \not\subset W$, and one can check that this implies that $W \subset y_\gamma^-$ (because W is γ-invariant). In other words, $W \subset \bigcap_{H\in\Lambda_{\Gamma^*}^{\mathrm{prox}}} H$ which is trivial because $\Lambda_{\Gamma^*}^{\mathrm{prox}}$ spans $\mathbf{V}^*$. □

Restricting and projecting properly convex open sets

As Crampon–Marquis observed in the strictly convex setting, Proposition 3.2.5 has interesting consequences: in many cases, given a non-elementary rank-one group Γ, one can, by "restricting to the spans of $\Lambda_\Gamma^{\mathrm{prox}}$ and $\Lambda_{\Gamma^*}^{\mathrm{prox}}$", project Γ onto a strongly irreducible rank-one group Γ', and then try to pull back properties of Γ' to obtain results on Γ. Let us formalise this idea.

Consider two subspaces $V_1, V_2 \subset \mathbf{V}$. Given a subgroup G of $\mathrm{GL}(\mathbf{V})$ (or of $\mathrm{PGL}(\mathbf{V})$), we denote by G^{V_1} (resp. G^{V_1,V_2}) the set of elements of G preserving $\mathbf{P}(V_1)$ (resp. $\mathbf{P}(V_1)$ and $\mathbf{P}(V_2)$); we naturally identify $V_1/(V_1\cap V_2)$ (resp. $(\mathbf{V}/V_2)^*$) as a subspace of $\mathbf{V}/V_2$ (resp. $\mathbf{V}^*$). To produce the natural map from $\mathrm{GL}(\mathbf{V})^{V_1,V_2}$ to $\mathrm{GL}(V_1/(V_1\cap V_2))$ (resp. from $\mathrm{GL}(\mathbf{V})^{V_2}$ to $\mathrm{GL}((\mathbf{V}/V_2)^*)$) one can equivalently go through $\mathrm{GL}(V_1)^{V_1\cap V_2}$ or $\mathrm{GL}(\mathbf{V}/V_2)^{V_1/(V_1\cap V_2)}$ (resp. $\mathrm{GL}(\mathbf{V}/V_2)$ or $\mathrm{GL}(\mathbf{V}^*)^{(\mathbf{V}/V_2)^*}$).

Consider a properly convex open set $\Omega \subset \mathbf{P}(\mathbf{V})$. We write $\Omega\cap V_1$ to denote $\Omega\cap\mathbf{P}(V_1)$ and Ω/V_2 to denote the projection of Ω in $\mathbf{P}(\mathbf{V}/V_2)$. Assume that $\Omega\cap V_1 \neq \varnothing$, that $\overline{\Omega}\cap V_2 = \varnothing$ (i.e. $\Omega^*\cap(\mathbf{V}/V_2)^* \neq \varnothing$), and that $(\Omega\cap V_1)/(\mathbf{V}_1\cap V_2)$ is equal to $(\Omega/V_2)\cap(V_1/(V_1\cap V_2))$. Then $(\Omega/V_2)^*$ naturally identifies with $\Omega^*\cap(\mathbf{V}/V_2)^*$. Denote by ρ the natural map from $\mathrm{Aut}(\Omega)^{V_1,V_2}$ to $\mathrm{Aut}((\Omega\cap V_1)/(V_1\cap V_2))$. Using Fact 2.2.9, one can find a constant $C > 0$ such that

$$C^{-1}\kappa(g) \leq \kappa(\rho(g)) \leq C\kappa(g) \tag{3.2.1}$$

for any $g \in \mathrm{Aut}(\Omega)^{V_1,V_2}$; in particular, this implies that the map ρ is proper. Note that to find C one can use a duality argument, in order to reduce to the case where V_2 is trivial.

Another useful formula (using Notation 2.2.2) will be the following: for any $g \in \mathrm{Aut}(\Omega)^{V_1,V_2}$,

$$\ell(\rho(g)) = \ell(g). \tag{3.2.2}$$

To see this, it suffices (using again a duality argument) to fix $g \in \mathrm{Aut}(\Omega)^{V_1}$ with $\ell(g) > 0$ and check that $\ell(\rho_1(g)) = \ell(g)$. That $\ell(\rho_1(g)) \leq \ell(g)$ is obvious. To establish the converse inequality, we only need to show that $g^+\cap V_1$ and $g^-\cap V_1$ are nonempty, where g^+ (resp. g^-) is the sum of all generalised eigenspaces associated to eigenvalues with maximal (resp. minimal) moduli. Observe that any accumulation point of $(g^n x)_n$ (resp. $(g^{-n}x)_n$), where $x \in \Omega\cap V_1$, belongs to $g^+\cap\mathbf{P}(V_1)$ (resp. $g^-\cap\mathbf{P}(V_1)$).

Let us apply the previous observations to non-elementary rank-one subgroups of $\Gamma \subset \mathrm{Aut}(\Omega)$. Then the span V_1 of the proximal limit set $\Lambda_\Gamma^{\mathrm{prox}}$ intersects Ω, and similarly $V_2' := \mathrm{Span}(\Lambda_{\Gamma^*}^{\mathrm{prox}})$ intersects $\Omega^* \neq \emptyset$. Furthermore, $(\Omega\cap V_1)/(V_1\cap V_2)$ is equal to $(\Omega/V_2)\cap(V_1/(V_1\cap V_2))$, where $V_2 = (\mathbf{V}^*/V_2')^*$ is the intersection of all hyperplanes in Λ_Γ^*. Thus, we obtain by Proposition 3.2.5, and (3.2.1) and (3.2.2), the following lemma.

Lemma 3.2.6. *Any non-elementary rank-one group Γ preserving a properly convex open set projects via a morphism ρ onto a strongly irreducible rank-one group preserving a properly convex open set, with $\ell(\rho(\gamma)) = \ell(\gamma)$ for any $\gamma \in \Gamma$. If moreover $\Gamma \subset \mathrm{PGL}(\mathbf{V})$ is closed (resp. discrete), then ρ may be taken proper (resp. with finite kernel and discrete image).*

Note that the restriction to $\Lambda_\Gamma^{\mathrm{prox}}$ of the projection $\mathrm{P}(\mathbf{V}) \to \mathrm{P}((\mathbf{V} \cap V_1)/(V_2 \cap V_1))$ is not in general injective, so we cannot identify $\Lambda_\Gamma^{\mathrm{prox}}$ with its image $\Lambda_{\rho(\Gamma)}^{\mathrm{prox}}$. However injectivity holds if Ω is strictly convex.

3.3 Rank-one convex projective orbifolds

3.3.1 The definition

We shall use the following terminology.

Definition 3.3.1. Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set and $\Gamma \subset \mathrm{Aut}(\Omega)$ a discrete subgroup; set $M = \Omega/\Gamma$. The orbifold M is said to be *rank-one* if Γ is rank-one. In this case, M is said to be non-elementary if Γ is.

Let $\gamma \in \Gamma$ be a biproximal (resp. rank-one) element whose axis meets Ω. Then the projection of $\mathrm{axis}(\gamma)$ on $T^1 M$ is a periodic $(\phi_t)_t$-orbit, which is said to be associated to γ, and to be *biproximal* (resp. *rank-one*); the unit tangent vectors along this geodesic are also said to be *biproximal* (resp. *rank-one*) periodic. Thus, M is rank-one if and only if it contains a rank-one periodic geodesic.

Let us also define rank-one geodesics which are not necessarily closed.

Definition 3.3.2. Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set. A unit tangent vector $v \in T^1\Omega$, and the geodesic it spans, are called *rank-one* if $\phi_\infty v$ and $\phi_{-\infty} v$ are $\mathcal{C}^1$ and strongly extremal points of $\partial\Omega$.

3.3.2 A characterisation of the non-elementary condition for rank-one convex projective orbifolds

Let us prove the following result, which is classical in several geometrical settings, such as hyperbolic geometry.

Proposition 3.3.3. *A rank-one convex projective orbifold $M = \Omega/\Gamma$ is non-elementary if and only if Γ does not contain $\mathbb{Z}$ as a finite-index subgroup, which is also equivalent to asking that $\#\Lambda_\Gamma^{\mathrm{bip}} \geq 3$.*

In other words, for any properly convex open set $\Omega \subset \mathrm{P}(\mathbf{V})$, a *discrete* rank-one subgroup of $\mathrm{Aut}(\Omega)$ is non-elementary if and only if it does not contain $\mathbb{Z}$ as a finite-index subgroup. Note that, if for instance Ω is an ellipsoid, then the stabiliser in $\mathrm{Aut}(\Omega)$ of any point of $\partial\Omega$ is a (non-discrete) elementary rank-one subgroup of $\mathrm{Aut}(\Omega)$, is not virtually isomorphic to $\mathbb{Z}$, and its proximal limit set is $\partial\Omega$.

How to use discreteness

The following lemma is a convex projective version of a result which is classical in the setting (for instance) of hyperbolic geometry. We use it to prove Proposition 3.3.3.

Lemma 3.3.4. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set and Γ a discrete subgroup of* $\mathrm{Aut}(\Omega)$. *Consider two infinite-order elements $g, h \in \Gamma$, and suppose that g (resp. h) fixes two points $x^-, x^+ \in \partial\Omega$ (resp. y^-, y^+) such that $[x^-, x^+]$ (resp. $[y^-, y^+]$) intersects Ω. If x^+ and y^+ are in the same open face, then g and h have a power in common, and x^- and y^- are in the same open face.*

Proof. Take $x \in [x^-, x^+] \cap \Omega$ (resp. $y \in [y^-, y^+] \cap \Omega$) and $v \in T^1\Omega$ (resp. w) such that $\pi v = x$ (resp. $\pi w = y$) and $\phi_\infty v = x^+$ (resp. $\phi_\infty w = y^+$). Without loss of generality we can assume that $(g^n x)_n$ (resp. $(h^n y)_n$) converges to $x^\pm$ (resp. $y^\pm$) as n goes to $\pm\infty$. By Lemma 2.1.4, $d_\Omega(\pi\phi_t v, \pi\phi_t w) \leq d_\Omega(x, y) + d_{\overline{\Omega}}(x^+, y^+)$ for any $t \geq 0$. Then

$$d_\Omega(x, g^{-n} h^{m_n} y) \leq d_\Omega(x, y) + d_{\overline{\Omega}}(x^+, y^+) + d_\Omega(y, hy)$$

for any $n \geq 1$, where $m_n := \left\lfloor n \frac{d_\Omega(x, gx)}{d_\Omega(y, hy)} \right\rfloor$ Since the action of Γ on Ω is properly discontinuous, this implies the existence of $N, M \geq 1$ such that $g^N = h^M =: k$. Then by lower semi-continuity of $d_{\overline{\Omega}}$,

$$d_{\overline{\Omega}}(x^-, y^-) \leq \lim_{n \to \infty} d_\Omega(k^{-n} x, k^{-n} y) = d_\Omega(x, y) < \infty,$$

hence x^- and y^- are in the same open face. Moreover, $d_{\overline{\Omega}}(x^+, y^+) \leq d_\Omega(x, y)$. □

Corollary 3.3.5. *Let $\Omega \subset \mathbb{R}\mathbf{P}^n$ be a properly convex open set and Γ a discrete subgroup of* $\mathrm{Aut}(\Omega)$. *Let g, $h \in \Gamma$ be rank-one elements. If $x_g^+ = x_h^+$, then $x_g^- = x_h^-$, and g and h have a common power.*

Proof of Proposition 3.3.3

Let $\Gamma \subset \mathrm{Aut}(\Omega)$ be a rank-one discrete subgroup, and $\gamma_0 \in \Gamma_0^{\mathbf{Z}}$ a rank-one element. If Γ is non-elementary, then we can apply Proposition 3.2.3. The fact that $\{f' \in \Lambda_\Gamma^{\mathrm{bip}} : f \pitchfork f'\}$ is dense in $\Lambda_\Gamma^{\mathrm{orb}}$ for any $f \in \Lambda_\Gamma^{\mathrm{orb}}$ implies that $\Lambda_\Gamma^{\mathrm{bip}}$ is perfect (i.e. has no isolated point) and hence infinite.

If Γ is virtually isomorphic to $\mathbb{Z}$, then any biproximal element of it has a common power with γ_0, hence shares the same axis, so $\Lambda_\Gamma^{\mathrm{bip}} = \{f_{\gamma_0}^+, f_{\gamma_0}^-\}$ has cardinality 2.

If Γ is elementary, then $\Gamma_0^{\mathbf{Z}}$ fixes a point of $x \in \overline{\Omega}$. In particular γ_0 fixes x, so $x \in \{x_{\gamma_0}^+, x_{\gamma_0}^-\}$, say $x = x_{\gamma_0}^+$. For any $\gamma \in \Gamma_0^{\mathbf{Z}}$, we have $x_{\gamma^{-1}\gamma_0\gamma}^+ = \gamma x = x = x_{\gamma_0}^+$. By Corollary 3.3.5, this means that $\gamma x_{\gamma_0}^- = x_{\gamma^{-1}\gamma_0\gamma}^- = x_{\gamma_0}^-$. We have proved that $\Gamma_0^{\mathbf{Z}}$ preserves the axis of γ_0, which implies by proper discontinuity of the action that γ_0 generates a finite-index subgroup of Γ.

3.4 Periodic geodesics and conjugacy classes

Let $M = \Omega/\Gamma$ be a convex projective orbifold. In this section we recall the link between periodic $(\phi_t)_t$-orbits of T^1M and conjugacy classes of Γ. In Definition 3.3.1, we have associated to any biproximal element $\gamma \in \Gamma$ whose axis meets Ω a periodic geodesic in M (and T^1M). To any non-trivial power or conjugate of γ is associated the same periodic geodesic. If M is compact hyperbolic and Γ is torsion-free (or more generally if Γ consists of rank-one elements), then this yields a (well-known) correspondence between the periodic $(\phi_t)_t$-orbits in T^1M (as subsets) and the conjugacy classes of primitive elements of Γ. (An element $\gamma \in \Gamma$ is called *primitive* if it does not belong to $\{g^k : g \in \Gamma,\ k \geq 2\}$.)

This correspondence holds more generally if Γ consists exclusively of rank-one elements, for instance if Γ is Gromov-hyperbolic and torsion-free, and acts convex cocompactly on Ω (by [DGKa, Th. 1.15] or [Zim20, Th. 1.22-27]). However the correspondence fails for general convex projective orbifolds, for several reasons which can combine.

3.4.1 Examples

Let us enumerate three (well-known) examples to illustrate how the correspondence between periodic $(\phi_t)_t$-orbits and conjugacy classes might fail.

1. If M is a non-compact hyperbolic manifold with finite-volume, then Γ contains an element γ such that $\ell(\gamma) = 0$, and which preserves no geodesic of Ω; i.e. it does not correspond to any $(\phi_t)_t$-periodic orbit in T^1M.

2. If M is a Benoist manifold (i.e. $M = \Omega/\Gamma$ is compact with Ω not strictly convex, Γ strongly irreducible, and $\dim(M) = 3$), then some (biproximal) elements of Γ do not preserve any straight geodesic of Ω. If M is one of the examples constructed by Benoist [Ben06a, §4.3] (or is a finite cover of one of these examples), then some elements of Γ preserve uncountably many geodesics.

3. Let Γ' be a cocompact torsion-free discrete subgroup of $\mathrm{PSO}(2, 1)$, which naturally embeds in $\mathrm{PSO}(3, 1)$, and $M' = \mathbb{H}^3/\Gamma'$; let $r \in \mathrm{PO}(3, 1)$ be the orthogonal reflection of $\mathbb{H}^3$ that preserves the natural embedding of $\mathbb{H}^2$ in $\mathbb{H}^3$, so that r commutes with Γ'; let Γ be the group generated by Γ' and r (i.e. $\Gamma \simeq \Gamma' \times \mathbb{Z}/2\mathbb{Z}$), and $M = \mathbb{H}^3/\Gamma$. Pick a periodic vector $v \in T^1M$. It lifts to exactly one periodic vector $v' \in T^1M'$, which corresponds to a conjugacy class $[\gamma'] \in [\Gamma']$ of a primitive element $\gamma \in \Gamma'$, which is also primitive in Γ. The element $r\gamma \in \Gamma$ is also primitive, has the same axis in $\mathbb{H}^3$ as γ, and the same translation length. Therefore, the periodic orbit $\{\phi_t v\}_t$ may be associated to two natural conjugacy classes of primitive elements in Γ. Note that $r\gamma^2$ is also primitive in Γ, although its translation length is twice that of γ.

3.4.2 Rank-one periodic orbits and rank-one strongly primitive conjugacy classes

In this section we examine the problem encountered in Example 3 of the previous section.

Let $M = \Omega/\Gamma$ be a rank-one non-elementary convex projective orbifold. Consider a rank-one element $\gamma \in \Gamma$, and a vector $v \in T^1\Omega$ tangent to the axis of γ with $\phi_\infty v = x_\gamma^+$. Let $G := \mathrm{Stab}_\Gamma(x_\gamma^-) \cap \mathrm{Stab}_\Gamma(x_\gamma^+)$, and $H := \mathrm{Stab}_\Gamma(v)$. We have a natural morphism $\alpha : G \to \mathbb{R}$ with kernel H such that $gv = \phi_{\alpha(g)}v$ for any $g \in G$. Let ℓ be the minimum of $\alpha(G) \cap \mathbb{R}_{>0}$.

The element γ is said to be *strongly primitive* if $\alpha(\gamma) = \ell$. The union of conjugacy classes of elements in $\alpha^{-1}(\ell)$ are called *the (strongly primitive) conjugacy classes associated to* $\{\phi_t \pi_\Gamma v\}_t$, where π_Γ is the natural projection from $T^1\Omega$ to T^1M.

The number of these conjugacy classes is bounded above by the cardinality of H. Let $g \in \alpha^{-1}(\ell)$ and $h \in \Gamma$ such that $hgh^{-1} \in \alpha^{-1}(\ell)$. Then $h \in G$, so we can find $n \in \mathbb{Z}$ and $h' \in H$ such that $h = h'g^n$, hence $hgh^{-1} = h'gh'^{-1}$. Therefore the number of strongly primitive conjugacy classes associated to $\{\phi_t \pi_\Gamma v\}_t$ is precisely the number of conjugacy classes in $\alpha^{-1}(\ell)$ under the action by conjugation of H. If for instance G is abelian then this number is exactly $\#H$.

More generally, consider $k \in \mathbb{Z}$. Then the number of conjugacy classes of Γ that intersects $\alpha^{-1}(k\ell)$ is the number of conjugacy classes in $\alpha^{-1}(k\ell)$ under the action by conjugation of H, is bounded above by $\#H$, and is equal to $\#H$ if G is abelian. These conjugacy classes are called *the conjugacy classes of length $k\ell$ associated to $\{\phi_t \pi_\Gamma v\}_t$*

In Example 3, every rank-one periodic geodesic is associated to exactly two strongly primitive conjugacy classes. In general two different rank-one periodic geodesics may be associated to a different number of conjugacy classes; let N be the minimal number of conjugacy classes associated to a rank-one periodic orbit. The idea is that "most rank-one periodic orbits" should have exactly N associated conjugacy classes.

Let $F \subset \Gamma$ be the set of elements that fixes every point of the span of $\Lambda_\Gamma^{\mathrm{prox}}$; it is a normal finite subgroup, called the *core-fixing subgroup of* Γ.

Observation 3.4.1. *Let $M = \Omega/\Gamma$ be a rank-one non-elementary convex projective orbifold, and let $F \subset \Gamma$ be the core-fixing subgroup. Then the set of points $x \in \Lambda_\Gamma^{\mathrm{prox}}$ with $\mathrm{Stab}_\Gamma(x) = F$ is G_δ-dense in $\Lambda_\Gamma^{\mathrm{prox}}$. If $T^1\Omega_{\mathrm{bip}}$ is the preimage by $\pi_\Gamma : T^1\Omega \to T^1 M$ of $T^1 M_{\mathrm{bip}}$, then the set of vectors $v \in T^1\Omega_{\mathrm{bip}}$ with $\mathrm{Stab}_\Gamma(v) = F$ is open and dense in $T^1\Omega_{\mathrm{bip}}$, and Γ-invariant.*

Proof. Consider $\gamma \in \Gamma \smallsetminus F$. The set of $x \in \Lambda_\Gamma^{\mathrm{prox}}$ such that $\gamma x \neq x$ is open. Let us show that it is dense, which will imply that the set of points $x \in \Lambda_\Gamma^{\mathrm{prox}}$ with $\mathrm{Stab}_\Gamma(x) = F$ is G_δ-dense in $\Lambda_\Gamma^{\mathrm{prox}}$. Let $U \subset \Lambda_\Gamma^{\mathrm{prox}}$ be non-empty open. By Proposition 3.2.2, we can find three distinct strongly extremal points $x, y, z \in U$. If $\gamma x = x$ and $\gamma y = y$, then γ is rank-one by Lemma 3.1.3, and hence $\gamma z \neq z$. Therefore not all three points x, y, z are fixed by γ.

In order to show that the set of vectors $v \in T^1\Omega_{\mathrm{bip}}$ with $\mathrm{Stab}_\Gamma(v) = F$ is open and dense in $T^1\Omega_{\mathrm{bip}}$, it is enough to show thanks to the previous point that this set is open. This is an immediate consequence of the fact that the map $v \in T^1\Omega \mapsto \mathrm{Stab}_\Gamma(v)$ is lower semi-continuous. □

In particular, this implies that the core-fixing subgroup contains the centre of Γ.

Question 3.4.2. *Let $M = \Omega/\Gamma$ be a rank-one non-elementary convex projective orbifold, and let $F \subset \Gamma$ be the core-fixing subgroup. Can we find an integer $N \geq 1$ and a Γ-invariant, open and dense subset U of $T^1\Omega_{\mathrm{bip}}$ such that for any $v \in U$, we have $\mathrm{Stab}_\Gamma(v) = F$, and if $\pi_\Gamma v \in T^1 M$ is rank-one periodic, then the number of conjugacy classes associated to $\{\pi_\Gamma \phi_t v\}_t$ is exactly N?*

If $\Lambda_\Gamma^{\mathrm{prox}}$ spans $\mathrm{P}(\mathbf{V})$, then F is trivial so the answer to this question is positive, and we can take $N = 1$.

3.4.3 Non-straight periodic geodesics

In this section we examine the problem encountered in Example 2 of Section 3.4.1, more precisely the fact that in compact rank-one convex projective manifold, we may find a closed curve that is not freely homotopic to a periodic $(\phi_t)_t$-orbit. However, by a compactness argument, one may still find in each free homotopy class a closed curve with minimal length, whose length is the translation length of the associated conjugacy class in the fundamental group; this observation generalises to convex projective orbifold with a non-empty compact convex core.

Fact 3.4.3 ([DGKa, Prop. 10.4])**.** *Consider a discrete subgroup $\Gamma \subset \mathrm{PGL}(\mathbf{V})$ that acts convex cocompactly on a properly convex open sets $\Omega \subset \mathrm{P}(\mathbf{V})$. Then for any $\gamma \in \Gamma$, there exists $x \in \mathcal{C}_\Omega^{\mathrm{cor}}(\Gamma)$ such that $d_\Omega(x, \gamma x) = \ell(\gamma)$.*

Let $M = \Omega/\Gamma$ be a convex projective manifold with non-empty compact convex core, and consider an infinite-order element $\gamma \in \Gamma$. According to the previous fact, we can find $x \in \mathcal{C}^{\mathrm{cor}}$ such that $d_\Omega(x, \gamma x) = \ell(\gamma)$; let $v \in T^1\Omega$ be such that $\pi v = x$ and $\pi\phi_\ell v = \gamma x$. The orbit segment $\{\phi_t v\}_{0\leq t\leq \ell}$ projects to a closed curve of M, but not necessarily to a closed curve of T^1M. This orbit still provide useful information, for instance because, according to the following result, we can choose v so that its entire orbit $\{\phi_t v\}_{t\in\mathbb{R}}$ lies in $\mathcal{C}^{\mathrm{cor}}$; this will be useful in order to count non-rank-one conjugacy classes in Section 7.4.3.

Lemma 3.4.4. *Let $\Gamma \subset \mathrm{PGL}(\mathbf{V})$ be a discrete subgroup, and $\Omega \subset \mathrm{P}(\mathbf{V})$ a Γ-invariant properly convex open set. If Γ acts convex cocompactly on Ω, then for any infinite-order element $\gamma \in \Gamma$ and vector $v \in T^1\Omega$ such that $\pi v \in \mathcal{C}^{\mathrm{cor}}$ and $\pi\phi_{\ell(\gamma)}v = \gamma\pi v$, the endpoints $\phi_{-\infty}v$ and $\phi_\infty v$ belong to $\Lambda^{\mathrm{orb}}_\Omega(\Gamma)$. In other words, v projects to a vector of T^1M_{cor}.*

Lemma 3.4.4 is a consequence of Facts 2.3.16 and 3.4.3, and the following result.

Lemma 3.4.5. *Let Ω be a properly convex open set. Let $g \in \mathrm{Aut}(\Omega)$ with $\ell(g) > 0$ and suppose there exists $x \in \Omega$ such that $d_\Omega(x, gx) = \ell(g)$. Consider $a, b \in \partial\Omega$ such that a, x, gx, b are aligned in this order. Then*

(1) the path $c : \mathbb{R} \to \Omega$, defined by $c(t) \in [g^n x, g^{n+1}x]$ and $d_\Omega(c(t), g^n x) = t - n$ for all $n \in \mathbb{Z}$ and $t \in [n, n+1]$, is a geodesic;

(2) the restriction of g to $x_g^+ \oplus x_g^-$ is diagonalisable over $\mathbb{C}$;

(3) the restriction of g to the span $\mathrm{P}(W)$ of $\{g^n x\}_n$ is biproximal;

(4) $x_g^+ \oplus x_g^0 \cap \mathrm{P}(W) \cap \overline{\Omega}$ (resp. $x_g^- \oplus x_g^0 \cap \mathrm{P}(W) \cap \overline{\Omega}$) is the smallest g-invariant closed face of Ω that contains b (resp. a);

(5) if x, gx, g^2x are not aligned, then $x_g^0 \cap \overline{\Omega}$ is non-empty;

(6) if g is not rank-one, then $d_{\mathrm{spl}}(a, b) = 2$.

Proof. Let us check that (1) holds. Consider three real numbers $r \leq s \leq t$, pick three integers $k \leq n \leq m$ such that $k \leq r \leq k+1$ and $n \leq s \leq n+1$ and $m \leq t \leq m+1$. By Fact 2.2.8, we have

$$\begin{aligned}
d_\Omega(c(r), c(t)) &\geq d_\Omega(g^k x, g^{m+1}x) - d_\Omega(g^k x, c(r)) - d_\Omega(c(t), g^{m+1}x)\\
&\geq \ell(g^{m+1-k}) - d_\Omega(g^k x, c(r)) - d_\Omega(c(t), g^{m+1}x)\\
&= \sum_{i=k}^{m} d_\Omega(g^i x, g^{i+1}x) - d_\Omega(g^k x, c(r)) - d_\Omega(c(t), g^{m+1}x)\\
&= d_\Omega(c(r), g^{k+1}x) + \sum_{i=k+1}^{m-1} d_\Omega(g^i x, g^{i+1}x) + d_\Omega(g^m x, c(t))\\
&\geq d_\Omega(c(r), c(s)) + d_\Omega(c(s), c(t)).
\end{aligned}$$

For all distinct pair of points $(y, z) \in \overline{\Omega}^2$, let us denote by $s_y^z \in \partial\Omega$ the point such that y, z, s_y^z are aligned in this order. Let $\mathrm{P}(W) \subset \mathrm{P}(V)$ be the span of $\{g^n x\}_{n\in\mathbb{Z}}$, with dimension $k \geq 1$; we set $\Omega' = \Omega \cap \mathrm{P}(W)$. By Fact 2.1.14, the smallest closed face of Ω that contains $\{g^n b\}_{n\in\mathbb{Z}}$ (resp. $\{g^n a\}_{n\in\mathbb{Z}}$), denoted by F_+ (resp. F_-), is proper, and therefore its span $\mathrm{P}(W_+)$ (resp. $\mathrm{P}(W_-)$) has dimension $k-1$. Moreover $s_{g^n x}^{g^m x} \in F_+$ (resp. F_-) for all $m > n$ (resp. $m < n$), and $(g^n x)_n$ converges to some point $\xi_+ \in F_+ \cap x_g^+$ (resp. $\xi_- \in F_- \cap x_g^-$) which is fixed by g, when n goes to $+\infty$ (resp. $-\infty$).

Consider a lift $\tilde{g} \in \mathrm{GL}(V)$ of g that preserves one connected component $C \subset V \smallsetminus \{0\}$ of the preimage of Ω, and such that $\lambda_1(\tilde{g}) = 1$. Let us examine the Jordan normal form of $\tilde{g}$: there exists $\ell \geq 0$ and a decomposition

$$\tilde{g} = u_1 + \cdots + u_\alpha - (u'_1 + \cdots + u'_{\alpha'}) + r_1^{\theta_1} v_1 + \cdots + r_\beta^{\theta_\beta} v_\beta + h, \tag{3.4.1}$$

which satisfies the following. The product of any two matrices in this sum is zero. The integers $\alpha, \beta \geq 0$ are not both zero. The sequence $(\frac{1}{n^\ell} h^n)_n$ tends to zero. The matrices $u_1, \dots, u_\alpha, u'_1, \dots, u'_{\alpha'}$ are all conjugate to the matrix with zeros everywhere except on the upper-left block of size $\ell+1$, which is the exponential of the upper-triangular matrix whose (i,j)-entry is 1 if $j = i+1$ and zero otherwise. For $\theta \in \mathbb{R}$ and $1 \leq i \leq \beta$, the matrices r_i^θ and v_i are simultaneously conjugate to the matrices with zeros everywhere except on the upper-left block of size $2\ell+2$, where they are respectively of the form

$$\begin{pmatrix} \mathrm{rot}^\theta & & \\ & \ddots & \\ & & \mathrm{rot}^\theta \end{pmatrix} \text{ and } \exp\begin{pmatrix} 0 & I_2 & & \\ & \ddots & \ddots & \\ & & \ddots & I_2 \\ & & & 0 \end{pmatrix}, \text{ with } \quad \mathrm{rot}^\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \text{ and } I_2 = \mathrm{rot}^0.$$

Finally $\theta_1, \dots, \theta_\beta \in \mathbb{R} \smallsetminus \pi\mathbb{Z}$. Let $\bar{u}_i = \lim_{n\to\infty} \frac{\ell!}{n^\ell} u_i^n$ and $\bar{u}'_i = \lim_{n\to\infty} \frac{\ell!}{n^\ell} {u'_i}^n$ for $1 \leq i \leq \alpha$ and $\bar{v}_i = \lim_{n\to\infty} \frac{\ell!}{n^\ell} v_i^n$ for $1 \leq i \leq \beta$; the set of accumulation points of $(r_i^{n\theta_i})_n$ is $\{r_i^\theta : \theta \in \overline{\theta_i\mathbb{Z} + 2\pi\mathbb{Z}}\}$ for $1 \leq i \leq \beta$. Let $\tilde{x} \in C$ be a lift of $x \in \Omega$. The accumulation points of $(\frac{\ell!}{n^\ell} \tilde{g}^n \tilde{x})_n$ are

$$\left\{ \sum_i \bar{u}_i \tilde{x} + (-1)^m \sum_i \bar{u}'_i \tilde{x} + \sum_i r_i^{\theta'_i} \bar{v}_i \tilde{x} : m \in \{0,1\},\ \theta'_i \in \overline{\theta_i(2\mathbb{Z} + m) + 2\pi\mathbb{Z}} \right\},$$

which are non-zero by Fact 2.2.3. Since $(g^n x)_n$ converges to ξ^+, these accumulation points are all in $\xi^+ \cap C$, and this imply that $\bar{u}'_i \tilde{x} = 0$ for $1 \leq i \leq \alpha'$ and $\bar{v}_i \tilde{x} = 0$ for $1 \leq i \leq \beta$. Up to considering another basis, we can assume that $\bar{u}_i \tilde{x} = 0$ for any $2 \leq i \leq \alpha$.

The element $\tilde{g}$ commutes with every matrix of its decomposition (3.4.1), and hence also with $\{\bar{u}_i\}_{1\leq i\leq\alpha}$, and $\{\bar{u}'_i\}_{1\leq i\leq\alpha'}$, and $\{r_i^\theta\}_{1\leq i\leq\beta,\ \theta\in\mathbb{R}}$ and $\{\bar{v}_i\}_{1\leq i\leq\beta}$. Thus $\{\bar{u}_i \tilde{g}^n \tilde{x}\}_{2\leq i\leq\alpha}$, $\{\bar{u}'_i \tilde{g}^n \tilde{x}\}_{1\leq i\leq\alpha'}$, and $\{\bar{v}_i \tilde{g}^n \tilde{x}\}_{1\leq i\leq\beta}$ are zero for any $n \geq 0$. By construction of W, all elements $\{\bar{u}_i\}_{2\leq i\leq\alpha}$, and $\{\bar{u}'_i\}_{1\leq i\leq\alpha'}$, and $\{\bar{v}_i\}_{1\leq i\leq\beta}$ are zero on W.

Suppose by contradiction that the restriction of g to x_g^+ is not diagonalisable over $\mathbb{C}$, i.e. that $\ell > 0$. Then $\bar{u}_1 \tilde{\xi}_+ = 0$ for any lift $\tilde{\xi}_+ \in W$ of ξ_+. This, and the fact that $\bar{u}_1 \tilde{x} \neq 0$, imply that $\bar{u}_1 \tilde{y} \neq 0$ for any lift $\tilde{y} \in C$ of $y := s^x_{\xi_+} \in F_-$. As a consequence, $(g^n y)_n$ converges to ξ_+. Since F_- is closed, it contains ξ_+, as well as x: contradiction. For the same reasons, the restriction of g to x_g^- is diagonalisable over $\mathbb{C}$, and this concludes the proof of (2).

The sequence of restrictions $(\tilde{g}^n_{|W})_n$ converges to $\bar{u}_{1|W}$ which is proximal because it is a projector with rank 1. Therefore $\tilde{g}^n_{|W}$ is proximal for n large enough, and so is $\tilde{g}_{|W}$. For the same reasons, $\tilde{g}^{-1}_{|W}$ is proximal, and this concludes the proof of (3).

In order to establish (4), it is enough to prove that F_+ (resp. F_-) is contained in $x_g^+ \oplus x_g^0$ (resp. $x_g^- \oplus x_g^0$), since its dimension is $k-1$. Pick $\xi \in F_+$; the sequence $(g^n \xi)_{n\geq 0}$ cannot converge to ξ_- since this point does not belong to F_+. That $g_{|\mathrm{P}(W)}$ is biproximal implies that $\xi \in x_g^+ \oplus x_g^0$. The proof of $F_- \subset x_g^- \oplus x_g^0$ is similar.

Suppose that x, gx, g^2x are not aligned. Then $b \in F_+ \smallsetminus \{\xi_+\} \subset x_g^+ \oplus x_g^0 \smallsetminus x_g^+$, hence $(g^{-n} b)_n$ accumulates in x_g^0. This proves (5).

Suppose that g is not rank-one. Since $x \in [a,b] \cap \Omega$, the simplicial distance between a and b is at least 2. If x, gx, g^2x are aligned then $a = \xi_-$ and $b = \xi_+$ are fixed by g, and $d_{\mathrm{spl}}(a,b) \leq 2$ by Lemma 3.1.3.(2). Otherwise, there is $\xi \in x_g^0 \cap \partial\Omega'$, and $d_{\mathrm{spl}}(a,b) \leq d_{\mathrm{spl}}(a,\xi) + d_{\mathrm{spl}}(\xi,b) \leq 2$ since $\xi \in F_+ \cap F_-$. This proves (6). ☐

Proof of Lemma 3.4.4. By Fact 2.3.16, we may assume that Γ acts convex cocompactly on Ω^*. By Lemma 3.4.5, $\phi_{\pm\infty}v \in x_\gamma^{\pm} \oplus x_\gamma^0 \cap \partial\Omega$, hence the segment between $\phi_{\pm\infty}v$ and any accumulation point of $(\gamma^{\pm n}\pi v)_n$ is contained in $\partial\Omega$. This implies that $\phi_{\pm\infty}v \in \Lambda^{\mathrm{orb}}$ since $\overline{\Omega} \smallsetminus \Lambda^{\mathrm{orb}}$ has bisaturated boundary by Fact 2.3.16. ☐

Another consequence of Facts 2.3.16 and 3.4.3 and Lemma 3.4.5 is a characterisation of the rank-one condition for convex projective orbifold with a non-empty compact convex core, which generalises results of Islam [Isl, Lem. 6.4] and Zimmer [Zim, Th. 7.1] for compact convex projective orbifolds; one may also compare this to a remark of Islam [Isl, Rem. A.1.C] which concerns not necessarily compact convex projective manifolds with a compact convex core.

Corollary 3.4.6. *Let $\Gamma \subset \mathrm{PGL}(\mathbf{V})$ be a discrete subgroup that acts convex cocompactly on a properly convex open set $\Omega \subset \mathrm{P}(\mathbf{V})$.*

- *Any biproximal element of Γ whose dual axis in $\mathrm{P}(\mathbf{V}^*)$ intersects Ω^* is rank-one; in particular, if $T^1(\Omega^*/\Gamma)_{\mathrm{bip}} \neq \emptyset$, then Ω/Γ is rank-one.*
- *Suppose that Γ acts convex cocompactly on Ω^*, then any biproximal element of Γ whose axis intersects Ω is rank-one; in particular, if $(T^1\Omega/\Gamma)_{\mathrm{bip}} \neq \emptyset$, then Ω/Γ is rank-one.*

Proof. The first point is an immediate consequence of Fact 3.4.3 and Lemma 3.4.5.(5). Let us establish the second point. Let $\gamma \in \Gamma$ be biproximal with $\mathrm{axis}(\gamma) \cap \Omega \neq \emptyset$. Since Γ acts convex cocompactly on Ω^*, the first point implies that γ, seen as a rank-one automorphism of Ω^*, is rank-one. Thus, γ is a rank-one automorphism of Ω by Lemma 3.1.2. ☐

3.5 The biproximal unit tangent bundle of reducible compact convex projective orbifolds

In this section we explain, for completeness, why the biproximal unit tangent bundle of a reducible compact convex projective orbifold is empty.

Vey [Vey70, Th. 3] (see [Ben08, § 5.1]) proved that, given any properly convex open set $\Omega \subset \mathrm{P}(\mathbf{V})$ divided by a discrete group $\Gamma \subset \mathrm{Aut}(\Omega)$, the group Γ is *not* strongly irreducible if and only if Ω is *reducible*, i.e. there exists a decomposition $V = V_1 \oplus V_2$ and convex open cones $\mathcal{C}_i \subset V_i$ with $\mathrm{P}(\mathcal{C}_i) \subset \mathrm{P}(V_i)$ properly convex for $i = 1, 2$, such that $\Omega = \mathrm{P}(\mathcal{C}_1 + \mathcal{C}_2)$; in this case we say that $M = \Omega/\Gamma$ is a *reducible* compact convex projective orbifold.

Lemma 3.5.1. *Suppose that $\dim(\mathbf{V}) = \mathtt{d} + 1 > 2$. Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a reducible properly convex open set, and $\Gamma \subset \mathrm{Aut}(\Omega)$ a discrete subgroup. Then the quotient $M = \Omega/\Gamma$ is higher-rank (i.e. not rank-one) and T^1M_{bip} is empty. In particular, reducible compact convex projective orbifolds are higher-rank and have an empty biproximal unit tangent bundle.*

Proof. Let us show that $\partial\Omega$ contains no strongly extremal point (this implies that M is higher-rank), and that $[x,y] \subset \partial\Omega$ for all extremal points of $\partial\Omega$ (this implies that T^1M_{bip} is empty since one can check that Λ_Γ is contained in the closure of the set of extremal points). Consider a decomposition $\Omega = \mathrm{P}(\mathcal{C}_1 + \mathcal{C}_2)$, where $\mathcal{C}_i \subset V_i$ is a convex cone for $i = 1, 2$, with $V = V_1 \oplus V_2$.

Observe that the boundary of Ω is equal to $\mathrm{P}(\partial\mathcal{C}_1+\overline{\mathcal{C}}_2)\cup\mathrm{P}(\overline{\mathcal{C}}_1+\partial\mathcal{C}_2)\cup\mathrm{P}(\overline{\mathcal{C}}_1+0)\cup\mathrm{P}(0+\overline{\mathcal{C}}_2)$. Take $x\in\partial\Omega$, and let us check that x is not strongly extremal. If $x=[tv+(1-t)w]$ for $v\in\partial\mathcal{C}_i$ and $w\in\overline{\mathcal{C}}_j$ and $0<t\leq 1$, with $i\neq j\in\{1,2\}$, then $\{[sv+(1-s)w : 0\leq s\leq 1\}$ is a non-trivial segment of $\partial\Omega$ that contains x. If $x\in\mathrm{P}(\overline{\mathcal{C}}_i)$, then we take $y\in\mathrm{P}(\partial\mathcal{C}_i)\smallsetminus\{x\}$, or in $\mathrm{P}(\partial\mathcal{C}_j)$, with $i\neq j\in\{1,2\}$ (y exists because $\dim(\mathbf{V})>2$), and $[x,y]\subset\partial\Omega$ is a non-trivial segment.

Observe that the set of extremal points of Ω is equal to $\mathrm{P}(\widetilde{E}_1+0)\cup\mathrm{P}(0+\widetilde{E}_2)$, where $\widetilde{E}_i$ is the preimage in V of the set of extremal points of $\mathrm{P}(\mathcal{C}_i)$ for $i=1,2$. If $x,y\in\mathrm{P}(\widetilde{E}_i)$ with $i=1,2$, then $[x,y]\subset\mathrm{P}(\overline{\mathcal{C}}_i)\subset\partial\Omega$. If $x\in\mathrm{P}(\widetilde{E}_1)$ and $y\in\mathrm{P}(\widetilde{E}_2)$, then either $x\in\mathrm{P}(\partial\mathcal{C}_1)$ or $y\in\mathrm{P}(\partial\mathcal{C}_2)$ (since $\dim(\mathbf{V})>2$), and $[x,y]\subset\mathrm{P}(\partial\mathcal{C}_1+\overline{\mathcal{C}}_2)\cup\mathrm{P}(\overline{\mathcal{C}}_1+\partial\mathcal{C}_2)\subset\partial\Omega$. □

Chapter 4

The proximal limit set of cocompact groups

4.1 Introduction

In this chapter, we prove Theorem 0.2.1, which says that the proximal limit set of any rank-one divisible convex set is the full projective boundary.

4.1.1 Structural results on divisible convex sets

The result that we discuss here continues a line of structural results on divisible convex sets Ω. These make the link between several kinds of regularity properties of the projective boundary $\partial\Omega \subset \mathrm{P}(\mathbf{V})$, of algebraic properties of $\mathrm{Aut}(\Omega)$ and its discrete cocompact subgroups, and of dynamical properties of the action of $\mathrm{Aut}(\Omega)$ and its subgroups on $\mathrm{P}(\mathbf{V})$.

One cornerstone of these structural results is the following result due to Vey [Vey70, Th. 3], which we discussed in Section 3.5. Consider a divisible convex set $\Omega \subset \mathrm{P}(\mathbf{V})$. Then

- either there exists two proper subspaces $V_1, V_2 \subset \mathbf{V}$ with $\mathbf{V} = V_1 \oplus V_2$ and two properly convex open cones $C_1 \subset V_1$ and $C_2 \subset V_2$ such that $\mathrm{P}(C_1) \subset \mathrm{P}(V_1)$ and $\mathrm{P}(C_2) \subset \mathrm{P}(V_2)$ are divisible convex sets and $\Omega = \mathrm{P}(C_1 + C_2)$ — in this case Ω is said to be *reducible*;

- or any cocompact closed subgroup of $\mathrm{Aut}(\Omega)$ is strongly irreducible; in this case Ω is said to be *irreducible*.

Let us assume that Ω is irreducible. Combining work of Koecher [Koe99], Vinberg [Vin65] and Benoist [Ben03] yields the following dichotomy:

- either $\mathrm{Aut}(\Omega) \subset \mathrm{PGL}(\mathbf{V})$ is a semi-simple Lie subgroup that acts transitively on Ω, in which case Ω is called *symmetric*;

- or $\mathrm{Aut}(\Omega) \subset \mathrm{PGL}(\mathbf{V})$ is a discrete Zariski-dense subgroup.

If Ω is symmetric, then it naturally identifies with the Riemannian symmetric space of $\mathrm{Aut}(\Omega$, and there is yet another natural dichotomy: namely, either $\mathrm{Aut}(\Omega)$ has real rank 1, in which case Ω is an ellipsoid and $\mathrm{Aut}(\Omega)$ is isomorphic to $\mathrm{PO}(n,1)$ for $n = \dim(\mathbf{V}) - 1$, or $\mathrm{Aut}(\Omega)$ has real rank greater than one, it is isomorphic to $\mathrm{PGL}(n, \mathbb{K})$ for some $n \geq 3$, and for $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$, or the classical division algebra of quaternions, or of octonions if $n = 3$ (see Section 5.5 for more details).

Recently, A. Zimmer proved the following higher-rank rigidity result [Zim, Th. 1.4], analogous to a celebrated result in Riemannian geometry by Ballmann [Bal85] and Burns–Spatzier [BS87]. If Ω is not symmetric, then $\mathrm{Aut}(\Omega)$ is rank-one (in the sense of Definition 3.1.1).

By Lemma 3.5.1 and Remark 5.5.2, reducible divisible convex sets and symmetric irreducible divisible convex sets with higher-rank automorphism groups are not rank-one, hence called higher-rank. On the other hand, ellipsoids are rank-one.

4.1.2 The proximal limit set

Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be an irreducible divisible convex set. The *proximal limit set* of Ω is defined as follows

$$\Lambda^{\mathrm{prox}}_{\Omega} := \Lambda^{\mathrm{prox}}_{\mathrm{Aut}(\Omega)}. \tag{4.1.1}$$

By Fact 2.2.3, the proximal limit set is contained in the closure of the set of extremal points of Ω. Vey [Vey70, Prop. 3] proved that we actually have equality.

Fact 4.1.1. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be an irreducible divisible convex set. Then its proximal limit set is the closure of the set of extremal points.*

Note that Proposition 2.3.15 is a generalisation of Vey's result to the convex cocompact case.

If Ω is an ellipsoid, i.e. a rank-one symmetric divisible convex set, then $\Lambda^{\mathrm{prox}}_{\Omega} = \partial\Omega$ and $\mathrm{Aut}(\Omega)$ acts transitively on it. If Ω is a higher-rank symmetric irreducible divisible convex set, then Λ^{prox} is an analytic submanifold of $\mathrm{P}(\mathbf{V})$ of dimension less than $\dim(\mathbf{V}) - 2$, and hence is a proper subset of $\partial\Omega$ (see [Bla, §7]), on which $\mathrm{Aut}(\Omega)$ acts transitively.

Our goal is to prove the following result.

Theorem 4.1.2. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a rank-one divisible convex set. Then $\Lambda^{\mathrm{prox}}_{\Omega} = \partial\Omega$.*

Combined with Zimmer's higher-rank rigidity theorem [Zim, Th. 1.4], Theorem 4.1.2 yields the following answer to a question of Benoist [Ben12, Prob. 5].

Corollary 4.1.3. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a non-symmetric irreducible divisible convex set. Then $\Lambda^{\mathrm{prox}}_{\Omega} = \partial\Omega$.*

Let Ω be a rank-one divisible convex set. The conclusion of Theorem 4.1.2 holds trivially if Ω is symmetric (i.e. is an ellipsoid). Thus we may assume that Ω is not symmetric, hence that $\mathrm{Aut}(\Omega)$ is discrete and Zariski-dense in $\mathrm{PGL}(\mathbf{V})$ (and finitely generated).

Benoist [Ben04, Th. 1.1] proved that $\mathrm{Aut}(\Omega)$ is Gromov-hyperbolic if and only if Ω is *strictly convex* (i.e. all points of $\partial\Omega$ are extremal), if and only if $\partial\Omega$ is $\mathcal{C}^1$. In this case, strict convexity implies that $\Lambda^{\mathrm{prox}}_{\Omega} = \partial\Omega$. One may find in [Ben04] more precise results on the regularity of $\partial\Omega$.

Benoist [Ben06a] also studied non-strictly convex 3-dimensional rank-one divisible convex sets. He constructed examples, and established a precise description of these (discussed in Section 0.1.4) which implies that $\Lambda^{\mathrm{prox}}_{\Omega} = \partial\Omega$.

Islam–Zimmer [IZ] generalised Benoist's description to higher-dimensional rank-one divisible convex sets, under the assumption that $\mathrm{Aut}(\Omega)$ is relatively hyperbolic, and their result implies that $\Lambda^{\mathrm{prox}}_{\Omega} = \partial\Omega$ in this case. M. Bobb [Bob] also generalised Benoist's result under the assumption that each non-trivial face of Ω (see Section 2.1.8) is contained in a properly embedded simplex of dimension $\dim(\mathbf{V}) - 2$, namely a closed simplex $S \subset \overline{\Omega}$ whose relative interior (see Section 2.1.8) is exactly $S \cap \Omega$; Bobb's result also implies that $\Lambda^{\mathrm{prox}}_{\Omega} = \partial\Omega$.

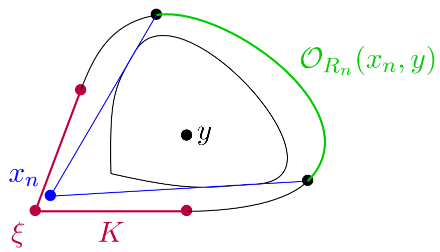

Figure 4.1: Illustration of the proof of Lemma 4.2.1. The green shadow $\mathcal{O}_{R_n}(x_n, y)$ fills more and more of $\partial\Omega \smallsetminus K$ as n tends to infinity

4.1.3 Organisation of the chapter

In Section 4.2 we establish a weak, convex projective version (Lemma 4.2.1) of Sullivan's celebrated Shadow lemma. This result can be seen as a consequence of Lemma 6.3.1, which is a more standard convex projective version of the Sullivan Shadow lemma. In Section 4.3 we establish two topological results (Lemmas 4.3.2 and 4.3.4) which concern the arrangement of faces on the boundary of a convex set. In Section 4.4 we use Sections 4.2 and 4.3 to prove Theorem 4.1.2.

4.2 A weak Shadow lemma

Let $\Omega \subset \mathrm{P}(V)$ be a properly convex open set. For $x \in \overline{\Omega}$, $y \in \Omega$ and $R > 0$, we consider the set

$$\mathcal{O}_R(x, y) = \{\xi \in \partial\Omega : [x, \xi] \cap B_\Omega(y, R) \neq \emptyset\},$$

interpreted as the shadow cast on $\partial\Omega$ by the balls of radius R around y with a light source at x.

Lemma 4.2.1. *Let $\Omega \subset \mathrm{P}(V)$ be a rank-one divisible convex set. Then there exists $R > 0$ such that $\mathcal{O}_R(x, y)$ contains a point of the proximal limit set $\Lambda_\Omega^{\mathrm{prox}}$ (see Section 4.1.2) for all $x, y \in \Omega$.*

Proof. Recall from Fact 4.1.1 that $\Lambda_\Omega^{\mathrm{prox}}$ is the closure of the set of extremal points of $\partial\Omega$. By contradiction, suppose that there is a diverging sequence of positive numbers $(R_n)_n$ and sequences of points $(x_n)_n, (y_n)_n$ in Ω such that for any $n \geq 0$, the set $\mathcal{O}_{R_n}(x_n, y_n)$ does not contain any extremal point of $\partial\Omega$. Since Ω is divisible, $\mathrm{Aut}(\Omega)$ acts cocompactly on Ω, and so we may assume that $(y_n)_n$ remains in a compact subset of Ω, and up to extracting, we may further assume that $(y_n)_n$ converges to a point $y \in \Omega$. Up to replacing R_n by $R_n + d_\Omega(y_n, y)$, we may actually assume that $(y_n)_n$ is constant equal to y.

Up to extraction, we assume that $(x_n)_n$ converges to some $\xi \in \overline{\Omega}$. If $\xi \in \Omega$, then for n such that $R_n \geq d_\Omega(o, \xi) + 1$ and $d_\Omega(x_n, \xi) < 1$, we have $\mathcal{O}_{R_n}(x_n, y) = \partial\Omega$, which is absurd; hence $\xi \in \partial\Omega$.

Let $K \subset \partial\Omega$ be the set of points η such that $[\xi, \eta] \subset \partial\Omega$. Then

$$\partial\Omega \smallsetminus K \subset \bigcup_n \bigcap_{k \geq n} \mathcal{O}_{R_k}(x_k, y).$$

See Figure 4.1. Let $\eta \in \partial\Omega \smallsetminus K$, and $z \in [\xi, \eta] \cap \Omega$. Since $(x_n)_n$ converges to ξ, we can find $z_n \in [x_n, \eta] \cap B_\Omega(z, 1)$ for any large enough n. On the other hand, $R_n \geq d_\Omega(y, z) + 2$ for n large. Thus, $z_n \in B_\Omega(y, R_n)$ and hence $\eta \in \mathcal{O}_{R_n}(x_n, y)$ for any large enough n.

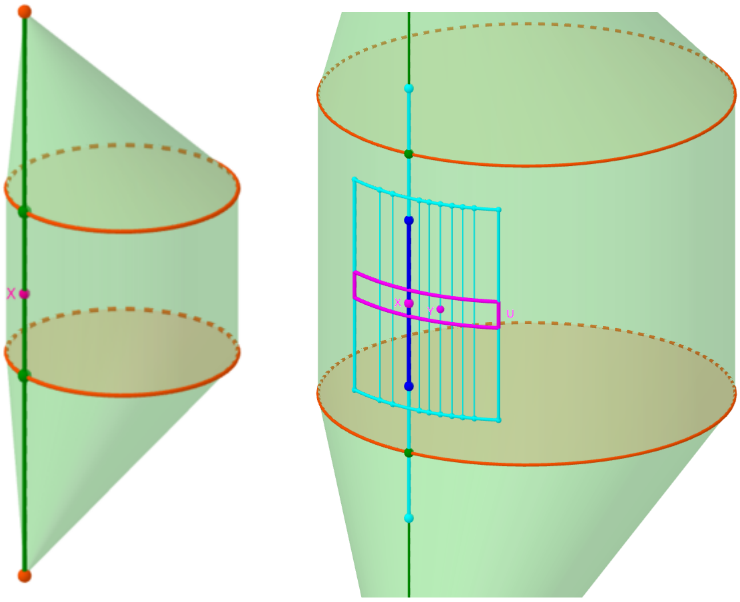

Figure 4.2: On the left: the whole set $\Omega_f \subset \mathrm{P}(\mathbb{R}^4)$ as in Section 4.3 for a 2π-periodic function f which is constant on $\mathbb{R} \smallsetminus 2\pi\mathbb{Z}$ and discontinuous on $2\pi\mathbb{Z}$. On the right: a zoom on Ω_f at point x, with blue vertical segments representing several $d_{\overline{\Omega}_f}$-balls.

By assumption, this implies that all extremal points are contained in K. Since $\mathrm{Aut}(\Omega)$ is rank-one (Definition 3.2.1) and irreducible, and by Lemma 3.1.2, $\partial\Omega$ contains a strongly extremal point which is different from ξ. Such a point cannot lie in K; this yields a contradiction. □

4.3 Two lemmas on general properly convex open sets

In this section we prove two lemmas on the arrangement of faces on the boundary of a general properly convex open subset of $\mathrm{P}(\mathbf{V})$, which is not necessarily divisible.

Let us first give a family of examples of non-divisible properly convex open subsets of $\mathrm{P}(\mathbb{R}^4)$ that one may wish to keep in mind while reading the lemmas of this section. Let $f : \mathbb{R} \to [1, \infty)$ be a 2π-periodic upper semi-continuous function. Let Ω_f be the interior of the convex hull in $\mathbb{R}^3$ of

$$\{(\cos(\theta), \sin(\theta), f(\theta)) : \theta \in \mathbb{R}\} \cup \{(\cos(\theta), \sin(\theta), -f(\theta)) : \theta \in \mathbb{R}\}. \tag{4.3.1}$$

Since f is upper semi-continuous and 2π-periodic, it is bounded, and so is Ω_f. Let us identify $\mathbb{R}^3$ with an affine chart of $\mathrm{P}(\mathbb{R}^4)$, so that Ω_f is a properly convex open subset of $\mathrm{P}(\mathbb{R}^4)$. One can check that (4.3.1) is exactly the set of extremal points of Ω_f, and that for any $\theta \in \mathbb{R}$, the set $\{(\cos(\theta), \sin(\theta), z) : z \in (-f(\theta), f(\theta))\}$ is an open face of of Ω_f.

4.3.1 More comparison results for Hilbert balls in a fixed affine chart

We will need the following elementary fact, in the spirit of Section 2.1.2.

Fact 4.3.1. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set and $\mathbb{A} \subset \mathrm{P}(\mathbf{V})$ an affine chart containing $\overline{\Omega}$. For all $a \in \mathbb{A}$ and $t > 0$, we denote by h_a^t the homothety of $\mathbb{A}$ with centre a and ratio t. Consider $x \in \overline{\Omega}$ and $0 < r < R$. Then*

1. $\overline{F}_\Omega(x) \subset \mathrm{h}_x^\lambda\left(\overline{B}_{\overline{\Omega}}(x,r)\right)$, *where*
$$\lambda = \frac{\mathrm{diam}_{\mathbb{A}}(F_\Omega(x))(e^{2r}+1)}{d_{\mathbb{A}}(x, \partial_{\mathrm{rel}} F_\Omega(x))(e^{2r}-1)} > 1;$$

2. $\mathrm{h}_x^\mu\left(\overline{B}_{\overline{\Omega}}(x,r)\right) \subset \overline{B}_{\overline{\Omega}}(x,R)$ *where* $\mu = (e^{2R}-1)/(e^{2r}-1) > 1$.

Proof. We see $\mathbb{A}$ as a vector space by setting $x = 0$. Let $y \in \partial_{\mathrm{rel}} B_{\overline{\Omega}}(x,r)$, and consider $a > 0$ and $b > 1$ such that $-ay$ and by lie in $\partial_{\mathrm{rel}} F_\Omega(x)$. To establish (1), it is enough to prove that
$$b \leq \frac{\max(a,b)(e^{2r}+1)}{\min(a,b)(e^{2r}-1)}.$$
This is an immediate consequence of (2.1.1), which implies that $(a+1)b = e^{2r}a(b-1)$, hence that
$$b = \frac{ae^{2r}+b}{a(e^{2r}-1)}.$$
Consider $t \in (1,b)$ such that $ty \in \partial_{\mathrm{rel}} B_{\overline{\Omega}}(x,R)$. By (2.1.1), we have
$$1 = \frac{ab(e^{2r}-1)}{ae^{2r}+b} \qquad \text{and} \qquad t = \frac{ab(e^{2R}-1)}{ae^{2R}+b}.$$
Thus,
$$t = \frac{(e^{2R}-1)(ae^{2r}+b)}{(e^{2r}-1)(ae^{2R}+b)} > \frac{e^{2R}-1}{e^{2r}-1},$$
and this proves (2). □

4.3.2 Existence of a point on the boundary with a sufficiently small Hilbert ball

Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set. We saw in Fact 2.1.10 that, for any $R > 0$, the map $\overline{B}_{\overline{\Omega}}(\cdot, R)$ is upper semi-continuous on $\overline{\Omega}$. However, it is not continuous in general. For instance, in Figure 4.2 on the left, each orange point $x \in \partial\Omega_f$ is extremal, hence $\overline{B}_{\overline{\Omega}_f}(x,R) = \{x\}$ for any $R > 0$, and orange points accumulate to a green point y which has a non-trivial face, hence $\overline{B}_{\overline{\Omega}_f}(y,R) \neq \{y\}$, and so $\overline{B}_{\overline{\Omega}_f}(\cdot,R)$ is discontinuous at y.

The goal of the next lemma is to show that in any open subset of $\partial\Omega$, one can find a point at which $\overline{B}_{\overline{\Omega}}(\cdot,R)$ is "almost continuous".

Lemma 4.3.2. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set, $0 < r < R$ and $U \subset \partial\Omega$ a non-empty open subset. Then one can find a point $x \in U$ such that $\overline{B}_{\overline{\Omega}}(x,r)$ is contained in any accumulation point of $\overline{B}_{\overline{\Omega}}(y,R)$ (for the Haudorff topology) when y tends to x.*

Note that if $x \in \partial\Omega$ is an extremal point, then $\overline{B}_{\overline{\Omega}}(x,r) = \{x\}$, and so $\overline{B}_{\overline{\Omega}}(\cdot,R)$ is continuous at x. Thus, the lemma is immediate when U contains an extremal point.

Suppose $\Omega = \Omega_f$ for some 2π-periodic upper semi-continuous function $f : \mathbb{R} \to [1,\infty)$, consider the open subset $U = \{(\cos(\theta), \sin(\theta), z) : z \in (-1,1),\ \theta \in \mathbb{R}\}$, and consider $\theta \in (0,2\pi)$, $z \in (-1,1)$ and $x = (\cos(\theta), \sin(\theta), z)$. Fix $R > 0$. Then $\overline{B}_{\overline{\Omega}}(\cdot,R)$ is continuous at x if and only if f is continuous at θ. In particular, if f is discontinuous everywhere, then $\overline{B}_{\overline{\Omega}_f}(\cdot,R)$ is discontinuous everywhere on U. Proving Lemma 4.3.2 in the case $\Omega = \Omega_f$ roughly amounts to proving that for any $\epsilon > 0$, we can find $\theta_\epsilon \in \mathbb{R}$ at which f is "ϵ-almost continuous", i.e. such that
$$f(\theta_\epsilon) - \epsilon \leq \liminf_{\theta \to \theta_\epsilon} f(\theta) \leq \limsup_{\theta \to \theta_\epsilon} f(\theta) \leq f(\theta_\epsilon).$$

Proof. Fix an affine chart $\mathbb{A}$ that contains $\overline{\Omega}$, and a Euclidean norm on $\mathbb{A}$ whose associated metric is denoted by $d_{\mathbb{A}}$, with associated balls denoted by $B_{\mathbb{A}}(x,t)$ for $x \in \mathbb{A}$ and $t > 0$. For the rest of this proof, we set $B_t(x) = \overline{B}_{\overline{\Omega}}(x,t)$ for $x \in \overline{\Omega}$ and $t > 0$, and denote by $\mathcal{B}_t(x)$ the set of accumulation points (for the Hausdorff topology) of $B_t(y)$ when y tends to x.

First step: We reduce U to control the dimension of faces.

Let k be the largest integer such that $\{x \in U : \dim F_\Omega(x) \geq k\}$ has non-empty interior in U. Let $x_0 \in U$ and $\epsilon > 0$ be such that $\overline{B}_{\mathbb{A}}(x_0, 2\epsilon) \cap \partial\Omega$ is contained in this interior, and $\dim F_\Omega(x_0) = k$. Note that $D := \{x : \dim F_\Omega(x) = k\} \cap \overline{B}_{\mathbb{A}}(x_0,\epsilon) \cap \partial\Omega$ is dense in $U' := \overline{B}_{\mathbb{A}}(x_0,\epsilon) \cap \partial\Omega$. Up to taking ϵ even smaller, we can assume that $\operatorname{diam}_{\mathbb{A}} \overline{\Omega} \leq \epsilon^{-1}$.

Second step: We bound from below the size of faces of dimension k.

Consider for this step $x \in D$. We denote by $\mathbb{A}_x$ the affine subspace of $\mathbb{A}$ spanned by $F_\Omega(x)$, which has dimension k. Any point in $\partial_{\mathrm{rel}} F_\Omega(x)$ has a face of dimension strictly less that k, hence is not in $\overline{B}_{\mathbb{A}}(x_0, 2\epsilon)$ by definition of x_0 and ϵ. By triangular inequality, this implies that

$$B_{\mathbb{A}}(x,\epsilon) \cap \mathbb{A}_x \subset F_\Omega(x) \subset \mathbb{A}_x.$$

Set $\lambda := \epsilon^{-2}(e^{2R}+1)/(e^{2R}-1) > 1$. For all $a \in \mathbb{A}$ and $t > 0$, we denote by h_a^t the homothety of $\mathbb{A}$ with centre a and ratio t. By Fact 4.3.1.1,

$$\overline{F}_\Omega(x) \subset \mathrm{h}_x^\lambda \left(B_R(x)\right).$$

As a consequence, we have

$$B_{\mathbb{A}}(x,\epsilon/\lambda) \cap \mathbb{A}_x \subset B_R(x) \subset \mathbb{A}_x. \tag{4.3.2}$$

By upper semi-continuity of B_R (Fact 2.1.10) and the above (4.3.2), any accumulation point of $B_{\mathbb{A}}(y,\epsilon/\lambda) \cap \mathbb{A}_y$ (for the Hausdorff topology) when $y \in D$ tends to x is contained in $B_R(x) \subset \mathbb{A}_x$. One may easily deduce that the map $y \in D \mapsto \overline{B}_{\mathbb{A}}(y,\epsilon/\lambda) \cap \mathbb{A}_y$ is continuous for the Hausdorff topology.

By upper semi-continuity of B_R and density of D, any element $K \in \mathcal{B}_R(x)$ contains the limit of some sequence $(B_R(x_n))_n$ where $(x_n)_n \subset D$ converges to x. By (4.3.2), this implies that

$$B_{\mathbb{A}}(x,\epsilon/\lambda) \cap \mathbb{A}_x \subset K \subset B_R(x) \subset \mathbb{A}_x, \tag{4.3.3}$$

hence K has dimension k.

Third step: We find a minimal element in $\mathcal{B}_R(x_0)$.

Let us show that $\mathcal{B}_R(x_0)$ contains an element which is minimal for inclusion; by the Zorn lemma, it is enough to show that for every totally ordered subset $\mathcal{A} \subset \mathcal{B}_R(x_0)$, the intersection K of all elements of $\mathcal{A}$ belongs to $\mathcal{B}_R(x_0)$.

The Hausdorff topology on the set of compact subsets of $\mathrm{P}(\mathbf{V})$ is metrisable, and K is in the closure of $\mathcal{A}$, so we can find a sequence $(K_n)_n$ in $\mathcal{A}$ that converges to K. If $K_n = K$ for some n, then $K \in \mathcal{B}_R(x_0)$; let us assume the contrary. For any n, we can find $m > n$ such that $K_m \subset K_n$ since, otherwise, $K \subsetneq K_n \subset K_m$ for any $m > n$ so $(K_m)_m$ would not converge to K. Thus, up to extraction, we may assume that $(K_n)_n$ is non-increasing.

For each n, let $(x_{n,k})_k$ be a sequence converging to x_0 such that $(B_R(x_{n,k}))_k$ converges to K_n. Then $(B_R(x_{n,n}))_n$ converges to K, which thus belongs to $\mathcal{B}_R(x_0)$.

Let $K \in \mathcal{B}_R(x_0)$ be a minimal element for inclusion, and let $(x_n)_n$ be a sequence in U' converging to x_0 such that $(B_R(x_n))_n$ converges to K. By density of D in U', upper semi-continuity of B_R and minimality of K, we may assume that $(x_n)_n$ is in D.

Fourth step: We prove that $B_r(x_n)$ is contained in any element of $\mathcal{B}_R(x_n)$ for n large enough.

Assume by contradiction that for each n there exists $K_n \in \mathcal{B}_R(x_n)$ that does not contain $B_r(x_n)$; since, by the previous step, K_n and $B_r(x_n)$ are convex subsets of $\mathbb{A}_{x_n}$ that contain x_n in their interior relative to $\mathbb{A}_{x_n}$, we may find $y_n \in \partial_{\mathrm{rel}} K_n \cap B_r(x_n)$.

Up to extraction, we can assume that $(K_n)_n$ converges to some K' and $(y_n)_n$ converges to some y. One can check that $K' \in \mathcal{B}_R(x)$. By (4.3.3), the compact convex sets K' and $\{K_n\}_n$ have dimension k. According to the following classical and elementary fact, y belongs to $\partial_{\mathrm{rel}} K'$.

Fact 4.3.3. *If $(A_n)_n$ is a sequence of k-dimensional compact convex subsets of $\mathbb{A}$ that converges to a k-dimensional compact convex set A for the Hausdorff topology, then $(\partial_{\mathrm{rel}} A_n)_n$ converges to $\partial_{\mathrm{rel}} A$.*

That $K_n \subset B_R(x_n)$ for each n implies that $K' \subset K$, which in turn implies, by minimality of K, and because $K' \in \mathcal{B}_R(x)$, that $K' = K$.

Let $\mu = (e^{2R}-1)/(e^{2r}-1) > 1$. By Fact 4.3.1.2, since $y_n \in B_r(x_n)$ for each n, we have $\mathrm{h}^{\mu}_{x_n} y_n \in B_R(x_n)$. As a consequence, $\mathrm{h}^{\mu}_{x_0} y \in K$, which contradicts the fact that $y \in \partial_{\mathrm{rel}} K$, $x_0 \in \mathrm{int}_{\mathrm{rel}}\, K$ and $\mu > 1$. □

4.3.3 The Grain of sand lemma

Consider a properly convex open set $\Omega \subset \mathrm{P}(\mathbf{V})$, positive numbers $r < R$, a point $x \in \partial\Omega$ at which $\overline{B}_{\overline{\Omega}}(\cdot, R)$ is "almost continuous" in the sense of Lemma 4.3.2, and a compact neighbourhood U of x in $\partial\Omega$.

The Grain of sand lemma (Lemma 4.3.4) says that the collection of balls $\overline{B}_{\overline{\Omega}}(y, R)$ centred at points $y \in U$ "foliates" a neighbourhood of $B_{\overline{\Omega}}(x, r)$, i.e. that no "grain of sand" is inserted between the convex "leaves" of this "foliation".

To illustrate this idea, we use again Figure 4.2, which represents the set $\Omega = \Omega_f$ (defined at the beginning of Section 4.3) for a 2π-periodic function f which is constant on $\mathbb{R} \smallsetminus 2\pi\mathbb{Z}$ and discontinuous on $2\pi\mathbb{Z}$. On the right of the figure, $U \subset \partial\Omega_f$ is the compact neighbourhood of the pink point x which is delimited by the pink rectangle on the cylinder. The vertical light blue segments are $d_{\overline{\Omega}_f}$-balls of radius R centred at points of U, while the dark blue segment is a ball of radius $r \in (0, R)$ centred at x. The union $B_{\overline{\Omega}_f}(U, R)$ of the balls $(B_{\overline{\Omega}_f}(y, R))_{y \in U}$ is the region delimited by the light blue rectangle, to which one must add the tall central light blue vertical segment. The set $B_{\overline{\Omega}_f}(U, R)$ is not open in $\partial\Omega$. Its relative interior $\mathrm{int}_{\partial\Omega}(B_{\overline{\Omega}_f}(U, R))$ is the region delimited by the light blue rectangle, and it is foliated by light blue balls for $d_{\overline{\Omega}_f}$. This relative interior contains the ball $B_{\overline{\Omega}_f}(x, r)$.

Lemma 4.3.4. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set, $0 < r < R$ and $x \in \partial\Omega$ such that $\overline{B}_{\overline{\Omega}}(x, r)$ is contained in any accumulation point of $\overline{B}_{\overline{\Omega}}(y, R)$ (for the Hausdorff topology) when y tends to x. Then for any compact neighbourhood $U \subset \partial\Omega$ of x,*

$$B_{\overline{\Omega}}(x, r) \subset \mathrm{int}_{\partial\Omega}(B_{\overline{\Omega}}(U, R)),$$

where $B_{\overline{\Omega}}(U, R) := \bigcup_{y \in U} B_{\overline{\Omega}}(y, R)$ is the uniform R-neighbourhood of U for the metric $d_{\overline{\Omega}}$.

As in the previous section, the lemma holds trivially if x is extremal, since

$$B_{\overline{\Omega}}(x, r) = \{x\} \subset \mathrm{int}_{\partial\Omega}\, U \subset \mathrm{int}_{\partial\Omega}(B_{\overline{\Omega}}(U, R)).$$

If x is not extremal, then the situation is more delicate. In fact, the problem is related to the Invariance of Domain theorem. For instance, Lemma 4.3.4 is a consequence of this classical theorem under the assumption that there exists a neighbourhood U' of x such that

$\dim(F_\Omega(y)) = \dim(F_\Omega(x))$ for any $y \in U'$ (more details on how to apply the Invariance of Domain in this particular case are given in the following proof). This assumption is satisfied when $\Omega = \Omega_f$ for some 2π-periodic upper semi-continuous function $f : \mathbb{R} \to [1, \infty)$, and $x = (\cos(\theta), \sin(\theta), z)$ for some $\theta \in \mathbb{R}$ and $z \in (-1, 1)$. In fact, in this last case, one can prove the lemma by hand.

In the general case, the strategy of proof of Lemma 4.3.4 is similar to one of those of the Invariance of Domain theorem.

Proof. We first embed U into a hyperplane of $\mathrm{P}(\mathbf{V})$.

Let $\mathbb{A}$ be an affine chart of $\mathrm{P}(\mathbf{V})$ containing $\overline{\Omega}$. Let $\mathrm{P}(V')$ be a supporting hyperplane of Ω at p, let $p \in \Omega$, let ψ be the projection from $\mathrm{P}(\mathbf{V}) \smallsetminus \{p\}$ to $\mathrm{P}(V')$. The map $\psi_{|\partial\Omega}$ is a local homeomorphism onto $\mathrm{P}(V')$, it is injective on $\overline{F}_\Omega(x)$, and $\psi(\overline{F}_\Omega(x)) \subset \mathbb{A}' := \mathbb{A} \cap \mathrm{P}(V')$. As a consequence, there exists a compact neighbourhood W of $\overline{F}_\Omega(x)$ in $\partial\Omega$ such that $\psi_{|W}$ is an open embedding whose image lies in $\mathbb{A}'$. Moreover, there exists a compact neighbourhood U_0 of x such that $\overline{B}_{\overline{\Omega}}(y, R) \subset W$ for any $y \in U_0$. We may assume that $U \subset U_0$.

For any $y \in \psi(U)$ and $0 < t \leq R$, we let $B_t(y) = \psi(\overline{B}_{\overline{\Omega}}(\psi^{-1}(y), t))$, $U' = \psi(U)$ and $x' = \psi(x)$. We want to prove that

$$\bigcup_{0<t<r} B_t(x') \subset \mathrm{int}_{\mathbb{A}'}\left(\bigcup_{y\in U} B_R(y)\right).$$

Fix any $t \in (0, r)$ and any affine subspace $\mathbb{A}_1 \subset \mathbb{A}'$ containing x' and transverse to the span of $B_R(x)$. For $s > 0$ we denote $B_{\mathbb{A}_1}(x, s) := \{z \in \mathbb{A}_1 : d_{\mathbb{A}'}(x, z) < s\}$. For any two points $p, q \in \mathbb{A}'$, the difference $p - q$ is a vector of the linear space associated to the affine space $\mathbb{A}'$, and for any subset $E \subset \mathbb{A}'$ we denote $E + p - q := \{e + p - q : e \in E\}$. To conclude the proof it is enough to find $\epsilon > 0$ such that for any $z \in B_t(x)$,

$$B_{\mathbb{A}_1}(x', \epsilon) + z - x' \subset \bigcup_{y\in U'\cap\mathbb{A}_1} B_R(y).$$

By assumption that any accumulation point of $B_R(y)$ (for the Hausdorff topology) as y tends to x' contains $B_r(x')$, and because $t < r$, we can find $\alpha > 0$ small enough so that $B_R(y)$ intersects $\mathbb{A}_1 + z - x$ for all $y \in \overline{B}_{\mathbb{A}_1}(x', \alpha)$ and $z \in B_t(x')$. Since B_R is upper semi-continuous, the map $(y, z) \in \overline{B}_{\mathbb{A}_1}(x', \alpha) \times B_t(x') \mapsto B_R(y) \cap (\mathbb{A}_1 + z - x)$ is also upper semi-continuous.

Let us explain how the rest of the proof works in a particular case, before we proceed to the general case. Let us assume that for any $y \in B_{\mathbb{A}_1}(x', \alpha)$, the dimension of $B_R(y)$ is the same as that of $B_R(x')$. Fix $z \in B_t(x')$. Up to taking α even smaller, we may further assume that, for any $y \in B_{\mathbb{A}_1}(x', \alpha)$, the intersection $B_R(y) \cap (\mathbb{A}_1 + z - x)$ is reduced to a singleton that we denote by $\{f(y)\}$. One can check that $y \mapsto B_R(y) \cap (\mathbb{A}_1 + z - x')$ being upper semi-continuous implies that the map f is continuous. Moreover, f is injective since two open faces of Ω intersect if and only if they coincide. We can conclude the proof of Lemma 4.3.4 by using the Invariance of Domain theorem, which says that $f(B_{\mathbb{A}_1}(x', \alpha))$ is a neighbourhood of $z = f(x')$ in $\mathbb{A}_1 + z - x'$.

We go back to the general case. For any open subset $\mathcal{O}$ of an affine space, we denote by $\mathrm{CvxCpt}(\mathcal{O})$ the topological space consisting of non-empty convex compact subsets of $\mathcal{O}$, endowed with the weakest topology making upper semi-continuous maps continuous. We consider the following continuous map:

$$\begin{array}{cccc} f : & \overline{B}_{\mathbb{A}_1}(x', \alpha) \times B_t(x') & \longrightarrow & \mathrm{CvxCpt}(\mathbb{A}_1) \\ & (y, z) & \longmapsto & (B_R(y) - z + x) \cap \mathbb{A}_1. \end{array}$$

Note that by definition of B_R, for all $z \in B_t(x)$ and $y \in \overline{B}_{\mathbb{A}_1}(x', \alpha) \smallsetminus \{x\}$, we have $f(x,z) = \{x\}$ while $x \notin f(y,z)$. Therefore we can consider $0 < \epsilon < \alpha$ such that $\epsilon < d_{\mathbb{A}_1}(x, f(y,z))$ for all $z \in \overline{B}(x)$ and $y \in \partial_{\mathrm{rel}}\overline{B}_{\mathbb{A}_1}(x, \alpha)$.

To conclude the proof of Lemma 4.3.4, it is enough to prove that for any $z \in B_t(x)$,

$$\overline{B}_{\mathbb{A}_1}(x', \epsilon) \subset \bigcup_{y \in \overline{B}_{\mathbb{A}_1}(x', \alpha)} f(y,z).$$

It will be a consequence of the following result, whose proof we postpone until the next section.

Lemma 4.3.5. *Let $\mathcal{O}$ be an open subset of an affine space. Then the map*

$$\begin{array}{rcl} \mathcal{O} & \longrightarrow & \mathrm{CvxCpt}(\mathcal{O}) \\ x & \longmapsto & \{x\} \end{array}$$

is an embedding and a weak homotopy equivalence.

Let us fix $z \in B_t(x')$ and $p \in \overline{B}_{\mathbb{A}_1}(x', \epsilon) \smallsetminus \{x'\}$, and assume by contradiction that p is not in $\bigcup_{y \in \overline{B}_{\mathbb{A}_1}(x', \alpha)} f(y,z)$. Then the continuous map

$$\begin{array}{rcl} \partial_{\mathrm{rel}} B_{\mathbb{A}_1}(x, \epsilon') & \longrightarrow & \mathrm{CvxCpt}(\mathbb{A}_1 \smallsetminus \{p\}) \\ y & \longmapsto & f(y,z) \end{array}$$

is homotopically trivial; it is also homotopic to $y \mapsto f(y, x')$, which is in turn homotopic to $y \mapsto \{y\}$. By Lemma 4.3.5, this means that the inclusion $\partial_{\mathrm{rel}} B_{\mathbb{A}_1}(x', \alpha) \hookrightarrow \mathbb{A}_1 \smallsetminus \{p\}$ is homotopically trivial. This is a contradiction because $p \in B_{\mathbb{A}_1}(x, \alpha)$. ☐

4.3.4 Proof of Lemma 4.3.5

We use the following fact, which is probably well known to experts. We recall its proof for the reader's convenience.

Fact 4.3.6. *Let $p \in Y \subset X$ be a topological space, a subspace and a point. Assume that for any integer $n \geq 0$, for any continuous map $f : [0,1]^n \to X$, there exists a continuous map $H : [0,1]^{n+1} \to X$ such that :*

- *$H(x, 0) = f(x)$ for any $x \in [0,1]^n$;*
- *$H([0,1]^n \times \{1\}) \subset Y$;*
- *for any face $F \subset [0,1]^n$ (i.e. of the form $F = F_1 \times \cdots \times F_n$ with $F_i \in \{[0,1], \{0\}, \{1\}\}$ for each $1 \leq i \leq n$), if $f(F) \subset Y$ (resp. $\{p\}$) then $H(F \times [0,1]) \subset Y$ (resp. $\{p\}$).*

Then the inclusion map $\iota : Y \hookrightarrow X$ is a weak homotopy equivalence.

Proof. Let n be a natural number. Let us prove that $\iota_* : \pi_n(Y, p) \to \pi_n(X, p)$ is surjective. We consider a continuous map $f : [0,1]^n \to X$, we want to prove that it is homotopic to a continuous map $[0,1]^n \to Y$. The homotopy is exactly given by the map $H : [0,1]^{n+1} \to X$ provided by our assumption.

Let us prove that $\iota_* : \pi_n(Y, p) \to \pi_n(X, p)$ is injective. We consider continuous map $f : [0,1]^n \to Y$ and a homotopy $h : [0,1]^{n+1} \to X$ from $f = h_{|[0,1]^n \times \{0\}}$ to $h_{|[0,1]^n \times \{1\}}$ which is constant equal to p. By assumption we can find a continuous map $H : [0,1]^{n+1} \to X$ such that:

- For any $x \in [0,1]^{n+1}$, $H(x,0) = h(x)$.
- $H([0,1]^{n+1} \times \{1\}) \subset Y$.
- For any face $F \subset [0,1]^{n+1}$ (i.e. of the form $F = F_1 \times \cdots \times F_{n+1}$ with $F_i \in \{[0,1],\{0\},\{1\}\}$), if $h(F) \subset Y$ (resp. $\{p\}$) then $H(F \times [0,1]) \subset Y$ (resp. $\{p\}$).

Since $h([0,1]^n \times \{0\}) \subset Y$, this means that $H([0,1]^n \times \{0\} \times [0,1]) \subset Y$. Then f is homotopic in Y to $H_{|[0,1]^n \times \{0\} \times \{1\}}$, which is homotopic in Y to $H_{[0,1]^n \times \{1\} \times \{1\}}$, which is constant equal to p because $h([0,1]^n \times \{1\}) = p$. □

Proof of Lemma 4.3.5. Consider an integer $n \geq 1$ and a continuous map $f : [0,1]^n \to \mathrm{CvxCpt}(\mathcal{O})$. By continuity there exists $N \geq 1$ such that for each $x \in \{0, \frac{1}{N}, \frac{2}{N}, \dots, 1\}^n$, there is a convex compact set $K_x \subset \mathcal{O}$ such that for any $y \in [0,1]^n$, if $\forall i \in \{1,\dots,n\}, |x_i - y_i| \leq \frac{1}{N}$ then $f(y) \subset K_x$. Fix for each $x \in \{0, \frac{1}{N}, \frac{2}{N}, \dots, 1\}^n$ a point $p_x \in K_x$. We define for each $x \in \{0, \frac{1}{N}, \frac{2}{N}, \dots, \frac{N-1}{N}\}^n$, for each $y \in [0,1]^n$ and for each $t \in [0,1]$,

$$H(x + \frac{y}{N}, t) = t \sum_{\epsilon \in \{0,1\}^n} \left(\prod_{1 \leq i \leq n} (1_{\epsilon_i = 1} y_i + 1_{\epsilon_i = 0}(1 - y_i)) \right) p_{x + \frac{\epsilon}{N}} + (1-t) f(x + \frac{y}{N}).$$

And finally we apply Fact 4.3.6. □

4.4 Proof of Theorem 4.1.2

Suppose by contradiction that there exists an open subset $U \subset \partial\Omega$ that does not contain any point of $\Lambda_\Omega^{\mathrm{prox}}$. Take $R > 0$ from Lemma 4.2.1 and fix $o \in \Omega$. By Lemmas 4.3.2 and 4.3.4, we can find $x \in U$ such that, given any compact neighbourhood $A \subset U$ of x, the ball $\overline{B}_{\overline{\Omega}}(x,R)$ is contained in the interior of $B_{\overline{\Omega}}(A, R+1)$ relative to $\partial\Omega$.

By Fact 2.1.10, any accumulation point of $\overline{B}_{\overline{\Omega}}(y,R)$ for the Hausdorff topology, as y tends to x, is contained in $\overline{B}_{\overline{\Omega}}(x,R)$ and hence in the interior of $B_{\overline{\Omega}}(A, R+1)$ relative to $\partial\Omega$.

The stereographic projection $\overline{\Omega} \smallsetminus \{o\} \to \partial\Omega$ sends $\overline{B}_\Omega(y,R)$ onto the closed shadow $\overline{\mathcal{O}}_R(o,y)$ for any $y \in \overline{\Omega} \smallsetminus \overline{B}_\Omega(o,R)$. By continuity of this stereographic projection, for any sequence $(y_n)_n$ in Ω converging to x, the sequence $(\overline{B}_\Omega(y_n,R))_n$ converges for the Hausdorff topology if and only $(\overline{\mathcal{O}}_R(o,y_n))_n$ converges, in which case they have the same limit.

Thus, for any $y \in \Omega$ close enough to x, the open shadow $\mathcal{O}_R(o,y)$ is contained in the interior of $B_{\overline{\Omega}}(A, R+1)$ relative to $\partial\Omega$, which contains no extremal point since A contains no extremal point. Since $\mathcal{O}_R(o,y) \subset \partial\Omega$ is open, it does not contain any point of $\Lambda_\Omega^{\mathrm{prox}}$ (which is the closure of the set of extremal points). This contradicts Lemma 4.2.1.

Part III

The dynamics of the geodesic flow on convex projective orbifolds

Chapter 5

Topological mixing of the geodesic flow on convex projective orbifolds

5.1 Introduction

The main result of this chapter concerns the topological mixing of the geodesic flow on the biproximal unit tangent bundle of a convex projective orbifold. Moreover, we collect some other results on topological recurrence properties of this geodesic flow. Finally, we study the dynamics of the geodesic flow of higher-rank (i.e. not rank-one) compact convex projective orbifolds on its non-wandering set, after observing that the biproximal unit tangent bundle is in this case is empty.

5.1.1 Topological mixing

The main result is the following.

Theorem 5.1.1. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set, Γ a discrete group of automorphisms of Ω, and denote by M the quotient Ω/Γ. Suppose that Γ is either strongly irreducible or non-elementary rank-one. Then the geodesic flow on T^1M_{bip} is topologically mixing.*

This theorem is due to a collaboration with F. Zhu in the case where Γ is non-elementary rank-one. In order to prove Theorem 5.1.1, we establish a more general and abstract result, namely Theorem 5.3.4, which concerns subgroups of $\mathrm{PGL}(\mathbf{V})$ that do not necessarily preserve a properly convex open set.

Let us recall previous results on this topic. Let $M = \Omega/\Gamma$ be a convex projective orbifold. Benoist [Ben04, Th. 1.2] proved that if M is compact and Ω is strictly convex, then the geodesic flow is topologically mixing on T^1M. In this case, $T^1M_{\mathrm{bip}} = T^1M$ by a result of Vey [Vey70, Prop. 3] saying that if M is compact, then $\Lambda_\Gamma^{\mathrm{prox}}$ is the closure of the set of extremal points of Ω.

Bray [Bra20b, Th. 5.7] extended this result to the case where M is compact and 3-dimensional, Γ is strongly irreducible, and Ω is not necessarily strictly convex. For this, Bray used — and this is where the assumption that Ω/Γ is compact and 3-dimensional is crucial — the description of these compact 3-orbifolds by Benoist [Ben06a, Th. 1.1] that we mentionned in Section 0.1.4; Benoist's work, combined in the result of Vey mentioned above, implies that $T^1M = T^1M_{\mathrm{bip}}$ (see Exemple 5.4.5). More generally, we proved in Chapter 4 that if M is compact and rank-one, then $T^1M_{\mathrm{bip}} = \mathrm{NW}(T^1M) = T^1M$.

When Ω is strictly convex, one can see that $T^1M_{\mathrm{bip}} = \mathrm{NW}(T^1M)$ (see [CM14b, §3.3] or Observation 5.4.1), and Crampon–Marquis [CM14b, Prop. 6.1] showed that in this case the geodesic flow is topologically mixing on $\mathrm{NW}(T^1M)$, if $\partial\Omega$ is $\mathcal{C}^1$. Thus, the point of Theorem 5.1.1 is to treat the non-strictly convex case. In this case, T^1M_{bip} is still contained in $\mathrm{NW}(T^1M)$ (Observation 5.4.1), but the inclusion might be strict (see (0.2.1) and the following discussion); the following result asserts that if the inclusion is strict then the geodesic flow on the non-wandering set is not topologically transitive.

Proposition 5.1.2. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set and $\Gamma \subset \mathrm{Aut}(\Omega)$ a discrete subgroup; set $M = \Omega/\Gamma$. If T^1M_{bip} is non-empty, then it is maximal for inclusion among all closed invariant subsets of T^1M on which the geodesic flow is topologically transitive and non-wandering.*

If M is compact and higher-rank, then T^1M_{bip} is empty by Corollary 3.4.6, while $\mathrm{NW}(T^1M)$ is non-empty.

5.1.2 Related results in Riemannian geometry

Let us briefly relate Theorem 5.1.1 to older results for non-positively curved Riemannian manifolds. Let $M = \Omega/\Gamma$ be a convex projective orbifold and X a non-positively curved Riemannian manifold.

Topological transitivity of $(\phi_t)_{t\in\mathbb{R}}$ on $\mathrm{NW}(T^1M)$ when Ω is strictly convex, $\partial\Omega$ is $\mathcal{C}^1$ and $\pi_1(M)$ is non-elementary [CM14b, Prop. 6.1] is analogous to that of $(\phi_t)_{t\in\mathbb{R}}$ on $\mathrm{NW}(T^1X)$ when X is negatively curved and $\pi_1(X)$ is non-elementary, proved by Eberlein [Ebe72, Th. 3.11].

The situation where Ω is not necessarily strictly convex but Γ is rank-one is analogous to X being non-positively curved and rank-one. By work of Ballmann [Bal82, Th. 3.5], if X is rank-one and $\mathrm{NW}(T^1X) = T^1X$ (e.g. if X is rank-one and compact), then $(\phi_t)_{t\in\mathbb{R}}$ is topologically mixing on T^1X. Coudène–Schapira studied the action of $(\phi_t)_{t\in\mathbb{R}}$ on $\mathrm{NW}(T^1X)$ without assuming that $\mathrm{NW}(T^1X) = T^1X$; they established [CS10, Th. 5.2] topological transitivity of $(\phi_t)_{t\in\mathbb{R}}$ on some invariant subset $\mathrm{NW}_1(T^1X)$ of $\mathrm{NW}(T^1X)$, defined in [CS10, §5.1], consisting of rank-one vectors with an extra recurrence condition; this set is analogous to T^1M_{bip} when Γ is rank-one.

In the Riemannian setting, if X is higher-rank (i.e. not rank-one), then $\mathrm{NW}_1(T^1X)$ is empty, while $\mathrm{NW}(T^1X)$ may be non-empty, for example if X is compact. If X is compact and rank-one, then $\mathrm{NW}_1(T^1X)$ is dense in $\mathrm{NW}(T^1X) = T^1X$ [CS10, §5]. However, Coudène–Schapira [CS10, §5.2] constructed an example where X is non-compact and $\mathrm{NW}_1(T^1X)$ is non-empty and not dense in $\mathrm{NW}(T^1X)$.

5.1.3 The higher-rank, irreducible and compact case

When M is compact, higher-rank and irreducible (in the sense that Γ is strongly irreducible), Theorem 5.1.1 does not tell us anything since T^1M_{bip} is empty. However, in this case, the investigation of dynamical properties of the geodesic flow happens to be easier, thanks to recent work of Zimmer [Zim, Th. 1.4], which classifies these orbifolds (this is similar to a classification of compact higher-rank non-positively curved Riemannian manifolds by Ballmann [Bal85, Cor. 1] and Burns–Spatzier [BS87, Th. 5.1]). More precisely, he proves, building on work of Benoist [Ben03], that universal covers in $\mathrm{P}(\mathbf{V})$ of higher-rank irreducible compact convex projective orbifolds belong to a narrow and explicit list of properly convex open sets, called *symmetric* (see Section 5.5). We use this to establish the following.

Proposition 5.1.3. *Let M be a higher-rank irreducible compact convex projective orbifold. Then the non-wandering set of the geodesic flow on T^1M has several (more than one) connected components, and the geodesic flow is topologically mixing on each of them.*

Proposition 5.1.3 is a direct consequence of Proposition 5.5.4, where the connected components of the non-wandering set are described more precisely.

5.2 Reminders on Benoist's work

In this section we recall results of Benoist on asymptotic properties of linear groups, and derive several consequences.

5.2.1 Schottky semi-groups

The definition of Schottky sub-semi-groups of $\mathrm{PGL}(V)$ that we use here is due to Benoist [Ben96, §6.2], building on previous work of Tits [Tit72] and others.

For all $x, y \in \mathrm{P}(\mathbf{V})$, set $d_{\mathrm{P}(\mathbf{V})}(x,y) = \sqrt{1-\langle v,w\rangle^2}$, where $\langle\cdot,\cdot\rangle$ is the canonical scalar product on $\mathbf{V} = \mathbb{R}^{d+1}$, and $v, w \in \mathbf{V}$ are lifts of x, y with norm 1. Identify $\mathbf{V}$ and $\mathbf{V}^*$ via the scalar product on $\mathbf{V}$, and transport to $\mathrm{P}(\mathbf{V}^*)$ the metric of $\mathrm{P}(\mathbf{V})$. For any $x \in \mathrm{P}(\mathbf{V})$ and $A \subset \mathrm{P}(\mathbf{V})$, we set $d_{\mathrm{P}(\mathbf{V})}(x, A) = \inf\{d_{\mathrm{P}(\mathbf{V})}(x,y) : y \in A\}$.

For any $x \in \mathrm{P}(\mathbf{V})$ and $H \in \mathrm{P}(\mathbf{V}^*)$, we have $d_{\mathrm{P}(\mathbf{V})}(x,H) = |\langle x,y\rangle| = d_{\mathrm{P}(\mathbf{V}^*)}(H,x)$, if y is orthogonal to H, and if we see H as a hyperplane of $\mathrm{P}(\mathbf{V})$ and x as a hyperplane of $\mathrm{P}(\mathbf{V}^*)$. Finally, set $d_{\mathrm{P}(\mathbf{V})\times\mathrm{P}(\mathbf{V}^*)}((x,H),(x',H')) = d_{\mathrm{P}(\mathbf{V})}(x,x') + d_{\mathrm{P}(\mathbf{V}^*)}(H,H')$ for all $x, x' \in \mathrm{P}(\mathbf{V})$ and $H, H' \in \mathrm{P}(\mathbf{V}^*)$.

The following uses Notation 2.2.2.

Definition 5.2.1. Consider ϵ, ϵ_0 with $\epsilon < \min(\epsilon_0/2, 1/2)$. An element $g \in \mathrm{PGL}(\mathbf{V})$ is said to be (ϵ,ϵ_0)-*proximal* if it is proximal, if $d_{\mathrm{P}(\mathbf{V})}(x_g^+, y_g^-) \geq \epsilon_0$, and if the restriction of g to $\{x \in \mathrm{P}(\mathbf{V}) : d_{\mathrm{P}(\mathbf{V})}(x, y_g^-) \geq \epsilon\}$ has its image in $\{x : d_{\mathrm{P}(\mathbf{V})}(x, x_g^+) \leq \epsilon\}$ and is ϵ-Lipschitz for $d_{\mathrm{P}(\mathbf{V})}$; if moreover g^{-1} is $(\epsilon, \epsilon_0$-proximal, then g is called (ϵ,ϵ_0)-*biproximal.*

A family of elements $F \subset \mathrm{PGL}(\mathbf{V})$ is said to be (ϵ,ϵ_0)-*Schottky* if each element is (ϵ,ϵ_0)-biproximal and for each $g \neq h \in F$, the distances $d_{\mathrm{P}(\mathbf{V})}(x_g^+, y_h^- \cup y_h^+)$ and $d_{\mathrm{P}(\mathbf{V})}(x_g^-, y_h^- \cup y_h^+)$ are greater than ϵ_0.

Note that the transpose ${}^t g \in \mathrm{PGL}(\mathbf{V}^*)$ (Notation 2.1.8) of a ϵ-proximal element $g \in \mathrm{PGL}(\mathbf{V})$ is also ϵ-proximal.

Fact 5.2.2 ([Ben96, §6])**.**

1. *Let $F \subset \mathrm{PGL}(\mathbf{V})$ be a finite family of biproximal elements such that the elements of $\{f_g^\alpha : g \in F,\ \alpha = \pm\}$ are pairwise transverse. Then there exists $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$, there is $N \geq 0$ such that for any $n \geq N$, the family $\{g^n : g \in F\}$ is (ϵ,ϵ_0)-Schottky.*

2. *Let ϵ, ϵ_0 be such that $0 < \epsilon < \min(\epsilon_0/8, 1/2)$. Let F be a (ϵ,ϵ_0)-Schottky family.*

   - *F generates a discrete free group Γ;*
   - *every non-trivial element of Γ is $(2\epsilon, \epsilon_0 - 2\epsilon)$-biproximal;*
   - *for every element of the form $g = g_1^{n_1} \cdots g_k^{n_k}$, with $g_i \in F$, $n_i > 0$ and $g_i \neq g_{i+1}$ for $i = 1 \dots k$, the distances $d_{\mathrm{P}(\mathbf{V})\times\mathrm{P}(\mathbf{V}^*)}(f_g^+, f_{g_1}^+)$ and $d_{\mathrm{P}(\mathbf{V})\times\mathrm{P}(\mathbf{V}^*)}(f_g^-, f_{g_k}^-)$ are bounded above by 2ϵ;*
   - *for any $g \neq h \in \Gamma \smallsetminus \{\mathrm{id}\}$ the flags $f_g^+, f_g^-, f_h^+, f_h^-$ are pairwise transverse.*

5.2.2 A criterion for strong irreducibility

Lemma 5.2.3. *Let $F \subset \mathrm{GL}(\mathbf{V})$ be a finite family of biproximal elements with $\#F \geq 2$ and such that:*

(a) $\mathrm{Span}\{x_g^{\pm} : g \in F\} = V$,

(b) $\bigcap_{g\in F} x_g^0 = 0$,

(c) the elements of $\{f_g^{\alpha} : g \in F,\ \alpha = \pm\}$ *are pairwise transverse.*

Then the group Γ generated by F acts strongly irreducibly on V.

Proof. Let us first check that the action of Γ on $\mathbf{V}$ is irreducible. Consider a non-empty Γ-invariant subspace $\mathrm{P}(W) \subset \mathrm{P}(\mathbf{V})$ and a point $x \in \mathrm{P}(W)$. Using assumption (b), we can find $g \in F$ such that $x \notin x_g^0$, and then $\alpha = \pm$ such that $x \notin y_g^{-\alpha}$, so that the sequence $(g^{\alpha n} x)_n$ converges to x_g^{α}. This means that $x_g^{\alpha} \in \mathrm{P}(W)$. Similarly, for any $h \in F \smallsetminus \{g\}$ and $\beta = \pm$, because $f_g^{\alpha} \pitchfork f_h^{-\beta}$ (by (c)), we have $x_h^{\beta} = \lim_n h^{\beta n} x_g^{\alpha} \in \mathrm{P}(W)$. Since $\#F \geq 2$ we deduce that $x_g^{-\alpha} \in W$. By (a) this means that $W = \mathbf{V}$.

Now let $\Gamma_1 \subset \Gamma$ be a finite-index subgroup. There exists an integer $N > 0$ such that Γ_1 contains the family $\{g^N : g \in F\}$, which satisfies conditions (a), (b) and (c), and therefore generates an irreducible group. Thus, Γ_1 is irreducible. We have proved that Γ is strongly irreducible. □

5.2.3 Density of the group generated by Jordan projections

A proof of the following result can be found in [CM14b, Prop. 6.5].

Fact 5.2.4 ([Ben00a, Rem. p. 17]). *Let $\Gamma \subset \mathrm{SL}(\mathbf{V})$ be a strongly irreducible discrete subgroup that contains a proximal element. Then the Zariski-closure of Γ in $\mathrm{SL}(\mathbf{V})$ is semi-simple and non-compact.*

The following fact uses the language of semi-simple Lie groups, see e.g. [BQ16, Ch. 6] for definitions.

Fact 5.2.5 ([Ben00b, Prop. p.2]). *Let G be a connected real semi-simple linear Lie group. Let $\mathfrak{a}_G$ be a Cartan subspace of its Lie algebra, let $\mathfrak{a}_G^+ \subset \mathfrak{a}_G$ be a closed Weyl chamber, and let $\lambda_G : G \to \mathfrak{a}_G^+$ be the associated Jordan projection. Let $\Gamma \subset G$ be a Zariski-dense sub-semi-group. Then the additive group generated by $\lambda_G(\Gamma)$ is dense in $\mathfrak{a}_G$.*

Note that, with the convention chosen in Notation 2.2.2, if $\mathfrak{a}_{\mathrm{SL}(V)}^+$ is the set of diagonal matrices in the Lie algebra of $\mathrm{SL}(V)$ with non-increasing diagonal entries, then $\lambda_{\mathrm{SL}(V)}(g)$ is the diagonal matrix with diagonal entries $\log(\lambda_1(g)), \dots, \log(\lambda_{\mathtt{d}+1}(g))$ for any $g \in \mathrm{SL}(V)$; recall also that $\ell(g) = 1/2 \log(\lambda_1(g)/\lambda_{\mathtt{d}+1}(g))$.

Corollary 5.2.6. *Let $\Gamma \subset \mathrm{SL}(\mathbf{V})$ be a sub-semi-group whose Zariski-closure in $\mathrm{SL}(\mathbf{V})$ is irreducible, semi-simple and non-compact. Then*

$$\overline{\langle \ell(\gamma),\ \gamma \in \Gamma \rangle} = \mathbb{R}.$$

Proof. Recall that $V = \mathbb{R}^{\mathtt{d}+1}$ so that $\mathrm{SL}(\mathbf{V})$ identifies with $\mathrm{SL}_{\mathtt{d}+1}(\mathbb{R})$. Let $\mathfrak{a}_{\mathrm{SL}(\mathbf{V})}$ (resp. $\mathfrak{a}_{\mathrm{SL}(\mathbf{V})}^+$) be the set of diagonal matrices (resp. diagonal matrices with non-increasing entries) in the Lie algebra of $\mathrm{SL}(\mathbf{V})$, and denote by $\mathfrak{o} : \mathfrak{a}_{\mathrm{SL}(\mathbf{V})} \to \mathfrak{a}_{\mathrm{SL}(\mathbf{V})}^+$ the reordering of the diagonal entries ($\mathfrak{o} = \lambda_{\mathrm{SL}(\mathbf{V})} \circ \exp$). Denote by $\epsilon_i : \mathfrak{a}_{\mathrm{SL}(\mathbf{V})} \to \mathbb{R}$ the linear form which gives the i-th entry of the diagonal for $i = 1, \dots, d+1$. Denote by G the Zariski-closure of Γ in $\mathrm{SL}(\mathbf{V})$, and $\rho : G \to \mathrm{SL}(\mathbf{V})$ the inclusion. The point is that we can choose a Cartan subspace $\mathfrak{a}_G$ of G such that $\mathrm{d}\rho(\mathfrak{a}_G) \subset \mathfrak{a}_{\mathrm{SL}(\mathbf{V})}$, but we cannot always choose a Weyl chamber $\mathfrak{a}_G^+ \subset \mathfrak{a}_G$ such that $\mathrm{d}\rho(\mathfrak{a}_G^+) \subset \mathfrak{a}_{\mathrm{SL}(\mathbf{V})}^+$. In other words, $\mathfrak{o} \circ \mathrm{d}\rho : \mathfrak{a}_G^+ \to \mathfrak{a}_{\mathrm{SL}(\mathbf{V})}^+$ is not always the restriction of a linear map, and the additive subgroup of $\mathfrak{a}_{\mathrm{SL}(\mathbf{V})}$ generated by $\lambda_{\mathrm{SL}(\mathbf{V})}(\Gamma)$ is not always the image under $\mathrm{d}\rho$ of the additive subgroup of $\mathfrak{a}_G$ generated by $\lambda_G(\Gamma)$.

It happens, however, that, for $i = 1$ (resp. $i = d+1$), the map $\epsilon_i \circ \mathfrak{o} \circ \mathrm{d}\rho : \mathfrak{a}_G^+ \to \mathbb{R}$ is the restriction of a linear form, namely the highest weight χ_+ of the representation ρ of G (resp. the highest weight χ_- of the dual representation in $\mathrm{SL}(\mathbf{V}^*)$). As a consequence, the group generated by $\ell(\Gamma)$ is the image under the linear form $\frac{1}{2}(\chi_+ - \chi_-) \circ \mathrm{d}\rho$ of the subgroup of $\mathfrak{a}_G$ generated by $\lambda_G(\Gamma)$, which is dense by Fact 5.2.5, and because G has finitely many connected components (as a real Lie group). □

Corollary 5.2.7. *Let $g, h \in \mathrm{PGL}(\mathbf{V})$ be two biproximal elements such that $f_g^+, f_g^-, f_h^+, f_h^-$ are pairwise transverse, and S the sub-semi-group generated by g and h. Then $\{\ell(s) : s \in S\}$ generates a dense subgroup of $\mathbb{R}$.*

Proof. Let us prove this by induction on the dimension of $\mathbf{V}$. By Fact 5.2.2.1, we may assume that g and h form an (ϵ, ϵ_0)-Schottky family, for some ϵ, ϵ_0 with $\epsilon < \min(\epsilon_0/8, 1/2)$.

- If $W = \mathrm{Span}\{x_s^\pm : s \in S\}$ is a proper subspace of $\mathbf{V}$, then, using notations from Section 3.2.2, let $\rho : \mathrm{PGL}(\mathbf{V})^W \to \mathrm{PGL}(W)$ be the restriction map. Any image $\rho(s)$ of an element $s \in S$ is biproximal with $\ell(\rho(s)) = \ell(s)$, and $f_{\rho(g)}^+, f_{\rho(g)}^-, f_{\rho(h)}^+, f_{\rho(h)}^-$ are pairwise transverse. Therefore we can apply the induction hypothesis.

- If $W = \bigcap_{s \in S} x_s^0$ is non-trivial, then, using again notations from Section 3.2.2, let $\rho : \mathrm{PGL}(\mathbf{V})^W \to \mathrm{PGL}(\mathbf{V}/W)$ be the natural projection. Any image $\rho(s)$ of an element $s \in S$ is biproximal with $\ell(\rho(s)) = \ell(s)$, and $f_{\rho(g)}^+, f_{\rho(g)}^-, f_{\rho(h)}^+, f_{\rho(h)}^-$ are pairwise transverse. Therefore we can apply the induction hypothesis.

- If $\mathrm{Span}\{x_s^\pm : s \in S\} = V$ and $\bigcap_{s \in S} x_s^0 = 0$, then g and h generate a strongly irreducible group by Lemma 5.2.3. We can apply Fact 5.2.4 and Corollary 5.2.6 to conclude. □

5.3 Topological recurrence properties

Let $M = \Omega/\Gamma$ be a (non-elementary) hyperbolic surface. Thanks to the work of Benoist [Ben96, Ben00a, Ben97, Ben00b], one can reformulate the classical proof of the topological mixing of the geodesic flow on $\mathrm{NW}(T^1M, (\phi_t)_t)$, and decompose it into several steps so that the geodesic flow itself only appears at the very last step, while all the other steps only involve the subgroup $\Gamma \subset \mathrm{PGL}(\mathbf{V})$ and its actions on $\mathrm{P}(\mathbf{V})$ and $\mathrm{P}(\mathbf{V}^*)$. In this section we try to establish each of these steps in the most general context we can think of. For this we use results from Section 2.3.3.

5.3.1 Abundance of closed geodesics

The following is due to Benoist for Γ strongly irreducible: see [Ben00a, Lem. 2.5.3.c] and [Ben97, Lem. 3.6]. Recall that we denote by Γ_0^{Z} the identity component of Γ for the Zariski topology.

Proposition 5.3.1. *Let $\Gamma \subset \mathrm{PGL}(\mathbf{V})$ be a subgroup. Assume that*

- Γ *contains a biproximal element;*
- *for all* $f, f' \in \Lambda_\Gamma^{\mathrm{bip}}$, *there exists* $g \in \Gamma_0^{\mathrm{Z}}$ *such that* gf *is tranverse to* f'.

Then for any non-empty open subset $U \subset (\Lambda_\Gamma^{\mathrm{bip}})^2$, *there exists* $g \in \Gamma$ *biproximal with* $(f_g^-, f_g^+) \in U$.

Proof. By Proposition 3.2.3, we can find non-empty open subsets $U_-, U_+ \subset \Lambda_\Gamma^{\mathrm{bip}}$ such that $U_- \times U_+$ is contained in U and consists of pairs of transverse flags. Let $g, h \in \Gamma$ be biproximal with $f_g^+ \in U_+$ and $f_h^+ \in U_-$. Take $k \in \Gamma_0^{\mathrm{Z}}$ such that $kf_h^- \pitchfork f_g^-$. Since $x_g^+ \notin y_h^+$ and $x_h^+ \notin y_g^+$, the element $\gamma = g^n k h^{-n}$ is biproximal with $(f_\gamma^-, f_\gamma^+) \in U$ for n large enough. □

5.3.2 Non-wandering and topological transitivity

Proposition 5.3.2. *Let* $\Gamma \subset \mathrm{PGL}(\mathbf{V})$ *be a closed subgroup. Assume that*

- Γ *contains a biproximal element;*
- *for all* $f, f' \in \Lambda_\Gamma^{\mathrm{bip}}$, *there exists* $g \in \Gamma_0^{\mathrm{Z}}$ *such that* gf *is tranverse to* f'.

Then the action of Γ *on* $\Lambda_\Gamma^{\mathrm{bip}} \times \Lambda_\Gamma^{\mathrm{bip}}$ *is non-wandering and topologically transitive.*

Proof. For any biproximal element γ of Γ, the point $(f_\gamma^-, f_\gamma^+) \in (\Lambda_\Gamma^{\mathrm{bip}})^2$ is non-wandering since it is fixed by the diverging sequence $(\gamma^n)_n$. By Proposition 5.3.1, such pairs are dense in $(\Lambda_\Gamma^{\mathrm{bip}})^2$, and $\mathrm{NW}((\Lambda_\Gamma^{\mathrm{bip}})^2, \Gamma)$ is closed. Therefore $\mathrm{NW}((\Lambda_\Gamma^{\mathrm{bip}})^2, \Gamma) = (\Lambda_\Gamma^{\mathrm{bip}})^2$, i.e. the action of Γ is non-wandering.

Let U and V be two non-empty open subsets of $(\Lambda_\Gamma^{\mathrm{bip}})^2$. By Proposition 3.2.3, up to reducing U and V, we can assume that for every $(f_1, f_2) \in U$ and $(f_1', f_2') \in V$, the flags f_1, f_2, f_1', f_2' are pairwise transverse. Let $g, h \in \Gamma$ be such that $(f_g^-, f_g^+) \in U$ and $(f_h^+, f_h^-) \in V$, and $f = (f_h^-, f_g^+)$. Then $(g^{-n} f)_n$ and $(h^n f)_n$ respectively converge to (f_g^-, f_g^+) and (f_h^-, f_h^+). As a consequence, $h^n g^n U \cap V$ is non-empty for n large enough. □

5.3.3 Local non-arithmeticity

Proposition 5.3.3. *Let* $\Gamma \subset \mathrm{PGL}(\mathbf{V})$ *be a subgroup. Assume that*

- Γ *contains a biproximal element;*
- *for all* $f, f' \in \Lambda_\Gamma^{\mathrm{bip}}$, *there exists* $g \in \Gamma_0^{\mathrm{Z}}$ *such that* gf *is tranverse to* f'.

Let $U \subset \Lambda_\Gamma^{\mathrm{bip}} \times \Lambda_\Gamma^{\mathrm{bip}}$ *be a non-empty open subset. Then we can find two elements* $g, h \in \Gamma$ *(a Schottky family in the sense of Definition 5.2.1) that generates a sub-semi-group* $S \subset \Gamma$ *consisting of biproximal elements* γ *with* $(f_\gamma^-, f_\gamma^+) \in U$*, and such that* $\ell(S)$ *generates a dense subgroup of* $\mathbb{R}$.

Proof. Let us find a Schottky family with attracting/repelling pair in U. By Proposition 5.3.1, there exists $g \in \Gamma$ biproximal with $(f_g^-, f_g^+) \in U$, and then we can find $h \in \Gamma$ biproximal with $(f_h^-, f_h^+) \in U$ and such that $f_h^+, f_h^-, f_g^-, f_g^+$ are pairwise transverse. Now apply Fact 5.2.2: large powers of g and h form a Schottky family and generate a free sub-semi-group $S \subset \Gamma$ which consists of biproximal elements γ with $(f_\gamma^-, f_\gamma^+) \in U$. To conclude, apply Corollary 5.2.7. □

5.3.4 Topological mixing

We are about to state the topological mixing in a setting which is a bit abstract. However the idea of the proof remains the same as in more classical settings such as for the geodesic flow on hyperbolic surfaces. Figure 5.1 illustrates the proof in the setting of the geodesic flow on a convex projective surface $M = \Omega/\Gamma$.

One crucial ingredient in the following proof is non-arithmeticity of the length spectrum, established in the previous section. The equivalence between non-arithmeticity of the length spectrum and topological mixing of the geodesic flow was established by Dal'bo [Dal00] in the negatively curve Riemannian setting.

Theorem 5.3.4. *Let $\Gamma \subset \mathrm{PGL}(\mathbf{V})$ be a closed subgroup. Assume that*

- *Γ contains a biproximal element;*
- *for all $f, f' \in \Lambda_\Gamma^{\mathrm{bip}}$, there exists $g \in \Gamma_0^{\mathbb{Z}}$ such that gf is tranverse to f'.*

Let $\Lambda = \Lambda_\Gamma^{\mathrm{bip}}$, or its projection in $\mathrm{P}(\mathbf{V})$. Let X be a properly metrisable space with a continuous proper action by Γ, and with a continuous flow $(\phi_t)_t$ without fixed point that commutes with the action of Γ. Assume that $X/(\phi_t)_t$ is isomorphic as a Γ-space to a Γ-invariant open subset $\mathcal{G} \subset \Lambda^2$, denote by $\pi_{\mathbb{R}} : X \to \mathcal{G}$ the natural projection, and assume that for any $\gamma \in \Gamma$ biproximal with $(f_\gamma^-, f_\gamma^+) \in \mathcal{G}$ and for any $x \in \pi_{\mathbb{R}}^{-1}(f_\gamma^-, f_\gamma^+)$, we have $\gamma x = \phi_{\ell(\gamma)} x$. Then the action of $(\phi_t)_t$ on X/Γ is topologically mixing.

Proof. Let us assume that $\Lambda = \Lambda_\Gamma^{\mathrm{bip}}$; the proof of Theorem 5.3.4 in the case where Λ is the projection of $\Lambda_\Gamma^{\mathrm{bip}}$ in $\mathrm{P}(\mathbf{V})$ is similar.

It is enough to prove that for any two non-empty open sets $U, V \subset X/\Gamma$ and for any $\epsilon > 0$, the set $\{t \geq T : \phi_t V \cap U \neq \emptyset\}$ is ϵ-dense in $[T, \infty)$ for some $T \in \mathbb{R}$. Indeed, if we do so, then for any two non-empty open sets $U, V \subset X/\Gamma$, we can find $\epsilon > 0$ and $V' \subset V$ such that $\phi_{[-\epsilon,\epsilon]} V' \subset V$, and then $\{t : \phi_t V \cap U \neq \emptyset\}$ contains the ϵ-neighbourhood of $\{t : \phi_t V' \cap U \neq \emptyset\}$.

Let $U, V \subset X$ be open, relatively compact and non-empty, let $0 < \epsilon < 1/2$, and let us show that $\{t : \phi_t V \cap \Gamma \cdot U \neq \emptyset\}$ is 5ϵ-dense in $[T, \infty)$ for some T.

Observe that $\mathcal{G}$ is an open dense subset of Λ^2 by topological transitivity of the action of Γ (Proposition 5.3.2). Using this and Proposition 3.2.3, and up to reducing U and V, we can assume that for any $(f_-, f_+) \in \pi_{\mathbb{R}} U$ and $(f'_-, f'_+) \in \pi_{\mathbb{R}} V$, the flags f_-, f_+, f'_-, f'_+ are pairwise transverse and $(f'_-, f_+) \in \mathcal{G}$.

By Proposition 5.3.1, we can find $g \in \Gamma$ such that $(f_g^-, f_g^+) \in \pi_{\mathbb{R}} U$ and $\ell(g) \geq 1$. By Proposition 5.3.3 and Observation 5.3.5, we can find $h \in \Gamma$ such that $(f_h^-, f_h^+) \in \pi_{\mathbb{R}} V$ and $\ell(g)\mathbb{Z} + \ell(h)\mathbb{Z}$ is ϵ-dense in $\mathbb{R}$. Let $N \geq 0$ be such that $\{\ell(g)n + \ell(h)m : |n|, |m| \leq N\}$ is 2ϵ-dense in $[0, \ell(g)]$.

There exists a basis of neighbourhoods U_- of f_g^- such that $g^{-1}U_- \subset U_-$. Using this, one can check that there exists a neighbourhood U_- (resp. V_+) of f_g^- (resp. f_h^+) such that $g^{-1}U_- \subset U_-$ (resp. $hU_+ \subset U_+$) and such that for any $0 \leq n \leq 2N$,

$$g^{-n}\phi_{n\ell(g)}\left(U \cap \pi_{\mathbb{R}}^{-1}(U_- \times \{f_g^+\})\right) \subset \phi_{[-\epsilon,\epsilon]} U \tag{5.3.1}$$

$$h^{n}\phi_{-n\ell(h)}\left(V \cap \pi_{\mathbb{R}}^{-1}(\{f_h^-\} \times U_-)\right) \subset \phi_{[-\epsilon,\epsilon]} V \tag{5.3.2}$$

Consider $x \in \pi_{\mathbb{R}}^{-1}(f_g^-, f_g^+) \cap U$, and $y \in \pi_{\mathbb{R}}^{-1}(f_h^-, f_h^+) \cap V$, and $z \in \pi_{\mathbb{R}}^{-1}(f_h^-, f_g^+)$. Let $k \geq 0$ (resp. $m \geq 0$) be such that $g^{-k} f_h^- \in U_-$ (resp. $h^m f_g^+ \in V_+$), and $\tau \in \mathbb{R}$ (resp. σ) such that $\phi_\tau g^{-k} z \in U$ (resp. $\phi_\sigma h^m z \in V$).

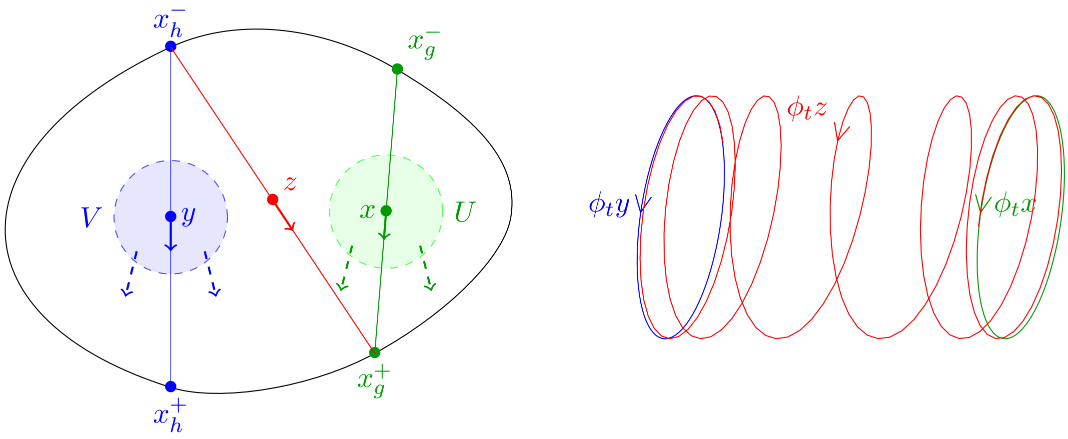

Figure 5.1: Proof of mixing. On the left: in Ω. On the right: in the quotient $M = \Omega/\Gamma$.

Let

$$\mathcal{T} = \{t \geq 0 : \phi_{t+\tau} g^{-k} z \in g^{\mathbb{N}} U\} \quad \text{and} \quad \mathcal{S} = \{s \geq 0 : \phi_{-s+\sigma} h^m z \in h^{-\mathbb{N}} V\}.$$

Then

$$\{t + s + \tau - \sigma : t \in \mathcal{T}, s \in \mathcal{S}\} \subset \{t : \phi_t V \cap \Gamma \cdot U \neq \emptyset\},$$

therefore to conclude the proof it is enough to prove that $\mathcal{T} + \mathcal{S} := \{t + s : t \in \mathcal{T}, s \in \mathcal{S}\}$ is 5ϵ-dense in $[T, \infty)$ for some T.

By definition of τ and σ, zero belongs in $\mathcal{T}$ and $\mathcal{S}$. According to (5.3.1) (resp. (5.3.2)), for any $t \in \mathcal{T}$ (resp. $s \in \mathcal{S}$) and any $0 \leq n \leq 2N$, we can find $\alpha_{t,n}$ (resp. $\beta_{s,n}$) with norm less than ϵ such that $t + n\ell(g) + \alpha_{t,n} \in \mathcal{T}$ (resp. $s + n\ell(h) + \beta_{s,n} \in \mathcal{S}$); by definition of N, this implies that $[t+s+T, t+s+T+\ell(g)]$ is contained in the 4ϵ-neighbourhood of $\mathcal{T}+\mathcal{S}$, where $T := N(\ell(g) + \ell(h))$. Let us conclude by proving that $\mathcal{T} + \mathcal{S}$ is 5ϵ-dense in $[T, \infty)$.

Define $(t_n)_n$ in $\mathcal{T}$ by induction: $t_0 = 0$ and $t_{n+1} = t_n + \ell(g) + \alpha_{t_n,1}$ for any $n \geq 0$, so that the segments $[t_n+T, t_n+T+\ell(g)+\epsilon]$ and $[t_{n+1}+T, t_{n+1}+T+\ell(g)+\epsilon]$ have non-empty intersection. Then the union $\bigcup_{n\geq 0}[t_n + T, t_n + T + \ell(g) + \epsilon]$ is connected. Moreover, it contains T and the sequence $(t_n + T)_n$, which goes to infinity (since $\ell(g) \geq 1/2 + \epsilon$). Thus, this union contains $[T, \infty)$, while it is contained in the 5ϵ-neighbourhood of $\mathcal{T} + \mathcal{S}$. □

Observation 5.3.5. *Let A be a subset of $\mathbb{R}$ which generates a dense additive subgroup G of $\mathbb{R}$. Let $x, \epsilon > 0$. Then there exists $g \in A$ such that $x\mathbb{Z} + g\mathbb{Z}$ is ϵ-dense in $\mathbb{R}$ (i.e. any point in $\mathbb{R}$ is at distance at most ϵ from $x\mathbb{Z} + g\mathbb{Z}$).*

Proof. Up to replacing A by A/x and ϵ by ϵ/x, we can assume that $x = 1$. Then there are two possibilities.

- The set A contains an irrational number g. Then $\mathbb{Z} + g\mathbb{Z}$ is dense in $\mathbb{R}$.
- The set A is contained in $\mathbb{Q}$. Let $q_0 \in \mathbb{N}^*$ be such that $\frac{1}{q_0} < \epsilon$. The subgroup $\frac{1}{q_0!}\mathbb{Z}$ is not dense in $\mathbb{R}$, so A must contain an element outside of it, of the form $\frac{p}{q}$ with p and q coprime and $q > q_0$. The group $\mathbb{Z} + \frac{p}{q}\mathbb{Z} = \frac{1}{q}\mathbb{Z}$ is ϵ-dense in $\mathbb{R}$. □

5.4 Consequences on the geodesic flow of convex projective orbifolds and further results

In this section we interpret the results of the previous section for the geodesic flow on convex projective orbifolds, and we give additional results, which can be used, among other things, to give another proof of the topological mixing for the geodesic flow on the biproximal unit tangent bundle, which is more in the spirit of the theory of Anosov flows, or of [CS10].

First observe that Theorem 5.1.1 is an immediate consequence of Theorem 5.3.4 with $\Lambda = \Lambda_\Gamma^{\mathrm{prox}}$, with $X = T^1M$, with $(\phi_t)_t$ being the geodesic flow on T^1M, and with $\mathcal{G} = \mathrm{Geod}(\Omega) \cap \Lambda^2$. Indeed, let us check that the assumptions of Theorem 5.3.4 are verified.

The group Γ contains a biproximal element by Fact 2.3.2 in the stronly irreducible case, and by definition of the rank-one property (Definition 3.1.1) in the rank-one case. The second assumption is obviously verified in the strongly irreducible case, and is a consequence of Lemma 3.2.4 in the rank-one case. The fact that $\Lambda_\Gamma^{\mathrm{bip}}$ projects onto $\Lambda_\Gamma^{\mathrm{prox}}$ is due to Fact 2.3.9 in the strongly irreducible case, and to Proposition 3.2.2 in the rank-one case. The fact that $\gamma v = \phi_{\ell(\gamma)} v$ for any biproximal γ and any v tangent to (x_γ^-, x_γ^+) is a consequence of Fact 2.2.8.

5.4.1 The non-wandering set for convex projective orbifolds

Observation 5.4.1. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set, let Γ be a discrete group of automorphisms of Ω, and denote by M the quotient Ω/Γ. Then $\mathrm{NW}(T^1M, (\phi_t)_{t\in\mathbb{R}}) \subset T^1M_{\mathrm{cor}}$. If moreover Γ is strongly irreducible or non-elementary rank-one, then $T^1M_{\mathrm{bip}} \subset \mathrm{NW}(T^1M, (\phi_t)_t)$.*

Proof. Consider a non-wandering vector $v \in T^1\Omega$ for the action of $\Gamma \times \mathbb{R}$. Let x be the footpoint of v. We want to show that $\phi_\infty v$ is an accumulation point of $\Gamma \cdot x$. Since v is non-wandering, we can find sequences of vectors $(v_n)_n$ in $T^1\Omega$ converging to v, of positive times $(t_n)_n$ going to infinity, and of automorphisms $(\gamma_n)_n$ in Γ such that $(d_{T^1\Omega}(\phi_{t_n} v_n, \gamma_n v))_n$ tends to zero. Since $(v_n)_n$ tends to v and $(t_n)_n$ goes to infinity, $(\pi\phi_{t_n} v_n)_n$ must converge to $\phi_\infty v$. By Section 2.1.2, the fact that $(d_\Omega(\pi\phi_{t_n} v_n, \gamma_n x))_n$ tends to zero implies that $(\gamma_n x)_n$ also converges to $\phi_\infty v$ in $\mathrm{P}(\mathbf{V})$.

If moreover Γ is strongly irreducible or non-elementary rank-one, the fact that T^1M_{bip} is contained in $\mathrm{NW}(T^1M, (\phi_t)_t)$ is an immediate consequence of Proposition 5.3.2. ☐

5.4.2 A criterion for being rank-one

Proposition 5.3.1 yields a criterion for a convex projective orbifold with strongly irreducible fundamental group to be rank-one.

Lemma 5.4.2. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be properly convex and open, and $\Gamma \subset \mathrm{Aut}(\Omega)$ a strongly irreducible subgroup. Then Γ is rank-one if and only if $\Lambda_\Gamma^{\mathrm{prox}}$ contains two points at simplicial distance at least* 3.

Proof. If Γ is rank-one, then the attracting/repelling pair of any rank-one element does the job. Conversely, if there exists $\xi, \eta \in \Lambda_\Gamma^{\mathrm{prox}}$ with $d_{\mathrm{spl}}(\xi, \eta) > 2$, then by lower semi-continuity of d_{spl} and by Proposition 5.3.1 (and Fact 2.3.2), there exists $\gamma \in \Gamma$ biproximal such that $d_{\mathrm{spl}}(x_\gamma^+, x_\gamma^-) > 2$. By Lemma 3.1.3, γ is rank-one. ☐

Using the previous result, combined with the recent work of Islam–Zimmer, we can see that many convex cocompact actions are rank-one. Recall that a properly embedded simplex (PES) in a properly convex open set $\Omega \subset \mathrm{P}(\mathbf{V})$ is a simplex $S \subset \overline{\Omega}$ whose relative interior is $S \cap \partial\Omega$.

Fact 5.4.3 ([IZ, Th.1.7 & 1.8])**.** *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set and $\Gamma \subset \mathrm{Aut}(\Omega)$ a strongly irreducible discrete subgroup. Assume that Γ acts convex cocompactly on Ω, and that Γ is relatively hyperbolic with respect to a collection of virtually abelian subgroups of rank at least two. Then the set of maximal (for inclusion) PES of Ω which are contained in the convex core C_Γ^{cor} is discrete and closed as a subset of the space of compact subsets of $\mathrm{P}(\mathbf{V})$ endowed with the Hausdorff topology. Moreover, two differents maximal PES have disjoint relative boundary, and any non-trivial segment in $\Lambda_\Gamma^{\mathrm{orb}}$ is contained in a PES.*

Fact 5.4.4 ([Isl, Prop. A.2])**.** *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set and $\Gamma \subset \mathrm{Aut}(\Omega)$ a strongly irreducible discrete subgroup. Assume that Γ acts convex cocompactly on Ω and Ω^*, and that Γ is relatively hyperbolic with respect to a collection of virtually abelian subgroups of rank at least two. Then $M = \Omega/\Gamma$ is rank-one.*

Proof. By Fact 5.4.3, any non-trivial segment of $\Lambda_\Gamma^{\mathrm{orb}}$ is contained in a PES, and that the relative boundaries of two distinct maximal PES are disjoint. This means, since $\overline{\Omega}\smallsetminus\Lambda_\Gamma^{\mathrm{orb}}$ has bisaturated boundary by Fact 2.3.16, that two distinct points of $\Lambda_\Gamma^{\mathrm{orb}}$ are at finite simplicial distance if and only if they lie in the same PES. Consider $\xi \in \Lambda_\Gamma^{\mathrm{prox}}$. By irreducibility, we can find $\gamma \in \Gamma$ such that ξ and $\gamma\xi$ do not belong to the same PES, and hence are at infinite simplicial distance. We conclude using Lemma 5.4.2. □

Exemple 5.4.5 below is a more concrete application of the Lemma 5.4.2, whose proof is exactly the same as the previous one, except that we use Benoist's work [Ben06a] instead of Islam–Zimmer's. It can also be seen as a consequence of Fait 5.4.4 and [Ben06a], or as a consequence of Zimmer's higher-rank rigidity [Zim, Th. 1.4].

Example 5.4.5. *Any 3-dimensional irreducible compact convex projective orbifold is rank-one.*

Proof. Let $M = \Omega/\Gamma$ be an irreducible compact convex projective orbifold of dimension 3. Benoist proved [Ben06a, Th. 1.1] that any segment of $\partial\Omega$ is contained in a properly embedded triangle (PET), and that two distinct PETs are disjoint. This means that two distinct points of $\partial\Omega$ are at finite simplicial distance if and only if they lie in the same PET. Consider $\xi \in \Lambda_\Gamma^{\mathrm{prox}}$. By irreducibility, we can find $\gamma \in \Gamma$ such that ξ and $\gamma\xi$ do not belong to the same PET, and hence are at infinite simplicial distance. We conclude using Lemma 5.4.2. □

5.4.3 Recurrent vectors of convex projective orbifolds

The following lemma can be seen as a generalisation of Lemma 3.1.3.(2). It is inspired by an analogous result of Knieper [Kni98, Prop. 4.1] in the non-positively curved Riemannian setting. Recall the definition of $\partial_{\mathrm{sse}}\Omega$ and $\partial_{\mathrm{sing}}\Omega$ from Section 2.1.8.

Lemma 5.4.6. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set, $\Gamma \subset \mathrm{Aut}(\Omega)$ a closed subgroup, and $v \in T^1\Omega$. If v is recurrent under the action of $\Gamma \times \mathbb{R}$ and $d_{\mathrm{spl}}(\phi_\infty v, \phi_{-\infty} v) > 2$, then $\phi_{\pm\infty} v \in \partial_{\mathrm{sse}}\Omega$.*

As a consequence, if Γ is non-elementary rank-one, then the subset $\Lambda_\Gamma^{\mathrm{prox}} \cap \partial_{\mathrm{sse}}\Omega \subset \Lambda_\Gamma^{\mathrm{prox}}$ contains a G_δ-dense set.

To prove it, we will need the following result, which is a particular case of a more general theorem of Benzécri.

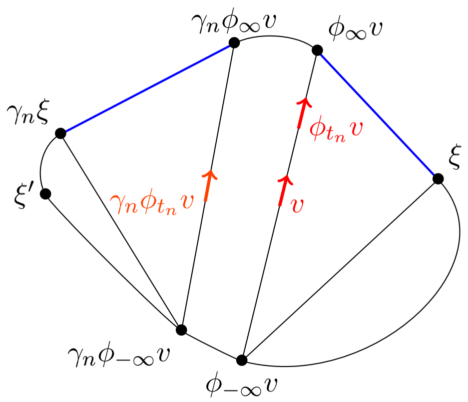

Figure 5.2: Illustration of the second part of the proof of Lemma 5.4.6

Fact 5.4.7 ([Ben60, Prop. 5.3.9]). *Suppose that* $\dim(\mathbf{V}) = 3$. *Let* $(x, \Omega) \in \mathcal{E}^{\bullet}_{\mathbf{V}}$ *and* $\xi \in \partial_{\text{sing}}\Omega$. *Let* $T \in \mathcal{E}_{\mathbf{V}}$ *be a triangle and* $o \in T$. *Then we have the following convergence in* $\mathcal{E}^{\bullet}_{\mathbf{V}}/\mathrm{PGL}(\mathbf{V})$.

$$[y, \Omega] \underset{\substack{y \to \xi \\ y \in [x,\xi)}}{\longrightarrow} [o, T].$$

Proof. We briefly recall the proof of this fact, which is very easy in this particular case. Consider a projective line $\mathrm{P}(W)$ that does not intersect $[x, \xi]$, and for each $y \in [x, \xi)$, denote by $g_y \in \mathrm{PGL}(\mathbf{V})$ the unique element that fixes ξ and $\mathrm{P}(W)$, and such that $g_y y = x$. Let $p : \mathrm{P}(\mathbf{V}) \smallsetminus \{\xi\} \to \mathrm{P}(W)$ be the stereographic projection. Since ξ is a singular point of $\partial\Omega$, the image $p(\Omega)$ is a properly convex open subset of $\mathrm{P}(W)$, i.e. the interior of a segment. As y tends to ξ, the properly convex open set $g_y\Omega$ converges to the convex hull of $p(\Omega)$ and ξ, which is a triangle containing x. □

Proof of Lemma 5.4.6. Since v is recurrent, there exist diverging sequences $(t_n)_n$ and $(\gamma_n)_n$ in Γ such that $(\gamma_n\phi_{t_n}v)_n$ tends to v. Up to flipping v, we may assume that $t_n > 0$ for any n.

Suppose by contradiction that $\phi_\infty v$ is singular. Then there exists a projective plane $\mathrm{P}(W) \subset \mathrm{P}(\mathbf{V})$ which contains v and such that ξ is a singular point of $\partial\Omega \cap \mathrm{P}(W)$. Up to extraction, we can assume that $(\gamma_n\mathrm{P}(W))_n$ converges to some $\mathrm{P}(W')$. By construction, $(\gamma_n\pi\phi_{t_n}v, \Omega \cap \gamma_n\mathrm{P}(W))_n$ converges to $(\pi v, \Omega \cap \mathrm{P}(W'))$. Since $\phi_\infty v$ is singular, we can apply Fact 5.4.7, and we obtain that $\Omega \cap \mathrm{P}(W')$ is a triangle that contains $\phi_{\pm\infty}v$, hence $d_{\text{spl}}(\phi_{-\infty}v, \phi_\infty v) \leq 2$, which is a contradiction.

Suppose by contradiction that there exists $\xi \in \partial\Omega \smallsetminus \{\phi_\infty v\}$ such that $[\xi, \phi_\infty v] \subset \partial\Omega$. We can take ξ extremal. Up to extraction we can assume that $(\gamma_n\xi)_n$ converges to some $\xi' \in \partial\Omega$. Observe that $[\phi_\infty v, \xi'] \subset \partial\Omega$, since it is the limit of the sequence of segments $([\gamma_n\phi_\infty v, \gamma_n\xi])_n$ that are contained in the boundary (see Figure 5.2). Since ξ is extremal, the Hilbert distance of $\pi\phi_{t_n}v$ to $[\phi_{-\infty}v, \xi] \cap \Omega$ tends to infinity with n; this implies that $[\phi_{-\infty}v, \xi'] \subset \partial\Omega$. Thus $d_{\text{spl}}(\phi_{-\infty}v, \phi_\infty v) \leq 2$, which is a contradiction.

Let us assume that Γ is non-elementary rank-one. Let $T^1\Omega_{\text{bip}} \subset T^1\Omega$ be the set of vectors v with $\phi_\infty v$ and $\phi_{-\infty}v$ in $\Lambda^{\text{prox}}_{\Gamma}$, let $A \subset T^1\Omega$ be the set of vectors v with $d_{\text{spl}}(\phi_{-\infty}v, \phi_\infty v) > 2$, and let $B \subset T^1\Omega_{\text{bip}}$ be the set of recurrent vectors for the action of $\Gamma \times \mathbb{R}$. By Proposition 5.3.2 and Fact 1.1.4, the action of $\Gamma \times \mathbb{R}$ on $T^1\Omega_{\text{bip}}$ is non-wandering and topologically transitive. $A \subset T^1\Omega_{\text{bip}}$ is open, Γ-invariant and non-empty (since Γ is rank-one), hence dense by topological transitivity. By Fact 1.1.1, the subset $B \subset T^1\Omega_{\text{bip}}$ contains a G_δ-dense set. Thus, $A \cap B \subset T^1\Omega_{\text{bip}}$ contains a G_δ-dense set. On the one hand,

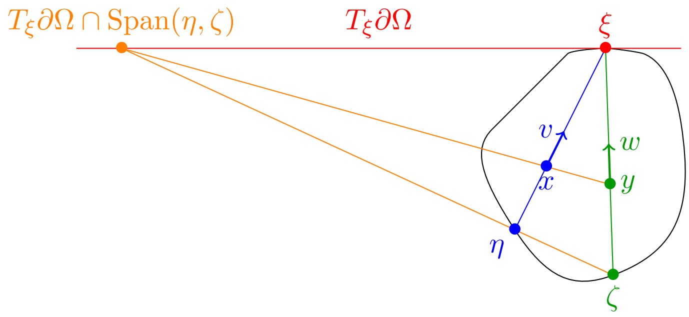

Figure 5.3: Vectors $v, w \in T^1\Omega$ in the same strong stable manifold, with footpoints x and y on the same horosphere

the map $\phi_\infty : T^1\Omega_{\mathrm{bip}} \to \Lambda_\Gamma^{\mathrm{prox}}$ that sends v to $\phi_\infty v$ is continuous, open and surjective, hence $\phi_\infty(A\cap B)$ contains a G_δ-dense set. On the other hand, the first part of Lemma 5.4.6 implies that $\phi_\infty(A \cap B) \subset \partial_{\mathrm{sse}}\Omega \cap \Lambda_\Gamma^{\mathrm{prox}}$. This concludes the proof. □

5.4.4 Ingredients for another proof of topological mixing

In this section, we collect three results that can be used to give a proof of Theorem 5.1.1 which focuses more on the geometrical properties of properly convex open set equipped with their Hilbert metric. More precisely, one can perform the same proof as Bray's [Bra20b, §5] in the particular case of Benoist manifolds (see Section 0.1.4).

In particular, we establish a specification property for the geodesic flow; see [CS10] for precise and general statements on this property.

Strong stable manifolds in convex projective geometry

The goal of this section is to establish the following geometrical description of the strong stable manifolds (in the sense of Section 1.2.4) of the unit tangent bundle $T^1\Omega$ of a properly convex open set Ω. Let us first recall the definition of horospheres of Ω centred at $\mathcal{C}^1$ points of $\partial\Omega$ (see Sections 1.4.1 and 6.1 for more details on horospheres).

Let $\xi \in \partial\Omega$ be $\mathcal{C}^1$ and $x \in \Omega$, take $\eta \in \partial\Omega$ such that $x \in [\eta, \xi]$. The *horosphere* centred at ξ and passing through x is the image of the map that sends $\zeta \in \partial\Omega \smallsetminus T_\xi\partial\Omega$ to the intersection point of $[\xi, \zeta]$ with $\mathrm{Span}(x, T_\xi\partial\Omega \cap \mathrm{Span}(\eta, \zeta))$; see Figure 5.3. Note that this map is the restriction of a projective transformation that fixes every point of $T_\xi\partial\Omega$.

Proposition 5.4.8. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set and $v, w \in T^1\Omega$; denote by $\mathrm{P}(W) \subset \mathrm{P}(\mathbf{V})$ the smallest subspace containing v and w (it has dimension 1 or 2 or 3), and $\Omega' = \Omega \cap \mathrm{P}(W)$. Then $(d_{T^1\Omega}(\phi_t v, \phi_t w))_t$ converges to zero as t tends to infinity if and only if $\phi_\infty v = \phi_\infty w$, if this point is a $\mathcal{C}^1$ point of $\partial\Omega'$, and if πv and πw belong to the same horosphere of Ω' centred at ξ.*

Proposition 5.4.8 is a consequence of the following more explicit result. See an illustration for the notation in Figure 5.4.

Lemma 5.4.9. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set and $v, w \in T^1\Omega$ with $\phi_\infty v = \phi_\infty w =: \xi$ and $\phi_{-\infty} v \neq \phi_{-\infty} w$. Let $\alpha, \beta \in \phi_{-\infty} v \oplus \phi_{-\infty} w$ be such that $\alpha \oplus \xi$ and $\beta \oplus \xi$ are tangent to $\partial\Omega$ at ξ and $\alpha, \phi_{-\infty} v, \phi_{-\infty} w, \beta$ are aligned in this order (α and β may coincide). Let c be the intersection point of $\pi v \oplus \pi w$ and $\phi_{-\infty} v \oplus \phi_{-\infty} w$. If $c \oplus \xi$ does not intersect*

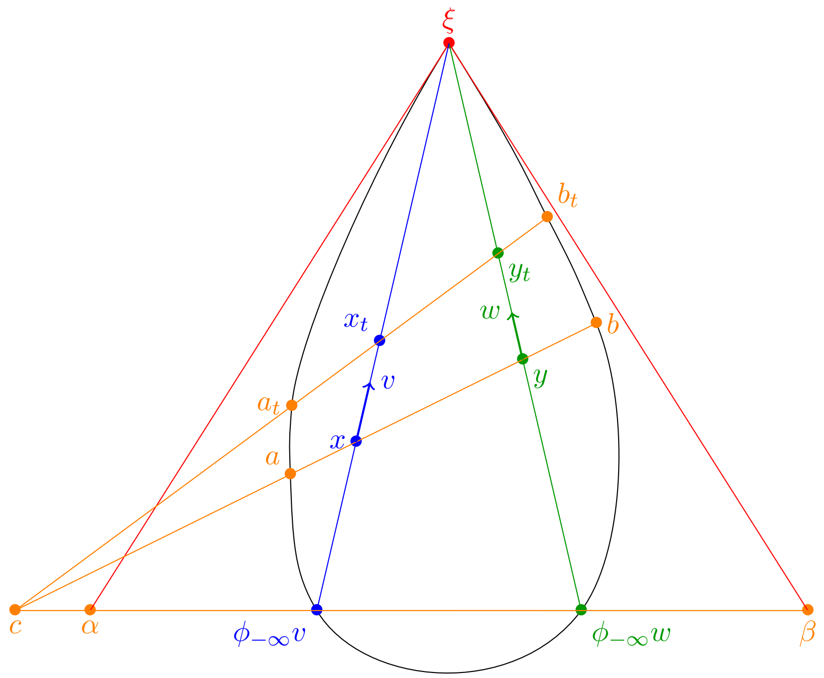

Figure 5.4: Illustration for the notation in Lemma 5.4.9

Ω, then $(d_{T^1\Omega}(\phi_t v, \phi_t w))_t$ converges, as t tends to infinity, to

$$\max\left(\frac{1}{2}\log[\alpha, \phi_{-\infty}v, \phi_{-\infty}w, \beta],\ \frac{1}{2}\log[c, \phi_{-\infty}v, \phi_{-\infty}w, \beta],\ \frac{1}{2}\log[\alpha, \phi_{-\infty}v, \phi_{-\infty}w, c]\right),$$

with the convention on the cross ratio that $[\infty, 0, 1, \infty] = 1$ in $\mathrm{P}(\mathbb{R}^2) = \mathbb{R} \cup \{\infty\}$.

Proof. We consider $x_t = \pi\phi_t v$ and $y_t = \pi\phi_t w$, and $x_0 = x$ and $y = y_0$. Since $d_\Omega(x, x_t) = t = d_\Omega(y, y_t)$ and by definition of the cross-ratio, we see that $y_t \in (y \oplus \xi) \cap (c \oplus x_t)$. Let $a_t, b_t \in \partial\Omega$ be such that the four points a_t, x_t, y_t, b_t are aligned in this order. Let α_t (resp. β_t) be the intersection point of $\phi_{-\infty}v \oplus \phi_{-\infty}w$ and $\xi \oplus a_t$ (resp. $\xi \oplus b_t$). By definition of the Hilbert metric and by Lemma 2.1.6,

$$d_{T^1\Omega}(\phi_t v, \phi_t w) = d_\Omega(x_t, y_t) = \frac{1}{2}\log[\alpha_t, \phi_{-\infty}v, \phi_{-\infty}w, \beta_t].$$

In conclusion, there are three possibilities:

- if $c, \alpha, \phi_{-\infty}v, \phi_{-\infty}w, \beta$ are in this order, then α_t (resp. β_t) converges to α (resp. β);
- if $\alpha, c, \phi_{-\infty}v, \phi_{-\infty}w, \beta$ are in this order, then α_t (resp. β_t) converges to c (resp. β);
- if $\alpha, \phi_{-\infty}v, \phi_{-\infty}w, c, \beta$ are in this order, then α_t (resp. β_t) converges to α (resp. c). □

Proof of Proposition 5.4.8. If $\phi_\infty v = \phi_\infty w$, if this point is a $\mathcal{C}^1$ point of $\partial\Omega'$, and if πv and πw belong to the same horosphere of Ω' centred at ξ, then the fact that $d_{T^1\Omega}(\phi_t v, \phi_t w)$ goes to zero as t goes to infinity is an immediate corollary of Lemma 5.4.9 (note that in this case $c = \alpha = \beta$ with the notation of Lemma 5.4.9).

If $d_{T^1\Omega}(\phi_t v, \phi_t w)$ goes to zero as t goes to infinity, then $d_\Omega(\pi\phi_t v, \pi\phi_t w)$ also goes to zero, as well as $d_{\mathrm{P}(\mathbf{V})}(\pi\phi_t v, \pi\phi_t w)$ by Section 2.1.2, whence $\phi_\infty v = \lim_{t\to\infty} \pi\phi_t v = \phi_\infty w$. Using the notations of Lemma 5.4.9, we have

$$0 = \frac{1}{2}\log[\alpha, \phi_{-\infty}v, \phi_{-\infty}w, \beta] = \frac{1}{2}\log[c, \phi_{-\infty}v, \phi_{-\infty}w, \beta] = \frac{1}{2}\log[\alpha, \phi_{-\infty}v, \phi_{-\infty}w, c],$$

therefore $c = \alpha = \beta$, and ξ is a smooth point of $\partial\Omega'$, and πv and πw belong to the same horosphere centred at ξ. □

A Weak orbit-glueing lemma

The idea of the following lemma is that if c is a (not necessarily continuous) concatenation of a finite sequence of straight geodesic segments $c_1, \dots, c_n$ such that the end of c_i is sufficiently close to the starting point of c_{i+1} for $i = 1, \dots, n-1$, then one can find one long straight geodesic that stays close to c.

Recall that in hyperbolic geometry, this fact holds for concatenations of an infinite number of geodesics (as long as the length of all geodesics is bounded below by some constant ϵ). This is not true for general properly convex open sets, for instance in the triangle.

Lemma 5.4.10. *For any $\eta > 0$ and $n > 0$ integer, there is $\epsilon > 0$ such that the following holds. Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set, and $v_1, \cdots, v_n \in T^1\Omega$, $t_1, \dots, t_n \in \mathbb{R}_{\geq 0}$. If $d_{T^1\Omega}(\phi_{t_i} v_i, v_{i+1}) \leq \epsilon$ for any $1 \leq i < n$, then there exists $v \in T^1\Omega$ such that for any $1 \leq i < n$ and any $t \in [0, t_i]$,*

$$d_{T^1\Omega}(\phi_{t_1+\cdots+t_{i-1}+t} v, \phi_t v_i) \leq \eta.$$

Proof. The proof is an induction on $n \geq 2$ (the case $n = 1$ is trivial).

Step 1 : Show Lemma for $n = 2$ and $\pi\phi_{t_1} v_1 = \pi v_2$. Let $\eta > 0$. It is enough find $\epsilon > 0$ so that for all $v \neq v' \in T^1\Omega$ and $t, t' > 2$, if $d_{T^1\Omega}(v, v') \leq \epsilon$ and if $\pi v = \pi v' = x$ then there is $u \in T^1\Omega$ such that $\pi u = x$ and $d_{T^1\Omega}(\phi_s u, \phi_s v) \leq \eta$ for any $0 \leq s \leq t$ and $d_{T^1\Omega}(\phi_{-s} u, \phi_{-s} v') \leq \eta$ for any $0 \leq s \leq t'$. We can assume without loss of generality that $\mathbf{V}$ has dimension 3.

Denote by a, b the points of $\partial\Omega$ such that $a, \pi\phi_1 v, \pi\phi_1 v', b$ are aligned in this order. Let Ω' be the interior of the convex hull of the points $\phi_\infty v$, $\phi_\infty v'$, b, $\phi_{-\infty} v$, $\phi_{-\infty} v'$ and a, so that $\Omega' \subset \Omega$ and $d_\Omega(y_1, y_2) \leq d_{\Omega'}(y_1, y_2)$ for any $y_1, y_2 \in \Omega'$. Consider an affine chart so that $x = (0, 0)$ and $\phi_{\pm\infty} v = (\pm 1, 0)$ and $\phi_{\pm\infty} v' = (0, \pm 1)$. The shape of Ω' depends only on the position of a and b. The smaller $d_{\Omega'}(\pi\phi_1 v, \pi\phi_1 v') = d_{T^1\Omega'}(v, v')$ (this equality is a consequence of Lemma 2.1.4), the further a and b. See Figure 5.5.

Suppose $d_{\Omega'}(\pi\phi_1 v, \pi\phi_1 v') < 1$, so that the segments $[\phi_{-\infty} v', \phi_\infty v]$ and $[\phi_{-\infty} v, \phi_\infty v']$ both intersect Ω'. Denote by B the intersection point of $\phi_\infty v \oplus \phi_\infty v'$ and $\phi_{-\infty} v \oplus \phi_{-\infty} v'$ (in the affine chart B is at infinity in direction $(1, -1)$). Denote by $y \in \Omega'$ the intersection point of $x \oplus B$ and $\phi_{-\infty} v \oplus \phi_\infty v'$, and w the unitary vector at y pointing at $\phi_\infty v$. Note that the orbit of w under the geodesic flow is the same in Ω' and in Ω. According to Lemma 2.1.4, it suffices to find ϵ small enough so that $d_{\Omega'}(x, y) \leq \eta$. Pick $c, d \in \partial\Omega'$ so that c, y, x, d are aligned in this order. Notice that c (resp. d) is the intersection point of $x \oplus B$ and $[\phi_{-\infty} v', \phi_\infty v]$ (resp. $[\phi_{-\infty} v, \phi_\infty v']$). When $d_{\Omega'}(\pi\phi_1 v, \pi\phi_1 v')$ goes to zero the points a and b converge to B, and so do c and d, hence $d_{\Omega'}(x, y)$ goes to zero.

One can give an explicit formula for ϵ, for instance $\log(e - e^{1-\frac{\eta}{2}} + e^{-\frac{\eta}{2}})$ should work.

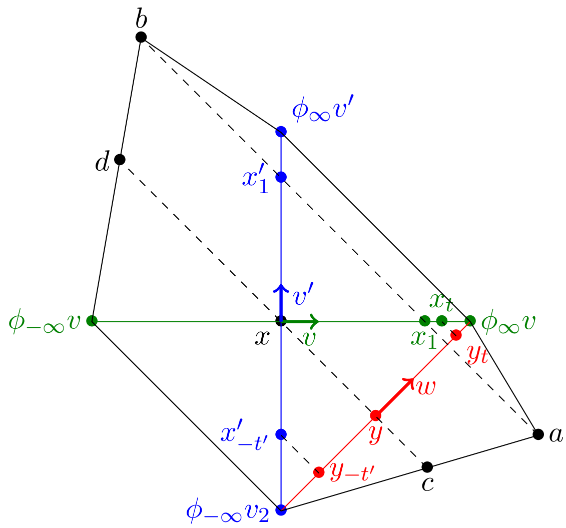

Figure 5.5: Illustration of the proof of the weak orbit-glueing lemma for $n = 2$, where $x_s = \pi\phi_s v$ and $x'_s = \pi\phi_s v'$ and $y_s = \pi\phi_s w$ for $s \in \mathbb{R}$

Step 2 : show Lemma for $n = 2$**.** Let $\eta > 0$. Let $\epsilon < \frac{\eta}{2}$ small enough as in step 1 for $\eta = \frac{\eta}{2}$. Let $v_1 \neq v_2 \in T^1\Omega$ and $t_1, t_2 > 2$, such that $d_{T^1\Omega}(v_1, v_2) \leq \frac{\epsilon}{2}$. Call v'_1 the unitary vector at $x_2 = \pi v_2$ pointing at $\phi_\infty v_1$. By Lemma 2.1.4, $d_{T^1\Omega}(v'_1, v_2) \leq d_{T^1\Omega}(v'_1, v_1) + d_{T^1\Omega}(v_1, v_2) \leq \epsilon$.

By step 1, there exists v such that the trajectory $(\phi_t v_2)_{-t_2 \leq t \leq t_1}$ $\frac{\eta}{2}$-shadows the concatenation of $(\phi_t v_2)_{-t_2 \leq t \leq 0}$ and $(\phi_t v'_1)_{0 \leq t \leq t_1}$. Since $(\phi_t v'_1)_{0 \leq t \leq t_1}$ and $(\phi_t v_1)_{0 \leq t \leq t_1}$ are ϵ-close by Lemma 2.1.4, we see that $(\phi_t v'_1)_{-t_2 \leq t \leq t_1}$ η-shadows the concatenation of $(\phi_t v_2)_{-t_2 \leq t \leq 0}$ and $(\phi_t v_1)_{0 \leq t \leq t_1}$.

Step 3 : Induction step. Assume the lemma is true for the $n \geq 2$. Let $\eta > 0$. Let $\eta > \epsilon > 0$ be as in the lemma for $(n = 2,\ \eta = \eta)$. Let $\epsilon/2 > \epsilon' > 0$ be as in the lemma for $(n = n,\ \eta = \epsilon/2)$. Let $v_1, \cdots, v_{n+1} \in T^1\Omega$, and $t_1, \ldots, t_{n+1} \in \mathbb{R}_{\geq 0}$, such that $d_{T^1\Omega}(\phi_{t_i} v_i, v_{i+1}) \leq \epsilon'$ for any $1 \leq i \leq n$. By the inductive hypothesis, there is $w \in T^1\Omega$ such that for any $1 \leq i \leq n$ and any $t \in [0, t_i]$,

$$d_{T^1\Omega}(\phi_{t_1+\cdots+t_{i-1}+t} w, \phi_t v_i) \leq \epsilon/2.$$

By Step 2, there is $v \in T^1\Omega$ such that $d_{T^1\Omega}(\phi_t v, \phi_t w) \leq \eta$ for any $t \in [0, t_1 + \cdots + t_n]$, and for any $t \in [0, t_{n+1}]$

$$d_{T^1\Omega}(\phi_{t_1+\cdots+t_n+t} v, \phi_t v_{n+1}) \leq \eta.$$

Then $d_{T^1\Omega}(\phi_{t_1+\cdots+t_{i-1}+t} v, \phi_t v_i) \leq 2\eta$ for any $1 \leq i \leq n+1$ and any $t \in [0, t_i]$. $\square$

A Closing lemma

In this section we state a closing lemma, generalising [Bra20b, Th. 4.4] and weaker than the classical one from Anosov [Ano67, Lem. 13.1]. We briefly recall the idea: whenever a geodesic segment comes back sufficiently close to its starting point (no matter how long it is), we can find closed geodesic which tracks it. The following version is a more geometrical formulation of the closing lemma. We state the dynamical version below.

We use several notations from Section 2.1, for instance we use closed faces, shadows, and the metrics $d_{\overline{\Omega}}$ and d_{spl}.

Lemma 5.4.11. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set, $x \in \Omega$ and $(\xi_-, \xi_+) \in \partial\Omega^2$. Assume that $d_{\mathrm{spl}}(\xi_+, \overline{F}_\Omega(\xi_-)) \geq 2$ and $d_{\mathrm{spl}}(\xi_-, \overline{F}_\Omega(\xi_+)) \geq 2$. Then there exists $R > 0$ such that for any neighbourhood W of $\overline{B}_{\overline{\Omega}}(\xi_-, R) \times \overline{B}_{\overline{\Omega}}(\xi_+, R)$ in $\overline{\Omega}^2$, there exists a neighbourhood U of (ξ_-, ξ_+) such that for any $g \in \mathrm{Aut}(\Omega)$, if $(g^{-1}x, gx) \in U$, then g is biproximal and $(x_g^-, x_g^+) \in W$.*

Proof. The case where $\dim(\mathbf{V}) = \mathtt{d} + 1 = 2$ is trivial, and we assume that $\dim(\mathbf{V}) \geq 3$.

(1) By assumption, we can find $R > 0$ large enough so that $\mathcal{O}_R(\xi_\pm, x)$ contains $\overline{F}_\Omega(\xi_\mp)$.

(2) By lower semi-continuity of $d_{\overline{\Omega}}$, we may find a neighbourhood $U_\pm \subset \overline{\Omega} \smallsetminus \overline{B}_\Omega(x, R)$ of $\xi_\pm$ such that $\mathcal{O}_R(\xi, x)$ contains $\overline{\mathcal{O}}_R(x, y)$ for all $\xi \in U_\pm$ and $y \in \Omega \cap U_\mp$.

(3) Our assumption ensures that $d_{\mathrm{spl}}(\xi, \overline{F}_\Omega(\xi_\mp)) \geq 2$ for any $\xi \in \overline{B}_{\overline{\Omega}}(\xi_\pm, R)$, so we can find $R' \geq R$ large enough so that $\mathcal{O}_{R'}(\xi, x)$ contains $\overline{F}_\Omega(\xi_\mp)$ for any $\xi \in \overline{B}_{\overline{\Omega}}(\xi_\pm, R)$.

(4) By lower semi-continuity of $d_{\overline{\Omega}}$, we may find a neighbourhood $W_\pm \subset \overline{\Omega} \smallsetminus \overline{B}_\Omega(x, R')$ of $\overline{B}_{\overline{\Omega}}(\xi_\pm, R)$ such that $\mathcal{O}_{R'}(\xi, x)$ contains $\overline{\mathcal{O}}_{R'}(\xi, y)$ for all $\xi \in W_\pm$ and $y \in \Omega \cap W_\mp$, and such that $W_- \times W_+ \subset W$.

(5) Take a neighbourhood $U'_\pm \subset U_\pm$ of $\xi_\pm$ such that $\overline{\mathcal{O}}_R(x, y)$ is contained in $W_\pm$ for any $y \in \Omega \cap U'_\pm$.

Consider $g \in \mathrm{Aut}(\Omega)$ such that $g^{\pm 1}x \in U'_\pm$, and let us show that g is biproximal with (x_g^-, x_g^+) in W. By (2) and since $gx \in U_+$ and $g^{-1}x \in U_-$, we have

$$g\overline{\mathcal{O}}_R(g^{-1}x, x) = \overline{\mathcal{O}}_R(x, gx) \subset \mathcal{O}_R(g^{-1}x, x).$$

Hence g fixes some point $\eta_+ \in \overline{\mathcal{O}}_R(x, gx)$ by the Brouwer fixed point theorem ($\overline{\mathcal{O}}_R(g^{-1}x, x)$ is homeomorphic to $[0, 1]^{\mathtt{d}-1}$, see Section 2.1.7). Symmetrically, g fixes some point $\eta_- \in \overline{\mathcal{O}}_R(x, g^{-1}x)$.

By (5), the point η_+ lies in W_+, and η_- lies in W_-. By (4), this implies that

$$g\overline{\mathcal{O}}_{R'}(\eta_-, x) = \overline{\mathcal{O}}_{R'}(\eta_-, gx) \subset \mathcal{O}_{R'}(\eta_-, x).$$

Therefore, according to Fact 2.2.7, the projection $g' \in \mathrm{PGL}(\mathbf{V}/\eta_-)$ of g is proximal, and its attracting fixed point corresponds in $\mathrm{P}(\mathbf{V})$ to a line of the form $\eta_- \oplus \zeta_+$, where $\zeta_+ \in \overline{\mathcal{O}}_{R'}(\eta_-, gx)$ is fixed by g.

By Fact 2.2.3, since $\ell(g) \geq \ell(g') > 0$ and since $\eta_- \oplus \zeta_+$ intersects Ω, we either have $(\eta_-, \zeta_+) \in x_g^- \times x_g^+$ or $(\eta_-, \zeta_+) \in x_g^+ \times x_g^-$. The latter case contradicts the fact that $\dim(\mathbf{V}) \geq 3$ and g' is proximal. Hence $\eta_- \in x_g^-$ and $\zeta_+ \in x_g^+$, and g is proximal with $\zeta_+ = x_g^+$. Symmetrically, g^{-1} is also proximal and $\eta_+ \in x_g^+$. We have proved that g is biproximal with $(x_g^-, x_g^+) = (\eta_-, \eta_+) \in W$. ☐

Corollary 5.4.12. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set. Let $x \in \Omega$, $(\xi_-, \xi_+) \in \partial\Omega^2$ and W a neighbourhood of (ξ_-, ξ_+). Then there exists a neighbourhood U of (ξ_-, ξ_+) such that for any $g \in \mathrm{Aut}(\Omega)$ with $(g^{-1}x, gx) \in U$,*

- *if $d_{\mathrm{spl}}(\xi_-, \xi_+) \geq 3$, then g is rank-one;*
- *if ξ_- and ξ_+ are extremal and $d_{\mathrm{spl}}(\xi_-, \xi_+) \geq 2$, then g is biproximal and $(x_g^-, x_g^+) \in W$;*
- *if ξ_- and ξ_+ are distinct and strongly extremal, then g is rank-one and $(x_g^-, x_g^+) \in W$.*

Recall that, given a convex projective orbifold M, we use $\tilde{B}^{(t)}_{T^1M}$ to denote the open balls for the metric $\tilde{d}^{(t)}_{T^1\Omega}$ (Section 2.1.1).

Lemma 5.4.13. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set and $\Gamma \subset \mathrm{Aut}(\Omega)$ a discrete subgroup; denote $M = \Omega/\Gamma$. Consider $\alpha > 0$ and $v_0 \in T^1M$ such that the endpoints $\phi_{-\infty}\tilde{v}_0$ and $\phi_\infty\tilde{v}_0$ of any lift $\tilde{v}_0$ are extremal (resp. strongly extremal). Then there exists $\epsilon > 0$ satisfying the following. For any $v \in B_{T^1M}(v_0, \epsilon)$, and for any time $t > \alpha$, if $\phi_t v \in B_{T^1M}(v_0, \epsilon)$, then one can find a biproximal (resp. rank-one) periodic vector $w \in \tilde{B}^{(t)}_{T^1M}(v, \alpha)$ with period in $[t-\alpha, t+\alpha]$.*

Proof. Let W be a neighbourhood of $(\phi_{-\infty}\tilde{v}_0, \phi_\infty v_0)$ such that $[\xi_-, \xi_+] \cap B_{T^1\Omega}(\phi_t\tilde{v}_0, \alpha/8) \neq \emptyset$ for any $0 \leq t \leq 1$. By Corollary 5.4.12, we can find a neighbourhood $U = U_- \times U_+ \subset W$ of $\phi_{\pm\infty}\tilde{v}_0$ such that for any $\gamma \in \Gamma$, if $(\gamma^{-1}\pi\tilde{v}_0, \gamma\pi\tilde{v}_0) \in U$ then γ is biproximal (resp. rank-one) and $(x^-_\gamma, x^+_\gamma) \in W$. Let $t_0 > 0$ and $\epsilon_1 < \alpha/8$ be such that $\overline{B}_\Omega(\pi\phi_{\pm t}\tilde{v}, \epsilon_1) \subset U_\pm$ for any $\tilde{v} \in \overline{B}_{T^1\Omega}(\tilde{v}_0, \epsilon_1)$ and $t \geq t_0$.

Now consider $t \geq t_0$ and $v \in B_{T^1M}(v_0, \epsilon_1)$ such that $\phi_t v \in B_{T^1M}(v_0, \epsilon_1)$. We can find a lift $\tilde{v} \in B_{T^1\Omega}(\tilde{v}_0, \epsilon_1)$ and an element $\gamma \in \Gamma$ such that $\phi_t\tilde{v} \in B_{T^1\Omega}(\gamma\tilde{v}_0, \epsilon_1)$. Then $(\gamma^{-1}\pi\tilde{v}_0, \gamma\pi\tilde{v}_0) \in U$, hence γ is biproximal (resp. rank-one) and $(x^-_\gamma, x^+_\gamma) \in W$. Be definition of W, we can find $\tilde{w} \in B_{T^1\Omega}(\tilde{v}_0, \alpha/8)$ tangent to the axis of γ. Then $d_{T^1\Omega}(\tilde{w}, \tilde{v}) \leq \alpha/4$ and $d_{T^1\Omega}(\gamma\tilde{w}, \phi_t\tilde{v}) \leq \alpha/4$; since $\gamma\tilde{w} = \phi_{\ell(\gamma)}\tilde{w}$, by triangular inequality we have $|\ell(\gamma) - t| \leq \alpha/2$, and $d^{(t)}_{T^1\Omega}(\tilde{w}, \tilde{v}) \leq \alpha$.

To finish the prooof, it remains to find $\epsilon < \epsilon_1$ such that for all $\alpha \leq t \leq t_0$ and $v \in B_{T^1M}(v_0, \epsilon)$, if $\phi_t v \in B_{T^1M}(v_0, \epsilon)$, then one can find a biproximal (resp. rank-one) periodic vector $w \in \tilde{B}^{(t)}_{T^1M}(v, \alpha)$ with period in $[t-\alpha, t+\alpha]$. If v_0 is not periodic, then we take $\epsilon < \epsilon_1$ small enough so that $B_{T^1M}(v_0, \epsilon) \subset B^{(t_0)}_{T^1M}(v_0, \epsilon_2/2)$, where $\epsilon_2 := \min_{\alpha \leq t \leq t_0} d_{T^1M}(\phi_t v_0, v_0)$. If v_0 is periodic, then it is biproximal (resp. rank-one) since $\phi_{\pm\infty}\tilde{v}_0$ are extremal (resp. strongly extremal). We then take $\epsilon < \epsilon_1$ small enough so that $\tilde{B}_{T^1M}(v_0, \epsilon) \subset B^{(t_0)}_{T^1M}(v_0, \epsilon_2/2)$, where $\epsilon_2 := \min_{t \in \mathcal{T}} d_{T^1M}(\phi_t v_0, v_0)$ and $\mathcal{T}$ is the set of times $\alpha \leq t \leq t_0$ such that $\phi_s v_0 \neq v_0$ for any $t - \alpha < s < t + \alpha$. □

Sketch of the other proof of topological mixing

Let us briefly explain what differences it would make to use Proposition 5.4.8 and Lemmas 5.4.10 and 5.4.13 in order to prove Theorem 5.1.1. Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set, $\Gamma \subset \mathrm{Aut}(\Omega)$ a strongly irreducible, rank-one, torsion-free and discrete subgroup, and set $M = \Omega/\Gamma$.

The Weak orbit-glueing lemma (Lemma 5.4.10) and the Closing lemma (Lemma 5.4.13), combined with the topological transitivity of the geodesic flow, may be used to upgrade a global non-arithmeticity property into a local non-arithmeticity property, similar to Proposition 5.3.3.

More precisely, suppose we know that the lengths of all rank-one periodic geodesics of M generate a dense subgroup of $\mathbb{R}$, fix any non-empty open subset U of T^1M_{bip}, and let us see how to find rank-one periodic geodesics passing through U whose lengths generate a dense subgroup of $\mathbb{R}$. We may assume that U is a small enough neighbourhood of a rank-one geodesic so that we may apply Lemma 5.4.13. Let $c : [0,1] \to T^1M$ be a rank-one periodic geodesic of M with length ℓ. By topological transitivity, there is a geodesic c_1 from U to near $c(0) = c(1)$ with length ℓ_1, and a geodesic c_2 from near $c(0)$ to U with length. By the Weak orbit-glueing Lemma and the Closing lemma, we may find a rank-one closed geodesic c_3 through U, shadowing the concatenation of c_1 and c_2 and with length ℓ_3

close to $\ell_1+\ell_2$. Similarly, we may find a rank-one closed geodesic c_4 through U, shadowing the concatenation of c_1, c and c_2 and with length ℓ_4 close to $\ell_1+\ell+\ell_2$. The difference $\ell_4-\ell_3$ is then close to ℓ.

Note that, in order to establish a global non-arithmeticity property, one would still need Benoist's work [Ben00b] on asymptotic properties of linear groups, and it would not be much easier that establishing directly the local non-arithmeticity property as in Proposition 5.3.3. It would be slightly simplified for instance if Γ is Zariski-dense in $\mathrm{PGL}(\mathbf{V})$ (which is true if M is compact, see Section 4.1.1).

Proposition 5.4.8 would ease the proof of topological mixing in the following way. Suppose $U,V\subset T^1M_{\mathrm{bip}}$ are two non-empty subsets, which have rank-one periodic vectors $x\in U$ and $y\in V$ whose periods ℓ and ℓ' generate an "almost" dense subgroup of $\mathbb{R}$, as in the proof of Theorem 5.3.4 and Figure 5.1. Consider a vector $z\in T^1M_{\mathrm{bip}}$ whose forward (resp. backward) endpoint in the universal cover is the same as x (resp. y). Since rank-one periodic vectors have smooth endpoints in the universal cover by Lemma 3.1.2, one may apply Proposition 5.4.8 and obtain that $\mathcal{T}:=\{t:\phi_t z\in U\}$ (resp. $\mathcal{S}:=\{s:\phi_{-s}z\in V\}$) contains $\{\tau+n\ell : n\in\mathbb{N}\}$ (resp. $\{\sigma+m\ell' : m\in\mathbb{N}\}$) for some $\sigma,\tau\in\mathbb{R}$. Finally one concludes that $\{t:\phi_tV\cap U\neq\emptyset\}\supset\mathcal{T}+\mathcal{S}$ is "almost" dense in $[T,\infty)$ for some T.

5.4.5 Maximality of the biproximal unit tangent bundle

In this section we give a proof of Proposition 5.1.2. For this, we need the following lemma, which is a consequence of Lemma 2.1.6.

Lemma 5.4.14. *Let $\Omega\subset\mathrm{P}(\mathbf{V})$ be a properly convex open set and $\Gamma\subset\mathrm{Aut}(\Omega)$ a discrete subgroup; set $M=\Omega/\Gamma$. Consider two vectors $v,w\in T^1M$ with w in the closure of the forward orbit $\{\phi_tv : t\geq 0\}$. Then $\phi_\infty\tilde v$ belongs to the closure of the orbit $\Gamma\cdot\phi_\infty\tilde w$ for all lifts $\tilde v,\tilde w\in T^1\Omega$.*

Proof. By assumption there exist sequences $(t_n)_n\in[0,\infty)^{\mathbb{N}}$ and $(\gamma_n)_n\in\Gamma^{\mathbb{N}}$ such that

$$\gamma_n\phi_{t_n}\tilde v \underset{n\to\infty}{\longrightarrow} \tilde w.$$

This implies that, for n large enough, $[\gamma_n\phi_{-\infty}\tilde v,\phi_\infty\tilde w]\cap\Omega$ is non-empty; let us consider $\tilde u_n\in T^1\Omega$ such that $\phi_{-\infty}\tilde u_n=\gamma_n\phi_{-\infty}\tilde v$, and $\phi_\infty\tilde u_n=\phi_\infty\tilde w$, and $\pi\tilde u_n$ is a closest point of $[\gamma_n\phi_{-\infty}\tilde v,\phi_\infty\tilde w]$ to $\pi\tilde w$ for the Hilbert distance. We easily observe that $(\tilde u_n)_n$ converges to $\tilde w$ as n tends to infinity. By Lemma 2.1.6, we obtain

$$\begin{aligned} d_{T^1\Omega}(\tilde v,\gamma_n^{-1}\phi_{-t_n}\tilde u_n) &\leq d_{T^1\Omega}(\phi_{t_n}\tilde v,\gamma_n^{-1}\tilde u_n)=d_{T^1\Omega}(\gamma_n\phi_{t_n}\tilde v,\tilde u_n)\\ &\leq d_{T^1\Omega}(\gamma_n\phi_{t_n}\tilde v,\tilde w)+d_{T^1\Omega}(\tilde w,\tilde u_n)\\ &\underset{n\to\infty}{\longrightarrow} 0. \end{aligned}$$

Therefore, $\gamma_n^{-1}\phi_\infty\tilde w=\gamma_n^{-1}\phi_\infty\tilde u_n$ tends to $\phi_\infty\tilde v$ as n goes to infinity. □

Proof of Proposition 5.1.2. Consider a closed invariant subset $A\subset T^1M$ containing T^1M_{bip} and on which $(\phi_t)_t$ is topologically transitive and non-wandering. By Fact 1.1.3, there exists $v\in A$ such that both $\{\phi_tv : t\geq 0\}$ and $\{\phi_tv : t\leq 0\}$ are dense in A.

Take $w\in T^1M_{\mathrm{bip}}$ (which is non-empty by assumption), and consider respective lifts $\tilde v,\tilde w\in T^1\Omega$ of v,w. By definition of v, the vector w is in the closure of the forward orbit $\{\phi_tv : t\geq 0\}$, so we can apply Lemma 5.4.14 and we obtain $\phi_\infty\tilde v\in\Lambda_\Gamma^{\mathrm{prox}}$. Using again Lemma 5.4.14, and the fact that w is in the closure of $\{\phi_t(-v) : t\geq 0\}=\{-\phi_tv : t\leq 0\}$, we see that $\phi_{-\infty}\tilde v\in\Lambda_\Gamma^{\mathrm{prox}}$. We have proved that $v\in T^1M_{\mathrm{bip}}$, therefore

$$A=\overline{\{\phi_tv : t\in\mathbb{R}\}}\subset T^1M_{\mathrm{bip}}\subset A,$$

and this concludes the proof. □

5.5 The geodesic flow in the higher-rank compact case

The goal of this section is to prove Proposition 5.1.3. We are actually going to prove a finer statement: that the connected components of the non-wandering set of the geodesic flow are quotients of homogeneous spaces whose Haar measure is mixing.

In this section we denote by $\mathbb{H}$ the classical division algebra of quaternions (and we stop calling $\mathbb{H}^2$ the poincaré disk), and by $\mathbb{O}$ the classical non-associative division algebra of octonions. Fix an integer $N \geq 3$ and the algebra $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or, if $N = 3$, $\mathbb{O}$. We shall use the following notation. (In the case $\mathbb{K} = \mathbb{R}$, conjugation is the identity and we abusively say Hermitian instead of symmetric.)

- For $x \in \mathbb{K}$, the element $\overline{x} \in \mathbb{K}$ is the conjugate of x.
- We consider the Hermitian bilinear form on $\mathbb{K}^N$ given by $\langle x, y\rangle = \sum_{i=1}^{N} \overline{x}_i y_i$.
- The real vector space $\mathbf{V} = V_{N,\mathbb{K}}$ consists of the Hermitian matrices of size N with entries in $\mathbb{K}$.
- The cone $C = C_{N,\mathbb{K}} \subset \mathbf{V}$ consists of the positive-definite Hermitian matrices.
- The properly convex open set $\Omega = \Omega_{N,\mathbb{K}} \subset \mathrm{P}(\mathbf{V})$ is the projectivisation of C.
- The group $\mathrm{Aut}(C) \subset \mathrm{GL}(\mathbf{V})$ consists of the transformations preserving C.
- The group $G = G_{N,\mathbb{K}} := \mathrm{Aut}(\Omega) = \mathrm{Aut}(C)/\mathbb{R}^*$ is the automorphism group of Ω, where $\mathbb{R}^*$ is seen as the group of homotheties of $\mathrm{GL}(\mathbf{V})$.
- The group $K \subset \mathrm{Aut}(C)$ is the stabiliser of the identity matrix; note that the map $K \to G$ is an embedding, and that K is a maximal compact subgroup of G.
- Finally the group A consists of the diagonal matrices of size N with entries in $\mathbb{R}_{>0}$; we see it embedded in $\mathrm{Aut}(C)$, acting on $\mathbf{V}$ by the following formula: $a \cdot X = aXa$ for $a \in A$ and $X \in \mathbf{V}$.

Let us be more explicit about the case $\mathbb{K} = \mathbb{R}$. The group $\mathrm{Aut}(C_{N,\mathbb{R}})$ identifies with the quotient $\mathrm{GL}_N(\mathbb{R})/\{\pm 1\}$, acting on $V_{N,\mathbb{R}}$ by the formula $g \cdot X = gXg^t$; the group $G_{N,\mathbb{R}}$ identifies with $\mathrm{PGL}_N(\mathbb{R})$; the group K identifies with $\mathrm{O}(N)/\{\pm 1\}$.

We come back to the general case. The spectral theorem (see [FK94, Th. V.2.5]) ensures that for every $X \in \mathbf{V}$ there exists $k \in K$ such that $k \cdot X$ is diagonal with real entries. This, using the action of A, has two consequences: $\mathrm{Aut}(C)$ acts transitively on C, and can be written as the product $KAK = \{k_1 a k_2 : k_1, k_2 \in K, a \in A\}$. Then, the quotient group G acts transitively on Ω, and can be written $K(A/\mathbb{R}_{>0})K$ — actually, the element of $A/\mathbb{R}_{>0}$ in the decomposition can be taken with non-increasing entries on the diagonal, and this yields the *Cartan decomposition* of G. The Lie algebra of G is $\mathfrak{sl}(N, \mathbb{K})$ when $\mathbb{K} \neq \mathbb{O}$, and $\mathfrak{e}_{6(-26)}$ if $\mathbb{K} = \mathbb{O}$ (see [FK94, p. 97]), therefore G is a non-compact real simple Lie group, with finitely many connected components, and with trivial centre. Observe that Ω identifies as a G-space with the Riemannian symmetric space of G.

Since $\Omega = G/K$, a discrete subgroup $\Gamma \subset G$ acts cocompactly on Ω if and only if G/Γ is compact, i.e. Γ is a uniform lattice of G; uniform lattices exist by a theorem of Borel [Bor63, Th. C]. The properly convex open sets $\Omega_{N,\mathbb{K}}$ are called the *symmetric divisible convex sets*. Zimmer [Zim, Th. 1.4] recently proved that the higher-rank irreducible closed convex projective orbifolds are exactly the quotients of the form $\Omega_{N,\mathbb{K}}/\Gamma$, where $N \geq 3$, the field $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$ (if $N = 3$), and Γ is a uniform lattice of $G_{N,\mathbb{K}}$.

5.5.1 The non-wandering set of $G \times \mathbb{R}$ on $T^1\Omega$

In this section we describe the non-wandering set $\mathrm{NW}(T^1\Omega, G \times \mathbb{R})$ and prove that G acts transitively on each of its connected components.

The boundary of Ω is the projectivisation of the cone of positive semi-definite Hermitian matrices. For $1 \leq i,j \leq N-1$, we denote by $T^1\Omega_{i,j}$ the set of unit tangent vectors $v \in T^1\Omega$ such that the respective ranks of $\phi_{-\infty}v$ and $\phi_\infty v$ (meaning the rank of any representative in $\mathbf{V}$) are i and j. Note that $T^1\Omega_{i,j}$ is non-empty if and only if $i+j \geq N$ (see Proposition 5.5.1.(1)). The subsets $T^1\Omega_{i,j}$, for $1 \leq i,j \leq N$ and $i+j \geq N$, are invariant under the automorphism group $\mathrm{Aut}(\Omega)$ and the geodesic flow $(\phi_t)_{t\in\mathbb{R}}$. They stratify $T^1\Omega$ in the following way:

- $T^1\Omega$ is the disjoint union of the $T^1\Omega_{i,j}$,
- the closure of $T^1\Omega_{i,j}$ is the union of the $T^1\Omega_{k,\ell}$ for $1 \leq k \leq i$ and $1 \leq \ell \leq j$,
- in particular, $T^1\Omega_{i,N-i}$ is closed for $1 \leq i \leq N-1$,
- $T^1\Omega_{N-1,N-1}$ is open and dense in $T^1\Omega$.

When $\mathbb{K} = \mathbb{R}$ we compute $\dim(T^1\Omega_{i,j}) = i(N-i) + \frac{i(i+1)}{2} + j(N-j) + \frac{j(j+1)}{2} - 1$.

We denote by $\mathrm{Geod}^\infty(\Omega)_{i,j}$ the quotient $T^1\Omega_{i,j}/(\phi_t)_{t\in\mathbb{R}}$. Observe that the space of all geodesics $\mathrm{Geod}^\infty(\Omega) := T^1\Omega/(\phi_t)_{t\in\mathbb{R}}$ identifies with the set of pairs (x,y) in $\partial\Omega^2$ such that $\mathrm{Ker}(x) \cap \mathrm{Ker}(y) = \emptyset$. We are going to prove that $\mathrm{NW}(\mathrm{Geod}^\infty(\Omega), G)$ is the union $\bigcup_{1\leq i\leq N-1} \mathrm{Geod}^\infty(\Omega)_{i,N-i}$. This exactly means, according to Section 1.1.2, that $\mathrm{NW}(T^1\Omega, G \times \mathbb{R})$ is $\bigcup_{1\leq i\leq N-1} T^1\Omega_{i,N-i}$. We choose a basepoint $v_{i,N-i} \in T^1\Omega_{i,N-i}$, such that $\pi v_{i,N-i}$, $\phi_{-\infty}v_{i,N-i}$ and $\phi_\infty v_{i,N-i}$ are the projectivisations of, respectively, the identity matrix, the orthogonal projection onto $\mathbb{K}^i \times \{0\}$ and the orthogonal projection onto $\{0\} \times \mathbb{K}^{N-i}$. We set

$$A_{i,N-i} := \left\{ a_t := \begin{bmatrix} e^{t/2} I_i & 0 \\ 0 & e^{-t/2} I_j \end{bmatrix} : t \in \mathbb{R} \right\} \subset A,$$

where I_k is the identity matrix of size k, and we observe that for any $t \in \mathbb{R}$, the image $a_t \cdot v_{i,N-i}$ is exactly $\phi_t v_{i,N-i}$. We denote by G_0 the identity component of G (for the topology induced by the structure of real Lie group) and by $K_{i,N-i}$ the stabiliser in G_0 of $v_{i,N-i}$; they are normalised by $A_{i,N-i} \subset G_0$.

Proposition 5.5.1. *Consider $N \geq 3$, the algebra $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$ (if $N = 3$), the vector space $\mathbf{V} = V_{N,\mathbb{K}}$, the properly convex open set $\Omega = \Omega_{N,\mathbb{K}} \subset \mathrm{P}(\mathbf{V})$, and the group $G = G_{N,\mathbb{K}}$, with identity component G_0.*

(1) For $1 \leq i,j \leq N-1$, the set $T^1\Omega_{i,j}$ is non-empty if and only if $i+j \geq N$.

(2) For $1 \leq i \leq N-1$, the group G_0 acts transitively on $T^1\Omega_{i,N-i}$. If we identify $T^1\Omega_{i,N-i}$ with $G_0/K_{i,N-i}$, then the geodesic flow identifies with the action by right multiplication of $A_{i,N-i}$ on $G_0/K_{i,N-i}$.

(3) The non-wandering set of G on $\mathrm{Geod}^\infty(\Omega)$ is

$$\mathrm{NW}(\mathrm{Geod}^\infty(\Omega), G) = \bigcup_{1\leq i\leq N-1} \mathrm{Geod}^\infty(\Omega)_{i,N-i}.$$

Proof. (1) Suppose there exists $v \in T^1\Omega_{i,j}$. Because G_0 acts transitively on Ω we can find $g_1 \in G_0$ such that $g_1\pi v$ is the projectivisation of the identity matrix. Then by the spectral theorem there exists an automorphism $g_2 \in K$ (i.e. fixing $g_1\pi v$) such that $\mathrm{Ker}(g_2g_1\phi_{-\infty}v) = \{0\} \times \mathbb{K}^{N-i}$; since the space of $(N-i)$-dimensional right $\mathbb{K}$-sub-modules of $\mathbb{K}^N$ is connected, we can take g_2 in G_0. We note that the subspaces $\mathrm{Ker}(g_2g_1\phi_{-\infty}v)$ and $\mathrm{Ker}(g_2g_1\phi_{\infty}v)$ are orthogonal. (Indeed, if T and T' are representatives in $\mathbf{V}$ of $g_2g_1\phi_{-\infty}v$ and $g_2g_1\phi_{\infty}v$ such that $T+T'$ is the identity matrix, and if $x \in \mathrm{Ker}(T)$ while $y \in \mathrm{Ker}(T')$, then $\langle x, y\rangle = \langle x, Ty + T'y\rangle = \langle x, Ty\rangle = \langle Tx, y\rangle = 0$.) This implies that $i + j \geq N$.

(2) Let $1 \leq i \leq N-1$. Let us show that there exists $g \in G_0$ such that $g\cdot v$ is the basepoint $v_{i,N-i}$ of $T^1\Omega_{i,N-i}$. We have already seen that there are $g_1, g_2 \in G$ such that πg_2g_1v is the projectivisation of the identity matrix, and $\mathrm{Ker}(g_2g_1\phi_{-\infty}v) = \{0\} \times \mathbb{K}^{N-i}$. Then $\mathrm{Ker}(g_2g_1\phi_{\infty}v) = \mathbb{K}^i \times \{0\}$, since $\mathrm{Ker}(g_2g_1\phi_{-\infty}v)$ and $\mathrm{Ker}(g_2g_1\phi_{\infty}v)$ are orthogonal. Moreover $g_2g_1\phi_{-\infty}v$ and $g_2g_1\phi_{\infty}v$ are the projectivisations of the orthogonal projections onto $\mathbb{K}^i \times \{0\}$ and $\{0\} \times \mathbb{K}^{N-i}$. (Indeed, consider representatives T and T' of $\phi_{-\infty}g_2g_1v$ and $\phi_{\infty}g_2g_1v$ in V such that $T+T'$ is the identity matrix; then T' and T are the orthogonal projections onto $\mathbb{K}^i \times \{0\}$ and $\{0\} \times \mathbb{K}^{N-i}$.)

(3) The stabiliser of $(\phi_{-\infty}v_{i,N-i}, \phi_{\infty}v_{i,N-i}) \in \mathrm{Geod}^{\infty}(\Omega)_{i,N-i}$ contains $A_{i,N-i}$, therefore the stabilisers of points in $\mathrm{Geod}^{\infty}(\Omega)_{i,N-i}$ are non-compact, hence the union $\bigcup_{i=1}^{N-1} \mathrm{Geod}^{\infty}(\Omega)_{i,N-i}$ is contained in $\mathrm{NW}(\mathrm{Geod}^{\infty}(\Omega), G)$. Let us prove the converse inclusion.

Suppose by contradiction the non-wandering set $\mathrm{NW}(\mathrm{Geod}^{\infty}(\Omega), G)$ is not contained in the union $\bigcup_{i=1}^{N-1} \mathrm{Geod}^{\infty}(\Omega)_{i,N-i}$. We may assume the existence of sequences of positive semi-definite Hermitian matrices $(S_n)_{n\in\mathbb{N}}$, $(T_n)_{n\in\mathbb{N}}$ in V, of automorphisms $(g_n)_{n\in\mathbb{N}}$ in $\mathrm{Aut}(C)$ and of positive scalars $(\mu_n)_{n\in\mathbb{N}}, (\nu_n)_{n\in\mathbb{N}}$, such that

- $(S_n)_n$, $(\mu_n g_n S_n)_n$, $(T_n)_n$, and $(\nu_n g_n T_n)_n$ respectively converge to S, S', T, and T',
- the rank of S and S' is i, the rank of T and T' is j, with $1 \leq i, j \leq N-1$ and $i + j > N$,
- $\mathrm{Ker}(S) \cap \mathrm{Ker}(T) = \mathrm{Ker}(S') \cap \mathrm{Ker}(T') = \{0\}$,
- $([g_n])_n \in G^{\mathbb{N}}$ leaves every compact subset of G.

Using $\mathrm{Aut}(C) = KAK$ and extracting, we may assume (up to renormalising) that $g_n = a_n \in A$ converge in $\mathrm{End}(\mathbf{V})$ to a non-invertible non-zero diagonal matrix a with non-negative coefficients. We extend to a the action of A on V, with the same notation: $a \cdot X = aXa$ for any $X \in V$.

Since $\mathrm{Ker}(S) \cap \mathrm{Ker}(T) = \{0\}$, up to exchanging S and T, we can assume that the image of a is not contained in $\mathrm{Ker}(S)$, and this implies that $a \cdot S \neq 0$. Both $(a_n \cdot S_n)_n$ and $(\mu_n a_n \cdot S_n)_n$ converge to a non-zero element of V, so $(\mu_n)_n$ must be bounded, and we may assume that it converges to 1, without loss of generality. Therefore, $a \cdot S = S'$, which means the rank of a is bounded below by i. Since $i + j > N$, the kernel of a is not contained in $\mathrm{Ker}(T)$, and $a \cdot T \neq 0$. As before, without loss of generality, we can assume that $a \cdot T = T'$. But now the kernel of a is contained in $\mathrm{Ker}(S') \cap \mathrm{Ker}(T') = \{0\}$, this is a contradiction. □

5.5.2 The non-wandering set of $(\phi_t)_{t\in\mathbb{R}}$ on T^1M

Let Γ be a lattice of G, not necessarily uniform. We set $M = \Omega/\Gamma$.

Remark 5.5.2. The biproximal unit tangent bundle T^1M_{bip} is empty. To see this, recall that the attracting fixed point of a proximal automorphism of Ω is always an extremal point of Ω, so by definition the proximal limit set of Γ is contained in the closure of the set of extremal points of Ω. Here, since Ω is symmetric, the set of extremal points is closed and consists of projectivisations of rank-1 positive semi-definite Hermitian matrices, so the set of straight geodesics between two extremal points is $\mathrm{Geod}^\infty(\Omega)_{1,1}$, which is empty since $N \geq 3$.

For $1 \leq i, j \leq N-1$, we denote by $T^1M_{i,j}$ the quotient $T^1\Omega_{i,j}/\Gamma$. In this section we use the following celebrated theorem of Howe–Moore to study the action of the geodesic flow on each $T^1M_{i,N-i}$, with $1 \leq i \leq N$.

Fact 5.5.3 ([HM79], see e.g. [Zim84, Th. 2.2.20])**.** *Let G be a connected non-compact simple Lie group with finite centre, let π be a unitary representation of G in a separable Hilbert space, without any non-zero G-invariant vector. Let x, y be two vectors in the Hilbert space. Then $\langle x, gy\rangle$ converges to zero when g goes to infinity, i.e. g leaves every compact subset of G.*

By Proposition 5.5.1 and Fact 1.1.2,

$$\mathrm{NW}(\mathrm{Geod}^\infty(\Omega), \Gamma) \subset \bigcup_{1\leq i\leq N-1} \mathrm{Geod}^\infty(\Omega)_{i,N-i},$$
$$\mathrm{NW}(T^1\Omega, \Gamma\times\mathbb{R}) \subset \bigcup_{1\leq i\leq N-1} T^1\Omega_{i,N-i},$$
$$\mathrm{NW}(T^1M, (\phi_t)_{t\in\mathbb{R}}) \subset \bigcup_{1\leq i\leq N-1} T^1M_{i,N-i}.$$

We are now going to see that we actually have equalities. Recall that a finite measure μ preserved by a measurable flow $(\phi_t)_{t\in\mathbb{R}}$ is mixing if, for any two functions $f, g \in \mathrm{L}^2(\mu)$ with zero integral, we have

$$\int f\cdot(g\circ\phi_t)\, d\mu \underset{t\to\infty}{\longrightarrow} 0.$$

Recall also that a continuous flow is topologically mixing on the support of a mixing invariant measure (Fact 1.2.9). Therefore Proposition 5.1.3 is an immediate consequence of the following proposition, and of Zimmer's rigidity theorem [Zim, Th. 1.4].

Proposition 5.5.4. *Consider $N \geq 3$, the algebra $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$ (if $N = 3$), the vector space $\mathbf{V} = V_{N,\mathbb{K}}$, the properly convex open set $\Omega = \Omega_{N,\mathbb{K}}$, and the group $G = G_{N,\mathbb{K}}$. Take a lattice Γ of G, not necessarily uniform, and denote by M the quotient Ω/Γ. Then for any $1 \leq i \leq N-1$, the (finite and fully supported) Haar measure on $T^1M_{i,N-i}$ is mixing under the geodesic flow; as a consequence the geodesic flow is topologically mixing on $T^1M_{i,N-i}$. Furthermore, $\mathrm{NW}(T^1M, (\phi_t)_{t\in\mathbb{R}})$ has exactly $N-1$ connected components, which are $\{T^1M_{i,N-i}\}_{1\leq i\leq N-1}$.*

Proof. Up to replacing Γ by a finite-index subgroup, we can assume that Γ is contained in G_0. Since Γ is a lattice, the Haar measure m on $\Gamma\backslash G_0$ is finite. Fix $1 \leq i \leq N-1$. By applying the Howe–Moore theorem (Fact 5.5.3) to the unitary representation of G_0 in $\mathrm{L}^2(\Gamma\backslash G_0, m)$, we obtain that m is mixing under the action of the (one-parameter) non-compact subgroup $A_{i,N-i}$ of G_0. According to Proposition 5.5.1.(2), it immediately follows that the induced measure on $T^1 M_{i,N-i} = \Gamma\backslash G/K_{i,N-i}$ is mixing under the action of the geodesic flow. Since the Haar measure in fully supported, the geodesic flow on $T^1 M_{i,N-i}$ is topologically mixing, and its non-wandering set is $T^1 M_{i,N-i}$. □

Chapter 6

The Hopf–Tsuji–Sullivan–Roblin dichotomy

The goal of this chapter is to define conformal densities on the projective boundary of properly convex open sets, to construct the Sullivan measures on the unit tangent bundle of non-elementary rank-one convex projective orbifolds, and then prove the HTSR dichotomy. Let us state now this dichotomy, even though we have not yet defined the Sullivan measures; recall that we gave a hint in Section 1.4.3 of how the Sullivan measures are constructed.

Theorem 6.0.1. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set and $\Gamma \subset \mathrm{Aut}(\Omega)$ a non-elementary rank-one discrete subgroup; set $M = \Omega/\Gamma$. Let $\delta \geq 0$ and $(\mu_x)_{x\in\Omega}$ a Γ-equivariant conformal density of dimension δ on $\partial\Omega$. Let m, m_Γ and $m_{\mathbb{R}}$ be Sullivan measures on $T^1\Omega$, T^1M and $\mathrm{Geod}^\infty(\Omega)$ associated to $(\mu_x)_x$. Fix $o \in \Omega$. Then there are two possibilities:*

1. *either $\sum_{\gamma\in\Gamma} e^{-\delta d_\Omega(o,\gamma o)} < \infty$, in which case $\nu_o(\Lambda^{\mathrm{con}}_\Gamma) = 0$, and the dynamical systems $(T^1\Omega, \mathbb{R}\times\Gamma, m)$, $(T^1M, (\phi_t)_t, m_\Gamma)$ et $(\mathrm{Geod}^\infty(\Omega), \Gamma, m_{\mathbb{R}})$ are dissipative and non-ergodic.*

2. *or $\sum_{\gamma\in\Gamma} e^{-\delta d_\Omega(o,\gamma o)} = \infty$, in which case $\delta = \delta_\Gamma$, and*
   - *$(\nu_x)_x$ is the only Γ-equivariant conformal density of dimension δ (up to multipliciation by a scalar);*
   - *$\nu_o(\partial_{\mathrm{sse}}\Omega \cap \Lambda^{\mathrm{prox}}_\Gamma \cap \Lambda^{\mathrm{con}}_\Gamma) = \nu_o(\partial\Omega)$ and ν_o has no atom: in particular the support of m_Γ is T^1M_{bip};*
   - *the dynamical systems $(T^1\Omega, \mathbb{R}\times\Gamma, m)$, $(T^1M, (\phi_t)_t, m_\Gamma)$ and $(\mathrm{Geod}^\infty(\Omega), \Gamma, m_{\mathbb{R}})$ are conservative and ergodic;*
   - *if m_Γ is finite, then it is mixing unde the action of $(\phi_t)_t$.*

The proof of this theorem is based on that of Roblin [Rob03] for $\mathrm{CAT}(-1)$-spaces.

In Section 6.1 we collect results on the horocompactification of properly convex open sets equipped with their Hilbert metric. In Section 6.2 we define the Hopf coordinates and the Gromov product in the setting of convex projective geometry, in order to make sense of Sullivan's formula [Sul79, Prop. 11], which defines Sullivan measures. In Section 6.3 we state and prove a convex projective version of the Shadow lemma, a fundamental lemma in the study of conformal densities. In Section 6.4 we establish the convergent case of the HTSR dichotomy. In Section 6.5 we assume that Γ is divergent, and follow closely Roblin's proof of HTSR dichotomy in order to produce the convex projective version of

Theorem 6.0.1.2. The proof is divided into several steps: proving that the conical limit set has full measure (Sections 6.5.1 and 6.5.2); proving that any δ_Γ-conformal density is Γ-ergodic, when restricted to a coarser σ-algebra of $\partial\Omega$ which does not distinguish two points belonging to the same face (Section 6.5.2); proving that the Sullivan measure is conservative (Section 6.5.3); proving that $\partial_{\mathrm{sse}}\Omega$ has full measure (Section 6.5.4); proving that the Bowen–Margulis measure is ergodic, and moreover mixing when finite (Section 6.5.7); and finally proving that Λ^{prox} has full measure (Section 6.5.8).

To establish the mixing property when m is finite, we use a general result of Coudène (see Section 1.2.4) that are inspired by work of Babillot [Bab02]. In particular, cross-ratios of quadruples of points on the boundary of a properly convex open set are a crucial component of the proof of the mixing property. Zhu proved the mixing property [Zhua, Th. 18] in the case where Ω is strictly convex with $\mathcal{C}^1$ boundary by using the same strategy.

6.1 The horoboundary of a properly convex open set

In this section we recall results on the horoboundary of a properly convex open set. Recall that the horoboundary was defined for general proper metric spaces in Section 1.4.1.

6.1.1 The horocompactification dominates the projective compactification

Lemma 6.1.1. *Let Ω be a properly convex open set, $\xi \in \overline{\Omega}$, $x \in \Omega$ and $y \in [x,\xi)$. Then $\mathtt{b}_z(x,y)$ converges to $d_\Omega(x,y)$ as $z \in \Omega$ tends to ξ.*

Proof. For any $z \in \Omega$, let $y_z \in \Omega$ be a closest point of $[x,z]$ to y for any fixed metric $d_{\mathrm{P}(\mathbf{V})}$ on $\mathrm{P}(\mathbf{V})$. It is clear that y_z converges to y in $\mathrm{P}(\mathbf{V})$ as z tends to ξ, hence $d_\Omega(y,y_z)$ tends to zero. Therefore by triangular inequality we have on one hand

$$\begin{aligned}\mathtt{b}_z(x,y) &= d_\Omega(z,x) - d_\Omega(z,y)\\ &\geq d_\Omega(z,y_z) + d_\Omega(y_z,x) - d_\Omega(z,y_z) - d_\Omega(y_z,y)\\ &\geq d_\Omega(y,x) - 2d_\Omega(y_z,y) \underset{z\to\xi}{\longrightarrow} d_\Omega(y,x),\end{aligned}$$

while on the other hand $\mathtt{b}_z(x,y) \leq d_\Omega(x,y)$ for any $z \in \Omega$, by another triangular inequality. □

A first consequence is the following fact, originally due to Walsh. Note that Walsh who gave in [Wal08] a description of the horoboundary of properly convex open sets, with important consequences on the group of isometries of properly convex open sets.

Fact 6.1.2 ([Wal08, Th. 1.3])**.** *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set. Then the horocompactification $\overline{\Omega}^{\mathtt{h}}$ dominates the projective compactification $\overline{\Omega}$. We denote by $\pi_{\mathtt{h}}$ the map $\overline{\Omega}^{\mathtt{h}} \to \overline{\Omega}$.*

Proof. Consider two distinct points $\xi, \eta \in \partial\Omega$, and two sequences $(\xi_n)_n$ and $(\eta_n)_n$ in Ω that respectively converge to ξ and η in $\overline{\Omega}$, and to p and q in $\overline{\Omega}^{\mathtt{h}}$, and let us show that $p \neq q$. Fix a two-dimensional subspace $\mathrm{P}(W) \subset \mathrm{P}(\mathbf{V})$ that contains ξ and η, and intersect Ω. It is clear that we can find two points ξ' and $\eta' \in \mathrm{P}(W) \cap \partial\Omega$ such that $\xi \oplus \xi'$ and $\eta \oplus \eta'$ intersect at some point $x \in \Omega$, and such that $\xi' \oplus \eta'$ intersect Ω. Let $y \in [x,\eta')$ at Hilbert distance 1 from x. By Lemma 6.1.1, we have $\mathtt{b}_q(y,x) = 1$.

For any two points $x', y' \in \overline{\Omega}$, let $a_{x'y'}, b_{x'y'} \in \partial\Omega$ be such that $a_{x'y'}, x', y', b_{x'y'}$ are aligned in this order. For any n, we set $a_n = a_{y\xi_n}$ and $b_n = b_{y\xi_n}$, and denote by a_n'

(resp. b'_n) the intersection point of $[\xi_n, a_{y\xi_n}]$ and $[a_{yx}, a_{x\xi_n}]$ (resp. $[\xi_n, b_{y\xi_n}]$ and $[b_{yx}, b_{x\xi_n}]$). Finally, fix a normed affine chart that contains $\overline{\Omega}$, with norm denoted by $\|\cdot\|$. By definition of the Hilbert metric,

$$1 + d_\Omega(x, \xi_n) = d_\Omega(y, x) + d_\Omega(x, \xi_n) = \frac{1}{2}\log\frac{\|b'_n - y\| \cdot \|a'_n - \xi_n\|}{\|b'_n - \xi_n\| \cdot \|a'_n - y\|}$$

for any n. As a consequence,

$$\begin{aligned} \mathtt{b}_{\xi_n}(y, x) &= 1 + \frac{1}{2}\log\frac{\|b_n - y\| \cdot \|a_n - \xi_n\|}{\|b_n - \xi_n\| \cdot \|a_n - y\|} - \frac{1}{2}\log\frac{\|b'_n - y\| \cdot \|a'_n - \xi_n\|}{\|b'_n - \xi_n\| \cdot \|a'_n - y\|} \\ &\leq 1 + \frac{1}{2}\log\frac{\|a'_n - y\| \cdot \|a_n - \xi_n\|}{\|a'_n - \xi_n\| \cdot \|a_n - y\|} \end{aligned}$$

Observe that $(a_n)_n$ and $(a'_n)_n$ converge in $\mathrm{P}(\mathbf{V})$ to respectively $a_{y\xi}$ and the intersection point of $[a_{y\xi}, y]$ with $[\eta', \xi']$; these two points are different because we have chosen ξ', η' so that $\xi' \oplus \eta' \cap \Omega \neq \emptyset$. Thus $\mathtt{b}_p(y, x) < 1$ and $p \neq q$. □

An important consequence of Lemma 6.1.1 is that, if we are given $\xi \in \partial_{\mathtt{h}}\Omega$ and one horosphere centred at ξ, then we can geometrically describe all horospheres centred at ξ in terms of the given horosphere and the projection $\pi_{\mathtt{h}}(\xi) \in \partial\Omega$. More precisely, for all $x, y \in \Omega$ with $y \in [x, \pi_{\mathtt{h}}(\xi)]$, the map sending $x' \in \partial\mathcal{H}_\xi(x)$ to the unique point $y' \in [x', \pi_{\mathtt{h}}(\xi))$ at distance $d_\Omega(x, y)$ from x' is a homeomorphism from $\partial\mathcal{H}_\xi(x)$ onto $\partial\mathcal{H}_\xi(y)$; we then see the horospheres centred at ξ foliate the properly convex open set Ω. Observe also that horoballs are convex, since Hilbert balls are convex by Section 2.1.4, and their boundaries are the horospheres, which are homeomorphic to $\mathbb{R}^{\mathtt{d}-1}$.

6.1.2 Horospheres at smooth points of the projective boundary

As observed by Crampon–Marquis [CM14a] and Bray [Bra20a], regularity properties on the projective boundary $\partial\Omega$ have repercussions on the geometry and regularity of the horospheres.

If ξ is smooth, then it has only one preimage by $\pi_{\mathtt{h}}$, and the horospheres centred at ξ coincide with the *algebraic horospheres* defined by Cooper–Long–Tillmann [CLT15, §3], which are images under a projective transformation of the set of points $\eta \in \partial\Omega$ such that $\xi \oplus \eta \cap \Omega \neq \emptyset$. We will from now on abuse notation by identifying any smooth point of the projective boundary with its preimage in the horoboundary ; if $\partial\Omega$ is $\mathcal{C}^1$ then it identifies entirely with $\partial_{\mathtt{h}}\Omega$.

Fact 6.1.3 ([Bra20a, Lem. 3.2])**.** *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set. Let ξ be a smooth point of $\partial\Omega$. Then it has only one preimage by $\pi_{\mathtt{h}} : \partial_{\mathtt{h}}\Omega \to \partial\Omega$. More precisely, let $\eta \in \partial\Omega \smallsetminus T_\xi\partial\Omega$ and $x \in [\xi, \eta] \cap \Omega$, then the horosphere $\partial\mathcal{H}_\xi(x)$ is the image of the map that sends $\zeta \in \partial\Omega \smallsetminus T_\xi\partial\Omega$ to the intersection point of $[\xi, \zeta]$ with* $\mathrm{Span}(x, T_\xi\partial\Omega \cap \mathrm{Span}(\eta, \zeta))$

Recall that we have already considered these horospheres in Section 5.4.4.

Proof. For any two points $y, z \in \overline{\Omega}$, let $a_{yz}, b_{yz} \in \partial\Omega$ be such that a_{yz}, y, z, b_{yz} are aligned in this order. Fix $\zeta \in \partial\Omega \smallsetminus T_\xi\partial\Omega$, let A be the intersection point of $T_\xi\partial\Omega$ and $\mathrm{Span}(\eta, \zeta)$, and let y be the intersection point of $[\xi, \zeta]$ with $\mathrm{Span}(x, A)$. Figure 6.1 illustrates the previous notation. Let us show that $\mathtt{b}_z(x, y)$ converges to zero as $z \in \Omega$ tends to ξ.

We may assume that x, y and ξ are not aligned. For each $z \in \Omega$, let $A_z \in \mathrm{P}(\mathbf{V})$ be the intersection point of the lines $b_{xz} \oplus b_{yz}$ and $a_{xz} \oplus a_{yz}$, and let $y_z \in \Omega$ be the intersection

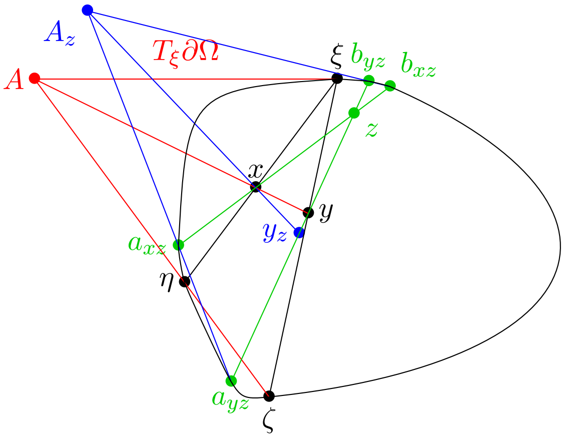

Figure 6.1: Illustration of the proof of Fact 6.1.3

point of $y \oplus z$ with $A_z \oplus x$. By definition of the Hilbert metric, $\mathtt{b}_z(x, y) = d_\Omega(y, y_z)$, so we only need to prove that y_z in Ω converges to y as z goes to ξ. It is enough for this to establish that A_z converges to A. Since a_{xz} tends to η and a_{yz} tends to ζ, any accumulation point of A_z belongs to $\eta \oplus \zeta$. Since b_{xz} and b_{yz} converge to ξ, any accumulation line of $b_{xz} \oplus b_{yz}$ contains ξ but does not intersect Ω, hence it is contained in $T_\xi\partial\Omega$, which therefore contains any accumulation point of A_z. □

One can refine the previous fact by showing that a point $\xi \in \partial\Omega$ has exactly one preimage in $\partial_{\mathtt{h}}\Omega$ if and only if it is a smooth point of $\partial\Omega$.

6.1.3 Conformal densities on the projective boundary

Definition 6.1.4. Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set and $\Gamma \subset \mathrm{Aut}(\Omega)$ a discrete subgroup. For $\delta \geq 0$, a *δ-conformal density on $\partial\Omega$* is the push-forward by $\pi_{\mathtt{h}}$ of a δ-conformal density on $\partial_{\mathtt{h}}\Omega$. Note that such a family is made of Γ-quasi-invariant measures.

Note that, if it was not true that $\partial_{\mathtt{h}}\Omega$ dominates $\partial\Omega$, we could have considered a common refinement of $\partial_{\mathtt{h}}\Omega$ and $\partial\Omega$, where the conformal densities are also well defined.

6.2 Construction of the Sullivan measures

In this section we apply the discussion in Section 1.4.3 to properly convex open sets equipped with their Hilbert metric, in order to define the Sullivan measures on the unit tangent bundle of convex projective orbifolds.

6.2.1 The Gromov product

Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set. Recall that the Gromov product of three points $x, \xi, \eta \in \Omega$ is defined by

$$\langle \xi, \eta \rangle_x := \frac{1}{2}\left(d_\Omega(x, \xi) + d_\Omega(x, \eta) - d_\Omega(\xi, \eta)\right) \geq 0, \tag{6.2.1}$$

and the following are three immediate properties of the Gromov product, for $y \in \Omega$:

$$x \in [\xi, \eta] \Rightarrow \langle \xi, \eta \rangle_x = 0; \tag{6.2.2}$$

$$\langle \xi, \eta \rangle_x = \langle \xi, \eta \rangle_y + \frac{1}{2}\mathtt{b}_\xi(x, y) + \frac{1}{2}\mathtt{b}_\eta(x, y); \tag{6.2.3}$$

$$|\langle \xi, \eta \rangle_x - \langle \xi, \eta \rangle_y| \leq d_\Omega(x, y). \tag{6.2.4}$$

We denote by $\mathrm{Geod}_{\mathtt{h}}(\Omega)$ (resp. $\mathrm{Geod}_{\mathtt{h}}^\infty(\Omega)$) the set of pairs (ξ, η) in $(\overline{\Omega}^{\mathtt{h}})^2$ (resp. $\partial_{\mathtt{h}}\Omega^2$) such that $[\xi, \eta] \cap \Omega \neq \emptyset$, where $[\xi, \eta] := [\pi_{\mathtt{h}}(\xi), \pi_{\mathtt{h}}(\eta)]$.

Proposition 6.2.1. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set. The map $(\xi, \eta, x) \mapsto \langle \xi, \eta \rangle_x$ defined on Ω^3 extends continuously to $\mathrm{Geod}_{\mathtt{h}}(\Omega) \times \Omega$, as well as (6.2.2), (6.2.3) and (6.2.4).*

Proof. Let $A := \{(\xi, \eta, x, y) \in \mathrm{Geod}_{\mathtt{h}}(\Omega) \times \Omega^2 : y \in [\xi, \eta]\}$. The projection $(\xi, \eta, x, y) \mapsto (\xi, \eta, x)$ from A to $\mathrm{Geod}_{\mathtt{h}}(\Omega) \times \Omega$ is continuous, surjective and open. Therefore, it is enough to prove that the function $(\xi, \eta, x, y) \mapsto \langle \xi, \eta \rangle_x$ defined on $A \cap \Omega^4$ extends continuously to A. This is a consequence of (6.2.2) and (6.2.3) combined, which yield, for any $(\xi, \eta, x, y) \in A \cap \Omega^4$,

$$\langle \xi, \eta \rangle_x = \frac{1}{2}\mathtt{b}_\xi(x, y) + \frac{1}{2}\mathtt{b}_\eta(x, y).$$

Note that (6.2.2) extends to $(\xi, \eta) \in \mathrm{Geod}_{\mathtt{h}}(\Omega)$ because the set $\{(\xi, \eta, x) \in \Omega^3 : x \in [\xi, \eta]\}$ is dense $\{(\xi, \eta, x) \in \mathrm{Geod}_{\mathtt{h}}(\Omega) \times \Omega : x \in [\xi, \eta]\}$. ☐

Recall that for general Gromov hyperbolic spaces, the extension to the boundary cannot always be made continuously, and requires taking a sup lim inf, (see [BH99, §III.H.3.15]). The fact that the Gromov product extends continuously to the projective boundary of $\mathcal{C}^1$ properly convex open sets was proved by Benoist [Ben06b, Lem. 5.2].

If Ω is strictly convex with $\mathcal{C}^1$ boundary, then we have identified $\partial_{\mathtt{h}}\Omega$ with $\partial\Omega$, and we see that the Gromov product is well defined on the set $\partial^2\Omega$ of pairs of distinct points of $\partial\Omega$.

6.2.2 The Hopf coordinates

Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set and $o \in \Omega$. The Hopf parametrisation based at o of the unit tangent bundle $T^1\Omega$ is the $(\phi_t)_t$-equivariant continuous surjective map

$$\mathrm{Hopf}_o \colon \mathrm{Geod}_{\mathtt{h}}^\infty(\Omega) \times \mathbb{R} \longrightarrow T^1\Omega,$$

that sends $(\xi, \eta, t) \in \mathrm{Geod}_{\mathtt{h}}^\infty(\Omega) \times \mathbb{R}$ to the vector $\mathrm{Hopf}_o(\xi, \eta, t)$ tangent to $[\xi, \eta] = [\pi_{\mathtt{h}}\xi, \pi_{\mathtt{h}}\eta]$ and such that $\mathtt{b}_\eta(o, \pi\mathrm{Hopf}_o(\xi, \eta, t)) = t$.

When the context is clear, we will simply write Hopf instead of Hopf_o. Changing the base-point for $x \in \Omega$ yields the following formula: for any $(\xi, \eta, t) \in \mathrm{Geod}_{\mathtt{h}}^\infty(\Omega) \times \mathbb{R}$,

$$\mathrm{Hopf}_x(\xi, \eta, t) = \mathrm{Hopf}_o(\xi, \eta, t + \mathtt{b}_\eta(o, x)).$$

We can lift to $\mathrm{Geod}_{\mathtt{h}}^\infty(\Omega) \times \mathbb{R}$ the three actions on the unit tangent bundle $T^1\Omega$ given by the geodesic flow, $\mathrm{Aut}(\Omega)$ and the flip involution (see Notation 2.1.1), so that the Hopf parametrisation is equivariant. Given $(\xi, \eta, t) \in \pi_{\mathtt{h}}^{-1}(\mathrm{Geod}^\infty(\Omega)) \times \mathbb{R}$ and $s \in \mathbb{R}$ and $\gamma \in \mathrm{Aut}(\Omega)$ these lifts may be written as

$$\phi_s(\xi, \eta, t) = (\xi, \eta, t + s), \tag{6.2.5}$$

$$\gamma \cdot (\xi, \eta, t) = (\gamma\xi, \gamma\eta, t + \mathtt{b}_\eta(\gamma^{-1}o, o)), \tag{6.2.6}$$

$$\iota(\xi, \eta, t) = (\eta, \xi, \langle \xi, \eta \rangle_o - t). \tag{6.2.7}$$

Observe that, apart from the action of the geodesic flow, these actions depends $o \in \Omega$.

6.2.3 The Sullivan measures

Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set, $\Gamma \subset \mathrm{Aut}(\Omega)$ a discrete subgroup, and set $M = \Omega/\Gamma$. Let $\delta \geq 0$ and let $(\mu_x^{\mathtt{h}})_x$ be a δ-conformal density on $\partial_{\mathtt{h}}\Omega$.

Using Sullivan's formula [Sul79, Prop. 11], we define the following measures, which are said to be *induced by* $(\mu_x^{\mathtt{h}})_x$.

- The *Sullivan measure* $m_{\mathbb{R}}^{\mathtt{h}}$ on $\mathrm{Geod}_{\mathtt{h}}^{\infty}(\Omega)$ is $\mathrm{d}m_{\mathbb{R}}^{\mathtt{h}}(\xi,\eta) = e^{2\delta\langle\xi,\eta\rangle_o}\,\mathrm{d}\mu_o^{\mathtt{h}}(\xi)\,\mathrm{d}\mu_o^{\mathtt{h}}(\eta)$ for $o \in \Omega$; this measure does not depend on o; it is Γ-invariant and invariant under $(\xi,\eta) \mapsto (\eta,\xi)$.
- The *Sullivan measure* $m_{\mathbb{R}}$ *on* $\mathrm{Geod}^{\infty}(\Omega)$ is the push-forward by $\pi_{\mathtt{h}}^2 : \partial_{\mathtt{h}}\Omega^2 \to \partial\Omega^2$ of the Sullivan measure on $\mathrm{Geod}_{\mathtt{h}}^{\infty}(\Omega)$; it is Γ-invariant and invariant under $(\xi,\eta) \mapsto (\eta,\xi)$.
- The *Sullivan measure* $m^{\mathtt{h}}$ on $\mathrm{Geod}_{\mathtt{h}}^{\infty}(\Omega)\times\mathbb{R}$ is defined as $\mathrm{d}m^{\mathtt{h}}(\xi,\eta,t) := \mathrm{d}m_{\mathbb{R}}^{\mathtt{h}}(\xi,\eta)\,\mathrm{d}t$, where $\mathrm{d}t$ is the Lebesgue measure on $\mathbb{R}$; this measure is invariant under the actions by $(\phi_t)_t$, Γ and ι given in (6.2.5), (6.2.6) and (6.2.7).
- The *Sullivan measure* m *on* $T^1\Omega$ is $m := (\mathrm{Hopf}_o)_* m^{\mathtt{h}}$, which does not depend on o; it is invariant under the actions by $(\phi_t)_t$, Γ and ι.
- The *Sullivan measure* m_Γ *on* T^1M is the quotient of m by Γ (see Section 1.2.3); it is invariant under the actions by ι and $(\phi_t)_t$.
- The *Sullivan measure* $m_\Gamma^{\mathtt{h}}$ *on* $\mathrm{Geod}_{\mathtt{h}}^{\infty}(\Omega)\times\mathbb{R}/\Gamma$ is the quotient of $m^{\mathtt{h}}$ by Γ; it is invariant under the natural actions by ι and $(\phi_t)_t$.

If $(\mu_x)_x$ is the puch-forward by $\pi_{\mathtt{h}}$ of $(\mu_x^{\mathtt{h}})_x$, then we say that the previous measures are *Sullivan measures associated to* $(\mu_x)_x$, although they are not a priori determined by $(\mu_x)_x$. (Although I do not know an example where $(\mu_x)_x$ is the push-forward of two different conformal densities on $\partial_{\mathtt{h}}\Omega$ that induce different Sullivan measures.)

A *Sullivan measure of dimension* δ is a Sullivan measures induced by a δ-conformal density.

Remark 6.2.2. The measure $m_{\mathbb{R}}$ (resp. $m_{\mathbb{R}}^{\mathtt{h}}$) is the quotient of m (resp. $m^{\mathtt{h}}$) by the action of $(\phi_t)_t$. This gives another way to define m, without using the Hopf parametrisation. Indeed for any measurable section $s : \mathrm{Geod}^{\infty}(\Omega) \to T^1\Omega$, the measure m is the push-forward of $m_{\mathbb{R}}$ times the Lebesgue measure on $\mathbb{R}$ by measurable isomorphism $f : \mathrm{Geod}^{\infty}(\Omega)\times\mathbb{R} \to T^1\Omega$, that sends (ξ,η,t) to $\phi_t(s(\xi,\eta))$. For instance, one can fix an affine chart containing $\overline{\Omega}$, and, for any $(\xi,\eta) \in \mathrm{Geod}^{\infty}(\Omega)$, take $s(\xi,\eta) \in T^1\Omega$ such that $\pi s(\xi,\eta)$ is the middle point of $[\xi,\eta]$ and $\phi_\infty s(\xi,\eta) = \eta$; in this case the section s is continuous, and the associated map $f : \mathrm{Geod}^{\infty}(\Omega)\times\mathbb{R} \to T^1\Omega$ is a homeomorphism.

Remark 6.2.3. Suppose $\Gamma' \subset \Gamma$ is a torsion-free, finite-index, normal subgroup. Then $\delta_{\Gamma'} = \delta_\Gamma$ and $(\mu_x)_x$ is Γ'-equivariant. Let $m_{\Gamma'}$ be the associated Sullivan measure on the quotient $T^1\Omega/\Gamma'$. Let $\pi_\Gamma^{\Gamma'} : T^1\Omega/\Gamma' \to T^1\Omega/\Gamma$ be the natural projection. Then

$$m_\Gamma = \frac{1}{[\Gamma:\Gamma']}(\pi_\Gamma^{\Gamma'})_* m_{\Gamma'}.$$

Observation 6.2.4. *Let* $\Omega \subset \mathrm{P}(\mathbf{V})$ *be a properly convex open set and* $\Gamma \subset \mathrm{Aut}(\Omega)$ *a discrete subgroup. Let* $\delta \geq 0$ *and let* $(\mu_x^{\mathtt{h}})_x$ *be a* δ*-conformal density on* $\partial_{\mathtt{h}}\Omega$*. Then the Sullivan measures on* $\mathrm{Geod}_{\mathtt{h}}(\Omega)\times\mathbb{R}$*,* $T^1\Omega$ *and* T^1M *are Radon. If* Γ *is strongly irreducible and* $T^1M_{\mathrm{bip}} \neq \emptyset$*, or if* M *is rank-one and non-elementary, then these measures are non-zero.*

Proof. The fact that these measures are Radon is due to the fact that $\mu_x^{\mathtt{h}}$ is finite for any $x \in \Omega$, and $\langle \xi, \eta \rangle_x$ depends continuously on ξ, η, x.

Suppose that Γ is strongly irreducible and $T^1 M_{\mathrm{bip}} \neq \emptyset$, or that M is rank-one and non-elementary. Then there exists $(\xi, \eta) \in \mathrm{Geod}^\infty(\Omega) \cap (\Lambda_\Gamma)^2$. Let U and V be neighbourhoods of ξ and η in $\partial\Omega$ such that $U \times V \subset \mathrm{Geod}^\infty(\Omega)$. Since $\mu_o = \pi_{\mathtt{h}*}\mu_o^{\mathtt{h}}$ is Γ-quasi-invariant, its support is Γ-invariant and hence contains Λ_Γ (see Remark 2.3.5 and Fact 3.2.2). Therefore, $\mu_o^2(U \times V) > 0$, and $m^{\mathtt{h}}(\pi_{\mathtt{h}}^{-1}(U) \times \pi_{\mathtt{h}}^{-1}(V) \times \mathbb{R}) > 0$. □

6.3 The Shadow lemma

In this section we establish the Shadow lemma (Lemma 6.3.1) which consists of estimates on the measures of shadows. The measure is a conformal density on $\partial\Omega$, and shadows are subsets of the projective boundary, defined as follows. Recall that the Shadow lemma is a classical result in the theory of conformal densities, and we adapt here its classical proof to the convex projective setting.

Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set. Take $x, y \in \Omega$ and $r > 0$. Set

$$\mathcal{O}_r(x,y) := \{\xi \in \partial\Omega : [x,\xi] \cap B_\Omega(y,r) \neq \emptyset\}; \tag{6.3.1}$$
$$\mathcal{O}_r^+(x,y) := \{\xi \in \partial\Omega : \exists z \in B_\Omega(x,r) \text{ such that } [z,\xi] \cap B_\Omega(y,r) \neq \emptyset\}; \tag{6.3.2}$$
$$\mathcal{O}_r^-(x,y) := \{\xi \in \partial\Omega : \forall z \in B_\Omega(x,r), [z,\xi] \cap B_\Omega(y,r) \neq \emptyset\}. \tag{6.3.3}$$

See in Figure 6.2 an example of a shadow. Note that

$$\mathcal{O}_r^-(x,y) \subset \mathcal{O}_r(x,y) \subset \mathcal{O}_r^+(x,y) \subset \mathcal{O}_{2r}(x,y). \tag{6.3.4}$$

The first two inclusions are immediate consequences of the definition; let us prove the last one. If $x' \in \overline{B}(x,r)$, $y' \in \overline{B}(y,r)$ and $\xi \in \partial\Omega$ are aligned in this order, then the point $y'' \in [x\xi)$ at distance $d_\Omega(x', y')$ from x is at distance at most $2r$ from y since by Lemma 2.1.4 it is at distance at most r from y'.

The Shadow lemma is the following.

Lemma 6.3.1. *Let $o \in \Omega \subset \mathrm{P}(\mathbf{V})$ be a pointed properly convex open set and $\Gamma \subset \mathrm{Aut}(\Omega)$ a discrete subgroup; set $M = \Omega/\Gamma$. Suppose that Γ is strongly irreducible and $T^1 M_{\mathrm{bip}}$ is non-empty, or that M is rank-one and non-elementary. Consider $\delta \geq 0$ and a δ-conformal density $(\mu_x)_{x \in \Omega}$ on $\partial\Omega$. Then there exists $R_0 > 0$ such that for any $R \geq R_0$, one can find $C = C(R) > 0$ such that for each $\gamma \in \Gamma$,*

$$C^{-1} e^{-\delta d_\Omega(o,\gamma o)} \leq \mu_o(\mathcal{O}_R(o,\gamma o)) \leq \mu_o(\mathcal{O}_R^+(o,\gamma o)) \leq C e^{-\delta d_\Omega(o,\gamma o)}.$$

We will actually need two more Shadow lemmas: Lemma 6.3.5 and Corollary 7.1.2. They both make stronger assumptions on the convex projective orbifold $M = \Omega/\Gamma$, and either are consequences of Lemma 6.3.1, or have a very similar proof.

6.3.1 Preliminaries

In this section we prove two classical intermediate lemmas, used in the proof of the Shadow lemma.

Lemma 6.3.2. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set. Consider $\xi \in \overline{\Omega}^{\mathtt{h}}$, $x, y \in \Omega$ and $r > 0$. If $\pi_{\mathtt{h}}(\xi) \in \mathcal{O}_r^+(x,y)$ (see (6.3.2)), then*

$$d_\Omega(x,y) - 4r \leq \mathtt{b}_\xi(x,y) \leq d_\Omega(x,y).$$

Proof. By definition of $\mathcal{O}_r^+(x,y)$, we can find $x' \in B_\Omega(x,r)$ and $y' \in B_\Omega(y,r) \cap [x',\xi]$. Then $\mathtt{b}_\xi(x',y') = d_\Omega(x',y')$ by Lemma 6.1.1, hence

$$\begin{aligned}\mathtt{b}_\xi(x,y) &= \mathtt{b}_\xi(x,x') + \mathtt{b}_\xi(x',y') + \mathtt{b}_\xi(y',y)\\ &\geq -d_\Omega(x,x') + d_\Omega(x',y') - d_\Omega(y',y)\\ &\geq d_\Omega(x,y) - 2d_\Omega(x,x') - 2d_\Omega(y,y')\\ &\geq d_\Omega(x,y) - 4r.\end{aligned}$$

□

Lemma 6.3.3. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set and $\Gamma \subset \mathrm{Aut}(\Omega)$ a discrete subgroup; set $M = \Omega/\Gamma$. Suppose that Γ is strongly irreducible and T^1M_{bip} is non-empty, or that M is rank-one and non-elementary. Consider a Γ-quasi-invariant finite Borel measure ν on $\partial\Omega$. Then there is $\epsilon > 0$ and $R > 0$ such that for all $x \in \Omega$,*

$$\nu(\mathcal{O}_R(x,o)) \geq \epsilon.$$

Proof. By contradiction suppose that there are sequences $(R_n)_n \in \mathbb{R}_{>0}^{\mathbb{N}}$ and $(x_n)_n \in \Omega$ such that

$$R_n \underset{n\to\infty}{\longrightarrow} \infty \text{ and } \nu(\mathcal{O}_{R_n}(x_n,o)) \underset{n\to\infty}{\longrightarrow} 0.$$

We can assume that $(x_n)_n$ converges to some $\xi \in \overline{\Omega}$. If $\xi \in \Omega$ then for n such that $R_n \geq d_\Omega(o,\xi) + 1$ and $d_\Omega(x_n,\xi) < 1$, we have $\mathcal{O}_{R_n}(x_n,o) = \partial\Omega$ which is absurd; hence $\xi \in \partial\Omega$. We claim that $\nu(\overline{B}_{\mathrm{spl}}(\xi,1)) = 1$ (recall the definition of d_{spl} from Section 2.1.10). It enough to prove that

$$\partial\Omega \smallsetminus \overline{B}_{\mathrm{spl}}(\xi,1) \subset \bigcup_n \bigcap_{k\geq n} \mathcal{O}_{R_k}(x_k,o).$$

See Figure 6.2. Let $\eta \in \partial\Omega \smallsetminus \overline{B}_{\mathrm{spl}}(\xi,1)$, and $z \in [\xi,\eta] \cap \Omega$. Since $(x_n)_n$ converges to ξ, we can find $z_n \in [x_n,\eta] \cap B_\Omega(z,1)$ for any large enough n. On the other hand, $R_n \geq d_\Omega(o,z)+2$ for n large. Thus, $z_n \in B_\Omega(o,R_n)$ and hence $\eta \in \mathcal{O}_{R_n}(x_n,o)$ for any large enough n, so $\nu(\overline{B}_{\mathrm{spl}}(\xi,1)) = 1$.

Since Λ_Γ is the smallest Γ-invariant closed subset of $\overline{\Omega}$ (Remark 2.3.5 and Proposition 3.2.2), and ν is Γ-quasi-invariant, we deduce that Λ_Γ is contained in the support of ν. In order to get a contradiction, let us prove that Λ_Γ is not contained in $\overline{B}_{\mathrm{spl}}(\xi,1)$. By assumption we can find $\eta,\eta' \in \Lambda_\Gamma$ such that $(\eta \oplus \eta') \cap \Omega \neq \emptyset$. Using again that $\Lambda_\Gamma \subset \overline{\Omega}$ is minimal for the action of Γ, we deduce the existence of a sequence $(\gamma_n)_n \in \Gamma^{\mathbb{N}}$ such that $(\gamma_n\xi)_n$ converges to η. But then $(\overline{B}_{\mathrm{spl}}(\gamma_n\xi,1))_n$ sub-converges to $\overline{B}_{\mathrm{spl}}(\eta,1)$ (i.e. any accumulation point for the Hausdorff topology is contained in $\overline{B}_{\mathrm{spl}}(\eta,1)$); hence $\overline{B}_{\mathrm{spl}}(\gamma_n\xi,1)$ does not contain η' for n large enough. □

We will need exactly twice a refined version of the previous lemma, which bounds from below the measure of *scarce shadows*. It will be used to prove the refined version of the Shadow lemma for scarce shadows (Lemma 6.3.5), and to prove Proposition 6.5.1. This version needs M to be rank-one, and the statement is a bit more technical. The proof is almost the same.

Lemma 6.3.4. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set and $\Gamma \subset \mathrm{Aut}(\Omega)$ a discrete subgroup. Suppose $M = \Omega/\Gamma$ is rank-one and non-elementary. Consider a Γ-quasi-invariant Borel finite measure ν on $\partial\Omega$. Then there exist $\epsilon > 0$ and $R_0 > 0$ such that $\nu(\mathcal{O}_R^-(x,o)) \geq \epsilon$ for all $R \geq R_0$ and $x \in \Omega \smallsetminus B_\Omega(o,2R)$.*

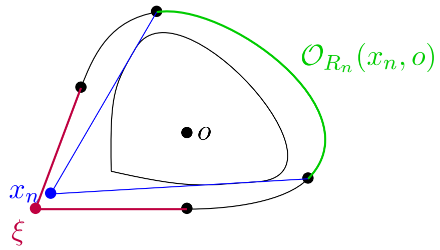

Figure 6.2: The sequence of increasing shadows in the proof of Lemma 6.3.3

Proof. By contradiction we suppose the existence of sequences $(R_n)_n \in \mathbb{R}_{>0}^{\mathbb{N}}$ and $(x_n)_n \in \Omega^{\mathbb{N}}$ such that for all $n \in \mathbb{N}$,

$$R_n \underset{n\to\infty}{\longrightarrow} \infty, \text{ while } d_\Omega(x_n, o) - R_n \underset{n\to\infty}{\longrightarrow} \infty \text{ and } \nu(\mathcal{O}^-_{R_n}(x_n, o)) \underset{n\to\infty}{\longrightarrow} 0.$$

We can assume, up to extracting, that $(x_n)_n$ converges to some point $\xi \in \partial\Omega$. We claim that $\nu(\overline{B}_{\mathrm{spl}}(\xi, 2)) = 1$. It is enough to prove that

$$\partial\Omega \smallsetminus \overline{B}_{\mathrm{spl}}(\xi, 2) \subset \bigcup_n \bigcap_{k\geq n} \mathcal{O}^-_{R_k}(x_k, o).$$

Let $\eta \in \partial\Omega \smallsetminus \overline{B}_{\mathrm{spl}}(\xi, 2)$. Fixing an affine chart containing $\overline{\Omega}$, we can consider for each n the following compact subsets of Ω:

$$K := \frac{1}{2}(\overline{B}_{\mathrm{spl}}(\xi, 1) - \eta) + \eta \text{ and } K_n := \frac{1}{2}(\overline{B}_\Omega(x_n, R_n) - \eta) + \eta,$$

where the map $x \mapsto (x-\eta)/2+\eta$ is defined on the affine chart as the homothety centred at η and with ratio one half. We observe that all accumulation points of the sequence $(K_n)_n$ are contained in K. Indeed, any accumulation point of $(\overline{B}_\Omega(x_n, R_n))_n$ is convex, contains ξ and is moreover contained in $\partial\Omega$ since x_n goes faster to infinity than R_n; hence it is contained in $\overline{B}_{\mathrm{spl}}(\xi, 1)$. Therefore we can find n large enough so that $B_\Omega(o, R_n)$ contains a neighbourhood U of K, and so that K_n is contained in U.

Since Λ_Γ is the smallest Γ-invariant closed subset of $\overline{\Omega}$ (Proposition 3.2.2), and ν is Γ-quasi-invariant, we deduce that Λ_Γ is contained in the support of ν. In order to get a contradiction, let us prove that Λ_Γ is not contained in $\overline{B}_{\mathrm{spl}}(\xi, 2)$. By assumption we can find two distinct points η, η' in $\Lambda_\Gamma \cap \partial_{\mathrm{sse}}\Omega$. Using again that $\Lambda_\Gamma \subset \overline{\Omega}$ is minimal for the action of Γ, we deduce the existence of a sequence $(\gamma_n)_n \in \Gamma^{\mathbb{N}}$ such that $(\gamma_n\xi)_n$ converges to η. But then $(\overline{B}_{\mathrm{spl}}(\gamma_n\xi, 2))_n$ sub-converges to $\overline{B}_{\mathrm{spl}}(\eta, 2) = \{\eta\}$; hence $\overline{B}_{\mathrm{spl}}(\gamma_n\xi, 2)$ does not contain η' for n large enough. □

6.3.2 Proof of Lemma 6.3.1 and another Shadow lemma

Proof of Lemma 6.3.1. Let $(\mu^{\mathtt{h}}_x)_{x\in\Omega}$ be a δ-conformal density whose push-forward is $(\mu_x)_x$. We compute for $\alpha \in \{\emptyset, +, -\}$:

$$\begin{aligned}
\mu_o(\mathcal{O}^\alpha_R(o, \gamma o)) &= \mu^{\mathtt{h}}_o(\pi_{\mathtt{h}}^{-1}(\mathcal{O}^\alpha_R(o, \gamma o))) \\
&= \mu^{\mathtt{h}}_{\gamma^{-1}o}(\pi_{\mathtt{h}}^{-1}(\mathcal{O}^\alpha_R(\gamma^{-1}o, o))) \\
&= \int_{\pi_{\mathtt{h}}^{-1}(\mathcal{O}^\alpha_R(\gamma^{-1}o, o))} e^{-\delta \mathtt{b}_\xi(\gamma^{-1}o, o)} d\mu^{\mathtt{h}}_o(\xi).
\end{aligned}$$

On one hand $\mathtt{b}_\xi(\gamma^{-1}o,o) \geq d_\Omega(o,\gamma^{-1}o) - 4R$ for $\xi \in \pi_{\mathtt{h}}^{-1}(\mathcal{O}_R^{\alpha}(\gamma^{-1}o,o))$ by Lemma 6.3.2, so

$$\begin{aligned}\mu_o(\mathcal{O}_R^+(o,\gamma o)) &\leq \int_{\pi_{\mathtt{h}}^{-1}(\mathcal{O}_R^+(\gamma^{-1}o,o))} e^{-\delta d_\Omega(o,\gamma o)+4\delta R} d\mu_o^{\mathtt{h}}(\xi) \\ &\leq e^{4\delta R} e^{-\delta d_\Omega(o,\gamma o)} \mu_o(\mathcal{O}_R^+(\gamma^{-1}o,o)) \\ &\leq e^{4\delta R} e^{-\delta d_\Omega(o,\gamma o)} \mu_o(\partial\Omega).\end{aligned}$$

On the other hand, we can use Lemma 6.3.3, to obtain $\epsilon > 0$ and R_0 such that for $R \geq R_0$ and for $\gamma \in \Gamma$ with $d_\Omega(o,\gamma o) \geq R_0$,

$$\begin{aligned}\mu_o(\mathcal{O}_R(o,\gamma o)) &\geq \int_{\pi_{\mathtt{h}}^{-1}(\mathcal{O}_R(\gamma^{-1}o,o))} e^{-\delta d_\Omega(o,\gamma o)} d\mu_o^{\mathtt{h}}(\xi) \\ &\geq \mu_o(\mathcal{O}_R(\gamma^{-1}o,o)) e^{-\delta d_\Omega(o,\gamma o)} \\ &\geq \epsilon e^{-\delta d_\Omega(o,\gamma o)}.\end{aligned}$$

□

As for Lemma 6.3.3, there exists a refined version of the Shadow lemma (Lemma 6.3.1) for scarce shadows. Its proof is exactly the same as that of Lemma 6.3.1, except that we use instead Lemma 6.3.3.

Lemma 6.3.5. *Let $o \in \Omega \subset \mathrm{P}(\mathbf{V})$ be a pointed properly convex open set and $\Gamma \subset \mathrm{Aut}(\Omega)$ a discrete subgroup. Suppose $M = \Omega/\Gamma$ is rank-one and non-elementary. Consider $\delta \geq 0$ and a δ-conformal density $(\mu_x)_{x\in\Omega}$ on $\partial\Omega$. Then there exists $R_0 > 0$ such that for any $R \geq R_0$, one can find $C = C(R) > 0$ such that for each $\gamma \in \Gamma$ satisfying $d_\Omega(o,\gamma o) \geq 2R$,*

$$C^{-1} e^{-\delta d_\Omega(o,\gamma o)} \leq \mu_o(\mathcal{O}_R^-(o,\gamma o)).$$

Proof. Let $(\mu_x^{\mathtt{h}})_{x\in\Omega}$ be a δ-conformal density whose push-forward is $(\mu_x)_x$. We make the same computation as in the proof of Lemma 6.3.1:

$$\mu_o(\mathcal{O}_R^-(o,\gamma o)) = \int_{\pi_{\mathtt{h}}^{-1}(\mathcal{O}_R^-(\gamma^{-1}o,o))} e^{-\delta \mathtt{b}_\xi(\gamma^{-1}o,o)} d\mu_o^{\mathtt{h}}(\xi).$$

We can use Lemma 6.3.4, to obtain $\epsilon > 0$ and R_0 such that for $R \geq R_0$ and for $\gamma \in \Gamma$ such that $d_\Omega(o,\gamma o) \geq 2R$,

$$\begin{aligned}\mu_o(\mathcal{O}_R^-(o,\gamma o)) &\geq \int_{\pi_{\mathtt{h}}^{-1}(\mathcal{O}_R^-(\gamma^{-1}o,o))} e^{-\delta d_\Omega(o,\gamma o)} d\mu_o^{\mathtt{h}}(\xi) \\ &\geq \mu_o(\mathcal{O}_R^-(\gamma^{-1}o,o)) e^{-\delta d_\Omega(o,\gamma o)} \\ &\geq \epsilon e^{-\delta d_\Omega(o,\gamma o)}.\end{aligned}$$

□

6.3.3 First consequences

In this section we deduce from the Shadow lemma that there is no conformal density with parameter $0 \leq \delta < \delta_\Gamma$. We also prove that conformal densities give zero measure to open faces of the conical limit set.

Proposition 6.3.6. *Let $o \in \Omega \subset \mathrm{P}(\mathbf{V})$ be a pointed properly convex open set and $\Gamma \subset \mathrm{Aut}(\Omega)$ a discrete subgroup; denote $M = \Omega/\Gamma$. Suppose that Γ is strongly irreducible and T^1M_{bip} is non-empty, or that M is rank-one and non-elementary. Consider $\delta \geq 0$ such that there exists a δ-conformal density on $\partial\Omega$. Then $\delta \geq \delta_\Gamma$, and there is some constant $C > 0$ such that*

$$\#\{\gamma \in \Gamma : d_\Omega(o,\gamma o) \leq r\} \leq C e^{\delta_\Gamma r}.$$

Proof. Let $(\mu_x)_{x\in\Omega}$ be a δ-conformal density on $\partial\Omega$. We consider R and $C>0$ from the Shadow lemma (Lemma 6.3.1), such that for each automorphism $\gamma\in\Gamma$, $\mu_o(\mathcal{O}_R(o,\gamma o))\geq C^{-1}e^{-\delta d_\Omega(o,\gamma o)}$. For each $r>0$ we give ourselves a maximal $(1+4R)$-separated subset F_r of $\Gamma\cdot o\cap B_\Omega(o,r+1)\smallsetminus B_\Omega(o,r)$. One can easily see that the shadows $(\mathcal{O}_R(o,x))_{x\in F_r}$ are pairwise disjoint, therefore

$$\begin{aligned}
1 &\geq \sum_{x\in F_r}\mu_o(\mathcal{O}_R(o,x))\\
&\geq C^{-1}\sum_{x\in F_r}e^{-\delta d_\Omega(o,x)}\\
&\geq C^{-1}e^{-\delta}e^{-\delta r}\#F_r\\
&\geq C^{-1}e^{-\delta}\#\{\gamma : d_\Omega(o,\gamma o)\leq 1+4R\}^{-1}e^{-\delta r}\#\{\gamma : \gamma o\in B_\Omega(o,r+1)\smallsetminus B_\Omega(o,r)\}.
\end{aligned}$$

This implies that $\#\{\gamma : d_\Omega(o,\gamma o)\leq r\}\leq C'e^{\delta r}$ for any $r>0$, for some $C'>0$ independent of r. By definition, this implies that $\delta\geq\delta_\Gamma$, and since by Fact 1.4.1 there exists a δ_Γ-conformal density, $\#\{\gamma : d_\Omega(o,\gamma o)\leq r\}\leq C''e^{\delta_\Gamma r}$ for any $r>0$, for some $C''>0$ independent of r. □

Proposition 6.3.7. *Let $\Omega\subset \mathrm{P}(\mathbf{V})$ be a properly convex open set and $\Gamma\subset\mathrm{Aut}(\Omega)$ a discrete subgroup; set $M=\Omega/\Gamma$. Suppose that Γ is strongly irreducible and T^1M_{bip} is non-empty, or that M is rank-one and non-elementary. Consider $\delta\geq 0$ and a δ-conformal density $(\mu_x)_{x\in\Omega}$ on $\partial\Omega$. Then $\mu_x(F_\Omega(\xi))=0$ for all $x\in\Omega$ and $\xi\in\Lambda^{\mathrm{con}}_\Gamma$.*

Proof. Let $o\in\Omega$. Observe that the open face $F_\Omega(\xi)$ is contained in $\Lambda^{\mathrm{con}}_\Gamma$ by [DGKa, Cor. 4.10]. It's enough to prove that $\mu_o(B_{\overline{\Omega}}(\xi,R))=0$ for any $R>0$ (recall the definition of $d_{\overline{\Omega}}$ from Section 2.1.10). By definition, there are sequences $(\gamma_n)_n\in\Gamma^{\mathbb{N}}$ and $(x_n)_n\in[o,\xi)^{\mathbb{N}}$ going to infinity such that $(d_\Omega(\gamma_n o,x_n)_n$ is bounded; denote $R'=\sup_n d_\Omega(\gamma_n o,x_n)$. Using the Shadow lemma (Lemma 6.3.1), we can find $C>0$ such that for any n,

$$\mu_o(B_{\overline{\Omega}}(\xi,R))\leq\mu_o(\mathcal{O}_R(o,x_n))\leq\mu_o(\mathcal{O}_{R+R'}(o,\gamma_n o))\leq Ce^{-\delta d_\Omega(o,\gamma_n o)}.$$

This last term goes to zero as n goes to infinity since $\delta\geq\delta_\Gamma>0$ (by Fact 2.3.18 and Proposition 6.3.6). □

6.4 The convergent case of the HTSR dichotomy

In this section, we establish the convergent case of the HTSR dichotomy (Theorem 6.0.1.1).

6.4.1 The conical limit set has zero measure

We prove that, in the convergent case of the HTSR dichotomy, any δ-conformal density gives zero measure to the conical limit set. Recall that, given $\Omega\subset\mathrm{P}(\mathbf{V})$ properly convex open, $\Gamma\subset\mathrm{Aut}(\Omega)$ and $o\in\Omega$, the conical limit set $\Lambda^{\mathrm{con}}_\Gamma$ is the union over $R>0$ of the sets $\Lambda^{\mathrm{con}}_R(\Gamma,\Omega,o)=\Lambda^{\mathrm{con}}_R$, consisting of points $\xi\in\partial\Omega$ for which there exists a sequence $(\gamma_n)_{n\in\mathbb{N}}$ in Γ going to infinity such that $d_\Omega([o,\xi),\gamma_n o)<R$ for each $n\in\mathbb{N}$.

Proposition 6.4.1. *Let $\Omega\subset\mathrm{P}(\mathbf{V})$ be a properly convex open set and $\Gamma\subset\mathrm{Aut}(\Omega)$ a discrete subgroup. Suppose that Γ is strongly irreducible and T^1M_{bip} is non-empty, or that M is rank-one and non-elementary. Let $\delta\geq\delta_\Gamma$ with $\sum_{\gamma\in\Gamma}e^{-\delta d_\Omega(o,\gamma o)}$ finite, and consider a δ-conformal density $(\mu_x)_{x\in\Omega}$ on $\partial\Omega$. Then $\mu_x(\Lambda^{\mathrm{con}}_\Gamma)=0$ for any $x\in\Omega$.*

Proof. We consider $R > 0$ and prove that $\mu_o(\Lambda_R^{\mathrm{con}}) = 0$. By definition, for any $r > 0$, the set $\Lambda_{\Gamma,R}^{\mathrm{con}}$ is contained in the union, over all $\gamma \in \Gamma$ such that $d_\Omega(o, \gamma o) \geq r$, of the shadows $\mathcal{O}_R(o, \gamma o)$. As a consequence, by the Shadow lemma (Lemma 6.3.1), we can find a constant $C > 0$ such that

$$\mu_o(\Lambda_R^{\mathrm{con}}) \leq C \sum_{\gamma : d_\Omega(o,\gamma o) \geq r} e^{-\delta d_\Omega(o,\gamma o)}$$

for any $r > 0$, and this last quantity converges to zero as r goes to infinity. □

6.4.2 Ergodic implies non-atomic

In this section we prove that, given $\Omega \subset \mathrm{P}(\mathbf{V})$ properly convex open, $\Gamma \subset \mathrm{Aut}(\Omega)$ discrete, $(\mu_x)_x$ conformal density on $\partial\Omega$, and $o \in \Omega$, if the action of Γ on $(\mathrm{Geod}^\infty(\Omega), \nu_o^2)$ is ergodic, then ν_o^2 is non-atomic. This will only be used to prove that in the convergent case of the HTSR dichotomy, the Sullivan measure is not ergodic under the geodesic flow. The proof is based on an idea of Roblin [Rob03, Item c p. 19].

Proposition 6.4.2. *Let $o \in \Omega \subset \mathrm{P}(\mathbf{V})$ be a pointed properly convex open set and $\Gamma \subset \mathrm{Aut}(\Omega)$ a discrete subgroup. Suppose $M = \Omega/\Gamma$ is rank-one and non-elementary. Consider $\delta \geq \delta_\Gamma$ and a δ-conformal density $(\mu_x)_{x\in\Omega}$ on $\partial\Omega$. Assume that the action of Γ on $(\mathrm{Geod}^\infty(\Omega), \mu_o^2)$ is ergodic. Then μ_o is non-atomic.*

Proof. We assume by contradiction that there exists $\xi \in \partial\Omega$ such that $\mu_o(\{\xi\}) > 0$. Since M is rank-one and non-elementary, there exist two distinct points $\eta, \eta' \in \Lambda_\Gamma \cap \partial_{\mathrm{sse}}\Omega \smallsetminus \{\xi\}$. Since the action of Γ on Λ_Γ is minimal (Proposition 3.2.2), we can find sequences $(g_n)_n$ and $(g'_n)_n$ in $\Gamma^{\mathbb{N}}$ such that $(g_n\xi)_n$ converges to η and $(g'_n\xi)_n$ converges to η'; furthermore, given an increasing sequence of compact subsets $(K_n)_n$ of Γ whose union covers Γ, we can find $(g_n)_n$ and $(g'_n)_n$ such that $\gamma g_n \xi \neq g'_n\xi$ for all $n \in \mathbb{N}$ and $\gamma \in K_n$.

For n large enough, since η and η' are strongly extremal, $(\xi, g_n\xi)$ and $(\xi, g'_n\xi)$ belong to $\mathrm{Geod}^\infty(\Omega)$. Moreover $\mu_o^2(\{(\xi, g_n\xi)\}) > 0$, so $\Gamma\cdot(\xi, g_n\xi)$ is a Γ-invariant subset of $\mathrm{Geod}^\infty(\Omega)$ with positive measure, by ergodicity it must have full measure. Since $\mu_o^2(\{(\xi, g'_n\xi)\})$ is also positive, $(\xi, g'_n\xi)$ must belong to $\Gamma\cdot(\xi, g_n\xi)$, i.e. there exists $\gamma_n \in \Gamma$ such that $\gamma_n\xi = \xi$ and $\gamma_n g_n \xi = g'_n\xi$. By construction, $\gamma_n \notin K_n$ for any n, so $(\gamma_n)_n$ goes to infinity.

Fix x (resp. x') on the geodesic (ξ, η') (resp. (ξ, η')). For any n, let x_n (resp. x'_n) be a closest point to x (resp. x') of the geodesic $(\xi, g_n\xi)$ (resp. $(\xi, g'_n\xi)$); the sequence $(x_n)_n$ (resp. $(x'_n)_n$) converges to x (resp. x').

The four points $\xi, x'_n, \gamma_n x_n, g'_n\xi$ are aligned for each n, either in this order or in the order $\xi, \gamma_n x_n, x'_n, g'_n\xi$. Up to extraction, and up to exchanging $(g_n)_n$ and $(g'_n)_n$, we may assume that $\gamma_n x_n \in [x'_n, g'_n\xi]$ for each n.

Since $(\gamma_n)_n$ goes to infinity, so does $(d_\Omega(x'_n, \gamma_n x_n))_n$; this implies that $(\gamma_n x_n)_n$ tends to η', as $(x'_n)_n$ tends to x' and $(g'_n\xi)_n$ tends to η'. Similarly, since $\gamma_n^{-1}x'_n \in [x_n, \xi]$ for each n, the sequence $(\gamma_n^{-1}x'_n)_n$ tends to ξ.

Thus $(\gamma_n^{-1}x')_n$ tends to ξ (since $(x'_n)_n$ tends to x'), and $(\gamma_n^{-1}x')_n$ tends to η' by lower semi-continuity of $d_{\overline{\Omega}}$, because η' is extremal. By Corollary 5.4.12 and since η' is strongly extremal distinct from ξ, this implies that γ_n is rank-one for n large enough, with $(x_{\gamma_n}^+)_n$ converging to η' and $x_{\gamma_n}^- = \xi$. By Corollary 3.3.5 and discreteness of Γ, this means that $x_{\gamma_n}^+ = \eta'$.

We have proved that for any pair of distinct points η, η' in $\Lambda_\Gamma \cap \partial_{\mathrm{sse}}\Omega \smallsetminus \{\xi\}$, there exists $\gamma \in \Gamma$ rank-one with $x_\gamma^- = \xi$ and $x_\gamma^+ \in \{\eta, \eta'\}$. This contradicts Corollary 3.3.5 and the fact that $\Lambda_\Gamma \cap \partial_{\mathrm{sse}}\Omega$ is infinite. □

6.4.3 Proof of Theorem 6.0.1.1

Let $\mu_o^{\mathtt{h}}$ be a δ_Γ-conformal density on $\partial_{\mathtt{h}}\Omega$ such that $(\pi_{\mathtt{h}})_*\mu_o^{\mathtt{h}} = \mu_o$, and let $\tilde{m}$ be the Sullivan measure on $T^1\Omega$ induced by $\mu_o^{\mathtt{h}}$. According to Proposition 6.3.6, we have $\delta \geq \delta_\Gamma$. Let us assume that $\sum_\gamma e^{-\delta d_\Omega(o,\gamma o)}$ is convergent.

By Fact 1.2.5, in order to prove that $(T^1M, (\phi_t)_t, m_\Gamma)$ and $(\mathrm{Geod}^\infty(\Omega), \Gamma, m_\mathbb{R})$ are dissipative, it is enough to prove that $(T^1\Omega, \Gamma \times \mathbb{R}, m)$ is dissipative. If by contradiction it is not the case, then the conservative part contains a compact subset K of positive measure. This means that for almost any vector $v \in K$,

$$\infty = \int_{\Gamma\times\mathbb{R}} 1_K(v) = \sum_{\gamma\in\Gamma}\int_{-\infty}^{\infty} 1_K(\gamma\phi_t v)\,\mathrm{d}t,$$

hence there exist diverging sequences $(\gamma_n)_n \in \Gamma^\mathbb{N}$ and $(t_n)_n \in \mathbb{R}^\mathbb{N}$ such that $\gamma_n\phi_{t_n}v \in K$; if for example $(t_n)_n$ tends to infinity, then $\phi_\infty v \in \Lambda_\Gamma^{\mathrm{con}}$, which contradicts Proposition 6.4.1 and the definition of m since for almost every vector v in $T^1\Omega$, the endpoints $\phi_{\pm\infty}v$ are not in $\Lambda_\Gamma^{\mathrm{con}}$.

Furthermore $(T^1M, \mathbb{R}, m_\Gamma)$, $(\mathrm{Geod}^\infty(\Omega), \Gamma, m_\mathbb{R})$ and $(T^1\Omega, \Gamma\times\mathbb{R}, m)$ are non-ergodic, otherwise μ_o^2 would be non-atomic by Proposition 6.4.2, and ergodicity and non-atomicity of Γ acting on $(\mathrm{Geod}^\infty(\Omega), m_\mathbb{R})$ implies conservativity by Remark 1.2.7, which contradicts dissipativity.

6.5 The divergent case of the HTSR dichotomy

In this section we adapt some proofs of Roblin [Rob03, p. 19-23] to our convex projective setting, in order to establish Theorem 6.0.1.2. Let $o \in \Omega \subset \mathrm{P}(\mathbf{V})$ be a pointed properly convex open set and $\Gamma \subset \mathrm{Aut}(\Omega)$ a discrete subgroup.

6.5.1 The conical limit set has non-zero measure

The proof of the fact that the conical limit set has full measure in the divergent case cuts into two steps: as Roblin, we first prove that it has non-zero measure. The proof of Roblin of this first step [Rob03, Item (*f*) p. 19] actually works verbatim; we rewrite it down for the convenience of the (non-french-speaking) reader.

Proposition 6.5.1. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set and $\Gamma \subset \mathrm{Aut}(\Omega)$ a discrete subgroup. Suppose Γ is divergent, and $M = \Omega/\Gamma$ is rank-one and non-elementary. Consider a δ_Γ-conformal density $(\mu_x)_{x\in\Omega}$ on $\partial\Omega$. Then $\mu_x(\Lambda_\Gamma^{\mathrm{con}}) > 0$ for any $x \in \Omega$.*

The idea of the proof of Proposition 6.5.1 is to find a compact subset $K \subset T^1M$ such that the set of vectors $v \in K$ whose geodesic come back infinitely often to K has positive m_Γ-measure where m_Γ is a Sullivan measure associated to $(\mu_x)_{x\in\Omega}$. There are two main ingredients. The first one is a generalisation of the Borel–Cantelli lemma, due to Rényi. One can find a proof of it in [AS84, Lem. 2]; it is the consequence of the following estimate : for any non-negative square-integrable function g on a probability space, for any $0 < a < E(g)$, where $E(g)$ is the expectation of g, one has

$$P(g > a) \geq \frac{E(g)^2}{E(g^2)}\left(1 + \frac{a}{E(g) - a}\right)^{-2}.$$

Fact 6.5.2 ([Rén70, p. 391])**.** *Let (X, ν) be a measurable space equipped with a finite positive measure. Let $(A_t)_{t\geq 0}$ be a family of subsets of X such that the function $(x,t) \in X \times \mathbb{R}_{\geq 0} \mapsto$*

$1_{A_t}(x)$ is measurable. Let us assume that $\int_0^\infty \nu(A_t)\,\mathrm{d}t = \infty$, and that, for some constant $C > 0$,

$$\int_0^T \int_0^T \nu(A_t \cap A_s)\,\mathrm{d}t\,\mathrm{d}s \leq C \left(\int_0^T \nu(A_t)\,\mathrm{d}t \right)^2, \tag{6.5.1}$$

for T large enough. Then the set of $x \in X$ such that $\int_0^\infty 1_{A_t}(x)\,\mathrm{d}t = \infty$ has ν-measure greater than or equal to $1/C$.

This lemma will be applied to $A_t = K \cap \phi_{-t}K$. The second ingredient consists of estimates that will allow us to check that the assumptions of Fact 6.5.2 are satisfied.

Lemma 6.5.3. *Let $o \in \Omega \subset \mathrm{P}(\mathbf{V})$ be a pointed properly convex open set and $\Gamma \subset \mathrm{Aut}(\Omega)$ a discrete subgroup. Suppose $M = \Omega/\Gamma$ is rank-one and non-elementary. Consider a δ_Γ-conformal density $(\mu_x)_{x\in\Omega}$ on $\partial\Omega$, with induced Sullivan measure m_Γ on T^1M. Let $v_o \in T^1_o\Omega$. For $R > 0$ large enough, if we denote by K the projection in T^1M of $\overline{B}_{T^1\Omega}(v_o, R)$, then there exist constants $C > 0$ and T_0 such that for any $T > T_0$, we have the estimates:*

$$\int_0^T m_\Gamma(K \cap \phi_{-t}K)\,\mathrm{d}t \geq C^{-1} \sum_{\substack{g\in\Gamma \\ d_\Omega(o,go)\leq T}} e^{-\delta_\Gamma d_\Omega(o,go)}, \tag{6.5.2}$$

$$\int_0^T \int_0^T m_\Gamma(K \cap \phi_{-t}K \cap \phi_{-t-s}K)\,\mathrm{d}s\,\mathrm{d}t \leq C \left(\sum_{\substack{g\in\Gamma \\ d_\Omega(o,go)\leq T}} e^{-\delta_\Gamma d_\Omega(o,go)} \right)^2. \tag{6.5.3}$$

Proof. We assume that μ_o is a probability measure. Let $(\mu^{\mathtt{h}}_x)_{x\in\Omega}$ be a δ_Γ-conformal density on $\partial_{\mathtt{h}}\Omega$ that induces $(\mu_x)_{x\in\Omega}$ and m_Γ; we denote by m the induced Sullivan measure on $T^1\Omega$. We fix $R > 0$ large enough so that we can apply the Shadow lemma (Lemmas 6.3.1 and 6.3.5) and Lemmas 6.3.3 and 6.3.4 to it and to $R' := (R-2)/6$. One can find a constant $C_1 > 0$ such that

$$m(K \cap \phi_{-t}K) \geq C_1^{-1} \sum_{g\in\Gamma} m(\tilde{K} \cap \phi_{-t}g\tilde{K}), \quad \text{and}$$
$$m(K \cap \phi_{-t}K \cap \phi_{-t-s}K) \leq \sum_{g,h\in\Gamma} m(\tilde{K} \cap \phi_{-t}g\tilde{K} \cap \phi_{-t-s}h\tilde{K})$$

for all $t, s \geq 0$. Indeed, we just have to recall the definition of m_Γ, which is the quotient of m under the action of Γ (Definition 1.2.3), then

$$\begin{aligned}\sum_{g\in\Gamma} m(\tilde{K} \cap \phi_{-t}g\tilde{K}) &= \int_{T^1\Omega} \sum_{g\in\Gamma} 1_{\tilde{K}} 1_{g\phi_{-t}\tilde{K}}\,\mathrm{d}m \\ &= \int_{T^1M} \sum_{h\in\Gamma}\sum_{g\in\Gamma} (1_{\tilde{K}} \circ h)\cdot(1_{g\phi_{-t}\tilde{K}} \circ h)\,\mathrm{d}m \\ &= \int_{T^1M} \underbrace{\left(\sum_{h\in\Gamma} 1_{h\tilde{K}}\right)}_{\leq C_1 1_K} \cdot \underbrace{\left(\sum_{g\in\Gamma} 1_{g\phi_{-t}\tilde{K}}\right)}_{\leq C_1 1_{\phi_{-t}K}} \mathrm{d}m_\Gamma,\end{aligned}$$

where $C_1 > 0$ is a constant which is independent of t. Similarly,

$$\sum_{g,h\in\Gamma} m(\tilde{K}\cap\phi_{-t}g\tilde{K}\cap\phi_{-t-s}h\tilde{K}) = \int_{T^1M}\sum_{k\in\Gamma}\sum_{g,h\in\Gamma}(1_{\tilde{K}}\circ k)\cdot(1_{g\phi_{-t}\tilde{K}}\circ k)\cdot(1_{h\phi_{-t-s}\tilde{K}}\circ k)\,\mathrm{d}m_\Gamma$$

$$= \int_{T^1M}\underbrace{\left(\sum_{k\in\Gamma}1_{k\tilde{K}}\right)}_{\geq 1_K}\cdot\underbrace{\left(\sum_{g\in\Gamma}1_{g\phi_{-t}\tilde{K}}\right)}_{\geq 1_{\phi_{-t}K}}\cdot\underbrace{\left(\sum_{h\in\Gamma}1_{h\phi_{-t-s}\tilde{K}}\right)}_{\geq 1_{\phi_{-t-s}K}}\mathrm{d}m_\Gamma$$

Therefore, in order to prove Lemma 6.5.3, it is enough to find a constant $C_2 > 0$ such that

$$a := \int_0^T\sum_{g\in\Gamma} m(\tilde{K}\cap\phi_{-t}g\tilde{K})\,\mathrm{d}t \geq C_2^{-1}\sum_{g:d_\Omega(o,go)\leq T} e^{-\delta_\Gamma d_\Omega(o,go)},$$

and

$$b := \int_0^T\int_0^T\sum_{g,h\in\Gamma} m(\tilde{K}\cap\phi_{-t}g\tilde{K}\cap\phi_{-t-s}h\tilde{K})\,\mathrm{d}s\,\mathrm{d}t \leq C_2\left(\sum_{g:d_\Omega(o,go)\leq T} e^{-\delta_\Gamma d_\Omega(o,go)}\right)^2.$$

We first establish the estimate of b. For all $0\leq t,s\leq T$ and $g,h\in\Gamma$, for any triple (ξ,η,τ) in $\mathrm{Hopf}^{-1}(\tilde{K}\cap\phi_{-t-s}h\tilde{K})$, one observes that $\eta\in\pi_{\mathtt{h}}^{-1}\mathcal{O}_R^+(o,ho)$ and $d_\Omega(o,\pi\mathrm{Hopf}(\xi,\eta,\tau))\leq R$; this last inequality implies that $|\tau|\leq R$ and $\langle\xi,\eta\rangle_o\leq R$; then by the Shadow lemma (Lemma 6.3.1), and by definition of m,

$$m(\tilde{K}\cap\phi_{-t}g\tilde{K}\cap\phi_{-t-s}h\tilde{K}) \leq 2Re^{2\delta_\Gamma R}\mu_0(\mathcal{O}_R^+(o,ho)) \leq 2Re^{2\delta_\Gamma R}Ce^{-\delta_\Gamma d_\Omega(o,ho)}.$$

Furthermore, by triangular inequality, if $\tilde{K}\cap\phi_{-t}g\tilde{K}\cap\phi_{-t-s}h\tilde{K}$ is non-empty, then

$$|d_\Omega(o,go)-t|, |d_\Omega(go,ho)-s|, |d_\Omega(o,ho)-t-s| \leq 2R, \quad \text{and}$$
$$d_\Omega(o,go)+d_\Omega(go,ho)\leq d_\Omega(o,ho)+6R.$$

Combining these estimates we obtain:

$$b \leq \int_0^T\int_0^T\sum_{g,h\in\Gamma} 2Re^{2\delta_\Gamma R}Ce^{-\delta_\Gamma d_\Omega(o,ho)}1_{\left\{\substack{|d_\Omega(o,go)-t|,|d_\Omega(go,ho)-s|\leq 2R\\ d_\Omega(o,go)+d_\Omega(go,ho)\leq d_\Omega(o,ho)+6R}\right\}}\,\mathrm{d}s\,\mathrm{d}t$$
$$\leq 2R2R2Re^{2\delta_\Gamma R}C\sum_{\substack{d_\Omega(o,go)\leq T+2R\\ d_\Omega(o,g^{-1}ho)\leq T+2R}} e^{-\delta_\Gamma(d_\Omega(o,go)+d_\Omega(o,g^{-1}ho)-6R)}$$
$$\leq 8R^3Ce^{8\delta_\Gamma R}\left(\sum_{d_\Omega(o,go)\leq T+2R} e^{-\delta_\Gamma d_\Omega(o,go)})\right)^2.$$

We end the estimation of b by noting that for all $T\geq 0$ and $A\geq 0$, using again the Shadow lemma (Lemma 6.3.1),

$$\sum_{T\leq d_\Omega(o,go)\leq T+A} e^{-\delta_\Gamma d_\Omega(o,go)} \leq C\sum_{T\leq d_\Omega(o,go)\leq T+A}\mu_o(\mathcal{O}_R(o,go))$$
$$\leq C\int_{\partial\Omega}\sum_{T\leq d_\Omega(o,go)\leq T+A} 1_{\mathcal{O}_R(o,go)}(\xi)\,\mathrm{d}\mu_o(\xi)$$
$$\leq C\#\{g : d_\Omega(o,go)\leq 4R+2A\}.$$

We now proceed to the minoration of a. Take $g \in \Gamma$, take $t \geq 0$ at distance less than R' from $d_\Omega(o, go)$; let us prove that

$$\mathrm{Hopf}\left(\pi_{\mathtt{h}}^{-1}(\mathcal{O}_{R'}^{-}(go, o)) \times \pi_{\mathtt{h}}^{-1}(\mathcal{O}_{R'}^{-}(o, go)) \times [0, R']\right) \subset \tilde{K} \cap \phi_{-t} g \tilde{K}.$$

Take $(\xi, \eta, \tau) \in \pi_{\mathtt{h}}^{-1}(\mathcal{O}_{R'}^{-}(go, o)) \times \pi_{\mathtt{h}}^{-1}(\mathcal{O}_{R'}^{-}(o, go)) \times [0, R']$. By Observation 6.5.4 below, there exist $s_1 < s_2$ such that $d_\Omega(\pi\mathrm{Hopf}(\xi, \eta, s_1), o) \leq R'$ and $d_\Omega(\pi\mathrm{Hopf}(\xi, \eta, s_2), go) \leq R'$. This implies that $|s_1| = |\mathtt{b}_\eta(\pi\mathrm{Hopf}(\xi, \eta, s_1), o)| \leq R'$, hence

$$d_\Omega(\pi\mathrm{Hopf}(\xi, \eta, \tau), o) \leq |\tau| + |s_1| + d_\Omega(\pi\mathrm{Hopf}(\xi, \eta, s_1), o) \leq 3R'.$$

Finally $d_{T^1\Omega}(\mathrm{Hopf}(\xi, \eta, \tau), v_o) \leq 3R' + 2 \leq R$, which means that $\mathrm{Hopf}(\xi, \eta, \tau) \in \tilde{K}$.

In order to prove the inclusion in $\phi_{-t} g \tilde{K}$, we note that since $s_2 - s_1$ is the distance between $\pi\mathrm{Hopf}(\xi, \eta, s_2)$ and $\pi\mathrm{Hopf}(\xi, \eta, s_1)$, then by triangular inequality $|s_2 - d_\Omega(o, go)| \leq 3R'$; therefore

$$\begin{aligned} d_\Omega(\pi\mathrm{Hopf}(\xi, \eta, t+\tau), go) &\leq |\tau| + |t - d_\Omega(o, go)| + |d_\Omega(o, go) - s_2| + d_\Omega(\pi\mathrm{Hopf}(\xi, \eta, s_2), go) \\ &\leq 6R'. \end{aligned}$$

Finally $d_{T^1\Omega}(\mathrm{Hopf}(\xi, \eta, t+\tau), gv_o) \leq 6R' + 2 = R$, which means that $\mathrm{Hopf}(\xi, \eta, \tau) \in \phi_{-t} g \tilde{K}$.

Thus, if $d_\Omega(o, go) \geq 2R'$, then by the Shadow lemma (Lemma 6.3.5) and Lemma 6.3.4,

$$m(\tilde{K} \cap \phi_{-t} g \tilde{K}) \geq \mu_o(\mathcal{O}_{R'}^{-}(o, go)) \mu_o(\mathcal{O}_{R'}^{-}(go, o)) R' \geq R' C^{-2} e^{-\delta_\Gamma d_\Omega(o, go)}.$$

We conclude that for $T \geq R'$,

$$\int_0^T \sum_{g \in \Gamma} m(\tilde{K} \cap \phi_{-t} g \tilde{K}) \,\mathrm{d}t \geq R'^2 C^{-2} \left(\sum_{g : d_\Omega(o, go) \leq T} e^{-\delta_\Gamma d_\Omega(o, go)} - \sum_{g : d_\Omega(o, go) \leq 2R'} e^{-\delta_\Gamma d_\Omega(o, go)} \right).$$

□

We then make the following elementary observation, before proceeding to the proof of Proposition 6.5.1.

Observation 6.5.4. *Let $n \geq 1$ be an integer, A and B be two non-empty disjoint compact subsets of $\mathbb{R}^n$, and $\xi, \eta \in \mathbb{R}^n$ be two points. If for all $a \in A$ and $b \in B$, the intersections $[a, \eta] \cap B$ and $[b, \xi] \cap A$ are non-empty, then one can find $a \in A$ and $b \in B$ such that ξ, a, b, η are aligned in this order.*

Proof of Proposition 6.5.1. We let m_Γ be a Sullivan measure associated to $(\mu_x)_{x \in \Omega}$ on T^1M. By definition of m_Γ and of the conical limit set, it is enough to find a compact set $K \subset T^1M$ large enough so that the set of vectors $v \in K$ such that $\int_0^\infty 1_K(\phi_t v) \,\mathrm{d}t = \infty$ has non-zero m_Γ-measure.

Let $R > 0$ be large enough so that we can apply Lemma 6.5.3, and let K be the projection in T^1M of $\overline{B}_{T^1\Omega}(v_o, R)$, where $v_o \in T^1_o\Omega$ and $o \in \Omega$. We want to apply Fact 6.5.2 for $(X, \nu) = (T^1M, (m_\Gamma)_{|K})$ and $A_t := K \cap \phi_{-t} K$ for all $t \geq 0$. The measure ν is finite because m is Radon and K is compact. Since Γ is divergent (this is important since it is the only place where we need this assumption), and by the estimate (6.5.2) of Lemma 6.5.3, the integral $\int_0^\infty \nu(A_t) \,\mathrm{d}t$ diverges. It remains to check that (6.5.1) is satisfied, but this is a direct consequence of the estimates (6.5.2) and (6.5.3) of Lemma 6.5.3, and of the fact that

$$\int_0^T \int_0^T \nu(A_t \cap A_s) \,\mathrm{d}t \,\mathrm{d}s \leq 2 \int_0^T \int_0^T \nu(A_t \cap A_{t+s}) \,\mathrm{d}t \,\mathrm{d}s.$$

□

6.5.2 The conical limit set has full measure

In this section we finish the proof of the fact that, if Γ is divergent, then the conical limit set has full measure. This proof is extremely similar to Roblin's ([Rob03, p. 22-23]), but there is a subtlety which makes our statement weaker: we have to consider a σ-algebra, denoted by $\mathfrak{S}_{\text{face}}$, which is coarser than the usual σ-algebra of Borel subsets, and which is defined as follows. A Borel subset $A \subset \partial\Omega$ belongs to $\mathfrak{S}_{\text{face}}$ if it contains the open faces of each of its points. For example the Borel subsets $\Lambda^{\text{con}}_\Gamma$, $\partial_{\text{smooth}}\Omega$, $\partial_{\text{sse}}\Omega$, or any subset consisting of extremal points, are all in $\mathfrak{S}_{\text{face}}$; however Λ_Γ does not always belong to $\mathfrak{S}_{\text{face}}$, although constructing a counter-example is not easy: this can be done by using the techniques of [DGKb]. While Roblin proves, for Γ' acting on a CAT(-1)-space, that any $\delta_{\Gamma'}$-conformal density on the visual boundary is ergodic under the action of Γ', we prove that any δ_Γ-conformal density on $\partial\Omega$, *restricted to* $\mathfrak{S}_{\text{face}}$, is ergodic. We will later obtain ergodicity on the full Borel algebra of $\partial\Omega$ by showing that non-extremal points have zero measure (Proposition 6.5.8).

Proposition 6.5.5. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set and $\Gamma \subset \mathrm{Aut}(\Omega)$ a divergent discrete subgroup. Suppose $M = \Omega/\Gamma$ is rank-one and non-elementary. Consider a δ_Γ-conformal density $(\mu_x)_{x\in\Omega}$ on $\partial\Omega$. Then $\mu_x(\partial\Omega \smallsetminus \Lambda^{\text{con}}) = 0$, and furthermore $(\mu_x)_{|\mathfrak{S}_{\text{face}}}$ is ergodic under the action of Γ, for any $x \in \Omega$.*

Proof. Consider $o \in \Omega$ and assume that μ_o has total mass 1. We first establish the following. Let $R > 0$ be large enough so that we can apply the Shadow lemma (Lemma 6.3.1). Then for any $A \in \mathfrak{S}_{\text{face}}$, for μ_o-almost any $\xi \in \Lambda^{\text{con}}_R$,

$$\frac{\mu_o(\mathcal{O}_R(o,\gamma o) \cap A)}{\mu_o(\mathcal{O}_R(o,\gamma o))} \underset{\substack{d_\Omega(o,\gamma o)\to\infty \\ \xi\in\mathcal{O}_R(o,\gamma o)}}{\longrightarrow} 1_A(\xi). \tag{6.5.4}$$

For any integrable function $f \geq 0$ and $\alpha = \sup$ or $\inf$, we set

$$f^\alpha(\xi) := 1_{\Lambda^{\text{con}}_R}(\xi) \lim_{r\to\infty} \underset{\substack{d_\Omega(o,\gamma o)\geq r \\ \xi\in\mathcal{O}_R(o,\gamma o)}}{\alpha} \frac{1}{\mu_o(\mathcal{O}_R(o,\gamma o))} \int_{\mathcal{O}_R(o,\gamma o)} f \, \mathrm{d}\mu_o.$$

In order to prove (6.5.4), it is enough to show that $1_A^{\sup} = 1_A^{\inf} = 1_{A\cap\Lambda^{\text{con}}_R}$ holds μ_o-almost surely for any $A \in \mathfrak{S}_{\text{face}}$. To this end we prove a maximal ergodic inequality: we find a constant $C > 0$ such that for any integrable function f, for any $\epsilon > 0$,

$$\mu_o(f^{\sup} > \epsilon) \leq \frac{C}{\epsilon}\|f\|_{L^1}. \tag{6.5.5}$$

Fix an integrable function f and $\epsilon > 0$. Let $E \subset \Gamma$ be a discrete subset such that $\{f^{\sup} > \epsilon\}$ is contained in $\bigcup_{g\in E} \mathcal{O}_R(o, go)$ and $\int_{\mathcal{O}_R(o,go)} f \, \mathrm{d}\mu_o > \epsilon\mu_o(\mathcal{O}_R(o, go))$ for any $g \in E$. We enumerate $E = \{g_1, g_2, \dots\}$ so that the sequence $(d_\Omega(o, g_n o))_n$ is non-decreasing. We define by induction a (possibly finite) increasing sequence of integers σ such that $\sigma(1) = 1$, such that $\bigcup_{g\in E} \mathcal{O}_R(o, go)$ is contained in $\bigcup_n \mathcal{O}_{5R}(o, g_{\sigma(n)}o)$, and such that the shadows $(\mathcal{O}_R(o, g_{\sigma(n)}o))_n$ are pairwise disjoint. The procedure is the following: given the first n terms $\sigma(1), \dots, \sigma(n)$, we take for $\sigma(n + 1)$ the smallest m such that $\mathcal{O}_R(o, g_m o) \not\subset \bigcup_{j\leq n} \mathcal{O}_{5R}(o, g_{\sigma(k)}o)$ (if such an m does not exist then the process stops). Then the Shadow lemma (Lemma 6.3.1) gives us $C > 0$, independent of f and ϵ, such that for each $\gamma \in \Gamma$, $\mu_o(\mathcal{O}_{5R}(o, \gamma o)) \leq C\mu_o(\mathcal{O}_R(o, \gamma o))$. An easy computation yields (6.5.5).

We now prove (6.5.4). Fix $A \in \mathfrak{S}_{\text{face}}$. Note that $1_K^{\sup} \leq 1_{A\cap\Lambda^{\text{con}}_R}$ for any compact subset $K \subset A$. Indeed, $1_K^{\sup} \leq 1_{\Lambda^{\text{con}}_R}$ by definition, let us prove that $1_K^{\sup} \leq 1_A$: pick $\xi \in \Lambda^{\text{con}}_R \smallsetminus A$.

Let $(\gamma_n)_n$ be any sequence of Γ going to infinity such that $\xi \in \mathcal{O}_R(o, \gamma_n o)$ for any n; we can extract it so that $(\gamma_n o)_n$ converges to some $\eta \in \partial\Omega$. By lower semi-continuity of $d_{\overline{\Omega}}$, for n large enough, $\mathcal{O}_R(o, \gamma_n o)$ is contained in arbitrary small neighbourhoods of $\overline{B}_{\overline{\Omega}}(\eta, R)$. As a consequence this ball must contain ξ, and so be contained in the larger ball $\overline{B}_{\overline{\Omega}}(\xi, 2R)$. But this larger ball, being contained in the open face of ξ, cannot intersect A, nor K; hence we can find a neighbourhood U of it, disjoint from K. For n large enough, $\mathcal{O}_R(o, \gamma_n o)$ is contained in U and so $\int_{\mathcal{O}_R(o,\gamma_n o)} 1_K d\mu_o = 0$. This conclude the proof of the estimate $1_K^{\sup} \leq 1_{A\cap\Lambda_R^{\mathrm{con}}}$, from which we derive the following.

$$1_A^{\sup} \leq 1_K^{\sup} + 1_{A\smallsetminus K}^{\sup} \leq 1_{A\cap\Lambda_R^{\mathrm{con}}} + 1_{A\smallsetminus K}^{\sup}.$$

This, combined with the interior regularity of μ_o, and the maximal ergodic inequality (6.5.5), yields for any $\epsilon > 0$:

$$\mu_o(1_A^{\sup} - 1_{A\cap\Lambda_R^{\mathrm{con}}} > \epsilon) \leq \mu_o(1_{A\smallsetminus K}^{\sup} > \epsilon) \leq \frac{C}{\epsilon}(\mu_o(A) - \mu_o(K)) \underset{\mu_o(K)\to\mu_o(A)}{\longrightarrow} 0.$$

Hence $1_A^{\sup} \leq 1_{A\cap\Lambda_R^{\mathrm{con}}}$. Similarly we prove $1_A^{\inf} \geq 1_{A\cap\Lambda_R^{\mathrm{con}}}$, and conclude the proof of (6.5.4).

Let us conclude the proof of Proposition 6.5.5. By Proposition 6.5.1, there exists $R > 0$ such that $\mu_o(\Lambda_R^{\mathrm{con}}) > 0$. We consider a Γ-invariant subset $A \in \mathfrak{S}_{\mathrm{face}}$ such that $\mu_o(A \cap \Lambda_R^{\mathrm{con}}) > 0$. Let us prove that $\mu_o(A) = 1$. We apply (6.5.4) to $B := \partial\Omega \smallsetminus A$. We obtain $R > 0$, a point $\xi \in A \cap \Lambda_R^{\mathrm{con}}$, and a sequence $(\gamma_n)_n \in \Gamma^{\mathbb{N}}$ such that

$$\frac{\mu_o(\mathcal{O}_R(o, \gamma_n o) \cap B)}{\mu_o(\mathcal{O}_R(o, \gamma_n o))} \underset{n\to\infty}{\longrightarrow} 0. \tag{6.5.6}$$

If by contradiction $\mu_o(B) > 0$, then the measure defined by $\mu_o'(E) = \frac{\mu_o(E\cap B)}{\mu_o(B)}$ for $E \subset \partial\Omega$ measurable is a δ-conformal density, to which applies the Shadow lemma (Lemma 6.3.1), and obtain a contradiction vis-à-vis (6.5.6).

We have proved in particular that $\mu_o(\Lambda_R^{\mathrm{con}}) = 1$, which implies that $\mu_o(A \cap \Lambda_R^{\mathrm{con}}) > 0$, and hence $\mu_o(A) = 1$, holds for any Γ-invariant $A \in \mathfrak{S}_{\mathrm{face}}$ with non-zero μ_o-measure. ☐

Remark 6.5.6. Using the density in L^1 of the vector space generated by indicator functions and (6.5.5), one can strengthen (6.5.4) into the following statement: $f^{\sup} = f^{\inf} = f 1_{\Lambda_R^{\mathrm{con}}}$ for any $\mathfrak{S}_{\mathrm{face}}$-measurable and integrable function f.

6.5.3 The geodesic flow is conservative

In this section we prove, that, in the setting of Theorem 6.0.1.2, the geodesic flow is conservative.

Proposition 6.5.7. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set and $\Gamma \subset \mathrm{Aut}(\Omega)$ a divergent discrete subgroup. Suppose $M = \Omega/\Gamma$ is rank-one and non-elementary. Let $(\mu_x)_{x\in\Omega}$ be a δ_Γ-conformal density on $\partial\Omega$ and m_Γ an induced Sullivan measure on T^1M. Then m_Γ is conservative under the action of $(\phi_t)_t$.*

Proof. Let $(\mu_x^{\mathtt{h}})_{x\in\Omega}$ be a δ_Γ-conformal density on $\partial_{\mathtt{h}}\Omega$ such that $(\pi_{\mathtt{h}})_*\mu_x^{\mathtt{h}} = \mu_x$ for any $x \in \Omega$, and let $\tilde{m}$ be the induced Sullivan measure on $T^1\Omega$.

According to Fact 1.2.5, it is enough to prove that $(T^1\Omega, \Gamma \times \mathbb{R}, m)$ is conservative. Let σ be an integrable, positive and continuous function on $T^1\Omega$, and let us prove that $\int_{\Gamma\times\mathbb{R}} \sigma = \infty$ almost surely (recall the notation from Section 1.2.1). For almost any vector $v \in T^1\Omega$, we have $\phi_\infty v \in \Lambda_\Gamma^{\mathrm{con}}$ by Proposition 6.5.5, and by definition of m; let v be such a vector. Then we can find $R > 0$ such that $\phi_\infty v \in \Lambda_R^{\mathrm{con}}$; in other words there exist $(\gamma_n)_n \in \Gamma^{\mathbb{N}}$ and $(t_n)_n \in \mathbb{R}^{\mathbb{N}}$ such that $\gamma_n \neq \gamma_k$ and $d_\Omega(\pi\phi_{t_n}\gamma_n v, o) \leq R$ for $n \neq k$. Let $\epsilon := \min\{\sigma(w) : d_\Omega(\pi w, o) \leq R+1\} > 0$. Then

$$\int_{\Gamma\times\mathbb{R}} \sigma(v) \geq \sum_{n\geq 0} \int_{t=t_n-1}^{t_n+1} \sigma(\phi_t\gamma v)\,\mathrm{d}t \geq \sum_{n\geq 0} 2\epsilon = \infty. \qquad \square$$

6.5.4 The smooth and strongly extremal points have full measure

In this section we use the conservativity of Sullivan's measures the smooth and strongly extremal points have full measure. This is inspired by work of Knieper [Kni98, §4] in the non-positively curved Riemannian setting.

Proposition 6.5.8. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set, and $\Gamma \subset \mathrm{Aut}(\Omega)$ a divergent discrete subgroup with $M = \Omega/\Gamma$ rank-one and non-elementary. Then any δ_Γ-conformal density on $\partial\Omega$ gives full measure to $\partial_{\mathrm{sse}}\Omega$. In particular the δ_Γ-conformal density is unique up to a scalar multiple.*

Proof. Let $o \in \Omega$ and $(\mu_x)_{x\in\Omega}$ a δ_Γ-conformal density on $\partial\Omega$. Observe that $\partial_{\mathrm{sse}}\Omega \in \mathfrak{S}_{\mathrm{face}}$ is Γ-invariant, hence, by Proposition 6.5.5, in order to show that $\partial_{\mathrm{sse}}\Omega$ has full μ_o-measure, it is enough to prove that $\mu_o(\partial_{\mathrm{sse}}\Omega) > 0$. Let us assume by contradiction that $\mu_o(\partial_{\mathrm{sse}}\Omega) = 0$. Let m be an induced Sullivan measure on $T^1\Omega$. By assumption, m gives zero measure to the set of vectors $v \in T^1\Omega$ such that $\phi_\infty v$ is smooth and strongly extremal. By Proposition 6.5.7 and Fact 1.2.6, the measure m gives full measure to the set vectors which are recurrent under the action of $\Gamma\times\mathbb{R}$. By Lemma 5.4.6, the measure m gives full measure to the set of recurrent vectors $v \in T^1\Omega$ with $d_{\mathrm{spl}}(\phi_{-\infty}v, \phi_\infty v) = 2$. Let $v \in T^1\Omega$ be a periodic rank-one vector; it is in the support of m and in the open set $T^1\Omega \smallsetminus \{v : d_{\mathrm{spl}}(\phi_{-\infty}v, \phi_\infty v) > 2\}$, which therefore has positive measure. This is a contradiction.

Let $(\mu'_x)_{x\in\Omega}$ be another δ_Γ-conformal density. The family $(\mu_x + \mu'_x)_{x\in\Omega}$ is also a conformal density, so it gives full measure to $\partial_{\mathrm{sse}}\Omega$; any Borel subset of $\partial_{\mathrm{sse}}\Omega$ belongs to $\mathfrak{S}_{\mathrm{face}}$, therefore $\mu_o + \mu'_o$ is ergodic under the action of Γ by Proposition 6.5.5. The measures μ_o and μ'_o are absolutely continuous with respect $\mu_o + \mu'_o$; the Radon–Nikodym derivatives are Γ-invariant, by definition of conformal densities, and hence constant $(\mu_o + \mu'_o)$-almost surely by ergodicity. Thus $\mu_o = \mu'_o$. $\square$

6.5.5 Smooth points and strong stable manifolds

Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set. Let us use Proposition 5.4.8 to describe, in terms of the Hopf parametrisation, the strong stable manifolds (Section 1.2.4) of the action of the geodesic flow on $T^1\Omega$. Fix $o \in \Omega$. Let $(\xi, \eta, t) \in \mathrm{Geod}_{\mathtt{h}}^\infty(\Omega) \times \mathbb{R}$. If $\pi_{\mathtt{h}}(\eta)$ is a smooth point of $\partial\Omega$, then by Proposition 5.4.8,

$$\mathrm{Hopf}_o(\xi', \eta, t) \in W^{ss}(\mathrm{Hopf}_o(\xi, \eta, t)) \tag{6.5.7}$$

for any $\xi' \in \partial_{\mathtt{h}}\Omega$ such that $(\xi', \eta) \in \mathrm{Geod}_{\mathtt{h}}^\infty(\Omega)$. The parametrisation of the unstable manifolds is more subtle. Recall that the Hopf parametrisation is equivariant under the flip action defined in (6.2.7). Since the unstable manifolds are the stable manifolds of the reversed flow, one can see that if $\pi_{\mathtt{h}}\xi$ is smooth, we get

$$\mathrm{Hopf}_o(\xi, \eta', t + \rho^o_{\xi,\eta}(\eta')) \in W^{su}(\mathrm{Hopf}_o(\xi, \eta, t)) \tag{6.5.8}$$

for any $\eta' \in \partial_{\mathtt{h}}\Omega$ such that $(\xi, \eta') \in \mathrm{Geod}_{\mathtt{h}}(\Omega)$, where

$$\rho^o_{\xi,\eta}(\eta') := 2\langle\xi, \eta'\rangle_o - 2\langle\xi, \eta\rangle_o. \tag{6.5.9}$$

6.5.6 A cross-ratio on the boundary

Following the article [Ota92], whose setting is that of negatively curved Riemannian manifolds, let us define a cross-ratio, denoted by B, for four points on the boundary of a properly convex open set $\Omega \subset \mathrm{P}(\mathbf{V})$. It should not be confused with the cross-ratio of four aligned points of the projective space, denoted with brackets, and used to defined the Hilbert metric d_Ω.

Recall that the cross-ratio of four points $\xi, \xi', \eta, \eta' \in \Omega$ is defined as

$$B(\xi, \xi', \eta, \eta') := d_\Omega(\xi, \eta) + d_\Omega(\xi', \eta') - d_\Omega(\xi, \eta') - d_\Omega(\xi', \eta). \tag{6.5.10}$$

Fix $o \in \Omega$. One can check that

$$B(\xi, \xi', \eta, \eta') = \rho^o_{\xi,\eta}(\eta') + \rho^o_{\xi',\eta'}(\eta), \tag{6.5.11}$$

and this implies that the B extends continuously to the set of quadruples $(\xi, \xi', \eta, \eta') \in (\overline{\Omega}^{\mathtt{h}})^4$ such that (ξ, η), (ξ', η), (ξ, η') and (ξ', η') belong to $\mathrm{Geod}_{\mathtt{h}}\,\Omega$. See Figure 6.3 for a geometrical interpretation using horospheres (which also works with spheres when ξ, ξ', η, η' belong to Ω).

The next lemma is classical and relates special values of B, called *periods*, with (4).

Lemma 6.5.9. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set. Let $g \in \mathrm{Aut}(\Omega)$ be a biproximal automorphism whose axis intersect Ω. Let $\xi \in \overline{\Omega}^{\mathtt{h}}$ such that (ξ, x_g^+) and (ξ, x_g^-) are in $\mathrm{Geod}_{\mathtt{h}}\,\Omega$ (recall that $x_g^\pm$ is a smooth point of $\partial\Omega$ by Lemma 3.1.2, so it identifies with its preimage in $\partial_{\mathtt{h}}\Omega$). Then $B(x_g^+, x_g^-, \xi, g\xi) = 2\ell(g)$.*

Proof. By continuity of the cross-ratio, we can assume that $\xi \in \Omega$. Then using (6.5.10) or (6.5.11), one can show that

$$B(x_g^+, x_g^-, \xi, g\xi) = \mathtt{b}_{x_g^+}(\xi, g\xi) + \mathtt{b}_{x_g^-}(g\xi, \xi).$$

Now consider $x \in \Omega$ on the axis of g. We compute

$$\begin{aligned}
\mathtt{b}_{x_g^+}(\xi, g\xi) &= \mathtt{b}_{x_g^+}(\xi, x) + \mathtt{b}_{x_g^+}(x, gx) + \mathtt{b}_{x_g^+}(gx, g\xi) \\
&= d_\Omega(x, gx) + \mathtt{b}_{x_g^+}(\xi, x) + \mathtt{b}_{g^{-1}x_g^+}(x, \xi) \quad \text{(by Lemma 6.1.1)} \\
&= \ell(g).
\end{aligned}$$

By taking the inverse we get:

$$\mathtt{b}_{x_g^-}(g\xi, \xi) = \mathtt{b}_{x^+_{g^{-1}}}(g\xi, g^{-1}g\xi) = \ell(g^{-1}) = \ell(g). \qquad \square$$

6.5.7 W^{ss} and W^{su}-invariant functions are essentially constant

In this section we prove that the W^{ss} and W^{su}-invariant functions on T^1M are essentially constant, and we derive as a corollary that the flow is ergodic and mixing.

Proposition 6.5.10. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set and $\Gamma \subset \mathrm{Aut}(\Omega)$ a discrete subgroup with $M = \Omega/\Gamma$ divergent rank-one and non-elementary. Let m_Γ be the Sullivan measure on T^1M induced by a δ_Γ-conformal density. Then any W^{ss} and W^{su}-invariant function on T^1M is essentially constant with respect to m_Γ.*

Proof. Let f be a W^{ss} and W^{su}-invariant function on T^1M. By Proposition 6.5.8, the measure m_Γ gives full measure to the (measurable) set $T^1M_{\mathrm{sse}} \subset T^1M$ of vectors v with $\phi_\infty v, \phi_{-\infty} v \in \partial_{\mathrm{sse}}\Omega$, hence it suffices to show that the restriction $f_{|T^1M_{\mathrm{sse}}}$ is essentially constant. We fix $o \in \Omega$ and lift $f_{|T^1M_{\mathrm{sse}}}$, via the Hopf parametrisation, to a function $\tilde{f}$ on $\{(\xi,\eta) \in \partial_{\mathrm{sse}}\Omega : \xi \neq \eta\} \times \mathbb{R}$ that we extend to $\partial_{\mathrm{sse}}\Omega^2 \times \mathbb{R}$ by setting $\tilde{f}(\xi,\xi) = 0$ for any ξ (recall that we abusively identify $\partial_{\mathrm{sse}}\Omega$ with its preimage in $\partial_{\mathrm{h}}\Omega$). Let us show that $\tilde{f}$ is μ_o^2 times Lebesgue-essentially constant, where $(\mu_x)_x$ is the δ_Γ-conformal density on $\partial\Omega$ such that μ_o is a probability measure. Recall that μ_o is non-atomic by Propositions 6.3.7 and 6.5.5, thus for μ_o-almost all $\xi, \eta \in \partial_{\mathrm{sse}}\Omega$ we have $\xi \neq \eta$.

The function f is W^{ss} and W^{su}-invariant, so by (6.5.7) and (6.5.8), for μ_o-almost $\xi, \xi', \eta, \eta' \in \partial_{\mathrm{sse}}\Omega$ and Lebesgue-almost any $t \in \mathbb{R}$, the quantity $\rho_{\xi,\eta}(\eta'))$ is well-defined and

$$\tilde{f}(\xi,\eta,t) = \tilde{f}(\xi',\eta,t) = \tilde{f}(\xi,\eta',t+\rho_{\xi,\eta}(\eta')).$$

This implies in particular that there exist $\xi_0 \neq \eta_0 \in \partial_{\mathrm{sse}}\Omega$ such that, if we denote $g(t) := \tilde{f}(\xi_0,\eta_0,t)$ for any $t \in \mathbb{R}$, then for μ_o-almost all $\xi,\eta \in \partial_{\mathrm{sse}}\Omega$ and Lebesgue-almost any $t \in \mathbb{R}$, the quantity $\rho_{\xi_0,\eta_0}(\eta)$ is well-defined and

$$\tilde{f}(\xi,\eta,t+\rho_{\xi_0,\eta_0}(\eta)) = g(t).$$

It is enough to establish that the measurable function g is essentially constant with respect to the Lebesgue measure. We denote by H the additive real subgroup consisting of numbers τ such that $g(t+\tau) = g(t)$ for Lebesgue-almost every $t \in \mathbb{R}$. A classical result says that H is a closed subgroup of $\mathbb{R}$. (To see this first reduce to the case where g is bounded and then note that H is the stabiliser of $g(t)\,\mathrm{d}t$ for the continuous action of $\mathbb{R}$ on the space of Radon measures on $\mathbb{R}$.)

To finish the proof of Proposition 6.5.10 it is enough, according to Proposition 5.3.3, to prove that $2\ell(\gamma) \in H$ for any rank-one element $\gamma \in \Gamma$. To this end we observe that many cross-ratios belong to H: for μ_o-almost all $\xi,\eta,\xi',\eta' \in \partial_{\mathrm{sse}}\Omega$ and Lebesgue-almost every $t \in \mathbb{R}$, the quantities $\rho_{\xi_0,\eta_0}(\eta)$, $\rho_{\xi,\eta}(\eta')$, $\rho_{\xi',\eta'}(\eta)$ and $B(\xi,\xi',\eta,\eta')$ are well-defined and we have the following serie of equalities (see Figure 6.3 for a geometrical interpretation).

$$\begin{aligned} g(t) &= \tilde{f}(\xi,\eta,t+\rho_{\xi_0,\eta_0}(\eta)) \\ &= \tilde{f}(\xi,\eta',t+\rho_{\xi_0,\eta_0}(\eta)+\rho_{\xi,\eta}(\eta')) \\ &= \tilde{f}(\xi',\eta',t+\rho_{\xi_0,\eta_0}(\eta)+\rho_{\xi,\eta}(\eta')) \\ &= \tilde{f}(\xi',\eta,t+\rho_{\xi_0,\eta_0}(\eta)+\rho_{\xi,\eta}(\eta')+\rho_{\xi',\eta'}(\eta)) \\ &= \tilde{f}(\xi,\eta,t+\rho_{\xi_0,\eta_0}(\eta)+B(\xi,\xi',\eta,\eta')) \\ &= g(t+B(\xi,\xi',\eta,\eta')). \end{aligned} \tag{6.5.12}$$

Consider a rank-one element $\gamma \in \Gamma$ and a point $\xi \in \partial_{\mathrm{sse}}\Omega \cap \mathrm{supp}(\mu_o) \smallsetminus \{x_g^-, x_g^+\}$. Then $2\ell(\gamma)$ is equal to $B(x_\gamma^+, x_\gamma^-, \xi, \gamma\xi)$ by Lemma 6.5.9, and furthermore this quantity belongs to H because H is closed, B is continuous and $x_\gamma^+, x_\gamma^-, \xi, \gamma\xi$ are in $\mathrm{supp}(\mu_o)$.

□

Corollary 6.5.11. *In the setting of Proposition 6.5.10, if Γ is divergent, then m is ergodic under the action of the geodesic flow. If moreover m_Γ is finite, then it is mixing.*

Proof. The ergodicity of m_Γ is a direct consequence of Fact 1.2.11, Proposition 6.5.7 and Proposition 6.5.10, which can be applied since m is Radon. If m_Γ is finite, then it is mixing by Fact 1.2.12 and Proposition 6.5.10. □

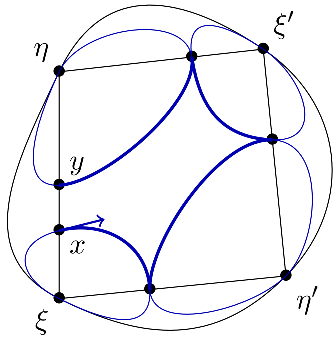

Figure 6.3: Illustration of computations (6.5.12), where $d_\Omega(x,y) = B(\xi,\xi',\eta,\eta')$

6.5.8 The support of conformal densities

Let us establish that the support of the δ_Γ-conformal density, for Γ divergent and $M = \Omega/\Gamma$ rank-one, is exactly the proximal limit set. This is a consequence of conservativity and ergodicity of the Bowen–Margulis measure, Fact 1.2.6, and Proposition 5.1.2.

Proposition 6.5.12. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set and $\Gamma \subset \mathrm{Aut}(\Omega)$ a divergent discrete subgroup with $M = \Omega/\Gamma$ rank-one and non-elementary. Then Λ_Γ is the support of any δ_Γ-confomal density and T^1M_{bip} is the support of the induce Sullivan measure.*

Proof. Let $(\mu_x)_{x\in\Omega}$ be the δ_Γ-conformal density on $\partial\Omega$, let m_Γ (resp. $m_\mathbb{R}$) be the induced Sullivan measure on T^1M (resp. $\mathrm{Geod}^\infty(\Omega)$), and let $o \in \Omega$. Since Λ_Γ is the smallest Γ-invariant closed subset of $\overline{\Omega}$ (Proposition 3.2.2) and μ_o is Γ-quasi-invariant, we have $\Lambda_\Gamma \subset \mathrm{supp}(\mu_o)$ and $T^1M_{\mathrm{bip}} \subset \mathrm{supp}(m_\Gamma)$. Since m_Γ is conservative and ergodic by Proposition 6.5.7 and Corollary 6.5.11, Fact 1.2.6 implies that the geodesic flow is topologically transitive on $\mathrm{supp}(m_\Gamma)$. According to Proposition 5.1.2, this implies that $\mathrm{supp}(m_\Gamma) = T^1M_{\mathrm{bip}}$, and hence $\mathrm{supp}(m_\mathbb{R}) = \mathrm{Geod}^\infty(\Omega) \cap \Lambda_\Gamma^2$.

Suppose by contradiction there is a point $\xi \in \mathrm{supp}(\mu_o) \smallsetminus \Lambda_\Gamma$. Since M is rank-one, we can find a strongly extremal point $\eta \in \Lambda_\Gamma$. Then we can find a neighbourhood $U \times V \subset \mathrm{Geod}^\infty(\Omega)$ of (ξ,η) such that $U \cap \Lambda_\Gamma$, and hence $(U \times V) \cap \Lambda_\Gamma^2$, are empty. Note that $\mu_o(U)\mu_o(V)$, and hence $m_\mathbb{R}(U \times V)$, are positive since $\xi, \eta \in \mathrm{supp}(\mu_o)$: this is a contradiction. ☐

6.5.9 Proof of Theorem 6.0.1.2

It is a direct consequence of Propositions 6.3.6, 6.3.7, 6.5.7, 6.5.8 and 6.5.12, Corollary 6.5.11 and Fact 1.2.5.

Chapter 7

The measure of maximal entropy

Let $M = \Omega/\Gamma$ be a non-elementary rank-one convex projective orbifold with non-empty compact convex core. The goal of this chapter is to prove that the geodesic flow on T^1M_{cor} admits a unique probability measure of maximal entropy, and use this to establish counting results on conjugacy classes of Γ. Let us first briefly apply the results of the previous chapter.

Proposition 7.0.1. *Let $\Omega \subset \mathrm{P}(V)$ be a properly convex open set and $\Gamma \subset \mathrm{Aut}(\Omega)$ a convex cocompact, non-elementary rank-one and discrete subgroup. Then Γ is divergent. Moreover, any Sullivan measure on T^1M of dimension δ_Γ (concentrated on $T^1M_{\mathrm{bip}} \subset T^1M_{\mathrm{cor}}$) has finite measure, and we call* Bowen–Margulis probability measure *the only Sullivan measure of dimension δ_Γ with total mass 1.*

Proof. Consider $o \in \mathcal{C}^{\mathrm{cor}}_\Omega(\Gamma)$. We set $\mathrm{Vol} := \sum_{\gamma\in\Gamma} \mathcal{D}_{\gamma o}$, where $\mathcal{D}_x$ denotes the Dirac mass at $x \in \Omega$; this defines a Γ-invariant Radon measure on Ω, which is supported on $\mathcal{C}^{\mathrm{cor}}_\Omega(\Gamma)$. We apply Fact 1.4.1 to Vol, in order to obtain a δ_Γ-conformal density $(\mu_x)_{x\in\Omega}$ on $\partial\Omega$ which is supported on Λ^{orb}. Recall that $\Lambda^{\mathrm{orb}} = \Lambda^{\mathrm{con}}$ (Fact 2.3.14), hence $\mu_o(\Lambda^{\mathrm{con}}) = 1$ and therefore Γ is divergent by Theorem 6.0.1. By the same theorem, the induced Sullivan measure m_Γ on T^1M is supported on T^1M_{cor}. Since m_Γ is Radon and T^1M_{cor} is compact, m_Γ must be finite. □

We can now state the main result of this chapter.

Theorem 7.0.2. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set and $\Gamma \subset \mathrm{Aut}(\Omega)$ a convex cocompact, non-elementary rank-one and discrete subgroup; set $M = \Omega/\Gamma$. Then the Bowen–Margulis probability measure is the unique measure of maximal entropy on T^1M_{cor}, with entropy δ_Γ.*

Following Knieper [Kni98], one can use the previous theorem and the fact that the Bowen–Margulis probability measure is mixing (Theorem 6.0.1) to obtain counting and equidistribution results on closed geodesics of M. Recall that $[\Gamma]$ denotes the set of conjugacy classes of Γ, that $[\Gamma]^{\mathrm{r1}}$ (resp. $[\Gamma]^{\mathrm{sing}}$) denotes the set of rank-one (resp. non-rank-one) conjugacy classes, and that for any subset $A \subset [\Gamma]$ and any $T \geq 0$, we set $A_T := \{c \in A : \ell(c) \leq T\}$. For any $c \in [\Gamma]^{\mathrm{r1}}$, we denote by $\mathcal{L}c$ the only $(\phi_t)_t$-invariant probability measure on the rank-one periodic $(\phi_t)_t$-orbit of T^1M associated to c.

Théorème 7.0.3. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set and $\Gamma \subset \mathrm{Aut}(\Omega)$ a convex cocompact, non-elementary rank-one and discrete subgroup. Then there exists $0 < \delta < \delta_\Gamma$ and $C > 1$ such that*

1. *$C^{-1}e^{\delta_\Gamma T} \leq \#[\Gamma]_T^{\mathrm{r1}} \leq Ce^{\delta_\Gamma T}$ for any large enough $T \geq 0$;*

2. *$\#[\Gamma]_T^{\mathrm{sing}} \leq Ce^{\delta T}$ for any $T \geq 0$;*

3. *for every $A \subset [\Gamma]^{\mathrm{r1}}$ with $\lim_{T\to\infty} T^{-1}\log(\#A_T) = \delta_\Gamma$, the family $(\frac{1}{\#A_T}\sum_{c\in A_T}\mathcal{L}c)_{T\geq 0}$ weak-$*$ converges to the Bowen–Margulis probability measure as T tends to infinity.*

We will establish in Corollary 8.0.4 (obtained in collaboration with F. Zhu) an estimate which is more precise than Theorem 7.0.3.(1), by using different techniques. We will also establish in Theorem 8.0.3 an equidistribution result which is similar to Theorem 7.0.3.(3).

In Section 7.1 we establish estimates on the measure of dynamical balls, sometimes refered to as the Gibbs property. In Section 7.2 we prove that the Bowen–Margulis probability measure has entropy δ_Γ. In Section 7.3 we conclude the proof of Theorem 7.0.2 by proving uniqueness of the measure of maximal entropy (Section 7.3.4). In Section 7.4 we prove Theorem 7.0.3. The main steps are: bounding from below the number of rank-one closed geodesics (Section 7.4.1); bounding from above the number of rank-one closed geodesics (Section 7.4.2); bounding from above the number of non-rank-one closed geodesics (Section 7.4.3); and finally proving the equidistribution of closed geodesics (Section 7.4.4).

Remark 7.0.4. In this chapter, we will use several times the following fact: a convex cocompact group Γ is finitely generated (see [BH99, Th. 8.10]), and hence contains a torsion-free finite-index normal subgroup by Selberg's lemma.

7.1 The Shadow lemma and the Gibbs property

7.1.1 A Shadow lemma for convex cocompact actions

In this section we refines the Shadow lemma (Lemma 6.3.1), in particular so that it yields estimates of the measure of *small* shadows. Then we apply if to the case where Γ is convex cocompact.

We denote $\mathrm{ray}_x(A) = \bigcup_{\xi\in A}[x,\xi)$ for $x \in \Omega$ and $A \subset \partial\Omega$.

Lemma 7.1.1. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set and $\Gamma \subset \mathrm{Aut}(\Omega)$ a discrete subgroup. Suppose that Γ is strongly irreducible and T^1M_{bip} is non-empty, or that M is rank-one and non-elementary. Consider $\delta \geq 0$ and a δ-conformal density $(\mu_x)_{x\in\Omega}$ on $\partial\Omega$. Let $K \subset \Omega$ be a compact subset and $\mathcal{C} = \Gamma \cdot K$. Then there exists $R_0 > 0$ such that for any $R > 0$, we can find a constant $C = C(R) > 0$ such that for all $x, y \in \mathcal{C}$,*

$$C^{-1}e^{-\delta d_\Omega(x,y)} \leq \mu_x(\mathcal{O}_{R_0+R}(x,y)) \leq \mu_x(\mathcal{O}^+_{R_0+R}(x,y)) \leq Ce^{-\delta d_\Omega(x,y)},$$

and if furthermore $y \in \mathrm{ray}_x(\mathrm{supp}(\mu_o))$, then

$$\mu_y(\mathcal{O}_R(x,y)) \geq C^{-1} \text{ and } \mu_x(\mathcal{O}_R(x,y)) \geq C^{-1}e^{-\delta d_\Omega(x,y)}.$$

Proof. Let $R > 0$. By definition of the conformal density, $\mu_x \leq e^{\delta d_\Omega(x,y)}\mu_y$ for any $x, y \in \Omega$. Using the triangular inequality and Lemma 2.1.4, it is elementary to see that for all $x, y, x', y' \in \Omega$,

$$\mathcal{O}^\alpha_R(x,y) \subset \mathcal{O}^\alpha_{R+d_\Omega(x,x')+d_\Omega(y,y')}(x',y') \text{ for } \alpha = \emptyset \text{ or } +.$$

By Γ-equivariance it is enough to prove the lemma for $x \in K$. Let D be the diameter of K for the Hilbert metric, and fix $o \in K$. For any $y \in \mathcal{C}$, there exists $\gamma \in \Gamma$ such that $d_\Omega(y, \gamma o) \leq D$, and then for any $x \in K$,

$$e^{-\delta_\Gamma D}\mu_o(\mathcal{O}_R(o,\gamma o)) \leq \mu_x(\mathcal{O}_{R+2D}(x,y)) \leq \mu_x(\mathcal{O}^+_{R+2D}(x,y)) \leq e^{\delta_\Gamma D}\mu_o(\mathcal{O}^+_{R+4D}(o,\gamma o)).$$

Now we can use the Shadow lemma (Lemma 6.3.1) to bound the right-most and left-most terms when R is greater than some R_0, and we obtain the first estimate.

Let us show the second estimate. We set

$$\epsilon := \inf\{\mu_x(\mathcal{O}_R(y,x')) : x \in K,\ y \in \Omega,\ x' \in K \cap \mathrm{ray}_y(\operatorname{supp}\mu_o)\} > 0.$$

We then make the same computations as in the Shadow lemma (Lemma 6.3.1). We take a δ-conformal density $(\mu_x)_{x\in\Omega}$ on $\partial_{\mathtt{h}}\Omega$ such that $(\pi_{\mathtt{h}})_*\mu_x = \mu_x$ for any $x \in \Omega$. Let $x \in K$ and $y \in \mathrm{ray}_x(\operatorname{supp}(\mu_o)) \cap \mathcal{C}$; by definition of K there exists $\gamma \in \Gamma$ such that $\gamma^{-1}y \in K$. Then

$$\begin{aligned}
\mu_x(\mathcal{O}_R(x,y)) &= \mu_x(\pi_{\mathtt{h}}^{-1}(\mathcal{O}_R(x,y)))\\
&= \mu_{\gamma^{-1}x}(\pi_{\mathtt{h}}^{-1}(\mathcal{O}_R(\gamma^{-1}x,\gamma^{-1}y)))\\
&= \int_{\pi_{\mathtt{h}}^{-1}(\mathcal{O}_R(\gamma^{-1}x,\gamma^{-1}y))} e^{-\delta \mathtt{b}_{\tilde{\xi}}(\gamma^{-1}x,\gamma^{-1}y)} d\mu_x(\tilde{\xi})\\
&\geq \int_{\pi_{\mathtt{h}}^{-1}(\mathcal{O}_R(\gamma^{-1}x,\gamma^{-1}y))} e^{-\delta d_\Omega(x,y)} d\mu_x(\tilde{\xi})\\
&\geq \mu_x(\mathcal{O}_R(\gamma^{-1}x,\gamma^{-1}y))e^{-\delta d_\Omega(x,y)}\\
&\geq \epsilon e^{-\delta d_\Omega(x,y)}. \qquad \square
\end{aligned}$$

As an immediate corollary, we obtain the following.

Corollary 7.1.2. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set and $\Gamma \subset \mathrm{Aut}(\Omega)$ a convex cocompact and discrete subgroup. Suppose that Γ is strongly irreducible and T^1M_{bip} is non-empty, or that M is rank-one and non-elementary. Consider $\delta \geq 0$ and a δ-conformal density $(\mu_x)_{x\in\Omega}$ on $\partial\Omega$. Then there exists $R_0 > 0$ such that for any $R > 0$, we can find a constant $C = C(R) > 0$ such that for all $x, y \in \mathcal{C}^{\mathrm{cor}}$,*

$$C^{-1}e^{-\delta d_\Omega(x,y)} \leq \mu_x(\mathcal{O}_{R_0+R}(x,y)) \leq Ce^{-\delta d_\Omega(x,y)},$$

and if furthermore $y \in \mathrm{ray}_x(\operatorname{supp}(\mu_o))$, then

$$\mu_y(\mathcal{O}_R(x,y)) \geq C^{-1} \text{ and } \mu_x(\mathcal{O}_R(x,y)) \geq C^{-1}e^{-\delta d_\Omega(x,y)}.$$

7.1.2 The measure of dynamical balls

In this section we derive from the previous Shadow lemma (Corollary 7.1.2) an estimate for the measure of balls of the form $\tilde{B}^{(t)}_{T^1M}$ from Section 2.1.1; these estimates can interpreted as a Gibbs property for the Bowen–Margulis probability measure. The idea is very similar to the computations in the proof of Proposition 6.5.1, which are actually Roblin's computations; the difference here is that our shadows are a priori *small*, and this is why we need the stronger Shadow lemma (Corollary 7.1.2). Note however that Corollary 7.1.2 only works for usual shadows $\mathcal{O}_R(x,y)$, and not for the scarce shadows of the form $\mathcal{O}^-_R(x,y)$; to overcome this issue we use the following lemma, which is a consequence of Benzécri's compactness theorem (Fact 2.1.11).

Lemma 7.1.3. *For any $\epsilon > 0$, there exists $\epsilon' > 0$ such that for any properly convex open set $\Omega \subset \mathrm{P}(\mathbf{V})$, for any $x, y \in \Omega$ at distance at least 1, if we have two points on the boundary $\xi \in \mathcal{O}_{\epsilon'}(y,x)$ and $\eta \in \mathcal{O}_{\epsilon'}(x,y)$, then the line $\xi \oplus \eta$ intersects both balls $B_\Omega(x,\epsilon)$ and $B_\Omega(y,\epsilon)$.*

Proof. By symmetry it is enough to prove that $\xi \oplus \eta$ meets $B_\Omega(x,\epsilon)$. We assume by contradiction that there exist sequence $(\epsilon'_n)_{n\in\mathbb{N}} \in \mathbb{R}_{>0}^{\mathbb{N}}$ converging to 0 and a sequence $(\Omega_n)_{n\in\mathbb{N}} \in \mathcal{E}_V^{\mathbb{N}}$ such that for each $n \in \mathbb{N}$ we can find $x_n, y_n \in \Omega_n$ at distance at least 1 and $\xi_n \in \mathcal{O}_{\epsilon'_n}(y_n, x_n)$ and $\eta_n \in \mathcal{O}_{\epsilon'_n}(x_n, y_n)$ such that $(\xi_n \oplus \eta_n) \cap B_\Omega(x_n, \epsilon) = \emptyset$.

By Benzécri's compactness theorem (Fact 2.1.11), we can assume, up to extraction, that $((\Omega_n, x_n))_n$ converges to some pointed properly convex open set $(\Omega, x) \in \mathcal{E}_V^\bullet$, bounded in some affine chart that we fix. Up to extraction we assume that Ω_n is also bounded in the affine chart for any $n \in \mathbb{N}$, and that $(y_n)_{n\in\mathbb{N}}$ (resp. $(\xi_n)_{n\in\mathbb{N}}$ and $(\eta_n)_{n\in\mathbb{N}}$) converges to $y \in \overline{\Omega} \smallsetminus B_\Omega(x,1)$ (resp. ξ and $\eta \in \partial\Omega$) such that $(\xi \oplus \eta) \cap B_\Omega(x,\epsilon) = \emptyset$.

Recall from Section 2.1.2 that

$$B_{\Omega_n}(x_n, \epsilon'_n) \subset (1 - e^{-2\epsilon'_n})(\Omega_n - x_n) + x_n,$$

hence $(B_{\Omega_n}(x_n, \epsilon'_n))_{n\in\mathbb{N}}$ converges to the singleton $\{x\}$ for the Hausdorff topology, and similarly $(B_{\Omega_n}(y_n, \epsilon'_n))_{n\in\mathbb{N}}$ converges to the singleton $\{y\}$. This implies that $x \in [\xi, \eta]$, which contradicts $(\xi \oplus \eta) \cap B_\Omega(x,\epsilon) = \emptyset$. □

Lemma 7.1.4. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set, and $\Gamma \subset \mathrm{Aut}(\Omega)$ a discrete subgroup; set $M = \Omega/\Gamma$. Suppose that Γ is strongly irreducible and $T^1M_{\mathrm{bip}} \neq \emptyset$, or that M is rank-one and non-elementary. Let m_Γ be a Sullivan measure on T^1M induced by a δ_Γ-conformal density. Then for any compact subset $K \subset T^1M$, for any $r > 0$, there exists a constant $C > 0$ such that given any time $t > 0$, for any $v \in \Gamma \cdot K$ such that $\phi_t v \in \Gamma \cdot K$,*

$$m_\Gamma(\tilde{B}^{(t)}_{T^1M}(v,r)) \leq Ce^{-\delta_\Gamma t},$$

and if $v \in T^1M_{\mathrm{bip}}$, then

$$C^{-1}e^{-\delta_\Gamma t} \leq m_\Gamma(\tilde{B}^{(t)}_{T^1M}(v,r)).$$

Proof. Let $(\mu_x)_{x\in\Omega}$ be a δ_Γ-conformal density on $\partial_{\mathtt{h}}\Omega$ which induces a δ_Γ-conformal density $(\mu_x)_{x\in\Omega}$ on $\partial\Omega$ and the Sullivan measures m on $T^1\Omega$ and m_Γ on T^1M. Let $\tilde{v} \in T^1\Omega$ be a lift of v. We have

$$C_0^{-1} m(B^{(t)}_{T^1\Omega}(\tilde{v},r)) \leq m_\Gamma(\tilde{B}^{(t)}_{T^1M}(v,r)) \leq m(B^{(t)}_{T^1\Omega}(\tilde{v},r))$$

for any $t \geq 0$, where

$$C_0 = \max_{\tilde{w}\in\pi^{-1}K} \#\{\gamma : d_\Omega(\pi\tilde{w}, \gamma\pi\tilde{w}) \leq 4r\}.$$

Let us prove the upper bound in the lemma. Consider $\tilde{w} \in B^{(t)}_{T^1\Omega}(\tilde{v},r)$. We make the following observations.

- The Lebesgue measure of the set of times $s \in \mathbb{R}$ such that $\phi_s\tilde{w} \in B^{(t)}_{T^1\Omega}(\tilde{v},r)$ is less than $2r$;
- $\phi_\infty\tilde{w} \in \mathcal{O}_r^+(\pi\tilde{v}, \pi\phi_t\tilde{v})$;
- $\langle \xi, \eta \rangle_{\pi\tilde{v}} \leq r$ for all $\xi \in \pi_{\mathtt{h}}^{-1}(\phi_{-\infty}\tilde{w})$ and $\eta \in \pi_{\mathtt{h}}^{-1}(\phi_\infty\tilde{w})$.

Combined with the definition of m (see Section 6.2.3), they yield:

$$m(B^{(t)}_{T^1\Omega}(\tilde{v},r)) \leq e^{2\delta_\Gamma r} \cdot \mu_{\pi\tilde{v}}(\partial\Omega) \cdot \mu_{\pi\tilde{v}}(\mathcal{O}_r^+(\pi\tilde{v}, \pi\phi_t\tilde{v})) \cdot 2r.$$

We deduce from this and the Shadow lemma (Lemma 7.1.1) the desired upper bound.

Let us prove the lower bound. We apply Lemma 7.1.3 to $\epsilon := r/16$ to obtain $\epsilon' > 0$. Then for all $\eta \in \mathcal{O}_{\epsilon'}(\pi\tilde{v}, \pi\phi_{t+1}\tilde{v})$ and $\xi \in \mathcal{O}_{\epsilon'}(\pi\phi_{t+1}\tilde{v}, \pi\tilde{v})$, one can find $\tilde{w} \in B^{(t)}_{T^1\Omega}(\tilde{v}, r/2)$ tangent to $\xi \oplus \eta$. Observe that the Lebesgue measure of the set of times $s \in \mathbb{R}$ such that $\phi_s\tilde{w} \in B^{(t)}_{T^1\Omega}(\tilde{v}, r)$ is greater than r. This means (remembering that the Gromov product is always non-negative) that

$$m(B^{(t)}_{T^1\Omega}(\tilde{v}, r)) \geq \mu_{\pi v}(\mathcal{O}_{\epsilon'}(\pi\tilde{v}, \pi\phi_{t+1}\tilde{v})) \cdot \mu_{\pi v}(\mathcal{O}_{\epsilon'}(\pi\phi_{t+1}\tilde{v}, \pi\tilde{v})) \cdot r.$$

We conclude thanks to the Shadow lemma (Lemma 7.1.1), and the fact that $\Lambda^{\mathrm{prox}} \subset \mathrm{supp}(\mu_o)$. □

In the convex cocompact case, this yields the following.

Corollary 7.1.5. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set, and $\Gamma \subset \mathrm{Aut}(\Omega)$ a convex cocompact discrete subgroup; set $M = \Omega/\Gamma$. Suppose that Γ is strongly irreducible and $T^1M_{\mathrm{bip}} \neq \emptyset$, or that M is rank-one and non-elementary. Let m_Γ be a Sullivan measure on T^1M induced by a δ_Γ-conformal density. Then for any $r > 0$, there exists a constant $C > 0$ such that given any time $t > 0$, for any $v \in T^1M_{\mathrm{cor}}$,*

$$m_\Gamma(\tilde{B}^{(t)}_{T^1M}(v, r)) \leq Ce^{-\delta_\Gamma t},$$

and if $v \in T^1M_{\mathrm{bip}}$, then

$$C^{-1}e^{-\delta_\Gamma t} \leq m_\Gamma(\tilde{B}^{(t)}_{T^1M}(v, r)).$$

7.2 A measure of maximal entropy

In this section we prove that the Bowen–Margulis probability measure on the unit tangent bundle of a non-elementary rank-one convex projective orbifold with compact convex core $M = \Omega/\Gamma$ has entropy δ_Γ.

7.2.1 Injectivity radius

Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set and $\Gamma \subset \mathrm{Aut}(\Omega)$ a discrete subgroup; set $M = \Omega/\Gamma$. Consider $t \geq 0$ and $v, w \in T^1M$ with lifts $\tilde{v}, \tilde{w} \in T^1\Omega$. Recall from Section 2.1.1 that

$$\tilde{d}^{(t)}_{T^1M}(v, w) := \min_{\gamma\in\Gamma} \max_{s\in[0,t+1]} d_\Omega(\pi\phi_s\tilde{v}, \gamma\pi\phi_s\tilde{w}).$$

We set $d_{T^1M} = \tilde{d}^{(0)}_{T^1M}$. According to Section 1.3.1, we may also define

$$d^{(t)}_{T^1M}(v, w) := \max_{s\in[0,t]} d_{T^1M}(\phi_t v, \phi_t w) = \max_{s\in[0,t]} \min_{\gamma\in\Gamma} \max_{r\in[0,1]} d_\Omega(\pi\phi_{s+r}\tilde{v}, \gamma\pi\phi_{s+r}\tilde{w})$$

Observe that $\tilde{d}^{(t)}_{T^1M} \geq d^{(t)}_{T^1M}$ and that we do not have equality in general; however we have equality on $\{d^{(t)}_{T^1M} < \mathrm{inj}(M)/2\}$, where $\mathrm{inj}(M)$ is the injectivity radius of M, namely

$$\mathrm{inj}(M) := \inf\{d_\Omega(x, \gamma x) : x \in \Omega, \gamma \in \Gamma \smallsetminus \{\mathrm{id}\}\}. \tag{7.2.1}$$

The injectivity radius is zero as soon as Γ has a torsion element, or more generally when Γ contains an element $\gamma \in \Gamma$ such that $\ell(\gamma) = 0$.

Also observe that $d^{(n)}_{T^1M} = \max_{k=0,\dots,n} d_{T^1M}(\phi_k v, \phi_k w)$ for any integer $n \geq 0$.

7.2.2 Manning's argument

In this section, we adapt to the convex projective setting an argument originally due to Manning [Man79]. Although our setting is not exactly the same as his, his proof works perfectly thanks to Lemma 2.1.4. We observe that only half of Manning's proof is given here, since the other half only works for compact manifolds (while we work with manifolds with a compact convex core).

Fact 7.2.1. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set and $\Gamma \subset \mathrm{Aut}(\Omega)$ a convex cocompact discrete subgroup. Then $\delta_\Gamma \geq h_{\mathrm{top}}(T^1M_{\mathrm{cor}}, (\phi_t)_{t\in\mathbb{R}}) \geq h_{\mathrm{top}}(T^1M_{\mathrm{bip}}, (\phi_t)_{t\in\mathbb{R}})$.*

Proof. The fact that $h_{\mathrm{top}}(T^1M_{\mathrm{cor}}, (\phi_t)_{t\in\mathbb{R}}) \geq h_{\mathrm{top}}(T^1M_{\mathrm{bip}}, (\phi_t)_{t\in\mathbb{R}})$ is immediate from the definition of topological entropy.

Consider $o \in \mathcal{C}^{\mathrm{cor}}$ and $R > 0$ large enough such that $\Gamma \cdot B_\Omega(o, R)$ contains $\mathcal{C}^{\mathrm{cor}}$ (which is acted on cocompactly by Γ). Let $T^1\Omega_{\mathrm{cor}} \subset T^1\Omega$ be the preimage of $T1M_{\mathrm{cor}}$. Fix $\epsilon > 0$, and fix a maximal $(\epsilon/4, d_\Omega)$-spanning set A of $B_\Omega(o, R)$. Let B_r be the set of vectors based at a point of A and pointing at a point of $\{\gamma a : a \in A,\ d_\Omega(o, \gamma o) \leq r + 2R\}$; for each $\tilde{v} \in B_r$ choose $\tilde{v}' \in T^1\Omega_{\mathrm{cor}}$ at distance less than $\epsilon/2$ of $\tilde{v}$ (when such a vector exists), and denote by B'_r the collection of $\tilde{v}'$. By Lemma 2.1.4, the projection of B'_r is a $(\epsilon, d^{(r)}_{T^1M})$-spanning set of T^1M_{cor} for any $r > 0$. Furthermore $\#B'_r \leq \#B_r \leq \#A^2 \cdot \#\{\gamma : d_\Omega(o, \gamma o) \leq r + 2R\}$, hence $\delta_\Gamma \geq h_{\mathrm{top}}(T^1M_{\mathrm{cor}}, (\phi_t)_{t\in\mathbb{R}})$. □

7.2.3 The Bowen–Margulis probability measure has maximal entropy

In this section we prove that any Sullivan measure induced by a δ_Γ-conformal density has maximal entropy, which is equal to δ_Γ.

Proposition 7.2.2. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set, and $\Gamma \subset \mathrm{Aut}(\Omega)$ a convex cocompact discrete subgroup with $M = \Omega/\Gamma$ rank-one and non-elementary. Let m_Γ be the Bowen–Margulis probability measure on T^1M. Then*

$$h_m(T^1M_{\mathrm{bip}}, (\phi_t)_t) \geq \delta_\Gamma.$$

Proof. We can assume without loss of generality that Γ is torsion-free by Observation 1.3.6 and since for any finite-index subgroup Γ' of Γ, the measure m_Γ is the push-forward of the Bowen–Margulis probability measure on $T^1\Omega/\Gamma'$. Let ϵ_0 be the injectivity radius of M. Let $\mathcal{P}$ be a finite measurable partition of T^1M_{cor} whose elements have diameter less than $\frac{\epsilon_0}{3}$. According to the definition of the measure-theoretic entropy, it is enough to prove that

$$H_m(\phi_1, \mathcal{P}) \geq \delta_\Gamma.$$

For $n \geq 1$, we observe that, according to Section 7.2.1, any element of $\mathcal{P}^{(n)}$ has by definition a diameter less than $\epsilon_0/3$ with respect to the metric $\tilde{d}^{(n)}_{T^1M}$. Therefore by Corollary 7.1.5, there exists a constant $C > 0$ such that for any $n \geq 1$, any element of $\mathcal{P}^{(n)}$ has an m_Γ-measure less than $Ce^{-\delta_\Gamma n}$. We now conclude the proof by the following computation.

$$\begin{aligned} H_{m_\Gamma}(\mathcal{P}^{(n)}) &= -\sum_{P\in\mathcal{P}^{(n)}} m_\Gamma(P)\log(m_\Gamma(P)) \\ &\geq -\sum_{P\in\mathcal{P}^{(n)}} m_\Gamma(P)\log(Ce^{-\delta_\Gamma n}) \\ &\geq \delta_\Gamma n - \log(C). \end{aligned}$$

The inequality $H_{m_\Gamma}(\phi_1, \mathcal{P}) \geq \delta_\Gamma$ follows immediately. □

7.3 Uniqueness of the measure of maximal entropy

In this section we conclude the proof of Theorem 7.0.2. In the case where M is compact, the proof below can be shorten by using Chapter 4. In particular we do not need Sections 7.3.1 and 7.3.3 in this case.

7.3.1 Separated sets in dynamical balls

In this section we prove a technical lemma which bound from above the size a separated set in a dynamical ball. To perform this estimate we will use the notion of proper densities (see Section 2.1.11).

Lemma 7.3.1. *Consider $0 < r < R$. Let $\Omega \in \mathcal{E}_V$, let $\Gamma \subset \mathrm{Aut}(\Omega)$ be a discrete subgroup, let $M = \Omega/\Gamma$ and let $t \geq 0$.*

(1) For any vector $v \in T^1M$, the cardinality of any $(\tilde{d}^{(t)}_{T^1M}, r)$-separated set of $\tilde{B}^{(t)}_{T^1M}(v, R)$ is less than $\chi_+(R + r/4)^2 \cdot \chi_-(r/4)^{-2}$ (see (2.1.2)).

(2) Consider a $(\tilde{d}^{(t)}_{T^1M}, r)$-separated subset $\{v_1, \dots, v_k\}$ of T^1M (and of size k), take a vector $w_i \in \tilde{B}^{(t)}_{T^1M}(v_i, R)$ for each $i = 1, \dots, k$. Then one can find a subset I of $\{1, \dots, k\}$ of size greater than $k \cdot \chi_+(2R + r/2)^{-2} \cdot \chi_-(r/4)^2$ such that $\{w_i : i \in I\}$ is $(\tilde{d}^{(t)}_{T^1M}, r)$-separated.

Proof. Let us prove (1) when Γ is trivial. Let $A \subset B^{(t)}_{T^1\Omega}(v, R)$ be a $(d^{(t)}_{T^1\Omega}, r)$-separated set. We set

$$B := \{(\pi w, \pi\phi_t w) : w \in A\} \subset B_\Omega(\pi v, R) \times B_\Omega(\pi\phi_t v, R)\}.$$

By Lemma 2.1.4, and since A is $(d^{(t)}_{T^1\Omega}, r)$-separated, we see that B is $(d, r/2)$-separated for the metric d on Ω^2 which is defined by $d((x, y), (x', y')) = \max(d_\Omega(x, x'), d_\Omega(y, y'))$ for $(x, y), (x', y') \in \Omega^2$. This exactly means that for all $(x, y) \neq (x', y') \in B$,

$$(B_\Omega(x, r/4) \times B_\Omega(y, r/4)) \cap (B_\Omega(x', r/4) \times B_\Omega(y', r/4)) = \emptyset.$$

As a consequence,

$$\begin{aligned}
\#A = \#B \\
&\leq \chi_-(r/4)^{-2} \sum_{(x,y)\in B} \mathrm{Vol}_\Omega(B_\Omega(x, r/4)) \,\mathrm{Vol}_\Omega(B_\Omega(y, r/4)) \\
&\leq \chi_-(r/4)^{-2} \,\mathrm{Vol}^2_\Omega \left(\bigsqcup_{(x,y)\in B} B_\Omega(x, r/4) \times B_\Omega(y, r/4) \right) \\
&\leq \chi_-(r/4)^{-2} \,\mathrm{Vol}^2_\Omega(B_\Omega(\pi v, R + r/4) \times B_\Omega(\pi\phi_t v, R + r/4)) \\
&\leq \chi_+(R + r/4)^2 \chi_-(r/4)^{-2}.
\end{aligned}$$

Let us prove (1) when Γ is not necessarily trivial. Let $A \subset \tilde{B}^{(t)}_{T^1M}(v, R)$ be a $(\tilde{d}^{(t)}_{T^1M}, r)$-separated set. Consider a lift $\tilde{v} \in T^1\Omega$ of v, and a lift $\tilde{A} \subset B^{(t)}_{T^1\Omega}(\tilde{v}, R)$ of A. Then $\tilde{A}$ is $(d^{(t)}_{T^1\Omega}, r)$-separated, therefore it has cardinality less than $\chi_+(R + r/4)^2 \chi_-(r/4)^{-2}$, and so do A.

Let us establish (2). We construct I by induction. We set $i_0 = 0$ and $I_0 := \{1, \dots, k\}$. For $j \geq 1$, if $(I_0, \dots, I_{j-1})$ and $(i_0, \dots, i_{j-1})$ have been constructed, we set $i_j := \min I_{j-1} > i_{j-1}$ and

$$I_j := \{i \in I_{j-1} : \tilde{d}^{(t)}_{T^1M}(w_{i_j}, w_i) \geq r\} \subsetneq I_{j-1}.$$

This process eventually stops, at the n-th step for some $n \in \{1, \dots, k\}$ such that $I_n = \emptyset$. The set $\{w_{i_j} : 1 \leq j \leq n\}$ is $(\tilde{d}^{(t)}_{T^1M}, r)$-separated by construction. In order to prove that k is bounded above by $n \cdot \chi_+(2R + r/2)^2 \cdot \chi_-(r/4)^{-2}$, it is enough to see that for each $0 \leq j \leq n-1$,

$$\#I_{j+1} \geq \#I_j - \chi_+(2R + r/2)^2 \cdot \chi_-(r/4)^{-2}.$$

This is a consequence of (1) and of the fact that $I_j \smallsetminus I_{j+1}$ is contained in the set of indices i such that $v_i \in \tilde{B}^{(t)}_{T^1\Omega}(v_{j+1}, r + 2R)$. □

7.3.2 Entropy-expansiveness

The estimates of the previous section can be used to show that the geodesic flow on a convex projective manifold is entropy-expansive if the injectivity radius is non-zero. This fact is due to Bray, who stated it for compact manifolds, but the proof works in general.

Fact 7.3.2 ([Bra20b, Th. 6.2])**.** *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set and $\Gamma \subset \mathrm{Aut}(\Omega)$ a discrete subgroup. Let us assume that the injectivity radius ϵ_0 of $M = \Omega/\Gamma$ is non-zero. Then the time-one map of the geodesic flow on T^1M is $(d_{T^1M}, \epsilon_0/3)$-entropy-expansive.*

It applies in particular if Γ acts convex cocompactly on Ω and *if Γ is torsion-free*. However, we will consider cases where Γ is not torsion-free, and the geodesic flow on T^1M is *not* entropy-expansive: for instance if Ω is the Poincaré disk and Γ is a cocompact triangle group. To overcome this issue, we will use Selberg's lemma.

7.3.3 A uniform neighbourhood of the biproximal unit tangent bundle

In order to prove the uniqueness of the measure of maximal entropy, we will use our estimates on the size of the dynamical balls Corollary 7.1.5. However, one of these estimates only holds for balls centred at vectors in T^1M_{bip}, whereas we would like the uniqueness of the measure of maximal entropy on T^1M_{cor}. This is why we will need Lemma 7.3.1 and the following lemma.

Lemma 7.3.3. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set, and $\Gamma \subset \mathrm{Aut}(\Omega)$ a convex cocompact discrete subgroup. Suppose $M = \Omega/\Gamma$ is rank-one and non-elementary. Then there exists $R > 0$ such that for every point $\xi \in \Lambda^{\mathrm{orb}}$ we can find $\eta \in \Lambda^{\mathrm{prox}}$ such that $d_{\overline{\Omega}}(\xi, \eta) \leq R$.*

In particular, for any $v \in T^1M_{\mathrm{cor}}$, we can find a vector $w \in T^1M_{\mathrm{bip}}$ such that for any $t \geq 0$;

$$\tilde{d}^{(2t)}_{T^1M}(\phi_{-t}v, \phi_{-t}w) \leq 2R.$$

Proof. Consider o in the convex hull of Λ^{orb} in Ω and a δ_Γ-conformal density $(\mu_x)_{x\in\Omega}$ on $\partial\Omega$. Let $R > 0$ be given by the Shadow lemma (Corollary 7.1.2). By lower semi-continuity of $d_{\overline{\Omega}}$, it is enough to show that for any $x \in [o, \xi)$, the shadow $\mathcal{O}_R(o, x)$ intersect Λ^{prox}; this is implied by the fact that $\mu_o(\Lambda^{\mathrm{prox}}) = 1$ (by Theorem 6.0.1 and Proposition 7.0.1) and $\mu_o(\mathcal{O}_R(o, x)) > 0$. □

7.3.4 Proof of Theorem 7.0.2

Let m_Γ be the Bowen–Margulis probability measure on T^1M. By Propositions 7.2.1 and 7.2.2, it is enough to prove that any $(\phi_t)_{t\in\mathbb{R}}$-invariant probability measure m' on T^1M_{cor} which is different from m_Γ has entropy strictly less that δ_Γ.

We can assume that Γ is torsion-free by Observation 1.3.6 and since for any finite-index subgroup Γ' of Γ, the measure m_Γ is the push-forward of the Bowen–Margulis probability measure on $T^1\Omega/\Gamma'$ (see Remark 6.2.3); let ϵ_0 be the injectivity radius of Ω/Γ. Since m_Γ is ergodic (Theorem 6.0.1 and Proposition 7.0.1), m' cannot be absolutely continuous with respect to m_Γ. By Radon–Nikodym Theorem we can decompose m' into a sum $tm'' + (1-t)m_\Gamma$ where $0 < t \leq 1$ and m'' is $(\phi_t)_{t\in\mathbb{R}}$-invariant and singular with respect to m_Γ. Then $h_{m'}(\phi) = th_{m''}(\phi) + (1-t)\delta_\Gamma$ (see [HK95, Cor. 4.3.17]), and we only need to prove that $h_{m''}(\phi) < \delta_\Gamma$. Note that since $\Lambda^{\text{orb}} \smallsetminus \Lambda^{\text{prox}}$ does not intersect $\partial_{\text{sse}}\Omega$. Without loss of generality we assume that $m' = m''$ is singular with respect to m_Γ. Let $A \subset T^1M_{\text{sse}}$ be a flow-invariant measurable subset such that $m_\Gamma(A) = 1$ while $m'(A) = 0$.

Fix $\epsilon > 0$ and let $K_1 \subset A \cap T^1M_{\text{cor}}$ and $K_2 \subset T^1M_{\text{cor}} \smallsetminus A$ be compact subsets such that $m_\Gamma(K_1) \geq 1-\epsilon$ and $m'(K_2) \geq 1-\epsilon$. Observe that

$$\min\{\tilde{d}^{(2t)}_{T^1M}(\phi_{-t}v, \phi_{-t}w) : v \in K_1, w \in K_2\} \underset{t\to\infty}{\longrightarrow} \infty. \tag{7.3.1}$$

Indeed, otherwise by compactness there would exist vectors $v \in K_1$ and $w \in K_2$ for which $\sup_{t>0} \tilde{d}^{(2t)}_{T^1M}(\phi_{-t}v, \phi_{-t}w) < \infty$. Then we would be able to find lifts $\tilde{v}, \tilde{w} \in T^1\Omega$ such that $\sup_{t>0} d^{(2t)}_{T^1\Omega}(\phi_{-t}\tilde{v}, \phi_{-t}\tilde{w}) < \infty$, which implies, since $\phi_\infty\tilde{v}$ and $\phi_{-\infty}\tilde{v}$ are extremal, that $\phi_{\pm\infty}\tilde{v} = \phi_{\pm\infty}\tilde{w}$ hence $w \in \phi_{\mathbb{R}}v \subset A$, which is a contradiction.

Let $n \geq 1$ and consider a maximal $(d^{(2n)}_{T^1M}, \epsilon_0/8)$-separated set $\{v_1, \dots, v_k\} \subset T^1M_{\text{cor}}$, which is ordered so that for any $i = 1, \dots, k$, the ball $B^{(2n)}_{T^1M_{\text{cor}}}(v_i, \epsilon_0/8)$ intersects $\phi_{-n}K_2$ if and only if $i \leq l$, where $1 \leq l \leq k$ is some integer.

Construct by induction the finite measurable partition $\mathcal{P} = \{P_1, \dots, P_k\}$ of T^1M_{cor} so that

$$B^{(2n)}_{T^1M_{\text{cor}}}(v_{i+1}, \epsilon_0/16) \subset P_{i+1} := B^{(2n)}_{T^1M_{\text{cor}}}(v_{i+1}, \epsilon_0/8) \smallsetminus (P_1 \cup \dots \cup P_i) \subset B^{(2n)}_{T^1M_{\text{cor}}}(v_{i+1}, \epsilon_0/8).$$

P_i diameter less than $\epsilon_0/3$ with respect to $d^{(2n)}_{T^1M}$ for each $1 \leq i \leq k$, therefore we can combine Remarks 1.3.1 and 1.3.3 with Facts 7.3.2 and 1.3.4 to obtain:

$$h_{m'}(\phi_1) = \frac{1}{2n} h_{m'}(\phi_{2n}) = \frac{1}{2n} H_{m'}(\phi_{2n}, \mathcal{P}) \leq \frac{1}{2n} H_{m'}(\mathcal{P}).$$

Now we use the classical fact that for all $q \in \mathbb{N}_{>0}$ and $a_1, \dots, a_q > 0$, if $s := a_1 + \dots + a_q \leq 1$ then

$$-\sum_i a_i \log(a_i) \leq -s\log s + s\log q \leq s\log(q) + 1/e,$$

and we compute:

$$\begin{aligned} H_{m'}(\mathcal{P}) &= -\sum_{i=1}^k m'(P_i)\log(m'(P_i)) \\ &\leq m'\left(\bigcup_{i=1}^l P_i\right)\log(l) + m'\left(\bigcup_{i=l+1}^k P_i\right)\log(k) + 2/e. \end{aligned}$$

Note that on one hand, $\phi_{-n}K_2$ is contained in $\bigcup_{i=1}^l P_i$, hence $m'\left(\cup_{i=1}^l P_i\right) \geq 1-\epsilon$. On the other hand $\phi_{-n}K_1$ does not intersect $\bigcup_{i=1}^l P_i$ for n large enough, indeed if there exist $i \in \{1,\dots,l\}$ and $v \in \phi_{-n}K_1 \cap P_i$, then we take a vector $w \in \phi_{-n}K_2 \cap B^{(2n)}_{T^1M}(v_i, \epsilon_0/8)$ and use the triangular inequality to obtain that $\tilde{d}^{(2n)}_{T^1M}(\phi_{-n}\phi_n v, \phi_{-n}\phi_n w) \leq \epsilon_0/4$, which is not compatible with (7.3.1) for n large. As a consequence, $m_\Gamma\left(\cup_{i=1}^l P_i\right) \leq \epsilon$.

We now need to bound from above k and l. If M is compact then it is easier to conclude the proof: we can use the fact that $T^1M = T^1M_{\mathrm{cor}} = T^1M_{\mathrm{bip}}$ (consequence of Theorem 4.1.2), and apply directly Corollary 7.1.5 to get:

$$l \leq \frac{m_\Gamma(\bigcup_{i=1}^l P_i)}{\min\{m_\Gamma(P_i) : 1 \leq i \leq l\}} \leq \epsilon C e^{2n\delta_\Gamma},$$

and similarly $k \leq Ce^{2n\delta_\Gamma}$, where C only depends on ϵ_0. Thus we obtain

$$2nh_{m'}(\phi_1) \leq (1-\epsilon)\log(\epsilon) + \log(C) + 2n\delta_\Gamma + 2/e,$$

which is strictly less than $2n\delta_\Gamma$ for ϵ small enough.

In the general case, we need to take into account that T^1M_{cor} and T^1M_{bip} might be different and that Corollary 7.1.5 only holds on T^1M_{bip}; this is where we need Lemma 7.3.1 and Lemma 7.3.3. Thanks to the latter, there exists $R>0$, which only depends on Γ and Ω, and such that for each $1 \leq i \leq k$, we can find $w_i \in \tilde{B}^{(2n)}_{T^1M}(v_i, R) \cap T^1M_{\mathrm{bip}}$. Then we can use Lemma 7.3.1.(2) to find $I \subset \{1,\dots,k\}$ such that $\{w_i : i \in I\}$ is $(d^{(2n)}_{T^1M}, \epsilon_0/8)$-separated and

$$\begin{aligned} k &\leq \#I \cdot \chi_+(2R+\epsilon_0/16)^2 \chi_-(\epsilon_0/32)^{-2} \\ &\leq \frac{m_\Gamma(\bigcup_{i\in I} B^{(2n)}_{T^1M}(w_i, \epsilon_0/16))}{\min\{m_\Gamma(B^{(2n)}_{T^1M}(w_i, \epsilon_0/16)) : i \in I\}} \chi_+(2R+\epsilon_0/16)^2 \chi_-(\epsilon_0/32)^{-2} \\ &\leq C' e^{2n\delta_\Gamma} \end{aligned}$$

where C' only depends on Γ and Ω. Similarly, we can find $w_i' \in \phi_{-n}K_2 \cap B^{(2n)}_{T^1M}(v_i, \epsilon_0/8)$ and $w_i'' \in \tilde{B}^{(2n)}_{T^1M}(w_i', R) \cap T^1M_{\mathrm{bip}}$ for each $i = 1,\dots,l$. Lemma 7.3.1.(2) gives us $I' \subset \{1,\dots,l\}$ such that $\{w_i'' : i \in I'\}$ is $(d^{(2n)}_{T^1M}, \epsilon_0/8)$-separated and

$$\begin{aligned} l &\leq \#I' \cdot \chi_+(2R+\epsilon_0/8)^2 \chi_-(\epsilon_0/32)^{-2} \\ &\leq \frac{m_\Gamma(\bigcup_{i\in I'} B^{(2n)}_{T^1M}(w_i'', \epsilon_0/16))}{\min\{m_\Gamma(B^{(2n)}_{T^1M}(w_i'', \epsilon_0/16)) : i \in I'\}} \chi_+(2R+\epsilon_0/8)^2 \chi_-(\epsilon_0/32)^{-2} \\ &\leq \epsilon C'' e^{2n\delta_\Gamma} \end{aligned}$$

where C'' only depends on Γ and Ω, and we have used the fact that $B^{(2n)}_{T^1M}(w_i'', \epsilon_0/16)$ does not intersect $\phi_{-n}K_1$ for any $i = 1,\dots,l$ and for n large enough (again because of (7.3.1)). As we explained in the compact case, this implies that

$$2nh_{m'}(\phi_1) \leq (1-\epsilon)\log(\epsilon) + \log\max(C', C'') + 2n\delta_\Gamma + 2/e,$$

which is strictly less that $2n\delta_\Gamma$ for ϵ small enough.

7.4 Counting closed geodesics

In this section we keep on adapting Knieper's article [Kni98] in order to prove Theorem 7.0.3, which gives asymptotic estimates for the number of closed geodesics of length less than t, when t goes to infinity.

These estimates do not all need the results of the previous sections. More precisely, to prove the upper bound on the number of rank-one conjugacy classes in 1 we do not need the HTSR dichotomy. To prove the lower bound in 1 we need the mixing property of the Bowen–Margulis probability measure, but not the uniqueness of the measure of maximal entropy. To establish the upper bound on the number of non-rank-one closed geodesics in 2 and the equidistribution of closed geodesics 3 we need uniqueness of the measure of maximal entropy.

7.4.1 The lower bound

In this section we use the mixing of the Bowen–Margulis measure to obtain a lower bound on the number of closed geodesics.

Proposition 7.4.1. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set, and $\Gamma \subset \mathrm{Aut}(\Omega)$ a convex cocompact discrete subgroup with $M = \Omega/\Gamma$ rank-one and non-elementary. Then there exists a constant $C > 0$ such that for any $T > C$,*

$$\#[\Gamma]^{\mathrm{r1}}_T \geq \frac{C}{T} e^{\delta_\Gamma T}.$$

Proof. Without loss of generality, we may assume that Γ is torsion-free, since for any finite-index subgroup $\Gamma' \subset \Gamma$, for any element $\gamma \in \Gamma'$, there at most $[\Gamma : \Gamma']$ conjugacy classes of Γ' inside the conjugacy class of γ in Γ. Let $\epsilon_0 > 0$ be the injectivity radius of M and m_Γ the Bowen–Margulis probability measure. Fix a compact subset $K \subset T^1 M_{\mathrm{sse}}$ whose measure $m_\Gamma(K)$ is positive. Using Lemma 5.4.13 we can find $0 < \epsilon < \epsilon_0/3$ such that for any vector $v \in K$, for any time $t \geq 1$, if $d_{T^1 M}(v, \phi_t v) \leq \epsilon$ then there exists a rank-one periodic vector $w \in B^{(t)}_{T^1 M}(v, \epsilon_0/6)$ with period in $[t-1, t+1]$. Let us denote by $R_t \subset K$ the subset of vectors $v \in K$ such that $d_{T^1 M}(v, \phi_t v) \leq \epsilon$; we are going to bound from below its measure. To that end take $\mathcal{P}$ a finite measurable partition of $T^1 M_{\mathrm{bip}}$ with diameter less than ϵ, and compute the following limit, using the mixing property, established in Theorem 6.0.1:

$$m_\Gamma(R_t) \geq \sum_{P \in \mathcal{P}} m_\Gamma(P \cap K \cap \phi_t(P \cap K)) \underset{t \to \infty}{\longrightarrow} \sum_{P \in \mathcal{P}} m_\Gamma(P \cap K)^2 \geq \frac{m(K)^2}{\#\mathcal{P}} > 0.$$

On the other hand we can bound from above this measure thanks to the closing lemma and Corollary 7.1.5. For any conjugacy class $c \in [\Gamma]^{\mathrm{r1}}$, fix a vector $v_c \in T^1 M$ tangent to the projection in $T^1 M$ of the axis of any element of c.

$$m_\Gamma(R_t) \leq \sum_{c \in [\Gamma]^{\mathrm{r1}}_{t+1}} \sum_{k=0}^{\lfloor \frac{6(t+1)}{\epsilon_0} \rfloor} m_\Gamma(B^{(t)}_{T^1 M}(\phi_{k\frac{\epsilon_0}{6}} v_c, \frac{\epsilon_0}{3})) \leq C e^{-\delta_\Gamma t} \left(\frac{6(t+1)}{\epsilon_0} + 1 \right) \#[\Gamma]^{\mathrm{r1}}_{t+1}.$$

This ends the proof. □

7.4.2 The upper bound on the number of rank-one closed geodesics

In this section we conclude the proof of Theorem 7.0.3.1.

Proposition 7.4.2. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set, and $\Gamma \subset \mathrm{Aut}(\Omega)$ a convex cocompact discrete subgroup with $M = \Omega/\Gamma$ rank-one and non-elementary. Then there exists a constant $C > 0$ such that for any $T > 0$,*

$$\#[\Gamma]^{\mathrm{r1}}_T \leq \frac{C}{T} e^{\delta_\Gamma T}.$$

Proof. Let $\Gamma' \subset \Gamma$ be a finite-index torsion-free subgroup; we set $M' = \Omega/\Gamma'$ and take $\epsilon_0 < 1$ smaller than the injectivity radius of M'. For each rank-one conjugacy class c of Γ, choose $\gamma_c \in c$ and $v_c \in \mathrm{axis}(\gamma_c)$. Consider an integer $n \geq 1$. One easily checks using the triangular inequality that any vector of $T^1\Omega$ belongs to at most

$$C_1 := \max_{x \in \Omega} \#\{\gamma \in \Gamma : d_\Omega(x, \gamma x) \leq 1 + 2\epsilon_0/3\}$$

balls of the family

$$\{B^{(n+1)}_{T^1\Omega}(\phi_k v_c, \epsilon_0/6) : c \in [\Gamma]^{\mathrm{r1}}_{[n,n+1]},\ 0 \leq k < n\}.$$

By Corollary 7.1.5, we can bound from below the $m_{\Gamma'}$-measure of the projection in T^1M' of these balls by $C^{-1}e^{-\delta_\Gamma n}$ for some constant $C > 0$, where $m_{\Gamma'}$ is the Bowen–Margulis probability measure on T^1M'. Since $m_{\Gamma'}$ is a probability measure we obtain:

$$n\#[\Gamma]^{\mathrm{r1}}_{[n,n+1]} \leq C_1 C e^{\delta_\Gamma n \epsilon_0/3}.$$

□

7.4.3 The upper bound on the number of non-rank-one closed geodesics

Let us bound from of above the number of non-rank-one conjugacy classes of Γ. The idea is that to each non-rank-one conjugacy class we can associate a closed geodesic which is contained in a flow-invariant closed subset of T^1M_{cor} to which the Bowen–Margulis measure gives zero measure; this implies that the topological entropy of the geodesic flow on this subset is smaller than δ_Γ.

Proposition 7.4.3. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set, and $\Gamma \subset \mathrm{Aut}(\Omega)$ a convex cocompact discrete subgroup with $M = \Omega/\Gamma$ rank-one and non-elementary. Let $K \subset T^1M_{\mathrm{cor}}$ be the $(\phi_t)_t$-invariant closed subset that consists of the vectors whose lifts $v \in T^1\Omega$ are such that $d_{\mathrm{spl}}(\phi_{-\infty}v, \phi_\infty v) \leq 2$. Then the topological entropy of $(\phi_t)_t$ on K is strictly smaller than δ_Γ.*

Proof. According to Remark 1.3.5 and Observation 1.3.6, there exists a probability measure m' on K whose entropy is the topological entropy on K. Observe that K is disjoint from the set T^1M_{sse} of vectors whose lifts $v \in T^1\Omega$ satisfy $\phi_{-\infty}v, \phi_\infty v \in \partial_{\mathrm{sse}}\Omega$, while the Bowen–Margulis probability measure m_Γ is concentrated on T^1M_{sse} by Theorem 6.0.1. Thus, m' and m_Γ are different. By Theorem 7.0.2, the entropy of m' must be strictly smaller than δ_Γ. □

Corollary 7.4.4. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set, and $\Gamma \subset \mathrm{Aut}(\Omega)$ a convex cocompact discrete subgroup with $M = \Omega/\Gamma$ rank-one and non-elementary. Then the exponential growth rate of the number of non-rank-one conjugacy classes of translation length shorter than t, when t grows, is strictly less than δ_Γ.*

Proof. Let $\Gamma' \subset \Gamma$ be a finite-index torsion free subgroup; we set $M' = \Omega/\Gamma'$ and ϵ_0 to be the injectivity radius of M'; we denote by ϵ_0 the injectivity radius of M'. It is easy to show that there are only finitely many conjugacy class c of Γ such that $\ell(c) \leq \epsilon_0$. Using

Lemma 3.4.4, for each conjugacy class c with $\ell(c) > 0$ we can find an element $\gamma_c \in c$ and a vector $v_c \in T^1\Omega$ such that $\phi_{\pm\infty} v_c \in \Lambda^{\mathrm{orb}}$ and $\gamma_c \pi v_c = \pi\phi_{\ell(c)} v_c$. Consider $t > 0$. Using the triangular inequality, one can check that any vector of $T^1\Omega$ belongs to at most

$$C_1 := \max_{x \in \mathcal{C}^{\mathrm{cor}}_{\Omega}(\Gamma)} \#\{\gamma \in \Gamma : d_\Omega(x, \gamma x) \leq 1 + 2\epsilon_0/3\}$$

balls of the family

$$\{B^{(t+1)}_{T^1\Omega}(v_c, \epsilon_0/6) : c \in [\Gamma]^{\mathrm{sing}}_{[t,t+1]}\}.$$

Therefore we can extract from $\{v_c : c \in [\Gamma]^{\mathrm{sing}}_{[t,t+1]}\}$ a $(d^{(t+1)}_{T^1\Omega}, \epsilon_0/6)$-separated family of size at least $C_1^{-1}\#[\Gamma]^{\mathrm{sing}}_{[t,t+1]}$. The projection in T^1M' of this family belongs to the set K in Proposition 7.4.3 by Lemma 3.4.5. By definition of the topological entropy, the exponential growth rate of the size of such a family, when t goes to infinity, is bounded above by the topological entropy on K, which is strictly less than δ_Γ by Proposition 7.4.3 above. □

7.4.4 Sums of uniform measures on closed geodesics

Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set and $\Gamma \subset \mathrm{Aut}(\Omega)$ a discrete subgroup; denote $M = \Omega/\Gamma$.

Proposition 7.4.5. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set, and $\Gamma \subset \mathrm{Aut}(\Omega)$ a convex cocompact discrete subgroup with $M = \Omega/\Gamma$ rank-one and non-elementary. Let $A \subset [\Gamma]^{\mathrm{r1}}$ be such that $\log(\#A_T)/T$ converges to δ_Γ when T tends to infinity. Then $\mathcal{L}A_T$ converges to the Bowen–Margulis probability measure when T goes to infinity.*

Proof. Let $\Gamma' \subset \Gamma$ be a torsion-free finite-index subgroup; let ϵ_0 be the injectivity radius of $M' = \Omega/\Gamma'$. For each conjugacy class $c \in [\Gamma]^{\mathrm{r1}}$, we choose a representative $\gamma_c \in \Gamma$, and we call $\mathcal{L}'c$ the unique $(\phi_t)_t$-invariant probability measure on the projection in T^1M' of the axis of γ_c. For any finite subset $B \subset [\Gamma]^{\mathrm{r1}}$, we set $\mathcal{L}'B = (\#B)^{-1}\sum_{c\in B}\mathcal{L}'c$. By Theorem 7.0.2, it is enough to show that any accumulation point $m' = \lim_{k\to\infty}\mathcal{L}'A_T$ on T^1M' has entropy bounded below by δ_Γ.

Let us give ourselves a finite measurable partition $\mathcal{P}$ of T^1M' of diameter less than $\epsilon_0/3$ and such that for any element $P \in \mathcal{P}$, we have $m'(\partial P) = 0$. Then

$$h_{m'}(\phi) \geq H_{m'}(\phi, \mathcal{P}) = \lim_{n\to\infty}\frac{1}{n}H_{m'}(\mathcal{P}^{(n)}).$$

Fix $n \geq 1$ and note that for each $P \in \mathcal{P}^{(n)}$, we have $m'(\partial P) = 0$. As a consequence

$$H_{m'}(\mathcal{P}^{(n)}) = \lim_{k\to\infty} H_{\mathcal{L}'A_{T_k}}(\mathcal{P}^{(n)}) \geq \liminf_{T\to\infty} H_{\mathcal{L}'A_T}(\mathcal{P}^{(n)}).$$

Consider $\alpha > 0$ and let us show that $\liminf_{T\to\infty} H_{\mathcal{L}'A_T}(\mathcal{P}^{(n)}) \geq n(\delta_\Gamma - \alpha)$. Let $T_0 > 0$ be large enough so that $\#A_T \geq e^{(\delta_\Gamma-\alpha)T}$ for any $T \geq T_0$. Take $T > T_0$ and decompose $[0,T]$ into disjoint intervals :

$$[0,T] = [0,T_0] \sqcup \bigsqcup_{I\in\mathcal{I}_T} I,$$

such that each $I \in \mathcal{I}_T$ has diameter less than 1, and $\#\mathcal{I}_T = \lceil T - T_0\rceil$. Then by [HK95, Prop. 4.3.3.6] we have

$$\begin{aligned}\liminf_{T\to\infty} H_{\mathcal{L}'A_T}(\mathcal{P}^{(n)}) &\geq \liminf_{T\to\infty} \frac{\#A_{T_0}}{\#A_T} H_{\mathcal{L}'A_{T_0}}(\mathcal{P}^{(n)}) + \sum_{I\in\mathcal{I}_T} \frac{\#A_I}{\#A_T} H_{\mathcal{L}'A_I}(\mathcal{P}^{(n)}) \\ &\geq \liminf_{T\to\infty} \sum_{I\in\mathcal{I}_T} \frac{\#A_I}{\#A_{]T_0,T]}} H_{\mathcal{L}'A_I}(\mathcal{P}^{(n)}) \\ &\geq \liminf_{T\to\infty} \frac{n}{T} \sum_{I\in\mathcal{I}_T} \frac{\#A_I}{\#A_{]T_0,T]}} H_{\mathcal{L}'A_I}(\mathcal{P}^{(\lceil T\rceil)}),\end{aligned}$$

where we have used the Euclidean division $\lceil T\rceil = q_T n + r_T$ with the following classical inequality (see e.g. [HK95, Prop. 4.3.3.1-4]):

$$H_{\lambda_I}(\mathcal{P}^{(n)}) \geq \frac{1}{q_T} H_{\lambda_I}(\mathcal{P}^{(\lceil T\rceil)}) - \frac{1}{q_T} H_{\lambda_I}(\mathcal{P}^{(r_T)}) \geq \frac{n}{T+1} H_{\lambda_I}(\mathcal{P}^{(\lceil T\rceil)}) - \frac{n}{T-n}\log(\#\mathcal{P}^n).$$

Using the triangular inequality, one checks that for any $P \in \mathcal{P}^{(\lceil T\rceil)}$ and any $I \in \mathcal{I}_T$, there are at most $C_1 = \max_{x\in\mathcal{C}^{\mathrm{cor}}_\Omega(\Gamma)} \#\{\gamma\in\Gamma : d_\Omega(x,\gamma x) \leq 1 + 2\epsilon_0/3\}$ conjugacy classes $c \in A_I$ such that $\mathcal{L}'c(P) > 0$; this implies that $\mathcal{L}'A_I(P) \leq C_1 \# A_I^{-1}$. Hence $H_{\mathcal{L}'A_I}(\mathcal{P}^{(\lceil T\rceil)}) \geq \log(\#A_I) - \log(C_1)$, and we resume our computation, using the concavity of the logarithm and Cauchy–Schwarz inequality:

$$\begin{aligned}\frac{n}{T}\sum_{I\in\mathcal{I}_T} \frac{\#A_I}{\#A_{]T_0,T]}} H_{\mathcal{L}'A_I}(\mathcal{P}^{(\lceil T\rceil)}) &\geq \frac{n}{T}\sum_{I\in\mathcal{I}_T} \frac{\#A_I}{\#A_{]T_0,T]}} \log(\#A_I) - \frac{n}{T}\log(C_1) \\ &\geq \frac{n}{T}\log\left(\frac{1}{\#A_{]T_0,T]}} \sum_{I\in\mathcal{I}_T} \#A_I^2\right) - \frac{n}{T}\log(C_1) \\ &\geq \frac{n}{T}\log\left(\frac{\#A_{]T_0,T]}}{\#\mathcal{I}_T}\right) - \frac{n}{T}\log(C_1) \\ &\geq \frac{n}{T}\log\left(\frac{\#A_T - \#A_{T_0}}{\lceil T-T_0\rceil}\right) - \frac{n}{T}\log(C_1) \\ &\geq \frac{n}{T}\log\left(\frac{e^{(\delta_\Gamma-\alpha)T} - \#A_{T_0}}{\lceil T-T_0\rceil}\right) - \frac{n}{T}\log(C_1).\end{aligned}$$

This last term converges to $n(\delta_\Gamma - \alpha)$ as T goes to infinity. This concludes the proof. □

We can now end the proof of Theorem 7.0.3.

Proof of Theorem 7.0.3. It is the immediate combination of Propositions 7.4.1, 7.4.2, 7.4.5, and Corollary 7.4.4. □

7.4.5 The number of conjugacy classes and of periodic geodesics

Finally, let us relate the number of strongly primitive rank-one conjugacy classes, the number of rank-one conjugacy classes, and the number of rank-one periodic $(\phi_t)_t$-orbits. For any convex projective manifold $M = \Omega/\Gamma$, let $[\Gamma]^{\mathrm{pr1}}$ denote the set of strongly primitive rank-one conjugacy classes, and $\mathcal{G}^{\mathrm{r1}}$ the set of rank-one periodic $(\phi_t)_t$-orbits in T^1M.

Observation 7.4.6. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set, and $\Gamma \subset \mathrm{Aut}(\Omega)$ a convex cocompact discrete subgroup with $M = \Omega/\Gamma$ non-elementary rank-one. Then $\#[\Gamma]_T^{\mathrm{pr1}} \underset{T\to\infty}{\sim} \#[\Gamma]_T^{\mathrm{r1}}$, and there exists $C > 0$ such that for any $T > C$,*

$$\frac{1}{CT} e^{\delta_\Gamma T} \leq \#\mathcal{G}_T^{\mathrm{r1}} \leq \frac{C}{T} e^{\delta_\Gamma T}.$$

Proof. This is a consequence of the discussion in Section 3.4.2. Let $\Gamma' \subset \Gamma$ be a finite-index normal subgroup. Then for any $T \geq 0$ we have

$$\#[\Gamma]_T^{\mathrm{pr1}} \leq \#[\Gamma]_T^{\mathrm{r1}} \leq \#[\Gamma]_T^{\mathrm{pr1}} + [\Gamma' : \Gamma]\#[\Gamma]_{\frac{T}{2}}^{\mathrm{r1}},$$

and we then apply Propositions 7.4.1 and 7.4.2. Using again the discussion in Section 3.4.2, we know that

$$\#\mathcal{G}_T^{\mathrm{r1}} \leq \#[\Gamma]_T^{\mathrm{pr1}} \leq [\Gamma' : \Gamma]\#\mathcal{G}_T^{\mathrm{r1}},$$

and we apply the previous point of Observation 7.4.6. □

The idea of the following result is exactly the same as that of Proposition 7.4.3, and has exactly the same proof,

Proposition 7.4.7. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set, and $\Gamma \subset \mathrm{Aut}(\Omega)$ a convex cocompact discrete subgroup with $M = \Omega/\Gamma$ rank-one and non-elementary. Let $F \subset \Gamma$ be the core-fixing subgroup (see Section 3.4.2). Let $K \subset T^1M_{\mathrm{bip}}$ be the set vectors of whose lifts $v \in T^1\Omega$ satisfy $\mathrm{Stab}_\Gamma(v) \neq F$. Then the topological entropy of the geodesic flow on K is strictly smaller than δ_Γ, as well as the exponential growth rate of the number of rank-one conjugacy classes of Γ with axis in K.*

Suppose further that F is the centre of Γ. Then $\#[\Gamma]_T^{\mathrm{r1}} \underset{T\to\infty}{\sim} \#F \cdot \#\mathcal{G}_T^{\mathrm{r1}}$. If Γ is strongly irreducible, then F is trivial and $\#[\Gamma]_T^{\mathrm{r1}} \underset{T\to\infty}{\sim} \#\mathcal{G}_T^{\mathrm{r1}}$.

Proof. The subset $K \subset T^1M_{\mathrm{bip}}$ is closed and $(\phi_t)_t$-invariant, and has empty-interior by Observation 3.4.1. By Remark 1.3.5 and Observation 1.3.6, we can find a probability measure on K whose entropy δ is the topological entropy of $(\phi_t)_t$; this measure is different from the Bowen–Margulis probability measure since the latter gives zero measure to K by Theorem 6.0.1. Hence $\delta < \delta_\Gamma$ by Theorem 7.0.2.

Let $\Gamma' \subset \Gamma$ be a finite-index torsion free subgroup; we set $M' = \Omega/\Gamma'$ and ϵ_0 to be the injectivity radius of M'; we denote by ϵ_0 the injectivity radius of M'. Recall that there are only finitely many conjugacy class c of Γ such that $\ell(c) \leq \epsilon_0$. For each rank-one conjugacy class $c \in [\Gamma]^{\mathrm{r1}}$ with $\ell(c) > 0$ we can find an element $\gamma_c \in c$ and a vector $v_c \in T^1\Omega_{\mathrm{bip}} := \pi_\Gamma^{-1}T^1M_{\mathrm{bip}}$ such that $\gamma_c\pi v_c = \pi\phi_{\ell(c)}v_c$. Let $A \subset [\Gamma]^{\mathrm{r1}}$ be the subset made of conjugacy classes c such that $\mathrm{Stab}_\Gamma(v_c) \neq F$. Consider $t > 0$. Using the triangular inequality, one can check that any vector of $T^1\Omega$ belongs to at most

$$C_1 := \max_{x\in\mathcal{C}_\Omega^{\mathrm{cor}}(\Gamma)} \#\{\gamma \in \Gamma : d_\Omega(x, \gamma x) \leq 1 + 2\epsilon_0/3\}$$

balls of the family

$$\{B_{T^1\Omega}^{(t+1)}(v_c, \epsilon_0/6) : c \in A_{[t,t+1]}\}.$$

Therefore we can extract from $\{v_c : c \in A_{[t,t+1]}\}$ a $(d_{T^1\Omega}^{(t+1)}, \epsilon_0/6)$-separated family of size at least $C_1^{-1}\#A_{[t,t+1]}$. The projection in T^1M' of this family is $(d_{T^1M'}^{(t+1)}, \epsilon_0/6)$-separated, and belongs to the preimage by $T^1M' \to T^1M$ of K. By definition of the topological entropy, the exponential growth rate of the size of such a family, when t goes to infinity, is bounded above by δ, which is strictly less than δ_Γ.

The fact that $\#[\Gamma]_T^{\mathrm{r}1} \underset{T\to\infty}{\sim} \#F \cdot \#\mathcal{G}_T^{\mathrm{r}1}$ if F is the centre of Γ is due to the discussion in Section 3.4.2. $\square$

Chapter 8

Equidistribution in Hilbert geometry

In this chapter, which is extracted from an article [BZ21] in collaboration with F. Zhu, we are interested in equidistribution questions which are similar to those treated in the last section of the previous chapter. However the answers brought here are not quite the same as those there. In particular, the setting is more general here, since we play with non-elementary rank-one convex projective orbifolds $M = \Omega/\Gamma$ which have a finite Sullivan measure of dimension δ_Γ, but whose convex core is not necessarily compact.

To start with, mixing gives us, via an argument of Babillot [Bab02], equidistribution of the unstable horospheres pushed forward by the geodesic flow; the precise statement is a bit technical and we refer the interested reader to Theorem 8.1.1 for this result.

Moreover, we have equidistribution of Γ-orbits in $\partial\Omega$. For any topological space X, we denote by $\mathcal{C}(X)^*$ and $\mathcal{C}_c(X)^*$ the respective duals to the space of continuous functions on X and to the space of compactly-supported continuous functions on X, endowed with the weak-$*$ topology; for any point $x \in X$, we denote by $\mathcal{D}_x \in \mathcal{C}(X)^*$ the Dirac mass at x, such that $\mathcal{D}_x(f) = f(x)$ for any function f.

Theorem 8.0.1. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set, and $\Gamma \subset \mathrm{Aut}(\Omega)$ a non-elementary rank-one discrete subgroup such that $T^1\Omega/\Gamma$ admits a finite Sullivan measure m_Γ associated to a Γ-equivariant conformal density $(\mu_x)_{x\in\Omega}$ of dimension δ_Γ on $\partial\Omega$. Then, for all $x, y \in \Omega$,*

$$\delta \|m_\Gamma\| e^{-\delta t} \sum_{\substack{\gamma\in\Gamma \\ d_\Omega(x,\gamma y)\leq t}} \mathcal{D}_{\gamma y} \otimes \mathcal{D}_{\gamma^{-1}x} \underset{t\to\infty}{\longrightarrow} \mu_x \otimes \mu_y$$

weakly in $\mathcal{C}(\overline{\Omega} \times \overline{\Omega})^*$.

Contrary to Section 7.4, counting results are obtained as corollaries to equidistribution results:

Corollary 8.0.2. *In the setting of Theorem 8.0.1, we have*

$$\delta \|m_\Gamma\| e^{-\delta t} \sum_{\substack{\gamma\in\Gamma \\ d_\Omega(x,\gamma y)\leq t}} \mathcal{D}_{\gamma y} \underset{t\to\infty}{\longrightarrow} \|\mu_y\|\, \mu_x$$

weakly in $\mathcal{C}(\overline{\Omega})^*$, and

$$\#\left\{\gamma \in \Gamma : d_\Omega(x, \gamma y) \leq t\right\} \underset{t\to\infty}{\sim} \frac{\|\mu_x\|\|\mu_y\|}{\delta\|m_\Gamma\|} e^{\delta t}.$$

Finally, we have equidistribution of rank-one closed geodesics in T^1M. Let $\mathcal{G}^{\mathrm{r1}}_\ell$ be the set of rank-one periodic $(\phi_t)_t$-orbits with period bounded above by $\ell \geq 0$, and $\mathcal{G}^{\mathrm{r1}} = \bigcup_\ell \mathcal{G}^{\mathrm{r1}}_\ell$. For any $c = \{\phi_t v\}_t \in \mathcal{G}^{\mathrm{r1}}$, we denote by $\ell(c)$ its period, and we denote by $\mathcal{L}c$ the only $(\phi_t)_t$-invariant probability measure on c, i.e. the push-forward by $t \mapsto \phi_t v$ of the renormalised Lebesgue measure on $[0, \ell(c)]$ with total mass 1.

Theorem 8.0.3. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set, and $\Gamma \subset \mathrm{Aut}(\Omega)$ a non-elementary rank-one discrete subgroup such that $T^1\Omega/\Gamma$ admits a finite Sullivan measure m_Γ associated to a Γ-equivariant conformal density $(\mu_x)_x$ of dimension δ_Γ. Then*

$$\delta\ell e^{-\delta\ell} \sum_{c \in \mathcal{G}^{\mathrm{r1}}_\ell} \mathcal{L}c \underset{\ell\to+\infty}{\longrightarrow} \frac{m_\Gamma}{\|m_\Gamma\|}$$

weakly in $\mathcal{C}_c(T^1\Omega/\Gamma)^*$.

If Ω is strictly convex, then all closed geodesics are rank-one. As a corollary to Theorem 8.0.3, we obtain the following counting result for closed rank-one geodesics:

Corollary 8.0.4. *In the setting of Theorem 8.0.3, if Γ acts convex cocompactly on Ω, then*

$$\#\mathcal{G}^{\mathrm{r1}}_\ell \underset{\ell\to\infty}{\sim} \frac{e^{\delta\ell}}{\delta\ell}.$$

Proof. Take $f \in \mathcal{C}_c(T^1\Omega/\Gamma)$ which is equal to 1 on the (compact) convex core. Integrating f against the measure $\delta\ell e^{-\delta\ell}\sum_{c\in\mathcal{G}^{\mathrm{r1}}_\ell} \mathcal{L}c$ gives $\delta\ell e^{-\delta\ell}\#\mathcal{G}^{\mathrm{r1}}_\ell$. From Theorem 8.0.3, this integral converges to 1 as $\ell \to \infty$. □

The proof of Theorem 8.0.1 follows the gist of Roblin's proof, making heavy use of mixing and of cones in the space and shadows on the boundary without reference to any notion of angle, which is not well-defined in our setting. We derive, as Roblin did, Theorem 8.0.3 from Theorem 8.0.1; an essential ingredient to perform this is the Closing lemma (more precisely Corollary 5.4.12).

Organisation

In Section 8.1 we prove Theorem 8.1.1, concerning equidistribution of generalised unstable horospheres. Section 8.2 describes the proof of Theorem 8.0.1, and Section 8.3 describes the proof of Theorem 8.0.3.

8.1 Equidistribution of unstable horospheres

Babillot obtains equidistribution of unstable horospheres as a consequence of mixing of the geodesic flow [Bab02, Th. 3], and we can do likewise here.

Generalised unstable horospheres of $T^1\Omega$ are sets of vectors tangent to geodesics which are backwards-asymptotic to a common point $\xi \in \partial\Omega$, and with foot-points along horospheres centred at a common preimage of ξ in $\partial_{\mathtt{h}}\Omega$. These are strong unstable manifolds (in the sense of Section 1.2.4) for the Hilbert geodesic flow if and only if ξ is smooth. The Hopf parametrisation we adopted in Section 6.2.2 is convenient to parametrise stable horospheres, but not for unstable horospheres; this issue is very easy to overcome: define the reversed Hopf parametrisation $\mathrm{Hopf}^- := \iota \circ \mathrm{Hopf} \circ \iota$ (recall that ι denotes the flip involution). Using the reversed Hopf parametrisation, we see subsets of unstable horospheres of $\mathrm{Geod}^\infty_{\mathtt{h}}(\Omega) \times \mathbb{R}$ as sets of the form $\{\xi\} \times J \times \{t\}$ with $\xi \in \partial_{\mathtt{h}}\Omega$ and $t \in \mathbb{R}$.

Theorem 8.1.1. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set and Γ be a non-elementary rank-one discrete group acting on Ω such that $T^1\Omega/\Gamma$ admits a finite Sullivan measure m_Γ associated to a Γ-equivariant conformal density $(\mu_x^{\mathtt{h}})_{x\in\Omega}$ of dimension δ_Γ on $\partial_{\mathtt{h}}\Omega$. Fix a basepoint $o \in \Omega$. Let $J \subset \partial_{\mathtt{h}}\Omega$ be a closed subset with $\mu_o^{\mathtt{h}}(J) \neq 0$, and $\xi_0 \in \mathrm{supp}(\mu_o^{\mathtt{h}})$ be such that $\{\xi_0\} \times J \subset \mathrm{Geod}_{\mathtt{h}}^{\infty}(\Omega)$. Then for any bounded and uniformly continuous function $f : T^1\Omega/\Gamma \to \mathbb{R}$, we have*

$$\frac{1}{c_J(\xi_0)} \int_J \tilde{f}(\xi_0, \eta, t) e^{-2\delta_\Gamma \langle \xi_0, \eta\rangle_o} d\mu_o^{\mathtt{h}}(\eta) \to \frac{1}{\|m_\Gamma\|} \int_{T^1\Omega/\Gamma} f \, \mathrm{d}m_\Gamma \quad \text{as } t \to +\infty$$

where $c_J(\xi_0) := \int_J e^{-2\delta_\Gamma \langle \xi_0, \eta\rangle_o} d\mu_o^{\mathtt{h}}(\eta)$ and the function $\tilde{f}$ is the Γ-invariant lift of f to $\mathrm{Geod}_{\mathtt{h}}(\Omega) \times \mathbb{R}$.

Recall that if Ω is strictly convex with $\mathcal{C}^1$ boundary, then the (reversed) Hopf parametrisation is a homeomorphism. Therefore $\mathrm{Geod}_{\mathtt{h}}^{\infty}(\Omega) \times \mathbb{R}$, $\partial^2\Omega \times \mathbb{R}$ and $T^1\Omega$ are all identified, the support of $\mu_0^{\mathtt{h}}$ is Λ_Γ, and the condition $\xi_0 \times J \subset \mathrm{Geod}_{\mathtt{h}}^{\infty}(\Omega)$ in the above statement is simply reformulated as $\xi_0 \notin J$.

Proof. We may suppose that f is non-negative. Consider a compact neighbourhood $I_0 \ni \xi_0$ sufficiently small such that $I_0 \times J \subset \mathrm{Geod}_{\mathtt{h}}^{\infty}(\Omega)$. Then for any $\epsilon > 0$, we can choose a compact neighbourhood $I \subset I_0$ of ξ_0 and $r > 0$ such that

(i) $1 - \epsilon \leq \frac{e^{-2\delta_\Gamma \langle \xi_0, \eta\rangle_o}}{e^{-2\delta_\Gamma \langle \xi, \eta\rangle_o}} \leq 1 + \epsilon$ for any $(\xi, \eta) \in I \times J$, and

(ii) $\left|\tilde{f}(\xi, \eta, t+s) - \tilde{f}(\xi_0, \eta, t)\right| \leq \epsilon$ for any $(\xi, \eta) \in I \times J$, $s \in [0, r]$, and $t > 0$.

The second property holds since choosing I and r sufficiently small ensures that, for $\xi \in I$ and $s \in [0, r]$, the vectors $\mathrm{Hopf}^{-}(\xi, \eta, s)$ and $\mathrm{Hopf}^{-}(\xi_0, \eta, 0)$ are uniformly (in $\eta \in J$) close. Moreover, since they both belong to the same weak stable leaf, given by the vectors pointing towards $\pi_{\mathtt{h}}(\eta)$, flowing them by the geodesic flow does not increase their distance by Lemma 2.1.6.

It then follows from properties (i) and (ii) that, for any fixed t, the integral

$$\frac{1}{c_J(\xi_0)} \int_J \tilde{f}(\xi_0, \eta, t) e^{-2\delta_\Gamma \langle \xi_0, \eta\rangle_o} d\mu_o^{\mathtt{h}}(\eta)$$

differs from

$$\begin{aligned} &\frac{1}{c_J(\xi_0) r \mu_o^{\mathtt{h}}(I)} \int_{I\times J\times[0,r]} \tilde{f}(\xi, \eta, t+s) e^{-2\delta_\Gamma \langle \xi, \eta\rangle_o} \, \mathrm{d}\mu_o^{\mathtt{h}}(\xi) \, \mathrm{d}\mu_o^{\mathtt{h}}(\eta) \, \mathrm{d}s \\ &= \frac{1}{c_J(\xi_0) r \mu_o^{\mathtt{h}}(I)} \int_{I\times J\times[0,r]} \tilde{f}(\xi, \eta, t+s) \, \mathrm{d}m^{\mathtt{h}} \end{aligned}$$

by at most $\epsilon\left(1 + \frac{2}{1-\epsilon}|f|_\infty\right)$. By the mixing property of the geodesic flow and Remark 1.2.10, we may choose t large enough so that

$$\left| \int_{I\times J\times[0,r]} \tilde{f}(\xi, \eta, t+s) \, \mathrm{d}m^{\mathtt{h}} - \frac{m^{\mathtt{h}}(I \times J \times [0, r])}{\|m_\Gamma\|} \cdot \int_{T^1\Omega/\Gamma} f \, \mathrm{d}m_\Gamma \right| \leq \epsilon r \mu_o^{\mathtt{h}}(I),$$

whence the conclusion follows, since

$$\frac{m^{\mathtt{h}}(I \times J \times [0, r])}{r \mu_o^{\mathtt{h}}(I)} = \frac{1}{\mu_o^{\mathtt{h}}(I)} \int_{I\times J} e^{-2\delta_\Gamma \langle \xi, \eta\rangle_o} \, \mathrm{d}\mu_o^{\mathtt{h}}(\xi) \, \mathrm{d}\mu_o^{\mathtt{h}}(\eta)$$

is bounded from below by $(1 - \epsilon) \cdot c_J(\xi_0)$ and from above by $(1 + \epsilon) \cdot c_J(\xi_0)$. ∎

8.2 Equidistribution in the projective boundary

In this section we prove Theorem 8.0.1.

8.2.1 Idea of the proof and notations

Let Ω be a properly convex open set and $\Gamma \subset \mathrm{Aut}(\Omega)$ a non-elementary rank-one discrete group such that $T^1\Omega/\Gamma$ admits a finite Sullivan measure m_Γ associated to a Γ-invariant conformal density (μ_x) of dimension δ_Γ.

For $t \geq 0$, we let $\nu^t_{x,y}$ denote the measure $\delta\|m_\Gamma\|e^{-\delta t}\sum_{\gamma\in\Gamma}\mathcal{D}_{\gamma y}\otimes\mathcal{D}_{\gamma^{-1}x}$. To prove the desired convergence, we need to show that for any $\varphi \in \mathcal{C}(\overline{\Omega}\times\overline{\Omega})$,

$$\int_{\overline{\Omega}\times\overline{\Omega}} \varphi \,\mathrm{d}\nu^t_{x,y} \xrightarrow[t\to\infty]{} \int_{\overline{\Omega}\times\overline{\Omega}} \varphi \,\mathrm{d}(\mu_x\otimes\mu_y).$$

The proof uses mixing of the geodesic flow applied to suitable geometrically-described sets: given $x \in \Omega$, $A \subset \partial\Omega$, and $R > 0$, define

$$\mathcal{C}^+_R(x, A) := \left\{y \in \Omega : \exists x' \in B_\Omega(x, r), \xi \in A \text{ with } B_\Omega(y, R) \cap [x', \xi) \neq \varnothing\right\},$$
$$\mathcal{C}^-_R(x, A) := \left\{y \in \Omega : B_\Omega(y, r) \subset \bigcap_{x'\in B_\Omega(x,R)} \bigcup_{\xi\in A} [x', \xi)\right\}.$$

These may be thought of as expanded or contracted cones from x to A, with the parameter R controlling the expansion or contraction. We can use mixing to show that the $(\mu_x\otimes\mu_y)$-measures of sufficiently small $\overline{\mathcal{A}}\times\overline{\mathcal{B}} \subset \overline{\Omega}\times\overline{\Omega}$ are uniformly well-approximated by $\nu^t_{x,y}$-measures of corresponding products of cones over A and B. Here "sufficiently small" means "contained in one of a system of neighbourhoods $\hat{V}\times\hat{W} \subset \overline{\Omega}\times\overline{\Omega}$, one for each $(\xi_0, \eta_0) \in \partial\Omega\times\partial\Omega$."

We can then approximate, topologically and hence in measure, any sufficiently small Borel subset of $\overline{\Omega}\times\overline{\Omega}$ by products of cones. From there, using standard arguments to approximate continuous positive functions using characteristic functions, we obtain the desired convergence of integrals if we replace the domain $\overline{\Omega}\times\overline{\Omega}$ with $\hat{V}\times\hat{W}$. We are then done by taking a finite subcover of the cover of the compact $\overline{\Omega}\times\overline{\Omega}$ by these neighbourhoods $\hat{V}\times\hat{W}$ and using a partition of unity subordinate to this subcover.

The proof estimates the $\nu^T_{x,y}$-measure of the product of cones uses a number of other geometric objects. For any subset $A \subset \Omega$, we denote by $T^1A \subset T^1\Omega$ the set of vectors v with $\pi v \in A$.

1. For $z \in \Omega$ and $(\xi, \eta) \in \mathrm{Geod}^\infty(\Omega)$, let $z_{\xi\eta}$ denote the point of $T^1\Omega$ tangent to $[\eta, \xi]$ with foot-point the middle point of the segment $[\eta, \xi] \cap \overline{B}(z, d_\Omega(z, (\eta\xi)))$. Given in addition $r > 0$ and $A \subset \partial\Omega$, define

   $$K^+(z, r, A) = \left\{g^s z_{\xi\eta} : -\frac{r}{2} \leq s \leq \frac{r}{2}, (\xi, \eta) \in \mathrm{Geod}^\infty(\Omega), \eta \in A, d_\Omega(z, (\xi\eta)) \leq r\right\}.$$

   We remark that when Ω is strictly convex with $\mathcal{C}^1$ boundary, the foot-point is also the nearest-point projection of z onto $[\xi, \eta]$.

   Inverting the role of ξ and η in the above definition yields $\iota K^+(z, r, A) =: K^-(z, r, A)$. We will also write $K(z, r)$ to denote $K^+(z, r, \partial\Omega) \cup K^-(z, r, \partial\Omega)$. We remark that $K(z, r) \subset T^1\overline{B}_\Omega(z, 3r/2)$ by construction.

2. Given $r > 0$ and $a, b \in \Omega$ with $d_\Omega(a,b) > 2r$, we will consider the following closed shadows:

$$\overline{\mathcal{O}}_r^+(a,b) := \{\xi \in \partial\Omega : \exists a' \in \overline{B}(a,r) :]a'\xi) \cap \overline{B}(b,r) \neq \varnothing\}$$
$$\overline{\mathcal{O}}_r^-(a,b) := \{\xi \in \partial\Omega : \forall a' \in \overline{B}_\Omega(a,r) :]a'\xi) \cap \overline{B}_\Omega(b,r) \neq \varnothing\}$$

Note also that, as in (6.3.4) we have $\overline{\mathcal{O}}_r^+(a,b) \subset \overline{\mathcal{O}}_{2r}(a,b)$.

3. For $r > 0$ and $a, b \in \Omega$ with $d_\Omega(a,b) > 2r$, we denote by $\mathcal{L}_r(a,b)$ the set of $(\xi,\eta) \in \mathrm{Geod}^\infty(\Omega)$ such that $[\xi,\eta]$ crosses first $\overline{B}_\Omega(a,r)$ and then $\overline{B}_\Omega(b,r)$. The reason we consider in this chapter closed shadows instead of open ones is so that the following holds:
$$\overline{\mathcal{O}}_r^-(b,a) \times \overline{\mathcal{O}}_r^-(a,b) \subset \mathcal{L}_r(a,b) \subset \overline{\mathcal{O}}_r^+(b,a) \times \overline{\mathcal{O}}_r^+(a,b). \tag{8.2.1}$$

8.2.2 The technical crux

Proposition 8.2.1. *In the setting of Theorem 8.0.1, fix $\epsilon > 0$, $\xi_0, \eta_0 \in \partial\Omega$ both extremal and smooth, $\xi_0', \eta_0' \in \Lambda_\Gamma$, and $x, y \in \Omega$ such that $x \in [\xi_0, \xi_0']$ and $y \in [\eta_0, \eta_0']$. Then there exist open neighbourhoods V and W of ξ_0 and η_0 (resp.) in $\partial\Omega$, such that for all Borel subsets $A \subset V, B \subset W$, we have*

$$\limsup_{T\to+\infty} \nu_{x,y}^T(\mathcal{C}_1^-(x,A) \times \mathcal{C}_1^-(y,B)) \leq e^\epsilon \mu_x(A)\mu_y(B)$$
$$\liminf_{T\to+\infty} \nu_{x,y}^T(\mathcal{C}_1^+(x,A) \times \mathcal{C}_1^+(y,B)) \geq e^{-\epsilon} \mu_x(A)\mu_y(B)$$

Proof. Set $\epsilon' = \epsilon/100$. Choose $r \in (0, \min\{1, \epsilon'/\delta\})$ such that both $\mu_x(\partial\overline{\mathcal{O}}_r(\xi_0,x))$ and $\mu_y(\partial\overline{\mathcal{O}}_r(\eta_0,y))$ are zero. Note $\partial\overline{\mathcal{O}}_r$ is the boundary of $\overline{\mathcal{O}}_r$ as a subset of $\partial\Omega$.

We have $\xi_0' \in \overline{\mathcal{O}}_r(\xi_0,x)$, and similarly $\eta_0' \in \overline{\mathcal{O}}_r(\eta_0,y)$; hence

$$M := r^2\mu_x(\overline{\mathcal{O}}_r(\xi_0,x))\mu_y(\overline{\mathcal{O}}_r(\eta_0,y)) > 0.$$

Take two open sets $\hat{V}_1, \hat{W}_1$ of $\overline{\Omega}$, containing ξ_0, η_0 respectively, and sufficiently small so that for all $a \in \hat{V}_1, b \in \hat{W}_1$, we have

$$e^{-\epsilon'}\mu_x(\overline{\mathcal{O}}_r(\xi_0,x)) \leq \mu_x(\overline{\mathcal{O}}_r^\pm(a,x)) \leq e^{\epsilon'}\mu_x(\overline{\mathcal{O}}_r(\xi_0,x)) \tag{8.2.2}$$
$$e^{-\epsilon'}\mu_y(\overline{\mathcal{O}}_r(\eta_0,y)) \leq \mu_y(\overline{\mathcal{O}}_r^\pm(b,y)) \leq e^{\epsilon'}\mu_y(\overline{\mathcal{O}}_r(\eta_0,y)) \tag{8.2.3}$$

This is possible because ξ_0 and η_0 are extremal. Indeed one can see that for any compact subset $K \subset \overline{\mathcal{O}}_r(\xi_0,x)$ and any neighbourhood U of $\overline{\mathcal{O}}_r(\xi_0,x)$, there exists a neighbourhood $\hat{U}$ of ξ_0 in $\overline{\Omega}$ such that for any $a \in \hat{U}$,

$$K \subset \overline{\mathcal{O}}_r^-(a,x) \subset \overline{\mathcal{O}}_r^+(a,x) \subset U.$$

Then, using again the fact that ξ_0 and η_0 are extremal, we can find two open sets $\hat{V}, \hat{W}$ of $\overline{\Omega}$, containing ξ_0, η_0 respectively, and sufficiently small so that for all $a \in \hat{V}, b \in \hat{W}$, we have $\overline{B}_{\overline{\Omega}}(a,1) \subset \hat{V}_1$ and $\overline{B}_{\overline{\Omega}}(b,1) \subset \hat{W}_1$.

Choose open neighbourhoods V, W of ξ_0, η_0 (resp.) in $\partial\Omega$, such that $\overline{V} \subset \hat{V} \cap \partial\Omega$ and $\overline{W} \subset \hat{W} \cap \partial\Omega$. In general, these will be the open neighbourhoods we desire.

Consider Borel subsets $A \subset V$, $B \subset W$. Let $K^+ = K^+(x,r,A)$, $K^- = K^-(y,r,B)$, and, for $T > 0$,

$$S_T^\pm := \{\gamma \in \Gamma : d_\Omega(x,\gamma y) \leq T,\ \gamma y \in \mathcal{C}_1^\pm(x,A),\ \gamma^{-1}x \in \mathcal{C}_1^\pm(y,B)\}.$$

We will estimate asymptotically, in two different ways, the quantity

$$\int_0^T e^{\delta t} \sum_{\gamma \in \Gamma} m(K^+ \cap g^{-t}\gamma K^-)\,\mathrm{d}t.$$

On the one hand this can be estimated using our mixing result; on the other hand we can obtain a different estimate by examining how the elements of Γ contribute to the various parts of the integral; we will observe that the elements which contribute are, asymptotically, those in $S_T^{\pm}$ as $T \to \infty$.

More precisely, we establish the following estimates for $T > 0$ large enough.

$$e^{\delta T} M \mu_x(A)\mu_y(B) \le e^{10\epsilon'} \delta \|m_\Gamma\| \int_0^{T-3r} e^{\delta t} \sum_{\gamma\in\Gamma} m(K^+ \cap g^{-t}\gamma K^-) dt + c_1 \tag{8.2.4}$$

$$\le e^{(10+5)\epsilon'} \delta \|m_\Gamma\| r^2 \sum_{\gamma \in S_T^+} \mu_x(\overline{\mathcal{O}}_r^+(\gamma y, x))\mu_x(\overline{\mathcal{O}}_r^+(x, \gamma y)) e^{\delta \cdot d_\Omega(x,\gamma y)} + c_1 + c_2 \tag{8.2.5}$$

$$\le e^{(10+5+6)\epsilon'} e^{\delta T} M \nu_{x,y}^T(\mathcal{C}_1^+(x, A) \times \mathcal{C}_1^+(y, B)) + c_1 + c_2 + c_3, \tag{8.2.6}$$

and

$$e^{\delta T} M \mu_x(A)\mu_y(B) \ge e^{-6\epsilon'} \delta \|m_\Gamma\| \int_0^{T+3r} e^{\delta t} \sum_{\gamma\in\Gamma} m(K^+ \cap g^{-t}\gamma K^-) dt - c_4 \tag{8.2.7}$$

$$\ge e^{-(6+3)\epsilon'} \delta \|m_\Gamma\| r^2 \sum_{\gamma \in S_T^-} \mu_x(\overline{\mathcal{O}}_r^-(\gamma y, x))\mu_x(\overline{\mathcal{O}}_r^-(x, \gamma y)) e^{\delta \cdot d_\Omega(x,\gamma y)} - c_4 - c_5 \tag{8.2.8}$$

$$\ge e^{-(6+3+2)\epsilon'} M e^{\delta T} \nu_{x,y}^T(\mathcal{C}_1^-(x, A) \times \mathcal{C}_1^-(y, B)) - c_4 - c_5 - c_6, \tag{8.2.9}$$

where $(c_i)_{1\le i \le 6}$ are constants independent of T.

(8.2.6) and (8.2.9): shadows to cones

(8.2.6) and (8.2.9) are consequences of the definition of $\nu_{x,y}^T$, of the conformality of $(\mu_z)_z$, and of (8.2.2) and (8.2.3).

Indeed, by Lemma 6.3.2 and conformality of $(\mu_z)_z$ we have, for any $\gamma \in \Gamma$,

$$\mu_y(\overline{\mathcal{O}}_r^{\pm}(\gamma^{-1}x, y)) \le \mu_x(\overline{\mathcal{O}}_r^{\pm}(x, \gamma y)) e^{\delta\cdot d_\Omega(x,\gamma y)} \le e^{4\epsilon'} \mu_y(\overline{\mathcal{O}}_r^{\pm}(\gamma^{-1}x, y)).$$

Denote $S^{\pm} = \bigcup_{T>0} S_T^{\pm}$ and $S := \{\gamma \in \Gamma : \gamma y \in \hat{V}_1, \gamma^{-1}x \in \hat{W}_1\}$. By (8.2.2) and (8.2.3) and by definition of $\nu_{x,y}^T$, on one hand we obtain

$$\begin{aligned}
&\sum_{\gamma \in S_T^+} \mu_x(\overline{\mathcal{O}}_r^+(\gamma y, x))\mu_x(\overline{\mathcal{O}}_r^+(x, \gamma y)) e^{\delta d(x,\gamma y)} \\
&\quad \le e^{4\epsilon'} \sum_{\gamma \in S_T^+} \mu_x(\overline{\mathcal{O}}_r^+(\gamma y, x))\mu_y(\overline{\mathcal{O}}_r^+(\gamma^{-1}x, y)) \\
&\quad \le e^{4\epsilon'} \sum_{\gamma \in S_T^+ \cap S} \mu_x(\overline{\mathcal{O}}_r^+(\gamma y, x))\mu_y(\overline{\mathcal{O}}_r^+(\gamma^{-1}x, y)) + e^{4\epsilon'} \|\mu_x\| \cdot \|\mu_y\| \cdot |S^+ \smallsetminus S| \\
&\quad \le e^{6\epsilon'} r^{-2} M \cdot |S_T^+ \cap S| + c_3' \le e^{6\epsilon'} r^{-2} M \cdot |S_T^+| + c_3' \\
&\quad = e^{6\epsilon'} r^{-2} M \cdot \frac{e^{\delta T}}{\delta \|m_\Gamma\|} \nu_{x,y}^T(\mathcal{C}_1^+(x, A) \times \mathcal{C}_1^+(y, B)) + c_3',
\end{aligned}$$

where one can check that $c_3' := e^{4\epsilon'}\|\mu_x\| \cdot \|\mu_y\| \cdot |S^+ \smallsetminus S|$ is finite. Indeed, since $\overline{V} \subset \hat{V}$ which is open in $\overline{\Omega}$, there must exist some $R > 0$ such that for any $x' \in \overline{B}_\Omega(x, 1)$, $\xi \in \overline{V}$ and $z \in [x', \xi)$, if $d_\Omega(x, z) \geq R$, then $z \in \hat{V}$; as a consequence, if $\gamma y \in \mathcal{C}_1^+(x, A) \smallsetminus \hat{V}_1$, then $d_\Omega(x, \gamma y) \leq R + 1$. Similarly one can find $R' > 0$ such that $d_\Omega(x, \gamma y) \leq R'$ whenever $\gamma^{-1}x \in \mathcal{C}_1^+(y, B) \smallsetminus \hat{W}_1$.

On the other hand,

$$\begin{aligned}\sum_{\gamma \in S_T^-} \mu_x(\overline{\mathcal{O}}_r^-(\gamma y, x))\mu_x(\overline{\mathcal{O}}_r^-(x, \gamma y))e^{\delta d(x,\gamma y)} &\geq \sum_{\gamma \in S_T^- \cap S} \mu_x(\overline{\mathcal{O}}_r^-(\gamma y, x))\mu_y(\overline{\mathcal{O}}_r^-(\gamma^{-1}x, y)) \\ &\geq e^{-2\epsilon'} r^{-2} M \cdot |S_T^- \cap S| \\ &\geq e^{-2\epsilon'} r^{-2} M \cdot |S_T^-| - e^{2\epsilon'} r^{-2} M \cdot |S^- \smallsetminus S| \\ &= e^{-2\epsilon'} r^{-2} M \cdot \frac{e^{\delta T}}{\delta \|m_\Gamma\|} \nu_{x,y}^T(\mathcal{C}_1^-(x, A) \times \mathcal{C}_1^-(y, B)) - c_6',\end{aligned}$$

where one can check that $c_6' := e^{-2\epsilon'} r^{-2} M \cdot |S^- \smallsetminus S|$ is finite.

(8.2.5): geodesic corridors to shadows, upper bound

We may verify, by recalling the definitions of m and $K^\pm$, that for $\gamma \in \Gamma$ with $d_\Omega(x, \gamma y) > 2r$, we have

$$m(K^+ \cap g^{-t}\gamma K^-) = \int \frac{d\mu_x(\xi)\, \mathrm{d}\mu_x(\eta)}{e^{-2\delta\langle \xi,\eta\rangle_x}} \int_{-r/2}^{r/2} \mathbf{1}_{K(\gamma y, r)}(g^{t+s} x_{\xi\eta})\, \mathrm{d}s \tag{8.2.10}$$

where the integral is supported on $\mathcal{L}_r(x, \gamma y) \cap (\gamma B \times A)$.

Then (8.2.5) is a consequence of the following facts:

(i) The following non-negative number is finite.

$$c_2' := \int_0^\infty e^{\delta t} \sum_{d_\Omega(x,\gamma y) \leq 2r} m(K^+ \cap g^{-t}\gamma K^-) dt$$

Indeed, if $\gamma \in \Gamma$ is such that $d_\Omega(x, \gamma y) \leq 2r$, then $K^+ \cap g^{-t}\gamma K^-$ is empty as soon as $t > 5r$.

(ii) For $(\xi, \eta) \in \mathcal{L}_r(x, \gamma y)$, $|s| \leq \frac{r}{2}$, and $T > 0$, we see, by examining the definition of $K(\gamma y, r)$, that

$$\int_0^{T-3r} e^{\delta t} \mathbf{1}_{K(\gamma y, r)}(g^{t+s} x_{\xi\eta}) dt \leq e^{3\delta r} r e^{\delta \cdot d_\Omega(x,\gamma y)} \leq e^{3\epsilon'} r e^{\delta \cdot d_\Omega(x,\gamma y)}$$

and also that this integral is zero if $d_\Omega(x, \gamma y) > T$.

(iii) $e^{-2\delta\langle \xi,\eta\rangle_x} \geq e^{-2\delta r} \geq e^{-2\epsilon'}$ for $(\xi, \eta) \in \mathcal{L}_r(x, \gamma y)$.

(iv) If $\mathcal{L}_r(x, \gamma y) \cap (\gamma B \times A) \neq \varnothing$, then $\gamma y \in \mathcal{C}_1^+(x, A)$ and $\gamma^{-1}x \in \mathcal{C}_1^+(y, B)$ (we have taken care to ensure $r < 1$).

(v) According to (8.2.1), $\mathcal{L}_r(x, \gamma y) \subset \overline{\mathcal{O}}_r^+(\gamma y, x) \times \overline{\mathcal{O}}_r^+(x, \gamma y)$.

(8.2.8): geodesic corridors to shadows, lower bound

(8.2.8) follows from (8.2.10), and from the following facts.

(i) $e^{-\langle\xi,\eta\rangle_x} \leq 1$.

(ii) For $(\xi,\eta) \in \mathcal{L}_r(x,\gamma y)$, $|s| \leq \frac{r}{2}$ and $T > 0$, we have

$$\int_0^{T+3r} e^{\delta t}\mathbf{1}_{K(\gamma y,r)}(g^{t+s}x_{\xi\eta})dt \geq e^{-3\epsilon'} r e^{\delta\cdot d_\Omega(x,\gamma y)}$$

if $3r \leq d_\Omega(x,\gamma y) \leq T$.

(iii) If $(\gamma y, \gamma^{-1}x) \in \mathcal{C}_1^-(x,A) \times \mathcal{C}_1^-(y,B)$, then $A \supset \overline{\mathcal{O}}_r^-(x,\gamma y)$ and $B \supset \overline{\mathcal{O}}_r^-(y,\gamma^{-1}x)$ i.e. $\gamma B \supset \overline{\mathcal{O}}_r^-(\gamma y, x)$. By (8.2.1), this yields

$$\mathcal{L}_r(x,\gamma y) \cap (\gamma B \times A) \supset \overline{\mathcal{O}}_r^-(\gamma y,x) \times \overline{\mathcal{O}}_r^-(x,\gamma y).$$

(iv) The following non-negative number is finite.

$$\begin{aligned} c_5' &:= e^{-3\epsilon'} r^2 \sum_{d_\Omega(x,\gamma y)\leq 3r} \mu_x(\overline{\mathcal{O}}_r^-(\gamma y,x))\mu_x(\overline{\mathcal{O}}_r^-(x,\gamma y))e^{\delta d_\Omega(x,\gamma y)} \\ &\leq e^{-3\epsilon'} r^2 e^{3\delta r}\|\mu_x\|^2 \#\{\gamma\in\Gamma : d_\Omega(x,\gamma y) \leq 3r\}. \end{aligned}$$

(8.2.4) and (8.2.7): the mixing step

Since the geodesic flow in the quotient is mixing (Theorem 6.0.1; see also Remark 1.2.10) with respect to m_Γ, we have, for T larger than some $T_0 > 0$,

$$e^{-\epsilon'} m(K^+)m(K^-) \leq \|m_\Gamma\| \sum_{\gamma\in\Gamma} m(K^+ \cap g^{-t}\gamma K^-) \leq e^{\epsilon'} m(K^+)m(K^-). \tag{8.2.11}$$

Recalling the definition of $K^+ = K^+(x,r,A)$, we see that

$$m(K^+) = r\int_A d\mu_x(\xi) \int_{\overline{\mathcal{O}}_r(\xi,x)} e^{\delta\langle\xi,\zeta\rangle_x} d\mu_x(\zeta).$$

Since $0 \leq \langle\xi,\zeta\rangle_x \leq r$ (see Section 6.2.1) and $A \subset \hat{V}_1$, by (8.2.2) we obtain

$$e^{-\epsilon'}\mu_x(\overline{\mathcal{O}}_r(\xi_0,x)) \leq \int_{\overline{\mathcal{O}}_r(\xi,x)} e^{2\delta\langle\xi,\zeta\rangle_x} d\mu_x(\zeta) \leq e^{3\epsilon'}\mu_x(\overline{\mathcal{O}}_r(\xi_0,x))$$

and hence

$$e^{-\epsilon'} r\mu_x(A)\mu_x(\overline{\mathcal{O}}_r(\xi_0,x)) \leq m(K^+) \leq e^{3\epsilon'} r\mu_x(A)\mu_x(\overline{\mathcal{O}}_r(\xi_0,x)). \tag{8.2.12}$$

Arguing similarly with $K^- = K^-(y,r,B)$, we obtain

$$e^{-\epsilon-} r\mu_y(B)\mu_y(\overline{\mathcal{O}}_r(\eta_0,y)) \leq m(K^-) \leq e^{3\epsilon'} r\mu_y(B)\mu_y(\overline{\mathcal{O}}_r(\eta_0,y)). \tag{8.2.13}$$

Combining (8.2.11), (8.2.12), and (8.2.13), we obtain

$$\begin{aligned}
&\delta\|m_\Gamma\| \int_0^{T-3r} e^{\delta t} \sum_{\gamma\in\Gamma} m(K^+ \cap g^{-t}\gamma K^-)dt \\
&\qquad \geq \delta\|m_\Gamma\| \int_{T_0}^{T-3r} e^{\delta t} \sum_{\gamma\in\Gamma} m(K^+ \cap g^{-t}\gamma K^-)dt \\
&\qquad \geq e^{-\epsilon'} m(K^+)m(K^-) \int_{T_0}^{T-3r} \delta e^{\delta t}dt \quad \text{(by (8.2.11))} \\
&\qquad = e^{-\epsilon'} m(K^+)m(K^-)(e^{\delta T}e^{-3\delta r} - e^{T_0}) \\
&\qquad \geq e^{-4\epsilon'} e^{\delta T} m(K^+)m(K^-) - c_1' \\
&\qquad \geq e^{-6\epsilon'} e^{\delta T} r^2 \mu_x(A)\mu_y(B)\mu_x(\overline{\mathcal{O}}_r(\xi_0,x))\mu_x(\overline{\mathcal{O}}_r(\eta_0,y)) - c_1' \\
&\qquad \geq e^{-6\epsilon'} e^{\delta T} M \mu_x(A)\mu_y(B) - c_1'
\end{aligned}$$

by (8.2.12) and (8.2.13), and where $c_1' := e^{T_0}e^{-\epsilon/3}m(K^+)m(K^-)$. Similarly,

$$\begin{aligned}
&\delta\|m_\Gamma\| \int_0^{T+3r} e^{\delta t} \sum_{\gamma\in\Gamma} m(K^+ \cap g^{-t}\gamma K^-)dt \\
&\qquad \leq \delta\|m_\Gamma\| \int_{T_0}^{T+3r} e^{\delta t} \sum_{\gamma\in\Gamma} m(K^+ \cap g^{-t}\gamma K^-)dt + c_4' \\
&\qquad \leq e^{\epsilon'} m(K^+)m(K^-) \int_{T_0}^{T+3r} \delta e^{\delta t} + c_4' \\
&\qquad = e^{\epsilon'} m(K^+)m(K^-)(e^{\delta T}e^{3\delta r} - e^{T_0}) + c_4' \\
&\qquad \leq e^{4\epsilon'} e^{\delta T} m(K^+)m(K^-) + c_4' \\
&\qquad \leq e^{10\epsilon'} e^{\delta T} r^2 \mu_x(A)\mu_x(B)\mu_x(\overline{\mathcal{O}}_r(\xi_0,x))\mu_x(\overline{\mathcal{O}}_r(\eta_0,y)) + c_4' \\
&\qquad \leq e^{10\epsilon'} e^{\delta T} M \mu_x(A)\mu_y(B) + c_4'
\end{aligned}$$

where $c_4' := \delta\|m_\Gamma\| \int_0^{T_0} e^{\delta t} \sum_{\gamma\in\Gamma} m(K^+ \cap g^{-t}\gamma K^-)dt$. □

8.2.3 Generalisation of the technical crux

Proposition 8.2.2. *In the setting of Theorem 8.0.1, fix $\epsilon > 0$, $\xi_0, \eta_0 \in \partial\Omega$ both extremal and smooth and $x, y \in \Omega$. Then there exist $R > 0$ and open neighbourhoods V and W of ξ_0 and η_0 (resp.) in $\partial\Omega$, such that for all Borel subsets $A \subset V, B \subset W$, we have*

$$\limsup_{T\to+\infty} \nu_{x,y}^T(\mathcal{C}_R^-(x,A) \times \mathcal{C}_R^-(y,B)) \leq e^{\epsilon}\mu_x(A)\mu_y(B),$$
$$\liminf_{T\to+\infty} \nu_{x,y}^T(\mathcal{C}_R^+(x,A) \times \mathcal{C}_R^+(y,B)) \geq e^{-\epsilon}\mu_x(A)\mu_y(B).$$

Proof. Choose $\zeta_0 \in \Lambda_\Gamma \smallsetminus \{\xi_0, \eta_0\}$ which is strongly extremal, and $x_0 \in [\xi_0, \zeta_0] \cap \Omega$ and $y_0 \in [\eta_0, \zeta_0] \cap \Omega$. Let us apply Proposition 8.2.1: we have neighbourhoods V_0, W_0 of ξ_0, η_0 (respectively) such that for all Borel subsets $A \subset V_0, B \subset W_0$, we have

$$\limsup_{T\to+\infty} \nu_{x,y}^T(\mathcal{C}_1^-(x_0,A) \times \mathcal{C}_1^-(y_0,B)) \leq e^{\epsilon}\mu_{x_0}(A)\mu_{y_0}(B),$$
$$\liminf_{T\to+\infty} \nu_{x,y}^T(\mathcal{C}_1^+(x_0,A) \times \mathcal{C}_1^+(y_0,B)) \geq e^{-\epsilon}\mu_{x_0}(A)\mu_{y_0}(B).$$

Since ξ_0 and η_0 are smooth, we can find $\widehat{V_0}$ and $\widehat{W_0}$ open subsets of $\overline{\Omega}$ containing respectively ξ_0 and η_0 such that $\widehat{V_0} \cap \partial\Omega \subset V_0$ and $\widehat{W_0} \cap \partial\Omega \subset W_0$, and for all $\xi \in \pi_{\mathtt{h}}^{-1}\widehat{V_0}$ and $\eta \in \pi_{\mathtt{h}}^{-1}\widehat{W_0}$,

$$|\mathtt{b}_\xi(x_0, x) - \mathtt{b}_{\xi_0}(x_0, x)| < \frac{\epsilon}{6\delta}$$
$$|\mathtt{b}_\eta(y_0, y) - \mathtt{b}_{\eta_0}(y_0, y)| < \frac{\epsilon}{6\delta}.$$

Set $R := 1 + \max\{d_\Omega(x, x_0), d_\Omega(y, y_0)\}$, and take a neighbourhood $\hat{V}_1$ (resp. $\hat{W}_1$) of ξ_0 (resp. η_0) such that $\overline{B}(z, R)$ is contained in $\hat{V}_0$ (resp. $\hat{W}_0$) for any z in $\hat{V}_1$ (resp. $\hat{W}_1$). Take two open neighbourhoods V and W of ξ_0 and η_0 (respectively) in $\partial\Omega$, such that $\overline{B}_{\overline{\Omega}}(\overline{V}, R) \subset \widehat{V_1} \cap \partial\Omega$ and $\overline{B}_{\overline{\Omega}}(\overline{W}, R) \subset \widehat{W_1} \cap \partial\Omega$. Consider $A \subset V$ and $B \subset W$. We now relate the orbit of y seen from x to that of y_0 seen from x_0, thanks to the following observations:

First, using the definition of R and $\mathcal{C}_R^\pm$, one can easily verify that, for any $\gamma \in \Gamma$,

$$(\gamma y, \gamma^{-1}x) \in \mathcal{C}_R^-(x, A) \times \mathcal{C}_R^-(y, B) \Rightarrow (\gamma y_0, \gamma^{-1}x_0) \in \mathcal{C}_1^-(x_0, A) \times \mathcal{C}_1^-(y_0, B),$$
$$(\gamma y_0, \gamma^{-1}x_0) \in \mathcal{C}_1^+(x_0, A) \times \mathcal{C}_1^+(y_0, B) \Rightarrow (\gamma y, \gamma^{-1}x) \in \mathcal{C}_R^+(x, A) \times \mathcal{C}_R^+(y, B).$$

Second, if $(\gamma y, \gamma^{-1}x) \in \widehat{V_1} \times \widehat{W_1}$, then $\gamma y_0 \in \widehat{V_0}$ and $\gamma^{-1}x \in \widehat{W_0}$, whence

$$\begin{aligned} d_\Omega(x_0, \gamma y_0) &= d_\Omega(x, \gamma y_0) + \mathtt{b}_{\gamma y_0}(x_0, x) \\ &\le d_\Omega(x, \gamma y_0) + \mathtt{b}_{\xi_0}(x_0, x) + \frac{\epsilon}{6\delta} \\ &= d_\Omega(y_0, \gamma^{-1}x) + \mathtt{b}_{\xi_0}(x_0, x) + \frac{\epsilon}{6\delta} \\ &\le d_\Omega(y, \gamma^{-1}x) + t_0 + \frac{\epsilon}{3\delta}, \end{aligned}$$

where $t_0 := \mathtt{b}_{\eta_0}(y_0, y) + \mathtt{b}_{\xi_0}(x_0, x)$. Symmetrically, if $(\gamma y_0, \gamma^{-1}x_0) \in \widehat{V_1} \times \widehat{W_1}$, then

$$d_\Omega(x_0, \gamma y_0) \ge d_\Omega(y, \gamma^{-1}x) + t_0 - \frac{\epsilon}{3\delta}.$$

Third, the sets $(\mathcal{C}_R^+(x, A) \cup \mathcal{C}_R^+(x_0, A)) \setminus \widehat{V_1}$ and $(\mathcal{C}_r^+(y, B) \cup \mathcal{C}_r^+(y_0, B)) \setminus \widehat{W_1}$ are bounded in Ω because any accumulation point of any sequence of $\mathcal{C}_r^+(x, A)$ which diverges in Ω must belong to $\overline{B}(\overline{A}, r) \subset \widehat{V_1}$. Therefore

$$\limsup_{t \to +\infty} \nu_{x,y}^t(\mathcal{C}_R^-(x, A) \times \mathcal{C}_R^-(y, B)) = \limsup_{t \to +\infty} \nu_{x,y}^t(\mathcal{C}_R^-(x, A) \times \mathcal{C}_R^-(y, B) \cap \widehat{V_1} \times \widehat{W_1}),$$
$$\liminf_{t \to +\infty} \nu_{x_0,y_0}^t(\mathcal{C}_R^+(x_0, A) \times \mathcal{C}_R^+(y_0, B)) = \liminf_{t \to +\infty} \nu_{x_0,y_0}^t(\mathcal{C}_R^+(x_0, A) \times \mathcal{C}_R^+(y_0, B) \cap \widehat{V_1} \times \widehat{W_1}).$$

i.e. when we are taking these limits we may restrict to looking at points inside $\widehat{V_1} \times \widehat{W_1}$, where the distance estimates from the previous observation hold.

Together, these observations imply that

$$\begin{aligned} \limsup_{t \to +\infty} \nu_{x,y}^t(\mathcal{C}_R^-(x, A) \times \mathcal{C}_R^-(y, B)) &\le e^{\delta \cdot (t_0 + \frac{\epsilon}{3\delta})} \limsup_{t \to +\infty} \nu_{x_0,y_0}^{t + t_0 + \frac{\epsilon}{3\delta}} \left(\mathcal{C}_1^-(x_0, A) \times \mathcal{C}_1^-(y_0, B)\right) \\ &\le e^{2\epsilon/3} e^{\delta t_0} \mu_{x_0}(A) \mu_{y_0}(B) \\ &\le e^{2\epsilon/3} (e^{\delta \cdot (\mathtt{b}_{\xi_0}(x_0, x)} \mu_{x_0}(A)) \cdot (e^{\mathtt{b}_{\eta_0}(y_0, y))} \mu_{y_0}(B)) \\ &\le e^\epsilon \mu_x(A) \mu_y(B), \end{aligned}$$

where we have used the conformality of the $(\mu_z)_z$ to say, for example, that $\frac{d\mu_{x_0}}{d\mu_x}(\xi) \leq e^{-\delta \mathrm{b}_{\xi_0}(x_0,x)+\epsilon/6}$ for any $\xi \in A$.

Similarly one shows that

$$\begin{aligned}\liminf_{t\to+\infty} \nu^t_{x,y}(\mathcal{C}^+_R(x,A)\times\mathcal{C}^+_R(y,B)) &\geq e^{\delta\cdot(t_0-\epsilon/3\delta)} \liminf_{t\to+\infty} \nu^{t+t_0-\frac{\epsilon}{3\delta}}_{x_0,y_0}\left(\mathcal{C}^+_1(x_0,A)\times\mathcal{C}^+_1(y_0,B)\right)\\ &\geq e^{-2\epsilon/3}e^{\delta t_0}\mu_{x_0}(A)\mu_{y_0}(B)\\ &\geq e^{-\epsilon}\mu_x(A)\mu_y(B).\end{aligned}$$

□

8.2.4 Negligibility from lack of extremal points

Here we pause to prove two lemmas which will together be used in the next step of the proof.

The first (Lemma 8.2.3) shows that a sum of Dirac masses at orbit points is comparable to a Patterson–Sullivan measure, and hence has negligible mass away from extremal points. The second (Lemma 8.2.4) establishes that the differences between measurable subsets of $\overline{\Omega}$ and certain associated cones do not contain extremal points.

Given $\Omega \subset \mathrm{P}(\mathbf{V})$ a properly convex open set and $\Gamma \leq \mathrm{Aut}(\Omega)$ a discrete subgroup, and $x, y \in \Omega$ and $t \geq 0$ we set

$$\alpha^t_{x,y} := \sum_{\substack{\gamma\in\Gamma\\ d_\Omega(x,\gamma y)\leq t}} \mathcal{D}_{\gamma y}.$$

If $M = \Omega/\Gamma$ is non-elementary and rank-one, then the Sullivan Shadow lemma implies that $(e^{-\delta_\Gamma t}\alpha^t_{x,y}(\overline{\Omega}))_t$ is bounded by Proposition 6.3.6.

Lemma 8.2.3. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set and $\Gamma \leq \mathrm{Aut}(\Omega)$ be a non-elementary rank-one divergent discrete subgroup; set $\delta = \delta_\Gamma$, and let $(\mu_x)_x$ be a δ-conformal density on $\partial\Omega$. Then there exists $C > 0$ such that $\alpha \leq C\mu_x$ for any accumulation point α of $(\alpha^t_{x,y})_{t\to\infty}$.*

In particular, if $K \subset \overline{\Omega}$ is compact and does not contain any extremal point, then

$$\alpha^t_{x,y}(K) \underset{t\to\infty}{\longrightarrow} 0.$$

Proof. Since $\partial_{\mathrm{sse}}\Omega$ has full μ_x-measure by Theorem 6.0.1, and by the interior regularity of finite measures, it is enough to find $C > 0$ and $R > 0$ such that for any compact subset $K \subset \overline{\Omega}$,

$$\alpha(K) \leq C\mu_x(\overline{B}_{\overline{\Omega}}(K,R)).$$

By Lemma 6.3.1, there exist $R > 0$ and $C_1 > 0$ such that for any $\gamma \in \Gamma$,

$$\mu_x(\overline{\mathcal{O}}_R(x,\gamma y)) \geq C_1^{-1}e^{-\delta d_\Omega(x,\gamma y)}.$$

Fix $\epsilon > 0$. Let U be a neighbourhood in $\overline{\Omega}$ of $\overline{B}_{\overline{\Omega}}(K,R)$ such that $\mu_x(U) \leq \mu_x(\overline{B}_{\overline{\Omega}}(K,R)) + \epsilon$. Let V be a neighbourhood in $\overline{\Omega}$ of K such that $\overline{\mathcal{O}}_R(x,z) \subset U$ for any $z \in V \cap \Omega$. Observe that for all $t \geq 0$ and $\xi \in \partial\Omega$,

$$\#\{\gamma : t-1 \leq d_\Omega(x,\gamma y) \leq t, \xi \in \overline{\mathcal{O}}_R(x,\gamma y)\} \leq \#\{g : d_\Omega(y,gy) \leq 4R+1\}. \tag{8.2.14}$$

Indeed, if $\gamma, g \in \Gamma$ are such that $t-1 \leq d_\Omega(x,\gamma y), d_\Omega(x,\gamma g y) \leq t$ and $\xi \in \overline{\mathcal{O}}_R(x,\gamma y) \cap \overline{\mathcal{O}}_R(x,\gamma g y)$, then let $y_1 \in [x,\xi) \cap B_\Omega(\gamma y, R)$ and $y_2 \in [x,\xi) \cap B_\Omega(\gamma g y, R)$, so that

$$\begin{aligned}d_\Omega(y,gy) = d_\Omega(\gamma y,\gamma g y) &\leq R + d_\Omega(y_1,y_2) + R\\ &= 2R + |d_\Omega(x,y_1) - d_\Omega(x,y_2)|\\ &\leq 4R + |d_\Omega(x,\gamma y) - d_\Omega(x,\gamma g y)| \leq 4R+1.\end{aligned}$$

Then we have

$$\begin{aligned}\alpha(K) &\le \limsup_{t\to\infty} e^{-\delta t}\alpha^t_{x,y}(V)\\ &:= \limsup_{t\to\infty} e^{-\delta t}\#\{\gamma : 0 \le d_\Omega(x,\gamma y) \le t, \gamma y \in V\}\\ &\le e^{\delta}\limsup_{t\to\infty} \sum_{1\le k\le \lfloor t+1\rfloor} e^{\delta(k-\lfloor t+1\rfloor)}e^{-\delta k}\#\{\gamma : k-1 \le d_\Omega(x,\gamma y)\le k, \gamma y\in V\}\\ &\le C_1 e^{\delta}\limsup_{n\to\infty}\sum_{1\le k\le n} e^{\delta(k-n)} \sum_{\substack{k-1\le d_\Omega(x,\gamma y)\le k\\ \gamma y\in V}} \mu_x(\overline{\mathcal{O}}_R(x,\gamma y)).\end{aligned}$$

Now recall that $\overline{\mathcal{O}}_R(x,\gamma y)\subset U$ for any $\gamma y\in V$. Hence by (8.2.14) we have

$$\begin{aligned}\alpha(K) &\le C_1 e^{\delta}\limsup_{n\to\infty}\sum_{1\le k\le n} e^{\delta(k-n)}\int_U \sum_{\substack{k-1\le d_\Omega(x,\gamma y)\le k\\ \gamma y\in V}} 1_{\overline{\mathcal{O}}_R(x,\gamma y)}(\xi)\mathrm{d}\mu_x(\xi)\\ &\le C_1 e^{\delta}\limsup_{n\to\infty}\sum_{1\le k\le n} e^{\delta(k-n)}\#\{\gamma : d_\Omega(y,\gamma y)\le 4R+1\}\cdot\mu_x(U)\\ &\le C\mu_x(\overline{B}_{\overline{\Omega}}(K,R)) + C\epsilon,\end{aligned}$$

where $C := \frac{C_1 e^{\delta}}{1-e^{-\delta}}\cdot\#\{\gamma : d_\Omega(y,\gamma y)\le 4R+1\}$. The previous estimates hold for any $\epsilon>0$, therefore

$$\alpha(K)\le C\overline{B}_{\overline{\Omega}}(K,R).$$

□

This lemma will be useful below in combination with the next one, which shows that certain sets we will want to have small measure do not contain extremal points:

Lemma 8.2.4. *Let $\Omega\subset \mathrm{P}(\mathbf{V})$ be a properly convex open set. Let $r>0$, $x\in\Omega$ and $\mathcal{A}\subset\overline{\Omega}$ be measurable. Consider an open neighbourhood A^+ of $\overline{\mathcal{A}}\cap\partial\Omega$ in $\partial\Omega$ and a compact subset $A^-\subset\mathrm{int}(\mathcal{A})\cap\partial\Omega$. Then the closures of $\mathcal{A}\cap\Omega\smallsetminus\mathcal{C}^-_r(x,A^+)$ and $\mathcal{C}^+_r(x,A^-)\smallsetminus\mathcal{A}$ do not contain any extremal point of $\overline{\Omega}$.*

Proof. Consider a sequence $(y_n)_n$ of points in $\mathcal{A}\cap\Omega\smallsetminus\mathcal{C}^-_r(x,A^+)$ that converges to $y\in\partial\Omega\cap\overline{\mathcal{A}}$. Let us show that y is not extremal. By definition, for each n there exists $x_n\in\overline{B}_\Omega(x,r)$ and $z_n\in\overline{B}_\Omega(y_n,r)\smallsetminus\mathcal{C}_r(x_n,A^+)$. Up to extraction, we can assume that $x_n\neq z_n$ for any n and that $(x_n)_n$ and $(z_n)_n$ converge respectively to $x'\in\overline{B}_\Omega(x,r)$ and $z\in\overline{B}_{\overline{\Omega}}(y,r)$. Since z is in the open face of y, it is enough to show that $z\neq y$. For each pair $(a,b)\in\Omega\times\overline{\Omega}$ such that $a\neq b$, denote by $\overline{\mathcal{O}}(a,b)$ the unique point $c\in\partial\Omega$ such that $b\in[a,c]$. The map $\overline{\mathcal{O}}$ is continuous, so $\overline{\mathcal{O}}^{-1}(\partial\Omega\smallsetminus A^+)$ is closed, hence it must contain (x',z), since it contains $\{(x_n,z_n)\}_n$. That $z\in\partial\Omega$ implies that $\overline{\mathcal{O}}(x',z)=z\notin A^+\ni y$, thus $z\neq y$.

Consider a sequence $(y_n)_n$ of points in $\mathcal{C}^+_r(x,A^-)\smallsetminus\mathcal{A}$ that converges to $y\in\partial\Omega\smallsetminus\mathrm{int}(\mathcal{A})$. Let us show that y is not extremal. By definition, for each n there exists $x_n\in\overline{B}_\Omega(x,r)$ and $z_n\in\overline{B}_\Omega(y_n,r)\smallsetminus\{x_n\}$ such that $\overline{\mathcal{O}}(x_n,z_n)\in A^-$. Up to extraction, we can assume that $(x_n)_n$ and $(z_n)_n$ converge respectively to $x'\in\overline{B}_\Omega(x,r)$ and $z\in\overline{B}_{\overline{\Omega}}(y,r)$. By continuity, $z=\overline{\mathcal{O}}(x',z)\in A^-$ which is closed. Thus, $z\neq y$, which is not extremal. □

8.2.5 End of proof

Proposition 8.2.5. *In the setting of Theorem 8.0.1, for any $x,y\in\Omega$, $\epsilon>0$ and $\xi_0,\eta_0\in\partial\Omega$ that are smooth and extremal, there exist open neighbourhoods $\hat{V}$ and $\hat{W}$ of ξ_0 and η_0*

in $\overline{\Omega}$ such that for any non-negative function φ supported on $\hat{V} \times \hat{W}$,

$$e^{-\epsilon} \int \varphi \, \mathrm{d}(\mu_x \otimes \mu_y) \leq \liminf \int \varphi \, \mathrm{d}\nu^t_{x,y} \leq \limsup \int \varphi \, \mathrm{d}\nu^t_{x,y} \leq e^{\epsilon} \int \varphi \, \mathrm{d}(\mu_x \otimes \mu_y).$$

Proof. Let $\hat{V}$ and $\hat{W}$ be the open neighbourhoods given by Proposition 8.2.2. It is enough to prove that for all measurable subsets $\mathcal{A}$ and $\mathcal{B}$ such that $\overline{\mathcal{A}} \subset \hat{V}$, $\overline{\mathcal{B}} \subset \hat{W}$ and $\mu_x(\partial \mathcal{A}) = \mu_y(\partial \mathcal{B}) = 0$, we have

$$e^{-\epsilon} \mu_x(\mathcal{A}) \mu_y(\mathcal{B}) \leq \liminf \nu^t_{x,y}(\mathcal{A} \times \mathcal{B}) \leq \limsup \nu^t_{x,y}(\mathcal{A} \times \mathcal{B}) \leq e^{\epsilon} \mu_x(\mathcal{A}) \mu_y(\mathcal{B}).$$

Let $\epsilon' > 0$. Let A^+ (resp. B^+) be an open neighbourhood of $\mathcal{A} \cap \partial\Omega$ (resp. $\mathcal{B} \cap \partial\Omega$) in $\partial\Omega$, and $A^- \subset \mathrm{int}(\mathcal{A}) \cap \partial\Omega$ (resp. $B^- \subset \mathrm{int}(\mathcal{B}) \cap \partial\Omega$) be compact such that

$$\mu_x(\mathrm{int}(\mathcal{A}) \smallsetminus A^-) + \mu_x(A^+ \smallsetminus \overline{\mathcal{A}}) + \mu_y(\mathrm{int}(\mathcal{B}) \smallsetminus B^-) + \mu_y(B^+ \smallsetminus \overline{\mathcal{B}}) < \epsilon'.$$

By Lemma 8.2.3 and Lemma 8.2.4,

$$e^{-\delta t} \alpha^t_{x,y}(\mathcal{A} \smallsetminus \mathcal{C}^-_r(x, A^+)),\ e^{-\delta t} \alpha^t_{x,y}(\mathcal{C}^+_r(x, A^-) \smallsetminus \mathcal{A}),$$
$$e^{-\delta t} \alpha^t_{y,x}(\mathcal{A} \smallsetminus \mathcal{C}^-_r(y, B^+)), \text{ and } e^{-\delta t} \alpha^t_{y,x}(\mathcal{C}^+_r(y, B^-) \smallsetminus \mathcal{B})$$

all converge to zero as $t \to \infty$. Since the projections on the first and second coordinates of $\nu^t_{x,y}$ are respectively $\delta \|m_\Gamma\| e^{-\delta t} \alpha^t_{x,y}$ and $\delta \|m_\Gamma\| e^{-\delta t} \alpha^t_{y,x}$, we have that

$$\limsup \nu^t_{x,y}(\mathcal{A} \times \mathcal{B}) \leq \limsup \nu^t_{x,y}(\mathcal{C}^-_r(x, A^+) \times \mathcal{C}^-_r(y, B^+))$$
$$\liminf \nu^t_{x,y}(\mathcal{A} \times \mathcal{B}) \geq \liminf \nu^t_{x,y}(\mathcal{C}^+_r(x, A^-) \times \mathcal{C}^+_r(y, B^-))$$

and hence

$$\begin{aligned}\limsup \nu^t_{x,y}(\mathcal{A} \times \mathcal{B}) &\leq \limsup \nu^t_{x,y}(\mathcal{C}^-_r(x, A^+) \times \mathcal{C}^-_r(y, B^+)) \\ &\leq e^{\epsilon} \mu_x(A^+) \mu_y(B^+) \\ &\leq e^{\epsilon} \mu_x(\mathcal{A}) \mu_y(\mathcal{B}) + \epsilon' e^{\epsilon} (\|\mu_x\| + \|\mu_y\|),\end{aligned}$$

and

$$\begin{aligned}\liminf \nu^t_{x,y}(\mathcal{A} \times \mathcal{B}) &\geq \liminf \nu^t_{x,y}(\mathcal{C}^+_r(x, A^-) \times \mathcal{C}^+_r(y, B^-)) \\ &\geq e^{-\epsilon} \mu_x(A^-) \mu_y(B^-) \\ &\geq e^{-\epsilon} \mu_x(\mathcal{A}) \mu_y(\mathcal{B}) - \epsilon' e^{\epsilon} (\|\mu_x\| + \|\mu_y\|).\end{aligned}$$

This concludes the proof of the proposition, since ϵ' can be taken arbitrarily small. □

Proof of Theorem 8.0.1. Let ν be an accumulation point of $(\nu^t_{x,y})_{t \to \infty}$. Let φ be a non-negative continuous function on $\overline{\Omega}^2$; it is enough to prove that for any $\epsilon > 0$,

$$e^{-\epsilon} \int \varphi \, \mathrm{d}(\mu_x \otimes \mu_y) - \epsilon \leq \int \varphi \, \mathrm{d}\nu \leq e^{\epsilon} \int \varphi \, \mathrm{d}(\mu_x \otimes \mu_y) + \epsilon.$$

As noted above, the projection on the first and second coordinates of ν are accumulation points of, respectively, $(\delta \|m_\Gamma\| e^{-\delta t} \alpha^t_{x,y})_t$ and $(\delta \|m_\Gamma\| e^{-\delta t} \alpha^t_{y,x})_t$. Thus, according to Lemma 8.2.3, ν (and $\mu_x \otimes \mu_x$) gives full measure to the set of pairs of smooth and extremal points of $\overline{\Omega}$. Let $K \subset \partial\Omega$ be a compact set of smooth and extremal points such that

$$\int_{\overline{\Omega}^2 \smallsetminus K^2} \varphi \, \mathrm{d}\nu + \int_{\overline{\Omega}^2 \smallsetminus K^2} \varphi \, \mathrm{d}(\mu_x \otimes \mu_y) \leq \epsilon.$$

By Proposition 8.2.5, K^2 can be covered by a finite number of open sets $(U_i)_{1\leq i\leq n}$, where $n\in\mathbb{N}_{>0}$, such that for any $1\leq i\leq n$, for any non-negative function ψ supported on U_i,

$$e^{-\epsilon}\int\psi\,\mathrm{d}(\mu_x\otimes\mu_y)\leq\int\psi\,\mathrm{d}\nu\leq e^{\epsilon}\int\psi\,\mathrm{d}(\mu_x\otimes\mu_y).$$

Set $U_0:=\overline{\Omega}^2\smallsetminus K^2$. Let $(\chi_i)_{0\leq i\leq n}$ be a partition of unity associated to $(U_i)_{0\leq i\leq n}$. Then

$$\begin{aligned}\int\varphi\,\mathrm{d}\nu&=\int\chi_0\varphi\,\mathrm{d}\nu+\sum_{i=1}^{n}\int\chi_i\varphi\,\mathrm{d}\nu\\&\leq\epsilon+e^{\epsilon}\sum_{i=1}^{n}\int\chi_i\varphi\,\mathrm{d}(\mu_x\otimes\mu_y)\\&\leq e^{\epsilon}\int\varphi\,\mathrm{d}(\mu_x\otimes\mu_y)+\epsilon,\end{aligned}$$

and similarly

$$\int\varphi\,\mathrm{d}\nu\geq e^{-\epsilon}\int\varphi\,\mathrm{d}(\mu_x\otimes\mu_y)-\epsilon.$$

□

8.3 Equidistribution of rank-one closed geodesics

In this section we prove Theorem 8.0.3.

8.3.1 Non-rank-one elements are negligible

We will need the following lemma, which also implies that Theorem 8.0.3 remains true if we replace $\mathcal{G}^{\mathrm{r1}}$ by any bigger set of closed straight geodesics with pairwise disjoint collections of conjugacy classes in Γ.

Lemma 8.3.1. *In the setting of Theorem 8.0.3, let $x\in\Omega$. Then*

$$e^{-\delta t}\cdot\#\{\gamma\in\Gamma\text{ not rank-one}: d_\Omega(x,\gamma x)\leq t\}\underset{t\to\infty}{\longrightarrow}0.$$

Moreover, for any $\epsilon>0$, we have,

$$e^{-\delta t}\cdot\#\{\gamma\in\Gamma\text{ rank-one}: d_\Omega(x,\gamma x)\leq t\text{ and }d_{\mathbf{P}(\mathbf{V})}(\gamma x,x_\gamma^+)\geq\epsilon\}\underset{t\to\infty}{\longrightarrow}0.$$

Proof. Fix $0<\epsilon'<\epsilon$. Since $\mu_x\otimes\mu_x$ gives full measure to the set A of distinct pairs of strongly extremal points of the boundary $\partial\Omega$, we can find a compact subset $K\subset A$ such that $\mu_x\otimes\mu_x(\overline{\Omega}^2\smallsetminus K)\leq\delta\|m_\Gamma\|\epsilon'$.

By Corollary 5.4.12 applied with W an ϵ-neighbourhood of K, we can find a neighbourhood U of K such that for any $\gamma\in\Gamma$, if $(\gamma^{-1}x,\gamma x)\in U$, then γ is rank-one. If furthermore $(\gamma^{-1}x,\gamma x)\in U\cap W$, we also have $d_{\mathbf{P}(\mathbf{V})}(\gamma x,x_\gamma^+)<\epsilon$.

For $t>0$ we denote by B_t the set of elements γ such that $d_\Omega(x,\gamma x)\leq t$ and such that either γ is not rank-one or $d_{\mathbf{P}(\mathbf{V})}(\gamma x,x_\gamma^+)\geq\epsilon$; then

$$\begin{aligned}\limsup_{t\to\infty}e^{-\delta t}\#B_t&\leq\limsup_{t\to\infty}e^{-\delta t}\left(\sum_{\substack{\gamma\in\Gamma\\ d_\Omega(x,\gamma x)\leq t}}\mathcal{D}_{\gamma^{-1}x}\otimes\mathcal{D}_{\gamma x}\right)\left(\overline{\Omega}^2\smallsetminus(U\cap W)\right)\\&\leq\delta^{-1}\|m_\Gamma\|^{-1}\mu_x\otimes\mu_x\left(\overline{\Omega}^2\smallsetminus(U\cap W)\right)\\&\leq\epsilon'.\end{aligned}$$

Since this holds for arbitrarily small ϵ', we obtain $e^{-\delta t}\#B_t\underset{t\to\infty}{\to}0$. □

8.3.2 To the universal cover

Let us take the setting of Theorem 8.0.3. The proof of this theorem will take place in the universal cover. Thus we need to interpret in the universal cover the Lebesgue measures on rank-one periodic orbits of T^1M.

Let $c = \{\phi_t v\}_t$ be a periodic orbit of period ℓ in T^1M; the measure $\ell \cdot \mathcal{L}c$ is by definition the push-forward by $t \mapsto \phi_t v$ of the Lebesgue measure on $[0,\ell]$. Choose a lift $\tilde{v} \in T^1\Omega$, so that $\ell \cdot \mathcal{L}c = \pi_{\Gamma *}\lambda$, where λ is the push-forward by $t \mapsto \phi_t\tilde{v}$ of the Lebesgue measure on $[0,\ell]$ By Remark 1.2.4, the measure $\ell \cdot \mathcal{L}c$ is the quotient by Γ of $\sum_{\gamma\in\Gamma}\gamma_*\lambda$.

For any rank-one element $\gamma \in \Gamma$, we denote by $\overline{\mathcal{L}}\gamma$ the push-forward of the Lebesgue measure on $\mathbb{R}$ by $t \mapsto \phi_t\tilde{w}$ for any $\tilde{w} \in T^1\Omega$ tangent to the axis of γ.

We consider the following objects

- $G := \{g \in \Gamma : g \cdot (\phi_{-\infty}\tilde{v}, \phi_\infty\tilde{v}) = (\phi_{-\infty}\tilde{v}, \phi_\infty\tilde{v})\}$;
- $H := \{h \in G : h\tilde{v} = \tilde{v}\}$;
- pick $g_0 \in G$ such that $\phi_\ell\tilde{v} = g_0\tilde{v}$;
- $A := g_0 \cdot H = \{g \in G : \phi_\ell\tilde{v} = g\tilde{v}\}$;
- $B := \{\gamma g\gamma^{-1} : g \in A,\ \gamma \in \Gamma\}$;
- $\mathcal{R} \subset \Gamma$ is a set of representatives of Γ/G that contains the identity.

The subset $B \subset \Gamma$ only depends on c; recall that we have called it the set of conjugacy classes associated to c in Section 3.3. We claim that

$$\sum_{\gamma\in\Gamma}\gamma_*\lambda = \sum_{\gamma\in B}\overline{\mathcal{L}}\gamma.$$

Indeed,

$$\begin{aligned}\sum_{\gamma\in\Gamma}\gamma_*\lambda &= \sum_{(r,n,h)\in\mathcal{R}\times\mathbb{Z}\times H} r_*g_{0*}^n h_*\lambda = \sum_{(r,n,h)\in\mathcal{R}\times\mathbb{Z}\times H} r_*g_{0*}^n\lambda \\ &= \sum_{(r,h)\in\mathcal{R}\times H} r_*\overline{\mathcal{L}}g_0 = \sum_{(r,h)\in\mathcal{R}\times H} r_*\overline{\mathcal{L}}(g_0h) \\ &= \sum_{(r,g)\in\mathcal{R}\times A}\overline{\mathcal{L}}(rgr^{-1}).\end{aligned}$$

Conclude by observing that the map $\mathcal{R}\times A \to B$ that sends (r,g) to rgr^{-1} is a bijection.

8.3.3 Proof of Theorem 8.0.3

Let $\Gamma^{\mathrm{pr1}} \subset \Gamma$ be the set of strongly primitive rank-one elements of Γ, and

$$\mathcal{E}^L := \delta L e^{-\delta L}\sum_{\substack{\gamma\in\Gamma^{\mathrm{pr1}}\\ \ell(\gamma)\leq L}}\frac{1}{\ell(\gamma)}\overline{\mathcal{L}}\gamma.$$

According to Section 8.3.2, the quotient in T^1M of $\mathcal{E}^L$ under the action of Γ is precisely $\delta Le^{-\delta L}\sum_{c\in\mathcal{G}_L^{\mathrm{r1}}}\mathcal{L}c$. Since m_Γ is the quotient of m, we need to prove that $(\mathcal{E}^L)_L$ converges weakly in $\mathcal{C}_c(T^1\Omega)^*$ to $\frac{m}{\|m_\Gamma\|}$ when $L \to +\infty$.

We will first use Theorem 8.0.1 to obtain a measure $\nu^L_{x,1}$ converging weakly to $m_{\mathbb{R}}$ (the Sullivan measure on $\mathrm{Geod}^\infty(\Omega)$) when $L \to +\infty$, then successively modify $\nu^L_{x,1}$ to form $\nu^L_{x,2}$ and $\nu^L_{x,3}$, so that $\nu^L_{x,3}$ will be supported on pairs of fixed points of rank-one elements, and that $\nu^L_{x,3}$ locally approaches the quotient $m_{\mathbb{R}}$ of m by the action of the geodesic flow. By taking the product of $\|m_\Gamma\|^{-1}\nu^L_{x,3}$ with the Lebesgue measure on $\mathbb{R}$, we obtain a measure $\mathcal{M}^L_{x,3}$ approaching $\|m_\Gamma\|^{-1}m$ locally (i.e. near the fibre over $x \in \Omega$ in $T^1\Omega$.) To finish, we relate $\mathcal{M}^L_{x,3}$ to the measure of equidistribution $\mathcal{E}^L$.

Fix $x \in \Omega$. By Theorem 8.0.1, the measure

$$\nu^L_{x,1} := \delta\|m_\Gamma\|e^{-\delta L} \sum_{d_\Omega(x,\gamma x)\le L} \mathcal{D}_{\gamma^{-1}x} \otimes \mathcal{D}_{\gamma x}$$

converges weakly in $\mathcal{C}(\overline{\Omega} \times \overline{\Omega})^*$ to $\mu_x \otimes \mu_x$ as $L \to +\infty$.

Write Γ^{r1} to denote the set of rank-one elements of Γ, and define the measure

$$\nu^L_{x,2} := \delta\|m_\Gamma\|e^{-\delta L} \sum_{\substack{\gamma\in\Gamma^{\mathrm{r1}}\\ d_\Omega(x,\gamma x)\le L}} \mathcal{D}_{\gamma^{-1}x} \otimes \mathcal{D}_{\gamma x}.$$

According to Lemma 8.3.1, we have $\nu^L_{x,1} - \nu^L_{x,2} \to 0$ weakly as $L \to +\infty$. We define the measure

$$\nu^L_{x,3} := \delta\|m_\Gamma\|e^{-\delta L} \sum_{\substack{\gamma\in\Gamma^{\mathrm{r1}}\\ d_\Omega(x,\gamma x)\le L}} \mathcal{D}_{x^-_\gamma} \otimes \mathcal{D}_{x^+_\gamma}.$$

Again by Lemma 8.3.1, we have $\nu^L_{x,3} - \nu^L_{x,2} \to 0$ weakly as $L \to +\infty$.

Fix now $r > 0$, and let $V(x, r)$ denote the open set of pairs $(a, b) \in \mathrm{Geod}(\Omega)$ such that the geodesic (a, b) intersects $B_\Omega(x, r)$. Since $0 \le \langle\xi, \eta\rangle_x \le r$ for $(\xi, \eta) \in V(x, r)$, we have,

$$e^{-2\delta r}\int \psi\, \mathrm{d}m_{\mathbb{R}} \le \lim_{L\to\infty}\int \psi\, \mathrm{d}\nu^L_{x,3} = \int \psi\, \mathrm{d}\mu_x \otimes \mu_x \le \int \psi\, \mathrm{d}m_{\mathbb{R}}$$

for all non-negative $\psi \in \mathcal{C}_c(V(x, r))$, and hence for all non-negative $\psi \in \mathcal{C}_c(\mathrm{Geod}^\infty(\Omega) \cap V(x, r))$, since the measures we are talking about are supported on $\partial\Omega^2$.

For $\gamma \in \Gamma^{\mathrm{r1}}$, we let

$$\mathcal{M}^L_{x,3} = \delta e^{-\delta L} \sum_{\substack{\gamma\in\Gamma^{\mathrm{r1}}\\ d_\Omega(x,\gamma x)\le L}} \overline{\mathcal{L}}\gamma.$$

In other words, $\mathcal{M}^L_{x,3}$ is the push-forward by the Hopf parametrisation of $\|m_\Gamma\|^{-1}\nu^L_{x,3} \otimes \mathrm{d}s$ (note that for any L, the measure $\nu^L_{x,3}$ is concentrated on $\partial^2_{\mathrm{sse}}\Omega$, which identifies with its preimage in $\pi_{\mathrm{h}}^{-1}(\mathrm{Geod}^\infty\,\Omega)$, thus the Hopf parametrisation is well defined $\nu^L_{x,3} \otimes ds$-almost surely).

Let $\widehat{V}(x, r) \subset S\Omega$ be the set of vectors v with $(v^-, v^+) \in V(x, r)$. From the preceding, we obtain, for any non-negative $\psi \in C_c(\widehat{V}(x, r))$,

$$e^{-2\delta r}\|m_\Gamma\|^{-1}\int \psi\, \mathrm{d}m \le \lim_{L\to\infty}\int \psi\, \mathrm{d}\mathcal{M}^L_{x,3} \le \|m_\Gamma\|^{-1}\int \psi\, \mathrm{d}m. \tag{8.3.1}$$

We now relate $\mathcal{M}^L_{x,3}$ to $\mathcal{E}^L$, via a slight modification $\mathcal{M}^L$ (which is in fact independent of x.) Let $\ell(\gamma)$ denote the translation length of $\gamma \in \Gamma^{\mathrm{r1}}$, and define

$$\mathcal{M}^L = \delta e^{-\delta L} \sum_{\substack{\gamma\in\Gamma^{\mathrm{r1}}\\ \ell(\gamma)\le L}} \overline{\mathcal{L}}\gamma.$$

Making the elementary observation that

$$\ell(\gamma) \leq d_\Omega(x, \gamma x) \leq \ell(\gamma) + 2d_\Omega(x, g_\gamma),$$

we deduce that

$$\mathcal{M}^L_{x,3} \leq \mathcal{M}^L \leq e^{2\delta r}\mathcal{M}^{L+2r}_{x,3} \tag{8.3.2}$$

when restricted to $\hat{V}(x,r)$.

Let us check that

$$\mathcal{M}^L = \delta e^{-\delta L} \sum_{\substack{\gamma\in\Gamma^{\mathrm{pr1}} \\ \ell(\gamma)\leq L}} \left\lfloor \frac{L}{\ell(\gamma)} \right\rfloor \overline{\mathcal{L}}\gamma \tag{8.3.3}$$

Indeed, let $\gamma \in \Gamma$ be rank-one with $\ell(\gamma) \leq L$. Let A be the set of rank-one elements γ' with the same attracting/repelling pair as γ and with $\ell(\gamma') \leq L$ (they incidentally satisfy $\overline{\mathcal{L}}\gamma' = \overline{\mathcal{L}}\gamma$), let $\gamma_0 \in A$ be strongly primitive, and let $H \subset \Gamma$ be the group of elements that fix every point of the axis of γ. Then A is precisely $\{\gamma_0^k h : 1 \leq k \leq \frac{L}{\ell(\gamma_0)},\ h \in H\}$, and has cardinality $\left\lfloor \frac{L}{\ell(\gamma_0)} \right\rfloor \cdot \#H$, while $A \cap \Gamma^{\mathrm{pr1}} = \{\gamma_0 h : h \in H\}$ has cardinality $\#H$. Thus

$$\sum_{\gamma'\in A} \overline{\mathcal{L}}\gamma' = \sum_{\gamma'\in A\cap\Gamma^{\mathrm{pr1}}} \left\lfloor \frac{L}{\ell(\gamma')} \right\rfloor \overline{\mathcal{L}}\gamma'.$$

It is then clear that

$$\mathcal{M}^L \leq \mathcal{E}^L = \delta L e^{-\delta L} \sum_{\substack{\gamma\in\Gamma^{\mathrm{pr1}} \\ \ell(\gamma)\leq L}} \frac{1}{\ell(\gamma)} \overline{\mathcal{L}}\gamma. \tag{8.3.4}$$

To obtain a complementary inequality, consider $\varphi \in \mathcal{C}_c(\hat{V}(x,r))$ non-negative. Fix $L \geq 2e^r$. Observe that $\lfloor \frac{L}{\ell} \rfloor \geq 1 \geq \frac{e^{-r}L}{\ell}$ for any $e^{-r}L < \ell \leq L$, while $\frac{1}{\ell} \leq \lfloor \frac{e^{-r}L}{\ell} \rfloor$ for any $\ell \leq e^{-r}L$ (because $e^{-r}L \geq 2$). Therefore,

$$\begin{aligned}
\int \varphi \,\mathrm{d}\mathcal{M}^L &\geq \delta e^{-\delta L} \sum_{\substack{\gamma\in\Gamma^{\mathrm{pr1}} \\ e^{-r}L<\ell(\gamma)\leq L}} \frac{e^{-r}L}{\ell(\gamma)} \int \varphi \,\mathrm{d}\overline{\mathcal{L}}\gamma \\
&= e^{-r} \int \varphi \,\mathrm{d}\mathcal{E}^L - e^{-r}L\delta e^{-\delta L} \sum_{\substack{\gamma\in\Gamma^{\mathrm{pr1}} \\ \ell(\gamma)\leq e^{-r}L}} \frac{1}{\ell(\gamma)} \int \varphi \,\mathrm{d}\overline{\mathcal{L}}\gamma \\
&\geq e^{-r} \int \varphi \,\mathrm{d}\mathcal{E}^L - e^{-r}Le^{\delta(1-e^{-r})L} \int \varphi \,\mathrm{d}\mathcal{M}^{e^{-r}L}.
\end{aligned}$$

By (8.3.1) and (8.3.2), $\int \varphi \,\mathrm{d}\mathcal{M}^{e^{-r}L}$ is bounded when L tends to infinity. Hence we have

$$\limsup \int \varphi \,\mathrm{d}\mathcal{M}^L \geq e^{-r} \limsup \int \varphi \,\mathrm{d}\mathcal{E}^L.$$

Combining the last inequality with (8.3.1), (8.3.2) and (8.3.4) above, we establish that for all non-negative $\varphi \in \mathcal{C}_c(\hat{V}(x,r))$, as $L \to +\infty$,

$$\begin{aligned}
e^{-2\delta r}\|m_\Gamma\|^{-1} \int \varphi \,\mathrm{d}m &\leq \liminf \int \varphi \,\mathrm{d}\mathcal{E}^L \\
&\leq \limsup \int \varphi \,\mathrm{d}\mathcal{E}^L \leq e^{(2\delta+1)r}\|m_\Gamma\|^{-1} \int \varphi \,\mathrm{d}m.
\end{aligned}$$

We now let $x \in \Omega$ vary (but keep $r > 0$ fixed). Appealing to a locally-finite partition of unity subordinate to a covering of $S\Omega$ by open sets of the form $\widehat{V}(x, r)$ with $x \in \Omega$, we extend the validity of the preceding inequalities to all functions $\varphi \in C_c^+(S\Omega)$. It remains only to take $r \to 0$ to conclude the proof.

8.3.4 The number of periodic geodesics and of conjugacy classes

Finally, as in Section 7.4.5, let us relate the number of strongly primitive rank-one conjugacy classes, the number of rank-one conjugacy classes, and the number of rank-one periodic $(\phi_t)_t$-orbits; the arguments are based on the discussion in Section 3.4.2.

Observation 8.3.2. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set, and $\Gamma \subset \mathrm{Aut}(\Omega)$ a divergent discrete subgroup with $M = \Omega/\Gamma$ non-elementary rank-one. Suppose any Sullivan measure m_Γ of dimension δ_Γ is finite. Then for any non-negative function $f \in \mathcal{C}_c(T^1M)$, there exists $C > 0$ such that for any $T > 0$,*

$$\sum_{c\in\ \mathcal{G}_T^{\mathrm{r1}}} \int f \,\mathrm{d}\mathcal{L}c \leq \sum_{c\in[\Gamma]_T^{\mathrm{pr1}}} \int f \,\mathrm{d}\mathcal{L}c \leq C \sum_{c\in\mathcal{G}_T^{\mathrm{r1}}} \int f \,\mathrm{d}\mathcal{L}c,$$

and if Γ contains a torsion-free finite-index subgroup Γ', then we can take $C = [\Gamma' : \Gamma]$. Moreover,

$$Te^{-\delta_\Gamma T}\left(\sum_{c\in[\Gamma]_T^{\mathrm{r1}}} \mathcal{L}c - \sum_{c\in[\Gamma]_T^{\mathrm{pr1}}} \mathcal{L}c\right) \underset{T\to\infty}{\longrightarrow} 0.$$

Proof. This is a consequence of the discussion in Section 3.4.2. Indeed, let $f \in \mathcal{C}_c(T^1M)$ be non-negative, let $K \subset T^1\Omega$ be compact such that its projection in T^1M contains the support of f, and let $A \subset \Gamma$ be the set of elements that fix a point of K, which is finite. We saw that the number of strongly primitive rank-one conjugacy classes associated to a rank-one periodic $(\phi_t)_t$-orbit intersecting the support of f is less than $\#A$. This implies the first assertion with $C = \#A$.

We also saw that, for any $\ell > 0$, the number of conjugacy classes of length ℓ associated to a rank-one periodic $(\phi_t)_t$-orbit intersecting the support of f is less than $\#A$. Therefore, for any $T > 0$,

$$\sum_{c\in[\Gamma]_T^{\mathrm{pr1}}} \int f \,\mathrm{d}\mathcal{L}c \leq \sum_{c\in[\Gamma]_T^{\mathrm{r1}}} \int f \,\mathrm{d}\mathcal{L}c \leq \sum_{c\in[\Gamma]_T^{\mathrm{pr1}}} \int f \,\mathrm{d}\mathcal{L}c + \#A \sum_{k\geq 2} \sum_{c\in[\Gamma]_{\frac{T}{k}}^{\mathrm{pr1}}} \int f \,\mathrm{d}\mathcal{L}c.$$

Let $\epsilon > 0$ be such that $\sum_{c\in[\Gamma]_\epsilon^{\mathrm{pr1}}} \int f \,\mathrm{d}\mathcal{L}c = 0$. Since

$$\sum_{c\in[\Gamma]_T^{\mathrm{pr1}}} \int f \,\mathrm{d}\mathcal{L}c \leq \frac{2\#A}{\delta_\Gamma T} e^{\delta_\Gamma T} \int f \,\mathrm{d}\frac{m_\Gamma}{\|m_\Gamma\|}$$

for T large enough, we obtain

$$\begin{aligned}\sum_{c\in[\Gamma]_T^{\mathrm{r1}}} \int f \,\mathrm{d}\mathcal{L}c - \sum_{c\in[\Gamma]_T^{\mathrm{pr1}}} \int f \,\mathrm{d}\mathcal{L}c &\leq [\Gamma' : \Gamma]\frac{T}{\epsilon} \sum_{c\in[\Gamma]_{\frac{T}{2}}^{\mathrm{pr1}}} \int f \,\mathrm{d}\mathcal{L}c \\ &\leq \frac{4[\Gamma' : \Gamma]^2}{\delta_\Gamma T}\frac{T}{\epsilon} e^{\delta_\Gamma T/2} \int f \,\mathrm{d}\frac{m_\Gamma}{\|m_\Gamma\|}\end{aligned}$$

□

Proposition 8.3.3. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly convex open set, and $\Gamma \subset \mathrm{Aut}(\Omega)$ a divergent discrete subgroup with $M = \Omega/\Gamma$ non-elementary rank-one. Consider a Sullivan measure m_Γ of dimension δ_Γ, and suppose it is finite. Let $F \subset \Gamma$ be the core-fixing subgroup (see Section 3.4.2). Let $K \subset T^1 M_{\mathrm{bip}}$ be the set vectors of whose lifts $v \in T^1\Omega$ satisfy $\mathrm{Stab}_\Gamma(v) \neq F$. Let $A \subset \mathcal{G}^{\mathrm{r1}}$ be the set of rank-one periodic orbits contained in K. Then*

$$\delta_\Gamma T e^{-\delta_\Gamma T} \sum_{c \in A_T} \mathcal{L}c \underset{T\to\infty}{\longrightarrow} 0.$$

Suppose further that F is the centre of Γ. Then

$$\frac{\delta_\Gamma T}{\# F} e^{-\delta_\Gamma T} \sum_{c \in [\Gamma]_T^{\mathrm{pr1}}} \mathcal{L}c \underset{T\to\infty}{\longrightarrow} \frac{m_\Gamma}{\|m_\Gamma\|}.$$

If Γ is strongly irreducible, then F is trivial and

$$\delta_\Gamma T e^{-\delta_\Gamma T} \sum_{c \in [\Gamma]_T^{\mathrm{pr1}}} \mathcal{L}c \underset{T\to\infty}{\longrightarrow} \frac{m_\Gamma}{\|m_\Gamma\|}.$$

Proof. Let us only give a proof of the first point, since the other are elementary consequences of it and of the discussion in Section 3.4.2.

Consider a non-negative function $f \in \mathcal{C}_c(T^1M)$. Fix $\epsilon > 0$. Since $m_\Gamma(K) = 0$ (recall that $K \subset T^1 M_{\mathrm{bip}}$ has empty interior by Observation 3.4.1, and m_Γ is ergodic with support $T^1 M_{\mathrm{bip}}$ by Theorem 6.0.1), we can find a non-negative function $\chi \in \mathcal{C}_c(T^1M)$ such that $\chi \geq 1$ on $\mathrm{supp}(f) \cap K$ and $\int \chi f \,\mathrm{d}m_\Gamma \leq \epsilon \|m_\Gamma\|$. Then

$$\delta_\Gamma T e^{-\delta_\Gamma T} \sum_{c \in A_T} \mathcal{L}c(f) \leq \delta_\Gamma T e^{-\delta_\Gamma T} \sum_{c \in \mathcal{G}_T^{\mathrm{r1}}} \mathcal{L}c(\chi f) \underset{T\to\infty}{\longrightarrow} \frac{1}{\|m_\Gamma\|} \int \chi f \,\mathrm{d}m_\Gamma \leq \epsilon.$$

This holds for any $\epsilon > 0$, so $(\delta_\Gamma T e^{-\delta_\Gamma T} \sum_{c \in A_T} \mathcal{L}c(f))_T$ converges to zero. □

Chapter 9

Equidistribution for geometrically finite convex projective orbifolds of "negatively curved" type

In this chapter, which is extracted from an article [BZ21] in collaboration with F. Zhu, we restrict our attention to convex projective orbifolds $M = \Omega/\Gamma$ such that Ω is strictly convex with $\mathcal{C}^1$ boundary, which we call *of "negatively curved" type*. The reason for this is that we are interested in geometrically finite orbifolds, and that the notion of geometrical finiteness has not yet been introduced for general convex projective orbifolds.

Geometrical finiteness can be seen as a weakening of convex cocompactness, where the convex core may be non-compact, but we ask that it decomposes into a compact part and finitely many non-compact elementary pieces. The simplest example is that of hyperbolic surfaces with finite volume, where the non-compact parts are subsurfaces with boundary, called cusps, which are homeomorphic to a cylinder, and are isomorphic, as hyperbolic surfaces with boundary, to a quotient of a horosphere $\{z \in \mathbb{C} : \Im(z) \geq a\}$ of the Poincaré half-plane $\{z \in \mathbb{C} : \Im(z) > 0\}$, by the cyclic subgroup of $\mathrm{PSL}_2(\mathbb{R})$ generated by $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

This notion arose first in the setting of Kleinian groups, and has subsequently been extended to higher dimensions and the more general setting of pinched negative curvature in [Bow95]; the group-theoretic notion of relative hyperbolicity, see e.g. [Bow12], may be seen as an extension of geometrical finiteness to a more general δ-hyperbolic setting.

Before we recall the definition of geometrically finite convex projective orbifolds of "negatively curved" type, which is due to Crampon–Marquis [CM14a], let us state the two main results of this chapter.

First we prove that we can apply the results of the previous section to geometrically finite orbifolds.

Theorem 9.0.1. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly and strictly convex open set with $\mathcal{C}^1$ boundary and $\Gamma \subset \mathrm{Aut}(\Omega)$ a non-elementary discrete subgroup acting geometrically finitely on $\partial\Omega$. Then any Sullivan measure m_Γ on $T^1\Omega/\Gamma$ of dimension δ_Γ is finite.*

Second, following the strategy of Roblin [Rob03], we enhance the result of the previous chapter on the equidistribution of closed geodesic (Theorem 8.0.3). Note that any geodesic of a convex projective orbifold of "negatively curved" type is rank-one, so we simply denote by $\mathcal{G}_\ell$ the set of periodic geodesic with period at most $\ell \geq 0$.

Theorem 9.0.2. *In the setting of Theorem 9.0.1,*

$$\delta\ell e^{-\delta\ell}\sum_{c\in\mathcal{G}_\ell}\mathcal{L}c \underset{\ell\to\infty}{\longrightarrow} \frac{m_\Gamma}{\|m_\Gamma\|}$$

in $\mathcal{C}_b(T^1\Omega/\Gamma)^*$, the dual to the space of bounded continuous functions on $T^1\Omega/\Gamma$.

Theorem 9.0.1 was proved in the case of surfaces by Crampon in his PhD thesis [Cra11], where he also explained how this could be generalised to higher dimension. The generalisation to arbitrary dimension is due to F. Zhu [Zhua], under a stronger geometrical finiteness assumption than the one in Theorem 9.0.1: Γ was required to act geometrically finitely on Ω instead of $\partial\Omega$; the two notions are defined in the next section. The more general version Theorem 9.0.1, as the rest of this chapter (and Chapter 9) was obtained during a collaboration between Zhu and the author.

Zhu also proved Theorem 9.0.2 under the assumption that Γ acts geometrically finitely on Ω.

As in the previous chapter, one can derive from the previous equidistribution resul the following counting result.

Corollary 9.0.3. *In the setting of Theorem 9.0.2, $\#\mathcal{G}_\Gamma(\ell) \underset{\ell\to\infty}{\sim} \frac{e^{\delta\ell}}{\delta\ell}$.*

Proof. The integral of the constant function 1 against the measure $\delta\ell e^{-\delta\ell}\sum_{c\in\mathcal{G}_\Gamma(\ell)}\mathcal{L}c$ is exactly $\delta\ell e^{-\delta\ell}\#\mathcal{G}_\Gamma(\ell)$. From Theorem 9.0.2, this integral converges to 1 as $\ell\to\infty$. □

Note that holonomies of geometrically finite strictly convex projective structures, in the sense of [CM14a], are not Anosov unless they are convex cocompact (see Fact 9.3.1, and [DGKa, Th. 1.4] or [Zim20, Th. 1.27]) . However they satisfy a relative version of the Anosov condition (see [KL18] or [Zhub]).

9.1 Geometrical finiteness

The faster way to define geometrical finiteness consists in generalising the characterisation of convex cocompactness in terms of the conical limit set, which we mentioned in Section 2.3.5. For convex projective orbifolds $M=\Omega/\Gamma$ of "negatively curved" type, the orbital limit set $\Lambda_\Gamma^{\mathrm{orb}}$ is simply the set of accumulation points in $\partial\Omega$ of *one* Γ-orbit in Ω, and M has a compact convex core if and only if $\Lambda_\Gamma^{\mathrm{orb}}=\Lambda_\Gamma^{\mathrm{con}}$. geometrically finite orbifolds allow non-conical limit points in $\Lambda_\Gamma^{\mathrm{orb}}$, be ask that satisfy good properties in order to have some control of the non-compact parts of the quotient of the convex hull of the limit set by Γ.

Let $\Omega\subset \mathrm{P}(\mathbf{V})$ be a properly and strictly convex open set with $\mathcal{C}^1$ boundary. Before we state the definition of geometrically finite subgroups of $\mathrm{Aut}(\Omega)$, we need to recall the classification of elements and of elementary discrete subgroups of $\mathrm{Aut}(\Omega)$, due to Crampon–Marquis [CM14a, Th. 3.3 & §3.5]. Any automorphism $g\in\mathrm{Aut}(\Omega)$ is

- either *elliptic*: it fixes a point of Ω;
- or *parabolic*: it fixes a unique point of $\overline{\Omega}$, which is in the boundary $\partial\Omega$;
- or *hyperbolic*: it fixes exactly two points of $\overline{\Omega}$, which are in the boundary.

Observe that g is rank-one if and only if it is hyperbolic. Any discrete subgroup $\Gamma\subset\mathrm{Aut}(\Omega)$ is

- either *elliptic*: $\#\Lambda_\Gamma^{\mathrm{orb}} = 0$, and Γ is finite, fixes a point of Ω, and consists of elliptic elements;
- or *parabolic*: $\#\Lambda_\Gamma^{\mathrm{orb}} = 1$, and Γ fixes a unique point of $\partial\Omega$ (the point of Λ_Γ), consists of elliptic and (at least one) parabolic elements, and act properly discontinuously on $\partial\Omega \smallsetminus \Lambda_\Gamma$ (see Section 9.2 for more advanced properties on these groups);
- or *elementary hyperbolic*: $\#\Lambda_\Gamma^{\mathrm{orb}} = 2$, and Γ consists of elliptic and (at least one) hyperbolic elements, and any hyperbolic element generates a finite-index subgroup;
- or *non-elementary*: $\#\Lambda_\Gamma^{\mathrm{orb}} = \infty$, and then Γ is rank-one and $\Lambda_\Gamma^{\mathrm{orb}} = \Lambda_\Gamma^{\mathrm{prox}}$, which implies that Γ acts minimally on Λ_Γ which is perfect and hence uncountable, and Γ contains a non-abelian free subgroup made of hyperbolic elements.

Definition 9.1.1. Let Ω be a properly and strictly convex open set with $\mathcal{C}^1$ boundary and $\Gamma \subset \mathrm{Aut}(\Omega)$ a discrete subgroup.

$\xi \in \Lambda_\Gamma^{\mathrm{orb}}$ is a *bounded parabolic point* if the stabiliser $\mathrm{Stab}_\Gamma(\xi)$ is parabolic and acts cocompactly on $\Lambda_\Gamma^{\mathrm{orb}} \smallsetminus \{\xi\}$. We say the action of Γ on $\partial\Omega$ is *geometrically finite* if every point in $\Lambda_\Gamma^{\mathrm{orb}}$ is either conical or bounded parabolic.

$\xi \in \Lambda_\Gamma^{\mathrm{orb}}$ is a *uniformly bounded parabolic point* if the stabiliser $\mathrm{Stab}_\Gamma(\xi)$ is parabolic and acts cocompactly on the closure of the set of points $\xi \in \partial\Omega \smallsetminus \{\xi\}$ such that $[\xi, \eta]$ intersects $\mathcal{C}_\Omega^{\mathrm{cor}}(\Gamma)$. We say the action of Γ on Ω is *geometrically finite* if every point in $\Lambda_\Gamma^{\mathrm{orb}}$ is either conical or uniformly bounded parabolic.

Observe that if Γ acts geometrically finitely on Ω, then it also does on $\partial\Omega$; the converse is not true (see [CM14a, §10.3]). We work in the present article with groups Γ that act geometrically finitely on $\partial\Omega$.

Crampon–Marquis gave a more concrete description [CM14a, Th. 1.2] of groups that acts geometrically finitely on Ω, in terms of geometrical and topological properties of the quotient Ω/Γ. For example, they proved that Γ acts geometrically finitely on Ω if and only if $\mathcal{C}_\Omega^{\mathrm{cor}}(\Gamma)$ admits a decomposition into a compact part and finitely many disjoint noncompact but well-understood parts (see the standard parabolic regions in [CM14a, Def. 7.22]), and this property implies that Ω/Γ is tame (i.e. the interior of a compact orbifold with boundary), that Γ is hyperbolic relative to the maximal parabolic subgroups, and that Γ is finitely presented.

We will need results of a similar nature for group acting geometrically finitely on the boundary $\partial\Omega$, although we establish only partial results in this direction here. More specifically, we will decompose into compact and noncompact parts a subset of $\mathcal{C}_\Omega^{\mathrm{cor}}(\Gamma)$, which is the quotient by Γ of the union of lines between points in $\Lambda_\Gamma^{\mathrm{orb}}$ (see Propositions 9.3.4 and Fact 9.3.1).

9.2 Parabolic groups

In this section we establish two useful properties of discrete parabolic groups of automorphisms of properly and strictly convex open set with $\mathcal{C}^1$ boundary:

Proposition 9.2.1. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly and strictly convex open set with $\mathcal{C}^1$ boundary. Then any discrete parabolic subgroup of* $\mathrm{Aut}(\Omega)$ *is finitely presented and divergent.*

The divergent property had already been established by Crampon–Marquis in the particular case where the parabolic subgroup is conjugate into $\mathrm{SO}(1, \mathtt{d})$, and they gave moreover a formula to compute the critical exponent.

Lemma 9.2.2 ([CM14b, Lem. 9.8]). *In the setting of Proposition 9.2.1, if P is conjugate into $\mathrm{SO}(1,\mathtt{d})$, then it contains $\mathbb{Z}^r$ as a finite-index subgroup for some $r \leq \mathtt{d}$, it is divergent, and $\delta_P = \frac{r}{2}$.*

9.2.1 The Zariski-closure of parabolic groups

The following important result, combined with [Rag72, Cor. 6.14], implies that parabolic groups are finitely presented; we will also use it to establish divergence of parabolic groups.

Fact 9.2.3 ([CM14a, Prop. 7.1 & Lem. 7.6]). *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly and strictly convex open set with $\mathcal{C}^1$ boundary, and $P \subset \mathrm{Aut}(\Omega)$ a discrete parabolic subgroup.*

Then P is a cocompact lattice of its Zariski-closure $\mathcal{N}$, which is nilpotent and equal to the direct product $K \times U$, where $K \subset \mathcal{N}$ is Zariski-closed, compact, abelian, and made of semi-simple elements, $U \subset \mathcal{N}$ is Zariski-closed and unipotent, and the map $(k,u) \mapsto ku$ from $K \times U$ to $\mathcal{N}$ is an isomorphism.

9.2.2 Unipotent groups are divergent

The idea to prove Proposition 9.2.1 is to use Fact 9.2.3 and prove that the Zariski-closure of our parabolic group is divergent in the sense of Section 2.3.8. Let us now prove that algebraic unipotent groups are divergent.

Lemma 9.2.4. *Any unipotent Zariski-closed subgroup of $\mathrm{SL}(\mathbb{R}^{\mathtt{d}+1})$ is divergent.*

Proof. Let $U \subset \mathrm{SL}(\mathbb{R}^{\mathtt{d}+1})$ be a Zariski-closed unipotent subgroup; denote by $\mathfrak{u}$ its Lie algebra. The exponential map $\exp : \mathfrak{u} \to U$ is a diffeomorphism such that the entries of $\exp(x)$ are polynomials in the entries of $x \in \mathfrak{u}$ (see [Bor66, §4]). Furthermore, the push-forward by exp of any Lebesgue measure on $\mathfrak{u}$ (let us fix one) is a Haar measure on U (see [CG90, Th. 1.2.10]).

Set $P(x) := \|\exp(x)\|^2 \cdot \|\exp(-x)\|^2 \geq 1$ for any $x \in \mathfrak{u}$, and observe that P is a polynomial on $\mathfrak{u}$, and is proper, in the sense that $P(x) \to \infty$ as $\|x\| \to \infty$. By definition, δ_U is the supremum of the $s \geq 0$ such that $\int_{\mathfrak{u}} P^{-s/4}$ diverges, where we integrate against the Lebesgue measure. To conclude the proof, it is enough to show that $\int_{\mathfrak{u}} P^{-\delta_U/4}$ diverges. This is a consequence of Lemma 9.2.5 below. □

Lemma 9.2.5. *Let $n \in \mathbb{N}_{\geq 1}$ and P a proper polynomial on n variables, with real coefficients, and such that $P \geq 1$ on $\mathbb{R}^n$. Let δ be the supremum of the set of $s \geq 0$ such that $\int_{\mathbb{R}^n} P^{-s}$ diverges. Then δ is finite and $\int_{\mathbb{R}^n} P^{-\delta}$ diverges.*

Proof. Denote by μ the Lebesgue measure on $\mathbb{R}^n$. For any $s > 0$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} P^{-s}(x)\,\mathrm{d}\mu(x) &= \int_{x\in\mathbb{R}^n}\int_{t=0}^{P^{-s}(x)} \mathrm{d}t\,\mathrm{d}\mu(x)\\ &= \int_{t=0}^{1} \mu(P^{-s} \geq t)\,\mathrm{d}t\\ &= s\int_{u\geq 1} \mu(P \leq u)u^{-s-1}\,\mathrm{d}u\end{aligned}$$

By [BO12, Prop. 7.2], there exist $a > 0$ and $r \in \mathbb{Q}_{>0}$ and $k \in \mathbb{N}$ such that $\mu(P \leq u)$ is equivalent to $au^r \log(u)^k$ when u goes to infinity. This concludes the proof since $\int_{u\geq 1} u^{r-s-1}\log(u)^k\,\mathrm{d}u$ is finite for any $s > r$, and is equal to $\lim_{u\to\infty} \frac{\log(u)^{k+1}}{k+1} = \infty$ for $s = r$. □

9.2.3 Proof of Proposition 9.2.1

Let $P \leq \mathrm{Aut}(\Omega)$ be a discrete parabolic subgroup. By Fact 9.2.3, the group P is a uniform lattice of its (nilpotent) Zariski-closure $\mathcal{N} = K \times U$, where K is compact, and U is Zariski-closed and unipotent. The group P is finitely presented by [Rag72, Cor. 6.14] ; let us prove that it is divergent.

The restriction to P of the projection onto U has finite kernel, and its image P' is a uniform lattice of U. By Lemma 9.2.4, the group U is divergent, and so are P' and P. Indeed, denote by μ a Haar measure on U and fix a relatively compact measurable subset $\mathcal{D} \subset U$ such that $(p, g) \in P' \times \mathcal{D} \mapsto pg \in U$ is a bijection. Then for any $s \geq 0$:

$$\int_U (\|g\| \cdot \|g^{-1}\|)^{-s} \, \mathrm{d}\mu(g) = \sum_{p \in P'} \int_{\mathcal{D}} (\|pg\| \cdot \|pg^{-1}\|)^{-s} \, \mathrm{d}\mu(g).$$

Set $C := \max\{\|g\| \cdot \|g^{-1}\| : g \in \mathcal{D}\}$, which is finite since $\mathcal{D}$ is relatively compact. The norm $\| \cdot \|$ we have chosen is submultiplicative, therefore

$$C^{-s} \sum_{p \in P'} (\|p\| \cdot \|p^{-1}\|)^{-s} \leq \frac{1}{\mu(\mathcal{D})} \int_U (\|g\| \cdot \|g^{-1}\|)^{-s} \, \mathrm{d}\mu(g) \leq C^s \sum_{p \in P'} (\|p\| \cdot \|p^{-1}\|)^{-s}.$$

These estimates conclude the proof.

9.3 Finiteness properties for boundary geometrically finite subgroups

The proofs of Theorems 9.0.1 and 9.0.2 for the general case of subgroups Γ acting geometrically finitely on $\partial\Omega$, but not geometrically finitely on Ω, will require some finiteness results for such subgroups. We will establish these in the present section.

9.3.1 Geometrically finite groups are relatively hyperbolic with respect to their maximal parabolic subgroups

The following result involves the notion of relatively hyperbolic group, and the work of several people on this topic. We do not recall precisely here the definition of relative hyperbolicity because it will not be used elsewhere; for details we refer to [Bow12, Yam06, Osi06, Hru10, CM14a] (the following proof simply consists in applying the results of these papers). The two consequences of the following result that we will need are the facts that geometrically finite groups are finitely generated and have finitely many orbits of parabolic points; these can proven by hand by using Proposition 9.3.4 (which does not use this section).

Fact 9.3.1 ([CM14a, Prop. 9.10]). *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly and strictly convex open set with $\mathcal{C}^1$ boundary and $\Gamma \leq \mathrm{Aut}(\Omega)$ a discrete subgroup that acts geometrically finitely on $\partial\Omega$. Then there are finitely many Γ-orbits of parabolic points, Γ is hyperbolic relative to its maximal parabolic subgroups, and Γ is finitely presented.*

Proof. We can assume that Γ is non-elementary. Since Γ is a discrete subgroup of $\mathrm{Aut}(\Omega)$, it is countable.

By Yaman's criterion [Yam06], Γ is hyperbolic relative to its maximal parabolic subgroups. In particular, it has finitely many classes of maximal parabolic subgroups (see also [Tuk98, Th. 1B].)

By Proposition 9.2.1, the maximal parabolic subgroups are finitely presented. Hence, by [Osi06, Cor. 2.4], which states that relatively hyperbolic groups, as defined in [Osi06], inherit finiteness properties from their peripheral subgroups, and [Hru10, Th. 5.1], which proves that the equivalence of several characterisations of countable relatively hyperbolic groups, including the definitions used in [Osi06] and in [CM14a], Γ is also finitely-presented. □

9.3.2 Sufficiently small horoballs at parabolic point are disjoint

The following elementary observation gives a necessary criterion to check whether two horoballs are disjoint.

Lemma 9.3.2. *Given a properly convex open set $\Omega \subset \mathbf{P}(\mathbf{V})$ and two distinct points $\xi, \eta \in \overline{\Omega}^{\mathtt{h}}$, if x is in $[\pi_{\mathtt{h}}(\xi), \pi_{\mathtt{h}}(\eta)]$ and Ω, then $\mathcal{H}_\xi(x)$ and $\mathcal{H}_\eta(x)$ are disjoint. In particular, any two horoballs $\mathcal{H}_\xi$ and $\mathcal{H}_\eta$, respectively centred at ξ and η, have non-empty intersection if and only if $(\xi\eta) \cap \mathcal{H}_\xi \cap \mathcal{H}_\eta \neq \varnothing$.*

Proof. Suppose by contradiction there is $y \in \mathcal{H}_\xi(x) \cap \mathcal{H}_\eta(x)$. Then we can find $(\xi', x', \eta') \in \Omega^3$ close enough to (ξ, x, η) such that $\mathtt{b}_{\xi'}(x', y) > 0$ and $\mathtt{b}_{\eta'}(x', y) > 0$ and $x' \in [\xi', \eta']$. This leads to the following contradiction:

$$\begin{aligned} d_\Omega(\xi', \eta') &= d_\Omega(\xi', x') + d_\Omega(x', \eta') \\ &= \mathtt{b}_{\xi'}(x', y) + d_\Omega(\xi', y) + \mathtt{b}_{\eta'}(x', y) + d_\Omega(y, \eta') \\ &> d_\Omega(\xi', \eta'). \end{aligned}$$

□

Lemma 9.3.3. *Let $\Omega \subset \mathbf{P}(\mathbf{V})$ be a properly and strictly convex open set with $\mathcal{C}^1$ boundary and $\Gamma \leq \mathrm{Aut}(\Omega)$ a discrete subgroup, p, p' two bounded parabolic points of the orbital limit set, and $\mathcal{H}'$ a horoball centred at p'. Then there exists a horoball $\mathcal{H}$ centred at p such that for any $\gamma \in \Gamma$, either $\mathcal{H}' \cap \gamma\mathcal{H} = \varnothing$ or $\gamma p = p'$.*

Proof. Suppose by contradiction that we can find a decreasing sequence of horoballs $(\mathcal{H}_n)_n$ centred at and converging to p, and a sequence of elements $(\gamma_n)_{n\in\mathbb{N}} \subset \Gamma$ such that $\mathcal{H}' \cap \gamma_n\mathcal{H}_n \neq \varnothing$ and $\gamma_n p \neq p'$. From Lemma 9.3.2, the intersection $[\gamma_n p, p'] \cap \mathcal{H} \cap \gamma_n \mathcal{H}_n$ is nonempty (and compact in Ω) for all n, and we can consider its closest point x_n to $\gamma_n p$, which belongs to $\partial\mathcal{H}'$. Similarly, for any n, we consider $y_n \in [p, \gamma_n^{-1}p'] \cap \partial\mathcal{H} \cap \gamma_n^{-1}\mathcal{H}'$.

Since p and p' are bounded parabolic, up to replacing $(\gamma_n)_n$ by a sequence of the form $(g_n\gamma_n h_n)_n$, where $(g_n)_n \subset \mathrm{Stab}_\Gamma(p')$ and $(h_n)_n \subset \mathrm{Stab}_\Gamma(p)$, we can assume that $(x_n)_n$ and $(y_n)_n$ stay in a compact subset of Ω; moreover, up to extraction, we can assume that these sequences converge respectively to x and $y \in \Omega$.

Let us prove that $\mathrm{Stab}_\Gamma(p)$ is finite, which will contradict the fact that p is bounded parabolic, and hence conclude the proof. Fix $\gamma \in \mathrm{Stab}_\Gamma(p)$. Observe that, $(d_\Omega(\gamma z_n, z_n))_n$ tends to zero as n goes to infinity, where $z_n = \gamma_n^{-1}x_n$. Indeed for each n let $v_n \in S_{y_n}\Omega$ be such that $v_n^+ = p$, and $t_n = d_\Omega(y_n, z_n)$, so that $\pi g^{t_n} v_n = z_n$. By construction $(t_n)_n$ diverges, hence by Lemma 2.1.4,

$$\limsup_{n\to\infty} d_\Omega(z_n, \gamma z_n) \leq \lim_{t\to\infty} \limsup_{n\to\infty} d_\Omega(\pi g^t v_n, \pi g^t \gamma v_n) = \lim_{t\to\infty} d_\Omega(\pi g^t v, \pi g^t \gamma v) = 0,$$

where $v \in S_y\Omega$ is such that $v^+ = p$. As a consequence,

$$\begin{aligned} d_\Omega(\gamma_n\gamma\gamma_n^{-1}x, x) &\leq d_\Omega(\gamma_n\gamma\gamma_n^{-1}x, \gamma_n\gamma\gamma_n^{-1}x_n) + d_\Omega(\gamma_n\gamma\gamma_n^{-1}x_n, x_n) + d_\Omega(x_n, x) \\ &\leq 2d_\Omega(x_n, x) + d_\Omega(\gamma z_n, z_n) \underset{n\to\infty}{\longrightarrow} 0 \end{aligned}$$

and $(\gamma_n\gamma\gamma_n^{-1}x)_n$ converges to x, hence $\gamma_n\gamma\gamma_n^{-1}$ stabilises x for n large enough by proper discontinuity. Thus $\mathrm{Stab}_\Gamma(p)$ is no larger than $\mathrm{Stab}_\Gamma(x)$, which is finite. □

9.3.3 The non-cuspidal part of the biproximal unit tangent bundle is compact

Proposition 9.3.4. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly and strictly convex open set with $\mathcal{C}^1$ boundary and $\Gamma \leq \mathrm{Aut}(\Omega)$ a discrete subgroup that acts geometrically finitely on $\partial\Omega$. For each parabolic point $\xi \in \Lambda_\Gamma$, fix an open horoball $\mathcal{H}_\xi$ centred at ξ such that $\mathcal{H}_{\gamma\xi} = \gamma\mathcal{H}_\xi$ for each $\gamma \in \Gamma$. Then the set of unit tangent vectors of $T^1 M_{\mathrm{bip}}$ whose footpoint does not belong to the projection of a horoball is compact.*

Proof. Pick $o \in \Omega$ and let $\mathcal{D} := \{x \in \Omega : d_\Omega(x, \gamma o) \geq d_\Omega(x, o)\ \forall \gamma \in \Gamma\}$ be the Dirichlet domain associated to o and Γ. It is enough to show that the set

$$A := \mathcal{D} \cap \bigcup_{\xi,\eta \in \Lambda_\Gamma} (\xi\eta) \smallsetminus \bigcup_{\xi \text{ parabolic}} \mathcal{H}_\xi \subset \Omega$$

is compact.

Assume that this is not the case, so that there is a sequence $(x_n)_{n\in\mathbb{N}} \subset A$ that converges to some $\xi \in \partial\Omega$. Observe that $\xi \in \Lambda_\Gamma$ since $(x_n)_n$ is contained in the convex hull of the limit set. Γ acts geometrically finitely on $\partial\Omega$ so ξ is either conical or bounded parabolic.

If ξ were conical, there would exists a sequence $(\gamma_k)_{k\in\mathbb{N}} \subset \Gamma$ such that $(\gamma_k o)_k$ converges to ξ while staying at bounded distance from $[o\xi)$, and

$$\begin{aligned}\infty = \lim_{k\to\infty} \mathsf{b}_\xi(o, \gamma_k o) &\leq \lim_{k\to\infty}\lim_{n\to\infty} \mathsf{b}_{x_n}(o, \gamma_k o) \\ &= \lim_{k\to\infty}\lim_{n\to\infty} d_\Omega(x_n, o) - d_\Omega(x_n, \gamma_k o) \leq 0,\end{aligned}$$

which is absurd! Thus ξ is bounded parabolic.

By the definition of A, we can find sequences $(\xi_n)_n$ and $(\eta_n)_n$ in Λ_Γ such that $x_n \in [\xi_n, \eta_n]$ for each n. Since $x_n \notin \mathcal{H}_\xi$ and $\mathcal{H}_\xi$ is convex, up to exchanging ξ_n and η_n we can assume that $[x_n, \eta_n] \cap \mathcal{H}_\xi$ is empty for all n. Up to extraction, we can assume that $(\eta_n)_n$ converges to $\eta \in \Lambda_\Gamma$: if η were different from ξ, then $[\xi, \eta]$ would intersect $\mathcal{H}_\xi$ nontrivially (because Ω is strictly convex and $\mathcal{H}_\xi$ is $\mathcal{C}^1$), and thus so would $[x_n, \eta_n]$ for n large enough; since $[x_n, \eta_n] \cap \mathcal{H}_\xi = \varnothing$ for all n, hence $\eta = \xi$.

Since ξ is bounded parabolic, we can find a diverging sequence $(\gamma_n)_{n\in\mathbb{N}} \subset \Gamma$ of parabolic elements fixing ξ such that, up to extraction, $(\gamma_n\eta_n)_n$ converges to some $\eta' \neq \xi$. Up to extraction, we can also assume that $(\gamma_n x_n)_n$ converges to some $x \in \overline{\Omega}$, which is different from ξ since, as before, $[\gamma_n x_n, \gamma_n\eta_n] \cap \mathcal{H}_\xi = \varnothing$ and so taking the limit as $n \to \infty$, $[x, \eta'] \cap \mathcal{H}_\xi$ is also empty. But then

$$\begin{aligned}\infty &= \lim_{n\to\infty} d_\Omega(o, \gamma_n o) \\ &\leq 2 \lim_{n\to\infty} \langle \gamma_n x_n, \gamma_n o\rangle_o \leq 2\langle x, \xi\rangle_o < \infty,\end{aligned}$$

which is a contradiction. □

9.4 Finiteness of a Sullivan measure

Let $M = \Omega/\Gamma$ be a geometrically finite convex projective orbifold of "negatively curved" type, and m_Γ a Sullivan measure of dimension δ_Γ. We would like to prove that m_Γ is finite. Since the support of the Sullivan measure m_Γ outside of the cusp neighbourhoods is compact (and since m_Γ is Radon), it suffices to check that the m_Γ-measure of (the unit tangent bundle over) each cusp neighbourhood is finite.

To obtain estimates in the cusp neighbourhoods, it will be useful to have the two lemmas below, the first establishing a gap between the critical exponent δ_Γ and the critical exponent of any parabolic subgroup, and the second showing that the Patterson–Sullivan measures have no atoms.

The proofs of Theorem 9.0.1 and of auxiliary results such as Proposition 9.4.2 and their consequences also take inspiration from those of analogous results of Dal'bo–Otal–Peigné in [DOP00], which characterise geometrically finite Riemannian manifolds of pinched negative curvature with finite Sullivan measure in terms of Poincaré series.

9.4.1 Parabolic gaps

Lemma 9.4.1. *Let Ω be a properly and strictly convex open set with $\mathcal{C}^1$ boundary. For any non-elementary discrete subgroup $\Gamma \subset \mathrm{Aut}(\Omega)$ containing a parabolic subgroup P, we have $\delta_\Gamma > \delta_P$.*

Proof. It follows from the definition of the critical exponent that $\delta_\Gamma \geq \delta_P$, and it suffices to show that the inequality is strict. Since Γ is non-elementary, we can use a ping-pong argument to find a free product subgroup $\langle h\rangle * P \leq \Gamma$ where $h \in \Gamma$ is a hyperbolic element. In particular, Γ contains all the distinct elements $g = h^{m_1}p_1 \cdots h^{m_k}p_k$ for $k \geq 1$, $n_k \in \mathbb{Z}_{\neq 0}$, $p_i \in P \smallsetminus \{\mathrm{id}\}$. Fix $x \in \Omega$. Then we have a lower bound for the Poincaré series

$$\sum_{\gamma\in\Gamma} e^{-sd_\Omega(x,\gamma x)} \geq \sum_{k\geq 1} \sum_{\substack{m_1,\dots,m_k\\ p_1,\dots,p_k}} e^{-s\cdot d_\Omega(x,h^{m_1}p_1\cdots h^{m_k}p_k x)}$$

and applying the triangle inequality

$$d_\Omega(x, h^{m_1}p_1\cdots h^{m_k}p_k x) \leq \sum_{i=1}^{k} d_\Omega(x, h^{m_i}x) + d_\Omega(x, p_i x)$$

to the right-hand side we obtain

$$\sum_{\gamma\in\Gamma} e^{-sd_\Omega(x,\gamma x)} \geq \sum_{k\geq 1} \left(\left(\sum_{n\in\mathbb{Z}\smallsetminus\{0\}} e^{-s\cdot d_\Omega(x,h^n x)}\right)\left(\sum_{p\in P\smallsetminus\{\mathrm{id}\}} e^{-s\cdot d_\Omega(x,px)}\right)\right)^k$$

$\sum_{p\in P} e^{-s\cdot d_\Omega(x,px)}$ converges for any $s > \delta_P$ and diverges at $s = \delta_P$ by Proposition 9.2.1. Hence there exists $s_0 > \delta_P$ such that

$$\left(\sum_{n\in\mathbb{Z}\smallsetminus\{0\}} e^{-s_0\cdot d_\Omega(x,h^n x)}\right)\left(\sum_{p\in P\smallsetminus\{\mathrm{id}\}} e^{-s_0\cdot d_\Omega(x,px)}\right) \geq 1,$$

so that $\sum_{\gamma\in\Gamma} e^{-s_0 d_\Omega(x,\gamma x)}$ diverges. Then $\delta_\Gamma \geq s_0 > \delta_P$. □

9.4.2 No atoms

Proposition 9.4.2 (cf. [Cra11, Prop 4.3.5])**.** *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly and strictly convex open set with $\mathcal{C}^1$ boundary, and $\Gamma \leq \mathrm{Aut}(\Omega)$ be a non-elementary discrete subgroup acting geometrically finitely on $\partial\Omega$. Then any δ_Γ-conformal density $(\mu_x)_{x\in\Omega}$ has no atoms, and Γ is divergent.*

Proof. Fix $x \in \Omega$. By Theorem 6.0.1, it is enough to prove the proposition for one specific δ_Γ-conformal density, and according to Fact 1.4.1, we can choose an accumulation point of the sequence $(\mu_{x,s})_s$ as s tends to δ_Γ, where

$$\mu_{x,s} = \frac{1}{f(s)} \sum_{\gamma \in \Gamma} \chi(d_\Omega(x, \gamma x)) e^{-s d_\Omega(x,\gamma x)} \mathcal{D}_{\gamma x},$$

with $f(s) = \sum_\gamma \chi(d_\Omega(x,\gamma x)) e^{-sd_\Omega(x,\gamma x)}$ and $\chi : [0,\infty) \to (0,\infty)$ is a non-decreasing function such that $f(\delta_\Gamma) = \infty$ and such that for any $\epsilon > 0$ there exists $R > 0$ with $\chi(r+t) \leq e^{\epsilon t}\chi(r)$ for $r \geq R$ and $r \geq 0$.

Consider $\xi \in \partial\Omega$. If ξ is conical, then $\mu_x(\{\xi\}) = 0$ by Proposition 6.3.7. We assume that ξ is not conical, hence that it is bounded parabolic. To show that $\mu_x(\{\xi\}) = 0$ it suffices to find a family $(V_n)_{n\in\mathbb{N}}$ of neighbourhoods of ξ such that $(\limsup_{s\searrow\delta_\Gamma} \mu_{x,s}(V_n))_n$ converges to zero.

By Lemma 9.3.3, we can find an open horoball $\mathcal{H}$ centred at ξ and which contains no point of the orbit $\Gamma \cdot x$. Since ξ is bounded parabolic, we can find a compact subset K of $\Lambda_\Gamma \smallsetminus \{\xi\}$ such that $P \cdot K = \Lambda_\Gamma \smallsetminus \{\xi\}$. Consider the compact set $K' := \{y \in \overline{\Omega} : [y,\xi] \cap H = \varnothing \ \forall \xi \in K\}$, which does not contain ξ, and observe that $\Gamma \cdot x \subset P \cdot K'$.

Enumerate $P = \{p_1, p_2, \dots\}$. The set $V_n := \overline{\Omega} \smallsetminus p_1 K' \cup \dots \cup p_n K'$ is a neighbourhood of ξ in $\overline{\Omega}$ for each $n \geq 1$; moreover $V_n \cap \Gamma \cdot x = \{p_k \gamma x : k > n, \gamma \in \Gamma'\}$, where $\Gamma' := \{\gamma \in \Gamma : \gamma x \in K'\}$. Thus

$$\mu_{x,s}(V_n) \leq \frac{1}{f(s)} \sum_{k>n} \sum_{\gamma\in\Gamma'} \chi(d_\Omega(x, p_k\gamma x)) e^{-s\cdot d_\Omega(x,p_k\gamma x)}$$

for each $n \geq 1$ and $s > \delta_\Gamma$.

Let us estimate $d_\Omega(x, p_k\gamma x) = d_\Omega(p_k^{-1}x, \gamma x)$ for $\gamma \in \Gamma'$ and for k large (independently of γ). Take a compact neighbourhood K'' of ξ in $\overline{\Omega}$ which is disjoint from K'. By strict convexity of Ω, we can find $R > 0$ such that $(\xi\eta) \cap B_\Omega(x,R)$ is non-empty for every $\xi \in K''$ and $\eta \in K'$; in particular, $\langle \xi, \eta\rangle_x \leq R$. Since $(p_n x)_n$ converges to ξ, there exists N such that $p_n x \in K''$ for every $n \geq N$. As a consequence, for all $k \geq N$ and $\gamma \in \Gamma'$,

$$\begin{aligned} d_\Omega(p_k^{-1}x, \gamma x) &= d_\Omega(x, p_k x) + d_\Omega(x, \gamma x) - 2\langle p_k^{-1}x, \gamma x\rangle_x \\ &\geq d_\Omega(x, p_k x) + d_\Omega(x,\gamma x) - 2R. \end{aligned}$$

Therefore we obtain, because χ is a non-decreasing function,

$$\mu_{x,s}(V_n) \leq \frac{e^{2sR'}}{f(s)} \sum_{k>n} e^{-sd_\Omega(x,p_k x)} \sum_{\gamma\in\Gamma'} \chi\left(d_\Omega(x,p_k x) + d_\Omega(x,\gamma x)\right) e^{-sd_\Omega(x,\gamma x)},$$

for all $s > \delta_\Gamma$ and $n \geq N$. By Lemma 9.4.1, $\epsilon := \frac{1}{2}(\delta_\Gamma - \delta_P) > 0$; by definition of χ, there is $C > 0$ such that $\chi(s+t) \leq Ce^{\epsilon s}\chi(t)$ for all $t, s \geq 0$. Hence

$$\mu_{x,s}(V_n) \leq Ce^{2sR'} \sum_{k>n} e^{-(s-\epsilon)d_\Omega(x,p_k x)},$$

for any $s > \delta_\Gamma$. Thus,

$$\mu_x(\{\xi\}) \leq \liminf_{n\to\infty} \limsup_{s\to\delta_\Gamma} \mu_{x,s}(V_n) \leq \liminf_{n\to\infty} Ce^{2\delta_\Gamma R'} \sum_{k>n} e^{-(\delta_\Gamma-\epsilon)d_\Omega(x,p_k x)} = 0. \qquad \square$$

9.4.3 Proof of Theorem 9.0.1

Thanks to Remark 6.2.3, we may assume that Γ is torsion-free.

Fix a Γ-invariant family of disjoint horoballs centred at the parabolic points of Λ_Γ. By Proposition 9.3.4, we have a decomposition of $T^1 M_{\text{bip}}$ into a compact core and a finite number of "cusp" neighbourhoods, which are the quotients of our fixed horoballs centred at the parabolic points. To prove the theorem, it suffices to show that the associated measure of each cusp neighbourhood is finite.

Let $P \subset \Gamma$ be a maximal parabolic subgroup that fixes some $\xi_P \in \Lambda_\Gamma$. Let $\mathcal{H} = \mathcal{H}_P$ be a horoball fixed by P (i.e. centred at ξ_P), and $\mathcal{C} = \mathcal{C}_P$ be a locally finite *strict* fundamental domain for the action of P on $\mathcal{H}$, in the sense that for any $x \in \mathcal{H}$, there exists a unique element $p \in P$ such that $px \in \mathcal{C}$.

Since the action of Γ on $\partial\Omega$ is geometrically finite, we can choose a relatively compact (measurable) strict fundamental domain $F \subset \Lambda_\Gamma \smallsetminus \{\xi_P\}$ for the action of P. Fix $x \in \Omega$. Since μ_x has no atoms (Proposition 9.4.2), we have

$$m_\Gamma(\pi_\Gamma S\mathcal{H}) = \sum_{p,q\in P} \int_{pF\times qF} e^{2\delta_\Gamma \langle \xi^-,\xi^+\rangle_x} \mu_x^2(d\xi^- d\xi^+) \int_{(\xi^-\xi^+)\cap\mathcal{C}} dt.$$

By using the Γ-invariance of μ and the definition of $\mathcal{C}$, we have

$$\begin{aligned} m_\Gamma(\pi_\Gamma S\mathcal{H}) &= \sum_{p,q\in P} \int_{F\times p^{-1}qF} e^{2\delta_\Gamma \langle \xi^-,\xi^+\rangle_x} \mu_x^2(d\eta^- d\eta^+) \int_{(\eta^-\eta^+)\cap p^{-1}\mathcal{C}} dt \\ &= \sum_{p\in P} \int_{F\times pF} e^{2\delta_\Gamma \langle \xi^-,\xi^+\rangle_x} \mu_x^2(d\eta^- d\eta^+) \int_{(\eta^-\eta^+)\cap\mathcal{H}} dt. \end{aligned}$$

From a geometrical point of view, any geodesic $(\eta^-\eta^+)$ intersecting $\mathcal{H}$ projects to a geodesic on Ω/Σ which makes an incursion into the cusp neighbourhood $\mathcal{C}$, and the term $\int_{(\eta^-\eta^+)\cap\mathcal{H}} \mathrm{d}t$ corresponds to the length of this incursion.

We will now bound the lengths of these incursions using a geometrical argument.

Let $U \subset \overline{\Omega}$ be an open neighbourhood of ξ_P such that $[y\eta) \cap \mathcal{H}$ is nonempty for all $\eta \in F$ and $y \in U$, and set $R := d_\Omega(x, \partial\mathcal{H} \smallsetminus U) < \infty$. For all $p \in P$ and $(\eta^-, \eta^+) \in F \times pF$, if $(\eta^-\eta^+) \cap \mathcal{H} \neq \varnothing$, then there exists $y, z \in \partial\mathcal{H}$ such that η^-, y, z, η^+ are aligned in this order, and by definition of U we observe that $d_\Omega(x, y)$ and $d_\Omega(px, z)$ are less than or equal to R; thus:

$$\int_{(\eta^-\eta^+)\cap\mathcal{H}} dt = d_\Omega(y,z) \le d_\Omega(x,px) + 2R, \text{ and } 0 \le \langle \eta^-, \eta^+\rangle_x \le R.$$

As a consequence,

$$m_\Gamma(\pi_\Gamma S\mathcal{H}) \le e^{2\delta_\Gamma R} \mu_x(F) \sum_{p\in P} (d_\Omega(x,px) + 2R)\mu_x(pF).$$

Since $\overline{F}$ and $P\cdot x \cup \{\xi_P\}$ are compact and disjoint, and Ω is strictly convex, we can find $R' > 0$ such that $[y,z] \cap B_\Omega(x, R') \neq \varnothing$ for all $(y,z) \in \overline{F} \times (P\cdot x \cup \{\xi_P\})$; this immediately implies that $F \subset \overline{\mathcal{O}}_{R'}(px, x)$ for any $p \in P$. Therefore, by Lemma 6.3.2,

$$\mu_x(pF) = \mu_{p^{-1}x}(F) = \int_{\xi\in F} e^{-\delta_\Gamma \mathsf{b}_\xi(x,p^{-1}x)} d\mu_x(\xi) \le e^{4\delta_\Gamma R'} e^{-\delta_\Gamma d_\Omega(x,px)} \mu_x(F).$$

We assemble these pieces to obtain

$$m_\Gamma(\mathcal{C}) = m_\Gamma(\pi_\Gamma S\mathcal{H}) \leq e^{2\delta_\Gamma(R+2R')}\mu_x(F)^2 \sum_{p\in P}(d_\Omega(x,px)+2R)e^{-\delta_\Gamma d_\Omega(x,px)}.$$

Together with the fact that the right hand side is finite since $\delta_\Gamma > \delta_P$ by Lemma 9.4.1, this concludes the proof of Theorem 9.0.1.

9.5 Proof of Theorem 9.0.2

Let $\mathcal{E}_\Gamma^L$ denote the measure $\delta L e^{\delta L}\sum_{c\in\mathcal{G}_L}\mathcal{L}c$ on $T^1\Omega/\Gamma$. By Theorem 8.0.3, we already know that $\mathcal{E}_\Gamma^L \to \frac{m_\Gamma}{\|m_\Gamma\|}$ weakly in $\mathcal{C}_c(T^1\Omega/\Gamma)^*$ when $L\to+\infty$. We start by replacing $\mathcal{E}_\Gamma^L$ by a nearby measure which will be better adapted to the argument to come, namely

$$\mathcal{M}_\Gamma^L := \delta e^{-\delta L}\sum_{c\in\mathcal{G}_L}\ell(c)\mathcal{L}c$$

We may verify, by arguing as in the proof of [Rob03, Th. 5.2], that we still have $\mathcal{M}_\Gamma^L \to \frac{m_\Gamma}{\|m_\Gamma\|}$ weakly in $\mathcal{C}_c(T^1\Omega/\Gamma)^*$ when $L\to+\infty$. and that it suffices to show that $\mathcal{M}_\Gamma^L$ converges weakly to $\frac{m_\Gamma}{\|m_\Gamma\|}$ in $\mathcal{C}_b(T^1\Omega/\Gamma)^*$ as $L\to+\infty$, to obtain the same (desired) conclusion for $\mathcal{E}_\Gamma^L$.

The rest of the proof consists in demonstrating that $\mathcal{M}_\Gamma^L$ converges weakly to $\frac{m_\Gamma}{\|m_\Gamma\|}$ in $\mathcal{C}_b(T^1\Omega/\Gamma)^*$ as $L\to+\infty$. We present this step in more detail since it more intimately involves the Hilbert geometry in the cusps.

Let us fix a Γ-invariant family of disjoint horoballs centred at all parabolic points of Λ_Γ. By Proposition 9.3.4, we have a decomposition of T^1M_{bip} into a compact core and a finite number of "cusp" neighbourhoods, which are the quotients of our fixed horoballs centred at the parabolic points. By Theorem 8.0.3, it suffices to show that $\int f d\mathcal{M}_\Gamma^L$ converges to $\int f d\frac{m_\Gamma}{\|m_\Gamma\|}$ as L tends to infinity for each bounded continuous function f which is supported on a cusp neighbourhood.

Fix a parabolic point $\xi\in\Lambda_\Gamma$, its stabiliser $P := \mathrm{Stab}_\Gamma(\xi)$ and a open horoball $\mathcal{H}$ centred at ξ such that $\gamma\mathcal{H}\cap\mathcal{H}$ is empty for any $\gamma\in\Gamma\smallsetminus P$. For each $r>0$, denote by $\mathcal{H}_r\subset\mathcal{H}$ the open horoball centred at ξ whose boundary is at distance r from that of $\mathcal{H}$. To prove the theorem, it is enough to prove that $\limsup_{L\to\infty}\mathcal{M}_\Gamma^L(\pi_\Gamma T^1\mathcal{H}_r)$ goes to zero as r goes to infinity.

The rest of the argument will resemble a more refined version of the argument in the proof of Theorem 9.0.1: whereas there we had a finite bound for the measure of the cusps, here we want a bound that goes to zero as $L\to\infty$.

Let $K\subset\Lambda_\Gamma\smallsetminus\{\xi\}$ be a compact subset such that $P\cdot K = \Lambda_\Gamma\smallsetminus\{\xi\}$. By the definition of $\mathcal{M}_\Gamma^L$, we have

$$\begin{aligned}\mathcal{M}_\Gamma^L(\pi_\Gamma T^1\mathcal{H}_r) &\leq \delta e^{-\delta L}\sum_{\substack{\gamma\in\Gamma^{\mathrm{pr1}}\\ \ell(\gamma)\leq L\ ,\ x_\gamma^-\in K}}\overline{\mathcal{L}}\gamma(T^1\mathcal{H}_r)\\ &\leq \delta e^{-\delta L}\sum_{p\in P}\sum_{\gamma\in\Gamma(L,p)}\overline{\mathcal{L}}\gamma(T^1\mathcal{H}_r),\end{aligned} \tag{9.5.1}$$

where, for $p\in P$ and $L\geq 0$, the subset $\Gamma(L,p)\subset\Gamma^{\mathrm{r1}}$ consists of the elements γ such that $\ell(\gamma)\leq L$ and $x_\gamma^-\in K$ and $x_\gamma^+\in pK$.

We now fix $r > 0$ and $p \in P$, and bound from above $\sum_{\Gamma(L,p)} \overline{\mathcal{L}}\gamma(T^1\mathcal{H}_r)$. In particular, we will bound from above the cardinality of the set $\Gamma(L,p,r)$ of $\gamma \in \Gamma(L,p)$ such that $\overline{\mathcal{L}}\gamma(T^1\mathcal{H}_r) > 0$, i.e. such that the axis of γ intersects $\mathcal{H}_r$.

Fix $\gamma \in \Gamma(L,p,r)$. Let $a, d \in \partial\mathcal{H}$ and $b, c \in \partial\mathcal{H}_r$ be such that $x_\gamma^-, a, b, c, d, x_\gamma^+$ are aligned along the axis of γ in this order. By definition, a belongs to the closed subset $A \subset \overline{\Omega}$ of points y for which there exists $\eta \in K$ with $(\eta, y]\cap\mathcal{H} = \varnothing$. The set $A \cap \partial\mathcal{H} \subset \Omega$ is compact, hence $d_\Omega(x,a) \le R_1 := \max\{d_\Omega(x,y) : y \in \partial\mathcal{H}\cap A\} < \infty$. As a first consequence, $d_\Omega(x,\gamma x) \le L{+}2R_1 \le N := \lceil L{+}2R_1 \rceil$. Moreover, $p^{-1}d \in A$ and $d_\Omega(px,d) \le R_1$, therefore

$$\overline{\mathcal{L}}\gamma(T^1\mathcal{H}_r) = d_\Omega(b,c) = d_\Omega(a,d) - d_\Omega(a,b) - d_\Omega(c,d) \le d_\Omega(x,px) + 2R_1 - 2r. \qquad (9.5.2)$$

Note, in particular, that $d_\Omega(x,px) \ge 2r - 2R_1$.

According to the Shadow lemma (Lemma 6.3.1), we can find $R_2 > 0$ such that for any $R \ge R_2$, there exists $C_R > 0$ so that

$$C_R^{-1} e^{-\delta d_\Omega(x,gx)} \le \mu_x(\mathcal{O}_R(x,gx)) \le \mu_x(\mathcal{O}_R^+(x,gx)) \le C_R e^{-\delta d_\Omega(x,gx)}$$

for any $g \in \Gamma$.

Since $\gamma\mathcal{H} \cap \mathcal{H} = \varnothing$ by definition of $\mathcal{H}$, we have $\gamma a \in [d, x_\gamma^+]$, and hence by (6.3.4)

$$\begin{aligned}\mathcal{O}_{R_2}(x,\gamma x) &\subset \mathcal{O}^+_{R_2+2R_1}(a,\gamma a) \subset \mathcal{O}_{2R_2+4R_1}(a,\gamma a) \subset \mathcal{O}_{2R_2+4R_1}(a,d)\\ &\subset \mathcal{O}^+_{R_3}(x,px),\end{aligned}$$

where $R_3 := 2R_2 + 4R_1$. We combine all these observations to produce:

$$\begin{aligned}
\#\Gamma(L,p,r) &= \sum_{0\le n\le N} \#\{\gamma \in \Gamma(L,p,r) : n-1 < d_\Omega(x,\gamma x) \le n\}\\
&\le \sum_{0\le n\le N} \sum_{\substack{\gamma\in\Gamma(L,p,r):\\ n-1<d_\Omega(x,\gamma x)\le n}} C_{R_2} e^{\delta n} \mu_x(\mathcal{O}_{R_2}(x,\gamma x))\\
&\le C_{R_2} \sum_{0\le n\le N} e^{\delta n} \int_{\xi\in\mathcal{O}^+_{R_3}(x,px)} \sum_{\substack{\gamma\in\Gamma(L,p,r):\\ n-1<d_\Omega(x,\gamma x)\le n}} 1_{\mathcal{O}_{R_2}(x,\gamma x)}(\xi)\, d\mu_x(\xi)\\
\text{(using (8.2.14))} &\le C_{R_2} \cdot \#\{g \in \Gamma : d_\Omega(x,gx) \le 4R_2 + 1\} \sum_{0\le n\le N} e^{\delta n} \mu_x(\mathcal{O}^+_{R_3}(x,px))\\
&\le C_{R_2} \cdot \#\{g \in \Gamma : d_\Omega(x,gx) \le 4R_2 + 1\} \frac{e^{\delta(N+1)}}{e^\delta - 1} C_{R_3} e^{-\delta d_\Omega(x,px)}\\
&\le C e^{\delta(L - d_\Omega(x,px))},
\end{aligned}$$

where $C := C_{R_2} C_{R_3} \frac{e^{\delta(2R_1+1)}}{e^\delta - 1} \cdot \#\{g \in \Gamma : d_\Omega(x,gx) \le 4R_2 + 1\}$.

Combining this with (9.5.1) and (9.5.2) yields

$$\mathcal{M}^L_\Gamma(\pi_\Gamma T^1\mathcal{H}_r) \le \delta C \sum_{\substack{p\in P\\ d_\Omega(x,px)>2r-2R_1}} (d_\Omega(x,px) - 2r + 2R_1) e^{-\delta\cdot d_\Omega(x,px)}.$$

By Lemma 9.4.1, $\sum_{p\in P} d_\Omega(x,px) e^{-\delta\cdot d_\Omega(x,px)}$ converges. Therefore,

$$\limsup_{L\to+\infty} \mathcal{M}^L_\Gamma(\pi_\Gamma T^1\mathcal{H}_r) \underset{r\to\infty}{\longrightarrow} 0.$$

9.5.1 The number of periodic geodesics and of conjugacy classes in the geometrically finite case

Finally, as in Sections 7.4.5 and 8.3.4, let us relate the number of strongly primitive rank-one conjugacy classes, the number of rank-one conjugacy classes, and the number of rank-one periodic $(\phi_t)_t$-orbits; the arguments are based on the discussion in Section 3.4.2.

Observation 9.5.1. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly and strictly convex open set with $\mathcal{C}^1$ boundary, and $\Gamma \subset \mathrm{Aut}(\Omega)$ a discrete subgroup which acts geometrically finitely on $\partial\Omega$. Let $\Gamma' \subset \Gamma$ be a torsion-free finite-index subgroup. Then*

$$\sum_{c \in \mathcal{G}_T^{\mathrm{r1}}} \int f \,\mathrm{d}\mathcal{L}c \leq \sum_{c \in [\Gamma]_T^{\mathrm{pr1}}} \int f \,\mathrm{d}\mathcal{L}c \leq [\Gamma' : \Gamma] \sum_{c \in \mathcal{G}_T^{\mathrm{r1}}} \int f \,\mathrm{d}\mathcal{L}c$$

for any $T > 0$, for any non-negative bounded continuous function f on T^1M, and

$$Te^{-\delta_\Gamma T}\left(\#[\Gamma]_T^{\mathrm{r1}} - \#[\Gamma]_T^{\mathrm{pr1}}\right) \underset{T\to\infty}{\longrightarrow} 0.$$

Proof. This is a consequence of the discussion in Section 3.4.2. Indeed, we saw there that the number of strongly primitive rank-one conjugacy classes associated to a rank-one periodic $(\phi_t)_t$-orbit is less than $[\Gamma' : \Gamma]$. This implies the first assertion.

We also saw that, for any $\ell > 0$, the number of conjugacy classes of length ℓ associated to a rank-one periodic $(\phi_t)_t$-orbit is less than $[\Gamma' : \Gamma]$. Therefore, for any $T > 0$,

$$\#[\Gamma]_T^{\mathrm{pr1}} \leq \#[\Gamma]_T^{\mathrm{r1}} \leq \#[\Gamma]_T^{\mathrm{pr1}} + [\Gamma' : \Gamma] \sum_{k \geq 2} \#[\Gamma]_{\frac{T}{k}}^{\mathrm{pr1}}.$$

Let $\epsilon > 0$ be such that $[\Gamma]_\epsilon^{\mathrm{r1}}$ is empty. Then for T large enough, $\#[\Gamma]_T^{\mathrm{pr1}} \leq \frac{2[\Gamma':\Gamma]}{\delta_\Gamma T} e^{\delta_\Gamma T}$, and

$$\begin{aligned}\#[\Gamma]_T^{\mathrm{r1}} - \#[\Gamma]_T^{\mathrm{pr1}} &\leq [\Gamma' : \Gamma] \frac{T}{\epsilon} \#[\Gamma]_{\frac{T}{2}}^{\mathrm{pr1}} \\ &\leq \frac{4[\Gamma' : \Gamma]^2}{\delta_\Gamma T} \frac{T}{\epsilon} e^{\delta_\Gamma T/2}\end{aligned}$$

□

Proposition 9.5.2. *Let $\Omega \subset \mathrm{P}(\mathbf{V})$ be a properly and strictly convex open set with $\mathcal{C}^1$ boundary, and $\Gamma \subset \mathrm{Aut}(\Omega)$ a discrete subgroup which acts geometrically finitely on $\partial\Omega$. Let $F \subset \Gamma$ be the core-fixing subgroup (see Section 3.4.2). Let $K \subset T^1M_{\mathrm{bip}}$ be the set vectors of whose lifts $v \in T^1\Omega$ satisfy* $\mathrm{Stab}_\Gamma(v) \neq F$. *Let $A \subset \mathcal{G}^{\mathrm{r1}}$ be the set of rank-one periodic orbits contained in K. Then*

$$Te^{-\delta_\Gamma T} \# A_T \underset{T\to\infty}{\longrightarrow} 0.$$

Suppose further that F is the centre of Γ. Then for any bounded continuous function f on T^1M,

$$\frac{\delta_\Gamma T}{\# F} e^{-\delta_\Gamma T} \sum_{c \in [\Gamma]_T^{\mathrm{pr1}}} \int_{T^1M} f \,\mathrm{d}\mathcal{L}c \underset{T\to\infty}{\longrightarrow} \int_{T^1M} f \,\mathrm{d}\frac{m_\Gamma}{\|m_\Gamma\|}.$$

If Γ is strongly irreducible, then F is trivial and

$$\delta_\Gamma T e^{-\delta_\Gamma T} \sum_{c \in [\Gamma]_T^{\mathrm{pr1}}} \int_{T^1M} f \,\mathrm{d}\mathcal{L}c \underset{T\to\infty}{\longrightarrow} \int_{T^1M} f \,\mathrm{d}\frac{m_\Gamma}{\|m_\Gamma\|}.$$

Proof. Let us only give a proof of the first point, since the other are elementary consequences of it and of the discussion in Section 3.4.2.

Consider a non-negative bounded function $f \in \mathcal{C}(T^1M)$. Fix $\epsilon > 0$. Since $m_\Gamma(K) = 0$ (recall that $K \subset T^1M_{\mathrm{bip}}$ has empty interior by Observation 3.4.1, and m_Γ is ergodic with support T^1M_{bip} by Theorems 6.0.1 and Proposition 7.0.1), we can find a non-negative bounded function $\chi \in \mathcal{C}(T^1M)$ such that $\chi \geq 1$ on $\mathrm{supp}(f) \cap K$ and $\int \chi f \, \mathrm{d}m_\Gamma \leq \epsilon \|m_\Gamma\|$. According to Theorem 9.0.2,

$$\delta_\Gamma T e^{-\delta_\Gamma T} \sum_{c \in A_T} \int f \, \mathrm{d}\mathcal{L}c \leq \delta_\Gamma T e^{-\delta_\Gamma T} \sum_{c \in \mathcal{G}_T^{\mathrm{r1}}} \int \chi f \, \mathrm{d}\mathcal{L}c \underset{T \to \infty}{\longrightarrow} \frac{1}{\|m_\Gamma\|} \int \chi f \, \mathrm{d}m_\Gamma \leq \epsilon.$$

This holds for any $\epsilon > 0$, so $(\delta_\Gamma T e^{-\delta_\Gamma T} \sum_{c \in A_T} \int f \, \mathrm{d}\mathcal{L}c)_T$ converges to zero. □

Part IV

Exposant critique des orbivariétés projectives convexes

Chapter 10

Exposant critique des surfaces projectives convexes de volume fini

Soit Σ une surface obtenue en enlevant un ensemble fini et non vide $\{\overline{p}_1, \dots, \overline{p}_k\}$ de points à une surface orientée fermée $\overline{\Sigma}$. Benoist–Hulin [BH13, Th. 1.1] ont construit un homéomorphisme entre l'espace des structures projectives convexes marquées de volume fini sur Σ et un certain fibré vectoriel au-dessus de l'espace des structures hyperboliques marquées de volume fini sur Σ; cette construction généralise les travaux (indépendants) de Labourie [Lab07] et Loftin [Lof01] sur l'espace des structures projectives convexes marquées compactes.

Notons $\widetilde{\Sigma}$ le revêtement universel de Σ. Fixons une structure hyperbolique marquée de volume fini S sur Σ, c'est-à-dire un morphisme injectif $\mathrm{hol} : \pi_1(\Sigma) \to \mathrm{PSL}_2(\mathbb{R})$, d'image discrète Γ, et un homéomorphisme $\pi_1(\Sigma)$-équivariant $\mathrm{dev} : \widetilde{\Sigma} \to \mathbb{H}^2 = \{z \in \mathbb{C} : \Re(z) > 0\}$, tel que $S = \mathbb{H}^2/\Gamma$ est de volume fini.

La fibre, notée $\mathcal{V}$, au-dessus de S dans le fibré mentionné ci-dessus est l'espace vectoriel des formes différentielles cubiques méromorphes sur $\overline{\Sigma}$ dont les pôles sont $\overline{p}_1, \dots, \overline{p}_k$ et sont d'ordre au plus 2, où l'on a mis sur $\overline{\Sigma}$ la structure complexe induite par S.

Chaque élément $v \in \mathcal{V}$ induit une structure projective convexe marquée sur Σ, c'est-à-dire un morphisme injectif $\mathrm{hol}_v : \Gamma \to \mathrm{PSL}_3(\mathbb{R})$ d'image discrète et un difféomorphisme Γ-équivariant $\mathrm{dev}_v : \mathbb{H}^2 \to \mathrm{P}(\mathbb{R}^3)$ dont l'image Ω_v est un ouvert proprement convexe; on note d_v le tiré en arrière sur $\mathbb{H}^2$ par dev_v de la distance de Hilbert sur Ω_v. En particulier, la distance d_0 est la distance usuelle sur $\mathbb{H}^2$. Rappelons que pour $v \in \mathcal{V}$, l'exposant critique de d_v, noté δ_v, est le suprémum des nombres s tels que la série $\sum_{\gamma\in\Gamma} e^{-s d_v(x,\gamma x)}$ diverge pour tout $x \in \mathbb{H}^2$. Fixons une norme $\|\cdot\|_{\mathcal{V}}$ sur $\mathcal{V}$, et notons $\mathcal{S} \subset \mathcal{V}$ l'ensemble des éléments de norme 1. Notre but est de démontrer le théorème suivant :

Théorème 10.0.1. *L'exposant critique δ_v de Γ sur $\mathbb{H}^2$ pour la distance d_v tend vers $1/2$ lorsque v sort de tout compact de $\mathcal{V}$. Plus précisément, il existe une constante $C \geq 1$ telle que*

$$\frac{1}{2} < \delta_v \leq \frac{1}{2} + Ce^{-\frac{\|v\|_{\mathcal{V}}}{C}}.$$

L'inégalité $1/2 < \delta_v$ est due à Crampon [Cra11, Lem. 4.3.4]. Ce résultat complète un théorème similaire de X. Nie [Nie15a] qui traite le cas des surfaces projectives convexes compactes à l'aide de la paramétrisation de Labourie et Loftin; dans ce cadre, l'exposant critique tendait vers zéro au lieu de $1/2$.

Notons que l'exposant critique est une fonction continue sur l'espace tout entier des structures projectives convexes marquées de volume fini sur Σ, d'après Crampon [Cra11, Prop. 5.4.1]. Ce résultat, combiné au théorème 10.0.1, au fait 2.3.17, et au fait que l'exposant critique d'une surface hyperbolique de volume fini est égal à 1, a pour corollaire le résultat suivant.

Corollaire 10.0.2. *L'ensemble des exposants critiques des surfaces projectives convexes non compactes de volume fini est égal à* $]1/2, 1]$.

Pour tout $v \in \mathcal{V}$, on note $\|\cdot\|_v$ la métrique finslérienne sur $\mathbb{H}^2$ associée à la distance de Hilbert d_v. La seule estimée dont nous aurons besoin pour contrôler comment d_v et $\|\cdot\|_v$ dépendent de v est la suivante. Tout élément $v \in \mathcal{V}$ se relève sur $\mathbb{H}^2$ en une forme différentielle cubique $z \mapsto f(z)\,\mathrm{d}z^3$, où f est une fonction holomorphe Γ-invariante sur $\mathbb{H}^2$, et l'on considère la fonction positive Γ-invariante λ_v dont la valeur en $z \in \mathbb{H}^2$ est

$$\lambda_v(z) = \frac{1}{2}|f(z)|^{\frac{1}{3}} \cdot \Re(z).$$

Fait 10.0.3 ([BH13, Prop. 3.4 & Lem. 5.7])**.** *Il existe une constante* $C_1 > 0$ *telle que pour tout* $v \in \mathcal{V}$*, tout* $z \in \mathbb{H}^2$ *et tout* $X \in T_z\mathbb{H}^2$*,*

$$\lambda_v(z)\|X\|_0 \leq C_1\|X\|_v.$$

Pour plus de détails, on pourra aussi consulter [Nie, Th. 6.1].

10.1 La partie non-cuspidale

Dans cette section, on choisit une décomposition de $\mathbb{H}^2$ et S en une partie cuspidale et une partie non-cuspidale, et l'on examine le comportement de la distance d_v sur la partie non-cuspidale quand v tend vers l'infini.

On note $P \subset \partial\mathbb{H}^2$ l'ensemble des points paraboliques pour l'action de Γ (c'est-à-dire les points fixés par un élément unipotent non trivial de $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$). Fixons pour chaque point $p \in P$ une horoboule ouverte H_p de $\mathbb{H}^2$ centrée en p, de sorte que $\overline{H_p} \cap \overline{H_q}$ soit vide pour tous $p, q \in P$ distincts et $H_{\gamma p} = \gamma H_p$ pour tous $p \in P$ et $\gamma \in \Gamma$. Pour chaque $p \in P$, on note $H'_p \subset H_p$ l'horoboule centrée en p dont le bord est à distance 1 du bord de H_p (pour la distance usuelle sur $\mathbb{H}^2$, i.e. d_0). On considère les fermés Γ-invariants $F = \mathbb{H}^2 \smallsetminus \bigcup_{p\in P} H_p$ et $F' = \mathbb{H}^2 \smallsetminus \bigcup_{p\in P} H'_p$, sur lesquels Γ agit librement et cocompactement.

Lemme 10.1.1. *Pour tout* $\epsilon > 0$*, il existe une constante* $C > 0$ *telle que pour tout* $v \in \mathcal{V}$*, pour tout chemin* $c : [0,1] \to \mathbb{H}^2$ *de classe* $\mathcal{C}^1$ *et contenu dans* F'*, si* $d_0(c(0), c(1)) \geq \epsilon$*, alors*

$$\int_0^1 \|c'(t)\|_v \,\mathrm{d}t \geq C^{-1} \cdot \|v\|_{\mathcal{V}}^{\frac{1}{3}} \cdot d_0(c(0), c(1)).$$

Preuve. Observons que $\lambda_{rv} = r^{1/3}\lambda_v$ pour tous $v \in \mathcal{V}$ et $r > 0$. D'après le fait 10.0.3, pour établir le lemme il suffit, ayant fixé $\epsilon > 0$, de trouver une constante $C > 0$ telle que pour tout $v \in \mathcal{S}$, pour tout chemin $c : [0,1] \to \mathbb{H}^2$ de classe $\mathcal{C}^1$ et contenu dans F', si $d_0(c(0), c(1)) \geq \epsilon$, alors

$$\int_0^1 \lambda_v(c(t)) \cdot \|c'(t)\|_0 \,\mathrm{d}t \geq C^{-1} d_0(c(0), c(1)).$$

Soit $v \in \mathcal{S}$. Comme f est holomorphe non constante, le sous-ensemble $\{\lambda_v = 0\} \cap F' \subset F'$ est fermé, discret et Γ-invariant, et les nombres suivants sont strictement positifs.

$$r_v = \frac{1}{5}\min\left(\epsilon,\ \min\left\{d_0(x,y) : x, y \in \{\lambda_v = 0\} \cap F' \text{ distincts}\right\}\right), \quad \text{et}$$
$$\alpha_v = \min\left\{\lambda_v(x) : x \in F' \text{ tel que } d_0\left(x, \{\lambda_v = 0\}\right) \geq r_v\right\}.$$

Considérons un voisinage U_v de v dans $\mathcal{S}$ tel que $\lambda_w(x) \geq \alpha_v/2$ pour tout $w \in U_v$ et $x \in F'$ tel que $d_0(x, \{\lambda_v = 0\}) \geq r_v$. Soit $w \in U_v$, et soit $c : [0,1] \to F'$ un chemin $\mathcal{C}^1$ tel que $d_0(c(0), c(1)) \geq \epsilon$. Soient $N \geq 1$ et $0 = t_0 < t_1 < \cdots < t_n = 1$ tels que $\epsilon \leq d_0(c(t_i), c(t_{i+1})) \leq 2\epsilon$ pour $0 \leq i \leq n-1$. Pour chaque $0 \leq i \leq n-1$, il existe $t_i < t_i^- < t_i^+ < t_{i+1}$ tels que $d_0(c(t_i^-), c(t_i^+)) \geq r_v$ et $d_0(c(t), \{\lambda_v = 0\}) \geq r_v$ pour tout $t_i^- \leq t \leq t_i^+$. Ainsi,

$$\begin{aligned}\int_0^1 \lambda_w(c(t)) \cdot \|c'(t)\|_0 \,\mathrm{d}t &\geq \sum_{i=0}^{n-1} \int_{t_i^-}^{t_i^+} \frac{\alpha_v}{2} \|c'(t)\|_0 \,\mathrm{d}t \\ &\geq \frac{\alpha_v}{2} \sum_{i=0}^{n-1} d_0(c(t_i^-), c(t_i^+)) \\ &\geq \frac{\alpha_v r_v}{2\epsilon} \sum_{i=0}^{n-1} d_0(c(t_i), c(t_{i+1})) \\ &\geq \frac{\alpha_v r_v}{2\epsilon} d_0(c(0), c(1)).\end{aligned}$$

Par compacité de $\mathcal{S}$, on peut trouver $A \subset \mathcal{S}$ fini tel que $S \subset \bigcup_{v \in A} U_v$. Pour conclure la démonstration, il suffit de considérer $C = \max_{v \in A} \frac{2\epsilon}{\alpha_v r_v}$. □

10.2 La partie cuspidale

Dans cette section, on s'intéresse à la longueur des chemins contenus dans une pointe.

Fixons pour chaque $p \in P$ un sous-groupe unipotent à un paramètre $u_p : \mathbb{R} \to \mathrm{PSL}_2(\mathbb{R})$ qui fixe p tel que $u_p(1)$ engendre $\mathrm{Stab}_\Gamma(p)$, et de sorte que $u_{\gamma p}(t) = \gamma u_p(t) \gamma^{-1}$ pour $t \in \mathbb{R}$ et $\gamma \in \Gamma$. On souhaite établir la proposition suivante.

Proposition 10.2.1. *Il existe une constante $C' > 0$ telle que pour tout $v \in \mathcal{V}$, tout $p \in P$, tout $x \in \partial H_p$ et tout $t \in \mathbb{R}$, on a*

$$d_v(x, u_p(t) \cdot x) \geq 2 \log(|t| + 1) - C'.$$

Pour ce faire, la première étape consiste en le lemme suivant.

Lemme 10.2.2. *Il existe $\alpha > 0$ tel que pour tout $v \in \mathcal{V}$, pour tout $p \in P$, pour tout $x \in \partial H_p$, on a*

$$d_v(x, u_p(1) \cdot x) \geq \alpha.$$

Preuve. L'infimum des $d_v(x, u_p(1)x)$, où $p \in P$, $x \in \partial H_p$ et $v \in \mathcal{V}$ avec $\|v\|_{\mathcal{V}} \leq 1$, est atteint et non nul, puisque cette quantité est Γ-invariante et continue en x, p, v, et car Γ agit cocompactement sur $\bigsqcup_{p \in P} \partial H_p$. Soit $\epsilon < 1$ plus petit que $d_0(x, u_p(1)x)$ pour tout $p \in P$ et $x \in \partial H_p$. Soient $p \in P$, $x \in \partial H_p$ et $v \in \mathcal{V}$ tel que $\|v\|_{\mathcal{V}} \geq 1$. Soit $T = d_v(x, u_p(1)x)$ et $c : [0, T] \to \mathbb{H}^2$ une géodésique de x vers $u_p(1)x$ pour la distance d_v. Si l'image de c est contenu dans F', alors le lemme 10.1.1 nous donne une constante $C > 0$ qui dépend de ϵ telle que

$$d_v(x, u_p(1)x) = \int_0^T \|c'(t)\|_v \,\mathrm{d}t \geq \frac{\|v\|_{\mathcal{V}}}{C} d_0(x, u_p(1)x) \geq \frac{\epsilon}{C}.$$

Si au contraire l'image de c n'est pas contenue dans F', alors il existe $0 \leq T' \leq T$ et $q \in P$ tels que $c(T') \in \partial H'_q$. Par définition, $d_0(x, c(T')) \geq 1 \geq \epsilon$ (donc $T' > 0$). Ainsi, en utilisant encore le lemme 10.1.1, on obtient

$$d_v(x, u_p(1)x) = \int_0^{T'} \|c'(t)\|_v \,\mathrm{d}t \geq \frac{\|v\|_{\mathcal{V}}}{C} d_0(x, c(t)) \geq \frac{1}{C}.$$ □

10.2.1 Lemmes techniques

Cette section regroupe deux lemmes techniques utilisés dans la démonstration de la proposition 10.2.1. On pose

$$\gamma_0 := \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \in \mathrm{PGL}_3(\mathbb{R}).$$

Lemme 10.2.3. *Pour tous $0 < a < b < \infty$, il existe $C_1' > 0$ tel que pour tout ouvert proprement convexe γ_0-invariant $\Omega \subset \mathrm{P}(\mathbb{R}^3)$ et tout point $x \in \Omega$, si $d_\Omega(x, \gamma_0 x) \in [a, b]$, alors pour tout $n \in \mathbb{Z}$,*

$$|d_\Omega(x, \gamma_0^n x) - 2\log(|n| + 1)| \leq C_1'.$$

Preuve. Soit $\Omega \subset \mathrm{P}(\mathbb{R}^3)$ un ouvert proprement convexe γ_0-invariant. On voit facilement Ω est contenu dans la carte affine $\mathrm{P}(\mathbb{R}^3) \smallsetminus \mathrm{Vect}(e_1, e_2)$ et $[e_1] \in \partial\Omega$, où e_1, e_2, e_3 est la base canonique de $\mathbb{R}^3$. De plus, pour tout point $x \in \Omega$, on peut trouver $g \in \mathrm{PGL}_3(\mathbb{R})$ tel que $g\gamma_0 = \gamma_0 g$ et $gx = x_0 = [e_3]$. Notons $\mathcal{E}^\bullet$ l'espace des ouvert proprement convexes pointés de $\mathrm{P}(\mathbb{R}^3)$, muni de la topologie de Hausdorff. Considérons le sous-espace $\mathcal{K} \subset \mathcal{E}^\bullet$ constitué des paires (x, Ω) telles que $x = x_0$ et $\gamma_0\Omega = \Omega$ ainsi que $d_\Omega(x_0, \gamma_0 x_0) \in [a, b]$, et démontrons qu'il est compact.

Soit $(\Omega_n)_n$ une suite d'ouverts proprement convexes tels que $(x_0, \Omega_n) \in \mathcal{K}$ pour tout n, et qui converge vers un ouvert convexe γ_0-invariant $\Omega \subset \mathrm{P}(\mathbb{R}^3)$. Si par l'absurde Ω n'est pas proprement convexe, alors $\overline{\Omega}$ contient un point dans $\mathrm{Vect}(e_1, e_2) \smallsetminus \{[e_1]\}$, et donc il contient $\mathrm{Vect}(e_1, e_2)$ par convexité et γ_0-invariance, et ainsi $(d_{\Omega_n}(x_0, \gamma_0 x_0))_n$ tend vers 0 : absurde ! Considérons pour chaque n les points $\xi_n, \eta_n \in \partial\Omega_n$ tels que $\xi_n, x_0, \gamma_0 x_0, \eta_n$ sont alignés dans cet ordre ; quitte à extraire on peut supposer que $(\xi_n)_n$ et $(\eta_n)_n$ convergent vers respectivement ξ et $\eta \in \partial\Omega \cap \mathrm{Vect}(e_2, e_3)$. Comme $(d_{\Omega_n}(x_0, \gamma_0 x_0))_n$ est bornée, on a $\xi \neq x_0$ et $\eta \neq \gamma_0 x_0$, si bien que l'enveloppe convexe de $\gamma_0^{\mathbb{Z}} \cdot \xi \cup \gamma_0^{\mathbb{Z}} \cdot \eta$ contient x_0 dans son intérieur. Ainsi, $x_0 \in \Omega$ et $d_\Omega(x_0, \gamma_0 x_0) = \lim_n d_{\Omega_n}(x_0, \gamma_0 x_0) \in [a, b]$. On a démontré que $(x_0, \Omega) \in \mathcal{K}$, ce qui conclut la preuve de la compacité de $\mathcal{K}$.

Le fait 2.2.9 nous fournit une constante $C_{11}' > 0$ telle que $|d_\Omega(x_0, \gamma_0^n x_0) - \frac{1}{2}\log(\|\gamma_0^n\| \cdot \|\gamma_0^{-n}\|)| \leq C_{11}'$ pour tout entier n et tout ouvert proprement convexe Ω tel que $(x_0, \Omega) \in \mathcal{K}$, où l'on a fixé une norme $\|\cdot\|$ sur $\mathrm{End}(\mathbb{R}^3)$. Or $(\log(\|\gamma_0^n\| \cdot \|\gamma_0^{-n}\|))_n$ est équivalente à $(4\log(|n| + 1))_n$, ce qui entraîne le lemme. □

Lemme 10.2.4. *Pour tout ouvert proprement convexe $\Omega \subset \mathrm{P}(\mathbb{R}^3)$, pour tous $x, y, y' \in \Omega$ et tout point $\xi \in \partial\Omega$ de classe $\mathcal{C}^1$ tels que ξ, y, y' sont alignés et $0 = \mathtt{b}_\xi(x, y) \leq \mathtt{b}_\xi(x, y')$, on a*

$$d_\Omega(x, y) \leq d_\Omega(x, y') + \log\frac{9}{2}.$$

Proof. Soient $\Omega \subset \mathrm{P}(\mathbb{R}^3)$ un ouvert proprement convexe, $x, y, y' \in \Omega$ et $\xi \in \partial\Omega$ de classe $\mathcal{C}^1$ tels que ξ, y, y' sont alignés et $0 = \mathtt{b}_\xi(x, y) < \mathtt{b}_\xi(x, y')$. Notons η et ζ les points de $\partial\Omega$ tels que $x \in [\xi, \eta]$ et $y \in [\xi, \zeta]$.

Considérons la carte affine $(x_1, x_2) \mapsto [x_1 e_1 + x_2 e_2 + e_3]$ de $\mathrm{P}(\mathbb{R}^3)$, où e_1, e_2, e_3 est la base canonique de $\mathbb{R}^3$. On fera l'abus de notation $(t, \infty) = (t, -\infty) = [e_2] \in \mathrm{P}(\mathbb{R}^3)$ pour $t \in \mathbb{R}$. Quitte à agir par un élément de $\mathrm{PGL}_3(\mathbb{R})$ bien choisi, on peut supposer que x est l'origine $(0, 0)$ de la carte affine considérée, que $y = (0, 1)$, que $\xi = (-1, 1)$, que $\eta = (1, -1)$ et que $T_\xi\partial\Omega = \{-1\} \times \mathbb{R}$, alors nécessairement $\zeta = (1, 1)$ (car $\mathtt{b}_\xi(x, y) = 0$) et $y' \in]0, 1[\times \{1\}$, par contre Ω n'est pas forcément contenu dans la carte affine.

Soient α, β (resp. α', β') dans $\partial\Omega$ tels que α, x, y, β (resp. α', x, y', β') sont alignés dans cet ordre. Il existe $a \in]0, \infty]$ et $b \in]1, \infty]$ (avec $(a, b) \neq (\infty, \infty)$) tels que $\alpha = (0, -a)$ et

$\beta = (0, b)$, de sorte que $d_\Omega(x, y)$ égale $\frac{1}{2} \log\left(\frac{b(1+a)}{(b-1)a}\right)$. On peut vérifier que $\alpha' \in [-1, 0[\times]-\infty, 0]$ et $\beta' \in]0, 1]\times]1, \infty[$ (en utilisant que $\eta, \zeta \in \partial\Omega$) ; soient $a', b' \in [0, \infty]$ tels que $(\xi, \alpha', (0, -a'))$ et $(\xi, (0, b'), \beta')$ sont alignés, de sorte que $d_\Omega(x, y') = \frac{1}{2} \log\left(\frac{b'(1+a')}{(b'-1)a'}\right)$; observons que $a' > a$ et $1 < b' \leq b$.

Deux cas de figures se présentent. Si $a \geq 1/2$, alors

$$\begin{aligned} 2d_\Omega(x, y') &\geq \log\left(\frac{b'}{b'-1}\right) \geq \log\left(\frac{b}{b-1}\right) \geq 2d_\Omega(x, y) - \log\left(\frac{1+a}{a}\right) \\ &\geq 2d_\Omega(x, y) - \log(3). \end{aligned}$$

Supposons donc le contraire : $a < 1/2$. Dès lors,

$$\begin{aligned} 2d_\Omega(x, y') &\geq \log\left(\frac{b(1+a')}{(b-1)a'}\right) = 2d_\Omega(x, y) + \log\left(\frac{(1+a')a}{a'(1+a)}\right) \\ &\geq 2d_\Omega(x, y) + \log\left(\frac{a}{a'}\right) - \log(3/2). \end{aligned}$$

Soit z le point d'intersection de $(0, 0) \oplus (1, 1)$ et $(0, -a) \oplus (1, -1)$. On voit facilement que α' appartient au triangle de sommets $(0, 0)$, $(0, -a)$ et z, lui-même contenu dans le triangle de sommets $(0, 0)$, $(0, -a)$ et $(-1/2, (1 - 3a)/2)$ (en utilisant que $a < 1/2$). Ceci entraîne que $a' \leq 3a$, et donc que $2d_\Omega(x, y') \geq 2d_\Omega(x, y) - \log(9/2)$. □

On remarquera, dans la preuve ci-dessus, que la constante $\log(9/2)$ n'est pas optimale, mais qu'il existe des situations où $d_\Omega(x, y) > d_\Omega(x, y')$.

10.2.2 Preuve de la proposition 10.2.1

Fixons $v \in \mathcal{V}$, $p \in P$, $x \in \partial H_p$, $t \in \mathbb{R}$ et $y = u_p(t)x$. Posons aussi $\phi = \mathrm{dev}_v$ et $\rho = \mathrm{hol}_v$ et $\Omega = \Omega_v$ pour simplifier les notations. D'après [Mar12b, Cor. 5.29], l'automorphisme $\gamma = \rho(u_p(1))$ est *parabolique*, c'est-à-dire qu'il est conjugué à γ_0, et il fixe un point $q \in \partial\Omega$ — en fait, l'espace (Ω, d_Ω) est Gromov-hyperbolique de bord à l'infini $\partial\Omega$, et l'application $\phi : (\mathbb{H}^2, d_0) \to (\Omega, d_\Omega)$ est une quasi-isométrie, donc elle s'étend continûment en un homéomorphisme Γ-équivariant de $\partial\mathbb{H}^2$ vers $\partial\Omega$, qui envoie p sur q (voir [CM14a, Cor. 9.6]).

Soient $X \in T^1_{\phi(x)}\Omega$ et $Y \in T^1_{\phi(y)}\Omega$ tels que $\phi_\infty X = \phi_\infty Y = q$. Comme Ω est γ-équivariant, et γ est parabolique, on vérifie que q est un point $\mathcal{C}^1$ du bord (en fait $\partial\Omega$ est tout entier $\mathcal{C}^1$ par [Mar12b, Th. 6.9]). Quitte à échanger x et y (et à changer t en $-t$), on peut supposer que $r = \mathtt{b}_q(\pi X, \pi Y) \geq 0$, de sorte que $d_v(x, y) \geq d_\Omega(\pi X, \pi\phi_r Y) - \log(9/2)$ au vu du lemme 10.2.4. D'après le lemme 2.1.6, $s \mapsto d_\Omega(\pi\phi_s X, \pi\phi_{s+r} Y)$ et $s \mapsto d_\Omega(\pi\phi_s X, \gamma\pi\phi_s X)$ sont décroissantes, si bien que, en utilisant le nombre α donné par le lemme 10.2.2, il existe $s \geq 0$ tel que, notant $x' = \pi\phi_s X$ et $y' = \pi\phi_{s+r} Y$, on a $d_\Omega(x', \gamma x') = \alpha$ et $d_v(x, y) \geq d_\Omega(x', y') - \log(9/2)$ ainsi que $\mathtt{b}_q(x', y') = 0$.

Le lemme 10.2.3 fournit une constante C_1' telle que $d_\Omega(x', \gamma^n x') \geq 2\log(|n| + 1) - C_1'$ pour tout entier n (la constante ne dépend que de α qui lui-même de ne dépend que de la surface hyperbolique S). Quitte considérer l'inverse de γ, on peut supposer que $t \geq 0$; posons $n = \lfloor t \rfloor$ et estimons la distance $d_\Omega(\gamma^n x', y')$. Le segment $[\gamma^n x', \gamma^{n+1} x']$ rencontre l'intervalle $]q, y']$ en un point y'' (car les horoboules sont convexes), et ceci entraîne que

$$\begin{aligned} d_\Omega(\gamma^n x', y') &\leq d_\Omega(\gamma^n x', y'') + d_\Omega(y'', y') \\ &\leq d_\Omega(\gamma^n x', \gamma^{n+1} x') + \mathtt{b}_q(y', y'') \\ &= \alpha + \mathtt{b}_q(\gamma^n x', y'') \\ &\leq 2\alpha. \end{aligned}$$

Pour finir, on obtient

$$\begin{aligned} d_v(x,y) &\geq d_\Omega(x',y') - \log(9/2) \\ &\geq d_\Omega(x',\gamma^n x') - d_\Omega(\gamma^n x', y') - \log(9/2) \\ &\geq \log(n+1) - C_1' - 2\alpha - \log(9/2) \\ &\geq \log(t+1) - C_1' - 2\alpha - \log(9). \end{aligned}$$

10.3 Démonstration du théorème 10.0.1

Fixons $x \in F$ et $s > 1/2$ et démontrons que $\delta_v < s$ pour v de norme assez grande ; cela suffira pour conclure la preuve du théorème 10.0.1 puisque $\delta_v > 1/2$ pour tout v d'après [Cra11, Lem. 4.3.4]. Il suffit de démontrer que pour tout $v \in \mathcal{V}$ de norme suffisamment grande,

$$\sum_{\gamma\in\Gamma} e^{-sd_v(x,\gamma x)} < \infty.$$

Soit $v \in \mathcal{V}$ et $\gamma \in \Gamma$; notons $T = d_v(x, \gamma x)$. Considérons une géodésique $c : [0, T] \to \mathbb{H}^2$ de x vers γx pour la distance de Hilbert d_v. On peut trouver un entier $n \geq 0$, une suite de points paraboliques $p_1, \ldots, p_n \in P$ distincts deux-à-deux et une suite de réels

$$0 = T_0^+ \leq T_1^- \leq T_1^+ < T_2^- \leq T_2^+ < \cdots < T_n^- \leq T_n^+ \leq T = T_{n+1}^-$$

tels que $c(T_i^\pm) \in \partial H_{p_i}$ et $c([T_i^+, T_{i+1}^-]) \cup c([0, T_1^-]) \subset F$ pour $1 \leq i \leq n$. Pour chaque $1 \leq i \leq n$, on se donne

- $t_i \in \mathbb{R}$ tel que $c(T_i^+) = u_{p_i}(t_i) \cdot c(T_i^-)$;
- $g_i \in \Gamma$ tel que $d_0(c(T_i^-), g_i x) = d_0(c(T_i^-), \Gamma x)$;
- $p_i' = g_i^{-1} p_i$, de sorte que $u_{p_i}(t) g_i = g_i u_{p_i'}(t)$ pour tout $t \in \mathbb{R}$;
- $\gamma_i = g_{i-1}^{-1} u_{p_{i-1}}(-\lfloor t_{i-1} \rfloor) g_i \in \Gamma$ (où $g_0 = \mathrm{id}$) et $\gamma_{n+1} = g_n^{-1} u_{p_n}(-\lfloor t_n \rfloor)\gamma$.

On peut vérifier par récurrence que

$$\gamma = \gamma_1 \cdot u_{p_1'}(\lfloor t_1 \rfloor) \cdot \gamma_2 \cdot u_{p_2'}(\lfloor t_2 \rfloor) \cdots \gamma_n \cdot u_{p_n'}(\lfloor t_n \rfloor) \cdot \gamma_{n+1}.$$

Puisque Γ agit cocompactement sur F', on peut trouver $R > 0$ assez grand pour que $F' \subset \Gamma \cdot \overline{B}_{d_0}(x, R)$. En particulier, on a par définition $d_0(c(T_i^-), g_i x) \leq R$ pour tout $1 \leq i \leq n$, et ainsi p_i' appartient au sous-ensemble fini $P' \subset P$ constitué des points paraboliques p tels que $\overline{H}_p$ rencontre $\overline{B}_{d_0}(x, R)$.

Soit $0 \leq i \leq n$. Posons $R' = \max\{d_0(x, u_p(t)x) : p \in P, x \in \partial H_p, 0 \leq t \leq 1\}$, de sorte que $c(T_i^+) \in \overline{B}_{d_0}(g_i u_{p_i'}(\lfloor t_i \rfloor)x, R + R')$. Soit $\epsilon > 0$ plus petit que $d_0(x, y)$ et $d_0(y, z)$ pour tous $y \in \overline{H}_p$ et $z \in \overline{H}_q$, où $p \neq q \in P$. Le lemme 10.1.1 nous donne une constante $C > 0$ telle que

$$\begin{aligned} d_v(c(T_i^+), c(T_{i+1}^-)) &\geq C^{-1} \|v\|_{\mathcal{V}} \cdot d_0(c(T_i^+), c(T_{i+1}^-)) \\ &\geq \frac{\|v\|_{\mathcal{V}}}{2C} \epsilon + \frac{\|v\|_{\mathcal{V}}}{2C} E_{\gamma_{i+1}}, \end{aligned}$$

où $E_\gamma = \min\{d_0(y, z) : y \in B_{d_0}(x, R + R'),\ z \in B_{d_0}(\gamma x, R + R')\}$ pour $\gamma \in \Gamma$.

De plus, la proposition 10.2.1 nous donne une constante $C' > 0$ telle que pour tout $1 \leq i \leq n$,

$$d_v(c(T_i^-), c(T_i^+)) \geq 2\log(|\lfloor t_i \rfloor| + 1) - C'.$$

En combinant tout ceci, on obtient

$$d_v(x, \gamma x) \geq \frac{\|v\|_\mathcal{V}}{2C} \sum_{i=1}^{n+1} E_{\gamma_i} + (n+1)\frac{\|v\|_\mathcal{V}}{2C}\epsilon + \sum_{i=1}^{n} 2\log(|\lfloor t_i \rfloor| + 1) - nC', \tag{10.3.1}$$

puis

$$\begin{aligned}
\sum_{\gamma\in\Gamma} e^{-sd_v(x,\gamma x)} &\leq \sum_{n\geq 0} \sum_{\substack{\gamma_1,\dots,\gamma_{n+1}\in\Gamma \\ p_1,\dots,p_n\in P' \\ k_1,\dots,k_n\in\mathbb{Z}}} \prod_{i=0}^{n} e^{-\frac{s\|v\|_\mathcal{V}}{2C}(\epsilon + E_{\gamma_{i+1}})} \cdot \prod_{i=1}^{n} e^{-2s\log(|k_i|+1)+sC'} \\
&\leq \sum_{n\geq 0} e^{-\frac{s\|v\|_\mathcal{V}}{2C}\epsilon(n+1)} \cdot \left(\sum_{\gamma\in\Gamma} e^{-\frac{s\|v\|_\mathcal{V}}{2C}E_\gamma}\right)^{n+1} \cdot e^{snC'} \cdot (\#P')^n \cdot 2^n \cdot \left(\sum_{k\geq 0} e^{-2s\log(k+1)}\right)^n \\
&\leq \sum_{n\geq 1} \left(e^{-\frac{\epsilon\|v\|_\mathcal{V}}{4C}} \cdot \chi\left(\frac{\|v\|_\mathcal{V}}{4C}\right) \cdot e^{sC'} \cdot 2\#P' \cdot \zeta(2s)\right)^n,
\end{aligned}$$

où $\chi(t) = \sum_{\gamma\in\Gamma} e^{-tE_\gamma}$ et $\zeta(t) = \sum_{k\geq 1} k^{-t}$ pour $t > 0$.

Quand t tend vers l'infini, $\chi(t)$ tend vers $\#\{\gamma \in \Gamma : E_\gamma = 0\} < \infty$; donnons-nous $T > 0$ tel que $\chi(t) \leq C_2 := \#\{\gamma : E_\gamma = 0\} + 1$ pour tout $t \geq T$. Donnons-nous également $C_3 > 0$ tel que $\zeta(t) \leq \frac{C_3}{t-1}$ pour tout $1 < t \leq 2$. Supposons $s \leq 2$, de sorte que d'après les calculs ci-dessus, $\sum_{\gamma\in\Gamma} e^{-sd_v(x,\gamma x)}$ converge dès que $\|v\|_\mathcal{V} \geq 4CT$ et que

$$s > \frac{1}{2} + C_3 C_2 e^{2C'} \#P' e^{-\frac{\epsilon\|v\|_\mathcal{V}}{4C}}.$$

Comme on sait par ailleurs que $\delta_v \leq 1$ (voir le fait 2.3.17), on en déduit l'existence d'une constante $C_4 > 1$ telle que $\delta_v \leq \frac{1}{2} + C_4 e^{-\frac{\|v\|_\mathcal{V}}{C_4}}$.

Chapter 11

Exposant critique et grosseur des réflectofolds projectifs convexes

Dans ce chapitre, correspondant à une collaboration avec Harrison Bray, on s'intéresse, comme dans le chapitre précédent, à l'exposant critique des orbivariétés projectives convexes (défini en (1.4.2)). Dans le chapitre précédent, on s'était restreint à l'ensemble des surfaces projectives convexes de volume fini, ensemble pour lequel on disposait d'une paramétrisation (celle de Benoist–Hulin [BH13]), commode pour lire certaines propriétés de l'exposant critique. Dans ce chapitre on s'intéresse à une classe différente d'orbivariétés projectives convexes, pour laquelle on dispose aussi d'une bonne paramétrisation, grâce aux travaux de Vinberg [Vin71] : ce sont les réflectofolds projectifs convexes, autrement dit les quotients de la forme Ω/Γ, où Γ est engendré par des réflexions projectives le long des faces d'un polytope P de l'ouvert proprement convexe Ω. Précisons qu'une face d'un polytope est de codimension 1, tandis qu'une facette est de codimension quelconque.

L'exposant critique de certaines de ces orbivariétés a déjà fait l'objet d'un examen approfondi par X. Nie [Nie15b]. Plus précisément, ce dernier a considéré le cas où le polytope P est un simplexe compact (dans Ω) et le groupe Γ est Gromov-hyperbolique, cas pour lequel Lanner [Lan50] a montré que Γ appartient à une liste précise (infinie) de groupes de Coxeter avec au plus cinq générateurs (voir aussi [Dav08, Th. 6.9.1 & Table 6.2]). Fixons une orbivariété topologique parmi celles considérées par Nie. L'espace des structures projectives convexes dessus est ou bien réduit à un point, ou bien paramétré par $[0, \infty[$. Dans le second cas, Nie a démontré que l'exposant critique tend vers zéro lorsque le paramètre tend vers l'infini. La théorie de Vinberg montre qu'il existe bien d'autres exemples de réflectofolds projectifs convexes, avec des espaces de paramètres qui peuvent être de dimension plus grande que 1.

Fixons un groupe de Coxeter Γ infini, irréductible, non affine et à N générateurs, ainsi qu'un espace vectoriel réel $\mathbf{V}$ de dimension inférieure ou égale à N. Les travaux de Vinberg [Vin71] fournissent un sous-ensemble semi-algébrique X de $\mathbb{R}^{N^2}$ et une application qui à $x \in X$ associe une représentation fortement irréductible, injective et d'image discrète ρ_x de Γ dans $\mathrm{GL}(\mathbf{V})$ telle que $\rho_x(\Gamma)$ préserve un ouvert proprement convexe de $\mathrm{P}(\mathbf{V})$ (dont le quotient est un réflectofold projectif convexe), et telle que chaque coefficient de la matrice $\rho_x(\gamma)$ dépend (de manière explicite) comme une fraction rationnelle des coefficients de x, ce pour tout $\gamma \in \Gamma$. (Voir [DGKLM, §3–4] pour plus de détails.)

Pour chaque élément γ, on peut accéder à certaines données algébriques de $\rho_x(\gamma)$, comme sa trace, sa norme, ses valeurs propres, $\ell(\rho_x(\gamma))$... A priori, cela ne permet toutefois pas de calculer l'exposant critique de ρ_x, qui dépend des images de tous les éléments de Γ à la fois. Notre but est de donner des estimées de l'exposant critique qui n'utilisent qu'un nombre fini d'éléments $\gamma \in \Gamma$.

Pour ce faire, on utilise de la façon suivante le fait qu'à ρ_x est associé un ouvert proprement convexe Ω_x et un polytope $P_x \subset \Omega_x$: on estime d'abord $\delta_{\rho_x(\Gamma)}$ en fonction de la taille de P_x par rapport à la métrique de Hilbert ; plus précisément, l'estimée fait intervenir la distance minimale entre deux points de P_x qui sont sur des facettes différentes, quantité que l'on appelle grosseur. Après quoi on majore la grosseur à l'aide de la longueur de translation d'un ensemble fini bien choisi d'éléments de $\rho_x(\Gamma)$. (Rappelons que la longueur de translation de $\rho_x(\gamma) \in \rho_x(\Gamma)$ est $\ell(\rho_x(\gamma))$ par le fait 2.2.8.)

11.1 Présentation des résultats

Soit $\Omega \subset \mathrm{P}(\mathbf{V})$ un ouvert proprement convexe. Un *hyperplan* de Ω est l'intersection avec Ω d'un hyperplan de $\mathrm{P}(\mathbf{V})$ (qui intersecte Ω) ; un *demi-espace* (fermé) de Ω est l'adhérence d'une composante connexe de Ω privé d'un de ses hyperplans. Une *réflexion* de Ω est une réflexion de $\mathrm{GL}(\mathbf{V})$ de la forme $r = \mathrm{id} - v \otimes \alpha$, où $v \in \mathbf{V}$ et $\alpha \in \mathbf{V}^*$ avec $\alpha(v) = 2$, telle que r préserve Ω et telle que Ω intersecte le noyau de α.

Remarquons que r est entièrement déterminée par Ω et $\mathrm{Ker}(\alpha)$, car deux réflexions distinctes qui fixent le même hyperplan de $\mathrm{P}(\mathbf{V})$ engendrent un groupe discret infini ne préservant aucun ouvert proprement convexe (ce groupe contient un élément unipotent conjugué à une matrice diagonale par bloc avec un bloc de taille 2 de la forme $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ et un bloc identité). De plus le point fixe isolé $[v]$ de r dans $\mathrm{P}(\mathbf{V})$ n'appartient pas à $\overline{\Omega}$, et l'on peut voir par dualité que r est déterminée par Ω et $[v]$.

On dit que Ω est *symétrique* par rapport à un de ses hyperplans H s'il existe une (nécessairement unique) réflexion de Ω qui fixe H. On note $\widetilde{\mathrm{Aut}}(\Omega) = \{g \in \mathrm{SL}^{\pm}(\mathbf{V}) : g\Omega = \Omega\}$, où $\mathrm{SL}^{\pm}(\mathbf{V}) \subset \mathrm{GL}(\mathbf{V})$ est constitué des éléments de déterminant plus ou moins un.

Un *polytope* P de Ω est l'intersection d'un nombre fini de demi-espaces de Ω, délimités par des hyperplans $H_1, \ldots, H_n$. On note $\mathring{P}$ (resp. $\overline{P}$) l'intérieur (resp. l'adhérence) dans $\mathrm{P}(\mathbf{V})$ de P. Une *facette* de P (dans Ω) est l'intersection un sous-ensemble non vide de la forme $P \cap \bigcap_{i \in I} H_i$ où $I \subset \{1, \ldots, n\}$; une *face* de P est une facette qui engendre un sous-espace de dimension $\mathtt{d} - 1$.

Soit $P \subset \Omega$ un polytope tel que Ω est symétrique par rapport aux hyperplans engendrés par les faces de P. Le *groupe* de P est le couple (Γ, S) où S est l'ensemble des réflexions de Ω le long des faces de P, et Γ est le groupe engendré par S.

Le polytope P est dit *de Coxeter* dans Ω, et le couple (P, Ω) est appelé polytope de Coxeter hilbertien, si Ω est symétrique par rapport aux hyperplans de P, et si les images de P sous l'action de son groupe (Γ, S) pavent Ω, dans le sens où $\mathring{P} \cap \gamma\mathring{P} = \emptyset$ pour tout élément non trivial $\gamma \in \Gamma$, et $\Omega = \bigcup_{\gamma \in \Gamma} \gamma P$.

L'exposant critique d'un polytope de Coxeter hilbertien est l'exposant critique de son groupe. Notre but est de comprendre l'asymptotique des exposants critiques de certaines suites de polytopes de Coxeter hilbertiens combinatoirement équivalents, dans le sens où leurs groupes sont isomorphes abstraitement. Le premier résultat est un critère suffisant pour que les exposants critiques tendent vers zéro, qui fait intervenir la *grosseur* d'un polytope de Coxeter hilbertien (P, Ω), définie comme le plus petit *écart* (de Hilbert) possible entre deux facettes de P disjointes, où l'écart entre deux sous-ensembles F et F' de Ω est défini comme $\inf\{d_\Omega(x, x') : x \in F,\ x' \in F'\}$.

Pour tout groupe Γ engendré en tant que semi-groupe par un sous-ensemble fini $S \subset \Gamma$, on note $|\gamma|$ la longueur de chaque élément $\gamma \in \Gamma$ pour la métrique des mots, c'est-à-dire le nombre minimal d'éléments $s_1, \ldots, s_n \in S$ tels que $\gamma = s_1 \cdots s_n$. Si de plus (Γ, S)

est un système de Coxeter, alors on note $\alpha_{\Gamma,S}$ le plus grand diamètre, pour la métrique des mots, des sous-groupes standards sphériques de Γ, c'est-à-dire les sous-groupes finis engendrés par un sous-ensemble de S (un sous-groupe est dit standard s'il est engendré par un sous-ensemble quelconque de S).

Théorème 11.1.1. *Soit (P, Ω) un polytope de Coxeter hilbertien, de groupe (Γ, S), d'exposant critique δ et de grosseur R. Alors pour tous $\gamma \in \Gamma$ et $x \in P$, on a*

$$d_\Omega(x, \gamma x) \geq \frac{R}{(2\alpha_{\Gamma,S}+1)^{\mathtt{d}}} \left(\frac{|\gamma|}{\alpha_{\Gamma,S}} - 1 \right).$$

Ceci entraîne en particulier que

$$\delta \leq \frac{\alpha_{\Gamma,S}(2\alpha_{\Gamma,S}+1)^{\mathtt{d}} \log \#S}{R}.$$

L'estimation de l'exposant critique dans le théorème 11.1.1 n'est bien sûr pas optimale : on trouve facilement une suite de polytopes de Coxeter dans le disque de Poincaré (qui engendrent par exemple un groupe convexe cocompact) dont la grosseur reste bornée mais dont l'exposant critique tend vers zéro.

Pour construire des exemples de polytopes de Coxeter hilbertiens dont les exposants critiques tendent vers zéro, on combine le théorème 11.1.1 avec l'observation élémentaire suivante servant à minorer la grosseur par une quantité plus facile à calculer dans certains cas.

Observation 11.1.2. *Soit (P, Ω) un polytope de Coxeter hilbertien de groupe (Γ, S) et F_1, F_2 deux facettes de P, soit de plus γ_1 (resp. γ_2) un élément du stabilisateur de F_1 (resp. F_2) dans Γ (le plus long pour la métrique des mots semble un bon choix), alors l'écart entre F et F' est plus grand que $\frac{1}{2}\ell(\gamma_1\gamma_2)$.*

Le dernier résultat de la thèse concerne l'exposant critique de certains polytopes de Coxeter hilbertiens dits *à pointes paraboliques de type* $\widetilde{A}_{\mathtt{d}-1}$. Ceux-ci sont définis de façon rigoureuse à la section 11.5.1 ; redonnons l'idée de la définition, déjà évoquée dans l'introduction. Un tel polytope P est l'intersection avec Ω d'un polytope de $\overline{\Omega}$ qui rencontre $\partial\Omega$ en un nombre fini de sommets appelés *sommets à l'infini* ; de plus, on demande que, pour chaque sommet à l'infini, le groupe engendré par les réflexions le long des faces adjacentes au sommet soit conjugué à un sous-groupe de $\mathrm{GL}(\mathbf{V})$ bien précis (celui donné par la représentation de Tits du groupe de Coxeter affine de type $\widetilde{A}_{\mathtt{d}-1}$).

La grosseur de ces polytopes est nulle car l'écart entre deux facettes disjointes adjacentes à un même sommet à l'infini est nul. C'est pourquoi nous définissons un autre type de grosseur. On montre dans la section 11.5.2 que, pour chaque sommet à l'infini de P, on peut choisir de façon canonique un voisinage dans Ω. La *grosseur non cuspidale* de (P, Ω) est définie comme l'infimum sur deux ensembles : d'une part l'ensemble des écarts entre deux facettes disjointes non adjacentes à un même sommet à l'infini ; d'autre part l'ensemble des écarts entre le voisinage canonique d'un sommet à l'infini et une facette non adjacente au sommet.

Théorème 11.1.3. *Soit (Γ, S) un système de Coxeter. Alors il existe une constante $C > 0$ telle que pour tout polytope de Coxeter hilbertiens à pointes paraboliques de type $\widetilde{A}_{\mathtt{d}-1}$, de groupe (Γ, S), d'exposant critique δ et de grosseur non cuspidale R, on a*

$$\frac{\mathtt{d}-1}{2} < \delta \leq \frac{\mathtt{d}-1}{2} + Ce^{-\frac{R}{C}}.$$

11.2 Minoration de l'exposant critique

Avant de se lancer dans la démonstration des théorèmes 11.1.1 et 11.1.3, on les complète à l'aide de l'observation suivante. Les groupes Γ qui nous intéressent contiennent généralement un groupe libre F_2 à deux générateurs a et b. Dans ce cas, l'exposant critique de n'importe quelle représentation ρ de Γ est plus grand ou égal à l'exposant critique de la restriction de ρ à F_2, et l'on peut estimer cette dernière quantité à l'aide de la fonction κ, définie en (2.2.1).

Lemme 11.2.1. *Soit $\rho : \mathrm{F}_2 \to \mathrm{PGL}_{\mathtt{d}+1}(\mathbb{C})$ un morphisme. Alors*

$$e^{-\delta_\rho \kappa(\rho(a))} + e^{-\delta_\rho \kappa(\rho(b))} \leq 1.$$

Preuve. Il suffit de démontrer qu'étant donné $\delta > 0$ tel que $e^{-\delta\kappa(\rho(a))} + e^{-\delta\kappa(\rho(b))} \geq 1$, la série $\sum_{\gamma \in \mathrm{F}_2} e^{-\delta\kappa(\rho(\gamma))}$ diverge. On utilise pour cela la sous-additivité de κ, c'est-à-dire le fait que $\kappa(gh) \leq \kappa(g) + \kappa(h)$ pour tous $g, h \in \mathrm{PGL}_{\mathtt{d}+1}(\mathbb{C})$. On a

$$\begin{aligned}
\sum_{\gamma \in \mathrm{F}_2} e^{-\delta\kappa(\rho(\gamma))} &\geq \sum_{k \geq 1} \sum_{\substack{n_1, \ldots, n_k \geq 1 \\ m_1, \ldots, m_k \geq 1}} \exp\left(-\delta\kappa(\rho(a^{n_1} b^{m_1} \cdots a^{n_k} b^{m_k}))\right) \\
&\geq \sum_{k \geq 1} \sum_{\substack{n_1, \ldots, n_k \geq 1 \\ m_1, \ldots, m_k \geq 1}} \exp\left(-\delta(n_1 + \cdots + n_k)\kappa(a)\right) \cdot \exp\left(-\delta(m_1 + \cdots + m_k)\kappa(b)\right) \\
&\geq \sum_{k \geq 1} \left(\sum_{n \geq 1} \exp\left(-\delta n \kappa(a)\right) \sum_{m \geq 1} \exp\left(-\delta m \kappa(b)\right) \right)^k \\
&\geq \sum_{k \geq 1} \left(\frac{1}{e^{\delta\kappa(\rho(a))} - 1} \cdot \frac{1}{e^{\delta\kappa(\rho(b))} - 1} \right)^k.
\end{aligned}$$

Cette dernière série diverge car $(e^{\delta\kappa(\rho(a))} - 1)(e^{\delta\kappa(\rho(a))} - 1) \geq 1$. □

Corollaire 11.2.2. *Soit $(\rho_t)_{0<t\leq 1}$ une famille de représentations de F_2 dans $\mathrm{GL}_{\mathtt{d}+1}(\mathbb{C})$, d'exposants critique $(\delta_t)_t$, telle que les coefficients de $\rho_t(a)$ et $\rho_t(b)$ sont des fractions rationnelles (ou plus généralement des séries de Puiseux) en t. Si $\ell(\rho_t(a))$ tend vers zéro quand t tend vers zéro, alors $\liminf_{t \to 0} \delta_t > 0$.*

Rappelons qu'une série de Puiseux en X à coefficients complexes est une série formelle de la forme $\sum_{k \geq -N} a_k X^{\frac{k}{n}}$, où $n, N > 0$ sont des entiers et $\{a_k\}_{k \geq -N}$ sont des nombres complexes.

Preuve. Il faut voir $\rho_t(a)$ comme une matrice à coefficients dans le corps des séries de Puiseux en t à coefficients complexes, qui est algébriquement clos d'après le théorème de Puiseux (voir par exemple [Eis95, Cor. 13.15]). Ainsi on peut trigonaliser $\rho_t(a)$, pour l'écrire sous la forme

$$\rho_t(a) = \lambda(t) g_t \begin{pmatrix} \lambda_1(t) e^{tN_1} & & & \\ & \ddots & & \\ & & \lambda_{r-1}(t) e^{tN_{r-1}} & \\ & & & e^{tN_r} \end{pmatrix} g_t^{-1},$$

où $\lambda(t), \lambda_1(t), \dots, \lambda_r(t)$ sont des séries de Puiseux en t telles que $|\lambda_1(t)| \geq \cdots \geq |\lambda_{r-1}(t)| \geq 1$ pour t assez petit, g_t est une matrice à coefficients dans le corps des séries de Puiseux, et $N_1, \dots, N_{r-1}$ sont des matrices nilpotentes à coefficients complexes.

Notons ρ'_t la représentation de F_2 donnée par $\rho'_t(a) = \lambda(t)^{-1} g_t^{-1} \rho_t(a) g_t$ et $\rho'_t(b) = \rho_t(b)$; elle a le même exposant critique que ρ_t. Comme $|\lambda_1(t)|$ tend vers 1 quand t tend vers zéro (puisque $\ell(\rho_t(a)) = \frac{1}{2}\log\lambda_1(t)$ tend vers zéro), les normes de $\rho'_t(a)$ et son inverse tendent vers 1, et l'on peut trouver des constantes $C > 0$ et $\alpha > 0$ telles que $\|\rho'_t(a)\| \cdot \|\rho'_t(a)^{-1}\| \leq 1 + Ct^\alpha$ pour t assez petit. On peut par ailleurs trouver des constante $C > 0$ et $\beta > 0$ telles que $\|\rho'_t(b)\| \cdot \|\rho'_t(b)^{-1}\| \leq C' t^{-\beta}$ pour t assez petit.

Pour tout $\delta > 0$, pour tout t assez petit, on a

$$\begin{aligned} e^{-\delta\kappa(\rho'_t(a))} + e^{-\delta\kappa(\rho'_t(b))} &\geq (1 + Ct^\alpha)^{-\delta/2} + \left(C' t^{-\beta}\right)^{-\delta/2} \\ &\geq 1 - \frac{\delta}{2} C t^\alpha + C'^{-\delta/2} t^{\beta\delta/2}, \end{aligned}$$

qui est plus grand que 1 pour t assez petit si $\delta < 2\frac{\alpha}{\beta}$. Donc $\liminf_{t\to 0} \delta_t \geq 2\frac{\alpha}{\beta}$. □

11.3 Exposant critique et grosseur

L'idée de la démonstration du théorème 11.1.1 est similaire à celle de Nie dans [Nie15b]. Reprenons les notations du théorème. On souhaite estimer $d_\Omega(x, \gamma x)$. On trace pour cela le segment $[x, \gamma x]$, et l'on note ses points d'intersection $x_1, \dots, x_n$ avec les facettes de P et de ses itérés sous l'action de Γ (qui pavent Ω par hypothèse).

Supposons que pour chaque $i = 1, \dots, n$, les points x_i et x_{i+1} sont dans des facettes disjointes d'un même polytope qu'on écrit sous la forme $\gamma_i P$, où $\gamma_i \in \Gamma$. Alors $d_\Omega(x_i, x_{i+1}) \geq R$ où R désigne la grosseur de P. Ainsi $d_\Omega(x, \gamma x)$ est plus grand que $(n-1)R$. De plus, $|\gamma| \leq \alpha_{\Gamma,S} n$ car chaque $\gamma_i^{-1}\gamma_{i+1}$ est dans un sous-groupe standard sphérique de Γ.

Malheureusement, il n'est pas toujours vrai que x_i et x_{i+1} sont dans des facettes disjointes. Typiquement, dans le cas de Nie, où P est un simplexe compact, deux faces de P ne sont jamais disjointes, et pour un point générique x, le segment $[x, \gamma x]$ ne rencontre aucune facette de codimension supérieure à 2.

Pour résoudre ce problème, l'astuce est que si les facettes de x_i et x_{i+1} s'intersectent, alors on peut trouver y dans l'intersection tel que

$$d_\Omega(x_i, z) + d_\Omega(z, x_{i+1}) \leq (2\alpha_{\Gamma,S} + 1) d_\Omega(x_i, x_{i+1});$$

ceci est démontré dans la section 11.3.2 (plus précisément dans le fait 11.3.6). Ainsi, on peut déformer par étapes le chemin de x à γx en chemins rectilignes par morceaux qui passent par des facettes de plus en plus petites, sans trop augmenter la longueur, de sorte qu'à la fin (après au plus $\mathtt{d}$ étapes), le chemin saute de facette en facette deux-à-deux disjointes ; on formalise cet argument dans la section 11.3.3, et ce de manière un peu abstraite car nous y aurons aussi recours pour démontrer le théorème 11.1.3.

11.3.1 Conséquences des travaux de Vinberg

Cette section contient quelques résultats de Vinberg. Afin de les énoncer, introduisons quelques terminologies supplémentaires.

Considérons un sous-ensemble fini $\tilde{S} \subset \mathbf{V} \times \mathbf{V}^*$ tel que $\alpha(v) = 2$ pour tout $(v, \alpha) \in \tilde{S}$. Soit $\tilde{K}$ le cône convexe, appelé *cône de polyédral fondamental*, constitué des vecteurs $v \in \mathbf{V}$ tels que $\alpha(v) \geq 0$ pour tout $(w, \alpha) \in \tilde{S}$, et Γ le groupe engendré par les réflexions de la forme $\mathrm{id} - v \otimes \alpha$ pour $(v, \alpha) \in \tilde{S}$. Suivant l'article [Vin71] de Vinberg, on dit que $\tilde{S}$ *définit un groupe de Coxeter linéaire* si :

- $\tilde{K}$ est d'intérieur $\mathrm{int}(\tilde{K})$ non vide ;
- $\tilde{K}$ a exactement $\#\tilde{S}$ faces ;
- $\mathrm{int}(\tilde{K}) \cap \gamma \cdot \mathrm{int}(\tilde{K})$ est vide pour tout élément non trivial $\gamma \in \Gamma$.

Énonçons à présent un des résultats principaux établis par Vinberg. Grossièrement parlant, l'idée est que pour que $\tilde{K}$ soit un domaine fondamental, c'est-à-dire que $\mathrm{int}(\tilde{K}) \cap \gamma\, \mathrm{int}(\tilde{K}) = \emptyset$ pour tout $\gamma \neq \mathrm{id}$, il suffit $\tilde{K}$ soit un domaine fondamental pour les sous-groupes de Γ engendrés par deux éléments de S.

Fait 11.3.1 ([Vin71, Th. 1, 2 & Prop. 5, 6, 13, 17, 21])**.** *Soit $\tilde{S} \subset \mathbf{V} \times \mathbf{V}^*$ un sous-ensemble fini tel que $\alpha(v) = 2$ pour tout $(v, \alpha) \in \tilde{S}$; notons $S = \{\mathrm{id} - v \otimes \alpha : (v, \alpha) \in \tilde{S}\}$. Alors $\tilde{S}$ définit un groupe de Coxeter linéaire si et seulement si le cône polyédral fondamental $\tilde{K}$ est d'intérieur non vide, et si pour tous $(v, \alpha) \neq (w, \beta) \in \tilde{S}$, en notant $s = \mathrm{id} - v \otimes \alpha$ et $t = \mathrm{id} - w \otimes \beta$, on a*

- *$\alpha(w)$ et $\beta(v)$ sont tous deux strictement négatifs ou tous deux nuls ;*
- *$\alpha(w)\beta(v)$ est ou bien supérieur à 4, alors on pose $m_{st} = \infty$, ou bien égal à $4\cos^2(\frac{\pi}{m_{st}})$ pour un certain entier $m_{st} \geq 2$.*

Supposons que c'est le cas, et considérons le groupe Γ engendré par S. Alors

1. *(Γ, S) est un système de Coxeter, où les relations sont données par $s^2 = \mathrm{id}$ pour tout $s \in S$, et $(st)^{m_{st}} = \mathrm{id}$ pour tous $s \neq t \in S$;*
2. *$\Gamma \cdot \tilde{K}$ est un cône convexe, appelé cône de Vinberg, sur l'intérieur duquel Γ agit proprement discontinûment ;*
3. *tout sous-ensemble $\tilde{S}' \subset \tilde{S}$ définit un groupe de Coxeter linéaire ;*
4. *Γ est fini si et seulement si $\Gamma \cdot \tilde{K} = \mathbf{V}$, auquel cas la famille $S \subset \mathbf{V}^*$ est libre ;*
5. *[DGKLM, Fait 3.15 & Rem. 4.3.1] supposons que (Γ, S) est infini, non affine et irréductible en tant que système de Coxeter abstrait, alors $\Gamma \cdot \tilde{K}$ est proprement convexe si et seulement si $\bigcap_{(v,\alpha) \in \tilde{S}} \mathrm{Ker}\, \alpha$ est triviale ;*
6. *Γ agit cocompactement sur le projeté dans $\mathrm{P}(\mathbf{V})$ de l'intérieur de $\Gamma \cdot \tilde{K}$ si et seulement si pour tout sous-ensemble fini $\tilde{S}' \subset \tilde{S}$ tel que $\bigcap_{(v,\alpha) \in \tilde{S}'} \mathrm{Ker}\, \alpha$ rencontre $\tilde{K}$, le sous-ensemble associé $S' \subset S$ engendre un groupe fini.*

Dans l'introduction de son papier [Vin71, §1.3], Vinberg fait le lien entre les groupes de Coxeter linéaires et les groupes engendrés par des réflexions orthogonales de l'espace hyperbolique de dimension $\mathtt{d}$. Cette discussion s'étend en fait immédiatement aux groupes engendrés par des réflexions d'un ouvert proprement convexe, plus précisément aux groupes des polytopes de Coxeter hilbertiens. Nous y ajoutons une caractérisation de la convexité d'une union de chambres, qui n'est pas explicitement énoncée ni démontrée dans l'article de Vinberg, mais qui se démontre avec les mêmes arguments que ceux qu'il met en œuvre dans [Vin71, §3].

Fait 11.3.2 ([Vin71, §1.3 & §3]). *Soit $\tilde{S} \subset \mathbf{V} \times \mathbf{V}^*$ un sous-ensemble fini tel que $\alpha(v) = 2$ pour tout $(v, \alpha) \in \tilde{S}$. Considérons le cône polyédral fondamental $\tilde{K}$ de $\tilde{S}$, l'ensemble $S = \{\mathrm{id} - v \otimes \alpha : (v, \alpha) \in \tilde{S}\}$, et le groupe Γ engendré par S. Soit $\Omega \subset \mathrm{P}(\mathbf{V})$ un ouvert proprement convexe Γ-invariant qui intersecte toutes les faces du projeté $K \subset \mathrm{P}(\mathbf{V})$ de $\tilde{K}$, et considérons le polytope $P = K \cap \Omega$.*

Alors le groupe de (P, Ω) est (Γ, S), et $\tilde{S}$ définit un groupe de Coxeter linéaire si et seulement si (P, Ω) est un polytope de Coxeter hilbertien. Supposons que c'est le cas, et notons $S' \subset \Gamma$ l'union des sous-groupes standards sphériques de (Γ, S). Alors :

- *pour tout sous-ensemble $A \subset \Gamma$, l'union des itérés $A \cdot P$ est connexe si et seulement si A induit un sous-graphe connexe du graphe de Cayley de (Γ, S') ;*
- *pour tous $s, t \in S$, qui engendrent le sous-groupe $\Gamma_{s,t} \subset \Gamma$, pour tout $A \subset \Gamma_{s,t}$, l'ensemble $A \cdot P$ est convexe si et seulement si A est ou bien vide, ou bien $\Gamma_{s,t}$ tout entier, ou bien de la forme*

$$\{\gamma, \gamma s, \gamma st, \gamma sts, \dots, \gamma(st)^k\} \quad ou \quad \{\gamma, \gamma s, \gamma st, \gamma sts, \dots, \gamma(st)^{k-1}s\}$$

  pour un certain $0 \leq k \leq \frac{1}{4}\#\Gamma_{s,t}$ et un certain $\gamma \in \Gamma_{s,t}$.
- *pour tout sous-ensemble $A \subset \Gamma$, l'union des itérés $A \cdot P$ est convexe si et seulement si A induit un sous-graphe connexe du graphe de Cayley de (Γ, S) et si pour tous $s, t \in S$ qui engendrent un sous-groupe $\Gamma_{s,t} \subset \Gamma$ fini, pour tout $\gamma \in A$, l'ensemble $((\gamma \cdot \Gamma_{s,t}) \cap A) \cdot P$ est convexe ;*
- *en particulier, pour tout sous-groupe standard $\Gamma' \subset \Gamma$, le sous-ensemble $\Gamma' \cdot P \subset \Omega$ est convexe.*

Preuve. Rappelons l'idée de Vinberg pour démontrer que $A \cdot P$ est convexe, sous les hypothèses :

(a) A induit un sous-graphe connexe du graphe de Cayley de (Γ, S) ;

(b) pour tous $s, t \in S$ qui engendrent un sous-groupe $\Gamma_{s,t} \subset \Gamma$ fini, pour tout $\gamma \in A$, l'ensemble $((\gamma \cdot \Gamma_{s,t}) \cap A) \cdot P$ est convexe.

L'hypothèse (b) se traduit par : l'union des chambres de $A \cdot P$ attenantes à une même facette de codimension 2 est convexe.

Enlevons à P ses facettes de codimension supérieure ou égale à 2, pour obtenir $P' \subset P$. D'après l'hypothèse (a), l'ensemble $A \cdot P'$ est connexe. Considérons le sous-ensemble $X \subset (A \cdot P')^2$ constitué des couples (x, y) tels que le segment $[x, y]$ ne rencontre aucune facette de codimension supérieure ou égale à 3, et rencontre l'union des facettes de codimension 2 en au plus un point. X est ouvert, dense et connexe. Considérons le sous-espace $X' \subset X$ constitué des couples (x, y) tels que $[x, y] \subset A \cdot P$. Pour conclure la preuve, il suffit de démontrer que $X' = X$; pour ce faire il suffit de prouver que $X' \subset X$ est ouvert et fermé. Le fait qu'il est fermé découle du fait que $A \cdot P \subset \Omega$ est fermé, et le fait qu'il est ouvert découle de l'hypothèse (b). □

11.3.2 Projections lipschitziennes

Les projections linéaires sont 1-lipschitzienne pour la distance de Hilbert

Le résultat suivante est une conséquence élémentaire de la définition de la distance de Hilbert.

Observation 11.3.3 ([Bus55, 18.6]). *Soient $W, W' \subset V$ deux sous-espaces supplémentaires, et $\mathrm{pr} : V \to W$ la projection linéaire surjective de noyau W', qui induit une projection de $\mathrm{P}(\mathbf{V}) \smallsetminus \mathrm{P}(W')$ sur $\mathrm{P}(W)$, également notée pr. Soit $\Omega \subset \mathrm{P}(\mathbf{V}) \smallsetminus \mathrm{P}(W')$ un ouvert proprement convexe tel que $\mathrm{pr}(\Omega) \subset \Omega$. Alors la restriction de pr à Ω est 1-lipschitzienne pour la distance de Hilbert.*

Les réflexions induisent des projections

On déduit de l'observation précédente, en suivant les idées de Nie, le résultat suivant.

Fait 11.3.4 ([Nie15b, Lem. 5-6]). *Soit $\Omega \subset \mathrm{P}(\mathbf{V})$ un ouvert proprement convexe symétrique par rapport à $k \geq 1$ hyperplans $H_1, \dots, H_k \subset \mathrm{P}(\mathbf{V})$ dont l'intersection $L \subset \mathrm{P}(\mathbf{V})$ est de dimension $\mathtt{d} - k$ et rencontre Ω. Notons $r_1, \dots, r_k \in \widetilde{\mathrm{Aut}}(\Omega)$ les réflexions associées, de points fixes isolés respectifs $x_1, \dots, x_k \in \mathrm{P}(\mathbf{V})$. Alors $L' = \mathrm{Vect}(x_1, \dots, x_k) \subset \mathrm{P}(\mathbf{V})$ est de dimension $k-1$ et est disjoint de L, et la projection $\mathrm{pr} : \mathrm{P}(\mathbf{V}) \smallsetminus L' \to L$ préserve Ω (i.e. $\mathrm{pr}(\Omega) \subset \Omega$).*

En particulier, d'après l'observation 11.3.3, la restriction de pr à Ω est 1-lipschitzienne pour la distance de Hilbert. De plus, la projection de tout point $x \in \Omega$ est exactement l'intersection de L avec l'enveloppe convexe des points $x, r_1(x), \dots, r_k(x)$.

Remarque 11.3.5. Dans le fait 11.3.4, si $k = 1$ alors la restriction de pr à Ω satisfait la propriété supplémentaire suivante : $d_\Omega(x, \mathrm{pr}(x)) \leq d_\Omega(x, y)$ pour tout $x \in \Omega$ et $y \in L \cap \Omega$. Cependant, cette propriété n'est en générale plus vraie lorsque $k \geq 2$. En effet, considérons la carte affine $(x, y, z) \in \mathbb{R}^3 \mapsto [1 : x : y : z] \in \mathrm{P}(\mathbb{R}^4)$, et l'intérieur Ω de l'enveloppe convexe des trois points $(1, 1, 0)$, $(1, -\frac{1}{2}, \frac{\sqrt{3}}{2})$ et $(1, -\frac{1}{2}, -\frac{\sqrt{3}}{2})$ et de leurs opposés $(-1, -1, 0)$, $(-1, \frac{1}{2}, -\frac{\sqrt{3}}{2})$ et $(-1, \frac{1}{2}, \frac{\sqrt{3}}{2})$. L'ouvert $\Omega \subset \mathrm{P}(\mathbb{R}^4)$ est proprement convexe et invariant sous une action du groupe dihédral de taille 6 qui préserve la carte affine et fixe la droite verticale passant par 0. On peut alors facilement se convaincre que 0 est plus proche que $(\frac{1}{2}, 0, 0)$ des points $(\frac{1}{2}, \frac{1}{2}, 0)$, $(\frac{1}{2}, -\frac{1}{4}, \frac{\sqrt{3}}{4})$ et $(\frac{1}{2}, -\frac{1}{4}, -\frac{\sqrt{3}}{4})$ (pour la distance de Hilbert).

Preuve. [Preuve du fait 11.3.4] Commençons par démontrer le fait pour $k = 1$. La seule chose à démontrer est que pr préserve Ω. Cela provient du fait que $\mathrm{pr}(x) \in [x, r_1 x] \subset \Omega$ pour tout $x \in \Omega$. Notons que pour tout $x \in \Omega$ et $y \in L \cap \Omega$, on a

$$d_\Omega(x, \mathrm{pr} x) = \frac{1}{2} d_\Omega(x, r_1 x) \leq \frac{1}{2}(d_\Omega(x, y) + d_\Omega(y, r_1 x)) = d_\Omega(x, y).$$

Passons au cas général $k \geq 1$. Soit $\Gamma \subset \mathrm{SL}^{\pm}(\mathbf{V})$ le groupe engendré par $r_1, \dots, r_k$. Comme $\widetilde{\mathrm{Aut}}(\Omega)$ agit proprement sur Ω et Γ fixe n'importe quel point de $L \cap \Omega \neq \emptyset$, ce dernier doit être relativement compact dans $\mathrm{GL}(\mathbf{V})$, et il préserve donc un produit scalaire sur V. On s'est ramené à un cadre classique :

- $x_1, \dots, x_k$ sont en position générale et engendrent l'orthogonal de L car ce sont les orthogonaux d'hyperplans en position générale ;
- pr préserve Ω car c'est la limite de la suite $((\mathrm{pr}_1 \cdots \mathrm{pr}_k)^N)_{N \to \infty}$, où $\mathrm{pr}_1, \dots, \mathrm{pr}_k$ sont les projections orthogonales respectives sur les hyperplans $H_1, \dots, H_k$, projections qui préservent toutes Ω. □

Applications aux polytopes de Coxeter hilbertiens

Le fait 11.3.4 s'applique en particulier aux polytopes de Coxeter hilbertiens (P, Ω) de groupe (Γ, S) fini. En effet, d'après les faits 11.3.1.4 et 11.3.2, l'intersection des hyperplans fixés par les éléments de S est de dimension $\mathtt{d} - \#S$.

Soit (P, Ω) un polytope de Coxeter hilbertien de groupe (Γ, S), et F une facette de P. On note $P_{|F}$ le polytope qui contient P et qui est délimité par les hyperplans engendrés par les faces de P qui contiennent F. Le sous-groupe $\mathrm{Stab}_\Gamma(F) \subset \Gamma$ est fini (c'est le groupe de $(P_{|F}, \Omega)$). Notons L le sous-espace de $\mathrm{P}(\mathbf{V})$ engendré par F, et pr_F la projection $\Omega \to L \cap \Omega$ fournie par le fait 11.3.4. D'après le fait 11.3.4, l'image de P par pr_F est contenue dans l'enveloppe convexe de $\mathrm{Stab}_\Gamma(F) \cdot P$, enveloppe qui est en fait égale à $\mathrm{Stab}_\Gamma(F) \cdot P$ d'après le fait 11.3.2. Ceci implique que $\mathrm{pr}_F(P) = L \cap P = F$. De plus, il existe une unique chambre de la forme γP pour un certain $\gamma \in \mathrm{Stab}_\Gamma(F)$ et dont l'intérieur est dans une composante connexe de $\Omega \smallsetminus \mathrm{Fix}(s)$ différente que celle l'intérieur de P pour tout $s \in \mathrm{Stab}_S(F)$; cette chambre est appelée la *chambre opposée à P par rapport à F*.

Fait 11.3.6 ([Nie15b, Lem. 1]). *Soit (P, Ω) un polytope de Coxeter hilbertien de groupe (Γ, S) et F une facette de P. Soit $\gamma \in \mathrm{Stab}_\Gamma(F)$, de longueur $|\gamma|$, tel que γP est la chambre opposée à P par rapport à F. Alors pour tous $x \in P$ et $y \in \gamma P$, il existe $z \in [\mathrm{pr}_F(x), \mathrm{pr}_F(y)]$ tel que*

$$d_\Omega(x, z) \leq d_\Omega(x, y).$$

Cela a pour consequence que pour tous $x, y \in P$ tels que F est l'intersection de leurs facettes, $d_\Omega(x, \mathrm{pr}_F(x)) \leq |\gamma| d_\Omega(x, y)$, et donc

$$d_\Omega(x, \mathrm{pr}_F(x)) + d_\Omega(\mathrm{pr}_F(x), \mathrm{pr}_F(y)) + d_\Omega(\mathrm{pr}_F(y), y) \leq (2|\gamma| + 1) d_\Omega(x, y).$$

Preuve. Pour démontrer le premier point, on raisonne par récurrence sur la dimension de la plus petite facette contenant x et F. Si cette dimension est $\dim(F)$, alors on peut choisir $z = \mathrm{pr}_F(x) = x \in F$. Supposons qu'on a démontré la propriété jusqu'au rang $k \geq \dim(F)$, et soit $x \in P$ tel que la plus petite facette contenant x et F, notée F', est de dimension $k + 1$, et soit $y \in \gamma P$. Soit L l'espace engendré par F', et considérons la facette $F'' = L \cap \gamma P$ de γP. Comme γP est la chambre opposée à P par rapport à F, la facette F'' engendre tout L, et $\mathrm{pr}_{F'}(\gamma P) = F''$. Le segment $[x, \mathrm{pr}_{F'} y]$ est contenu dans $L \cap K$, où $K = \mathrm{Stab}_\Gamma(F) \cdot P$. On considère le point x_1 de $[x, \mathrm{pr}_{F'} y] \cap F'$ le plus loin de x. Observons que F' est un voisinage, relativement à K, de tous ses points x' tels que F' est la plus petite facette contenant x' et F. Ainsi la plus petite facette contenant x_1 et F est strictement plus petite que F', et l'on peut lui appliquer l'hypothèse de récurrence : il existe $z \in [\mathrm{pr}_F x_1, \mathrm{pr}_F \mathrm{pr}_{F'} y]$ tel que $d_\Omega(x_1, z) \leq d_\Omega(x_1, \mathrm{pr}_{F'} y)$. D'après le fait 11.3.4, on a $\mathrm{pr}_F \circ \mathrm{pr}_{F'} = \mathrm{pr}_F$, et de plus pr_F envoie tout convexe sur un convexe, donc $z \in [\mathrm{pr}_F x, \mathrm{pr}_F y]$. Toujours d'après le fait 11.3.4, on a $d_\Omega(x, \mathrm{pr}_{F'} y) \leq d_\Omega(x, y)$, si bien que

$$d_\Omega(x, z) \leq d_\Omega(x, x_1) + d_\Omega(x_1, z) \leq d_\Omega(x, x_1) + d_\Omega(x_1, \mathrm{pr}_{F'} y) = d_\Omega(x, \mathrm{pr}_{F'} y) \leq d_\Omega(x, y).$$

Établissons le second point : soient $x, y \in P$ tels que l'intersection de leurs facettes est F, autrement dit tels que,

$$F = \bigcap_{s \in \mathrm{Stab}_S(x) \cup \mathrm{Stab}_S(y)} \mathrm{Fix}(s).$$

Comme les éléments de $\mathrm{Stab}_S(F)$ sont linéairement indépendants d'après les faits 11.3.1.4 et 11.3.2, on en déduit que $\mathrm{Stab}_S(F) = \mathrm{Stab}_S(x) \cup \mathrm{Stab}_S(y)$. Donnons-nous $s_1, \dots, s_{|\gamma|} \in \mathrm{Stab}_S(F)$ tels que $\gamma = s_1 \cdots s_{|\gamma|}$. Pour tout $1 \leq i \leq |\gamma| - 1$, l'union

$$[s_1 \cdots s_i x, s_1 \cdots s_i y] \cup [s_1 \cdots s_{i+1} x, s_1 \cdots s_{i+1} y] = s_1 \cdots s_i \left([x, y] \cup [s_{i+1} x, s_{i+1} y]\right)$$

est connexe par arcs puisque s_{i+1} fixe x ou y. Ainsi $\bigcup_{1\leq i\leq|\gamma|}[s_1\cdots s_i x, s_1\cdots s_i y]$ est connexe par arcs et contient x et γx, donc il existe un chemin de x vers γx qui est une concaténation d'au plus $|\gamma|$ segments de longueur inférieure à $d_\Omega(x,y)$, si bien que $d_\Omega(x,\gamma x)\leq|\gamma|d_\Omega(x,y)$. Comme $\mathrm{pr}_F(x)=\mathrm{pr}_F(\gamma x)$, le premier point nous dit que $d_\Omega(x,\mathrm{pr}_F(x))\leq d_\Omega(x,\gamma x)\leq|\gamma|d_\Omega(x,y)$. □

11.3.3 De facettes en facettes disjointes

Cette section contient un lemme technique, que l'on formule de manière un peu abstraite car on y aura recours deux fois, dans des situations légèrement différentes.

Soit X un ensemble muni d'un système de facettes, c'est-à-dire un ensemble $\mathcal{F}$ de parties de X stable par intersection, dont les éléments, appelés facettes, recouvrent X. La dimension d'une facette $F\in\mathcal{F}$, notée $\dim F$, est le supremum des entiers $n\geq 0$ tel qu'il existe une suite $F_0,\dots,F_n\in\mathcal{F}$ strictement décroissante pour l'inclusion avec $F=F_0$. La facette d'un point $x\in X$, notée $\mathrm{face}(x)$, est l'intersection des facettes qui contiennent x.

Munissons X d'une distance d_X qui vérifie la propriété suivante : il existe une constante $C\geq 1$ telle que pour tous points $x,y\in X$ dans une même facette et dont les facettes s'intersectent, on peut trouver un point z dans l'intersection tel que

$$d_X(x,z)+d_X(z,y)\leq Cd_X(x,y).$$

Lemme 11.3.7. *Soit deux entiers $n\geq 0$ et $k\geq 1$, et une suite de points $x_1,\dots,x_k\in X$ tels que $\dim\mathrm{face}(x_i)\leq n$ pour tout $1\leq i\leq k$, et x_i et x_{i+1} sont dans une même facette si $i<k$. Alors on peut trouver $m\geq 1$ et $y_1,\dots,y_m$ tels que $y_1\in\mathrm{face}(x_1)$ et $y_m\in\mathrm{face}(x_k)$, tels que y_i et y_{i+1} sont dans une même facette mais on des facettes disjointes pour tout $0\leq i<m$, et enfin tels que*

$$d_X(x_1,y_1)+\sum_{i=1}^{m-1}d_X(y_i,y_{i+1})+d_X(y_m,x_k)\leq C^n\sum_{i=1}^{k-1}d_X(x_i,x_{i+1}).$$

Preuve. On raisonne par récurrence sur n. Si $n=0$, alors il suffit d'extraire une sous-suite $x_{i_1},\dots,x_{i_m}$ telle que les facettes de x_{i_j} et $x_{i_{j+1}}$ sont distinctes, et donc disjointes, pour tout $1\leq j<m$, puis appliquer l'inégalité triangulaire.

Soit $n\geq 0$, et supposons le lemme vrai dès que $\dim\mathrm{face}(x_i)\leq n$ pour $1\leq i\leq k$. Soit $k\geq 1$ et $x_1,\dots,x_k\in X$ une suite de points telle que $\dim\mathrm{face}(x_i)\leq n+1$ pour tout $1\leq i\leq k$, et telle que x_i et x_{i+1} sont dans une même facette pour tout $1\leq i<k$; quitte à extraire, on peut supposer que les facettes de x_i et x_{i+1} sont différentes.

On se donne $q\geq 0$ et $1\leq k_1<k'_1<k_2<k'_2<\cdots<k_q<k'_q\leq k$ tels que pour tout $1\leq i<k$, les facettes de x_i et x_{i+1} s'intersectent si et seulement si il existe $1\leq j\leq q$ tel que $k_j\leq i<k'_j$, auquel cas on peut trouver, par hypothèse, un point y_i dans cette intersection tel que $d_\Omega(x_i,y_i)+d_\Omega(y_i,x_{i+1})\leq Cd_\Omega(x_i,x_{i+1})$; puisque les facettes de x_i et x_{i+1} sont différentes, la facette de y_i a une dimension inférieure à n. Par hypothèse de récurrence, on peut trouver pour chaque $1\leq j\leq q$ une suite $z_1^j,\dots,z_{m_j}^j$ telle que z_1^j (resp. $z_{m_j}^j$) est dans la facette de y_{k_j} (resp. $y_{k'_j-1}$), telle que z_i^j et z_{i+1}^j sont dans des facettes disjointes d'une même facette pour $1\leq i<m_j$, et enfin telle que

$$d_X(y_{k_j},z_1^j)+\sum_{i=1}^{m_j-1}d_X(z_i^j,z_{i+1}^j)+d_X(z_{m_j}^j,y_{k'_j-1})\leq C^n\sum_{i=k_j}^{k'_j-2}d_X(y_i,y_{i+1}).$$

Pour conclure la preuve du lemme, il suffit de considérer $(x_1,\dots,x_k)$ et remplacer pour chaque $1\leq j\leq q$ la sous-suite $(x_{k_j},\dots,x_{k'_j})$ par $(z_1^j,\dots,z_{m_j}^j)$.

La démonstration formelle qui suit est très fastidieuse. On note $w_{i\dots j} = w_i \dots w_j$, qui est vide si $i \geq j$, et l'on note $(uv)_{i\dots j} = u_i v_i u_{i+1} v_{i+1} \dots u_j v_j$. On note $\ell(w_{1\dots k}) = \sum_{i=1}^{k-1} d_X(w_i, w_{i+1})$. La liste construite ci-dessus est

$$z = x_{1\dots k_1-1} \cdot z^1_{1\dots m_1} \cdot x_{k'_1+1\dots k_2-1} \cdot z^2_{2\dots m_2} \cdot x_{k'_2+1\dots k_3-1} \cdots x_{k'_{q-1}+1\dots k_q-1} \cdot z^q_{1\dots m_q} \cdot x_{k'_q+1\dots k}.$$

On sait que pour tout $1 \leq j \leq q$,

$$\ell(y_{k_j} \cdot z^j_{1\dots m_j} \cdot y_{k'_j-1}) \leq C^n \ell(y_{k_j\dots k'_j-1}), \text{ et}$$
$$\ell((xy)_{k_j\dots k'_j-1} x_{k'_j}) \leq C\ell(x_{k_j\dots k'_j}).$$

Par l'inégalité triangulaire, il vient

$$\begin{aligned}
\ell(z) &\leq \ell(x_{1\dots k_1} \cdot y_{k_1} \cdot z^1_{1\dots m_1} \cdot y_{k'_1-1} \cdot x_{k'_1\dots k_2} \cdot y_{k_2} \cdot x_{k'_2\dots k_3} \cdots x_{k'_{q-1}\dots k_q} \cdot y_{k_q} \cdot z^q_{1\dots m_q} \cdot y_{k'_q-1} \cdot x_{k'_q\dots k}) \\
&\leq C^n \ell(x_{1\dots k_1} \cdot y_{k_1\dots k'_1-1} \cdot x_{k'_1\dots k_2} \cdot y_{k_2\dots k'_2-1} \cdot x_{k'_2\dots k_3} \cdots x_{k'_{q-1}\dots k_q} \cdot y_{k_q\dots k'_q-1} \cdot x_{k'_q\dots k}) \\
&\leq C^n \ell(x_{1\dots k_1-1} \cdot (xy)_{k_1\dots k'_1-1} x_{k'_1} \cdot x_{k'_1+1\dots k_2} \cdots (xy)_{k_q\dots k'_q-1} x_{k'_q} \cdot x_{k'_q+1\dots k}) \\
&\leq C^{n+1} \ell(x_{1\dots k_1-1} \cdot x_{k_1\dots k'_1} \cdot x_{k'_1+1\dots k_2} \cdots x_{k_q\dots k'_q} \cdot x_{k'_q+1\dots k}) \\
&= C^{n+1} \ell(x_{1\dots k}).
\end{aligned}$$

□

11.3.4 Démonstration du théorème 11.1.1

On trouve facilement une suite de points $x_1, x_2, \dots, x_k$ alignés dans cet ordre sur $[x, \gamma x]$ telle que $x = x_1$ et $\gamma x = x_k$, telle que x_i et x_{i+1} sont dans une même chambre pour tout $1 \leq i < k$, et telle que $d_\Omega(x, \gamma x) = d_\Omega(x_1, x_2) + \cdots + d_\Omega(x_{k-1}, x_k)$. Le fait 11.3.6 et le lemme 11.3.7 nous fournissent alors $m \geq 1$ et une suite $y_1, \dots y_m$ telle que $y_1 \in P$ et $y_m \in \gamma P$, et y_i et y_{i+1} sont dans des facettes disjointes d'une même chambre $\gamma_i P$ pour tout $1 \leq i < m$, ainsi que

$$(m-1)R \leq d_\Omega(y_1, y_2) + \cdots + d_\Omega(y_{m-1}, y_m) \leq (2\alpha_{\Gamma,S} + 1)^{\mathtt{d}} d_\Omega(x, \gamma x).$$

Notons $\gamma_0 = \mathrm{id}$ et $\gamma_m = \gamma$, et observons que $\gamma_i^{-1}\gamma_{i+1}$ fixe une facette de P pour tout $0 \leq i < m$, et a donc une longueur plus petite que $\alpha_{\Gamma,S}$ pour la métrique des mots ; par conséquent $|\gamma| \leq \alpha_{\Gamma,S} m$.

11.4 Exemples

Dans cette section, nous considérons deux exemples de familles de polytopes de Coxeter hilbertiens en dimension $\mathtt{d} = 3$, choisis de la forme (P, Ω) où P est compact, n'est pas un simplexe, et son groupe Γ est fortement irréductible mais pas Gromov hyperbolique ; rappelons que Nie [Nie15b] avait considéré *tous* les exemples où P est un simplexe compact et Γ est Gromov hyperbolique.

Le premier exemple que nous allons présenter en section 11.4.1 ne va pas aboutir, au sens où nous ne pourrons pas lui appliquer le théorème 11.1.1. En fait, pour cette famille de polytopes de Coxeter hilbertiens, l'exposant critique peut être minoré par une contante strictement positive d'après la section 11.2 et un résultat de Crampon [Cra11, Prop. 5.4.1]. Nous appliquerons le théorème 11.1.1 au deuxième exemple, présenté en section 11.4.2. Celui-ci sera élaboré en appliquant au premier exemple la technique de pliage, suggérée par L. Marquis, qu'on remercie donc chaleureusement.

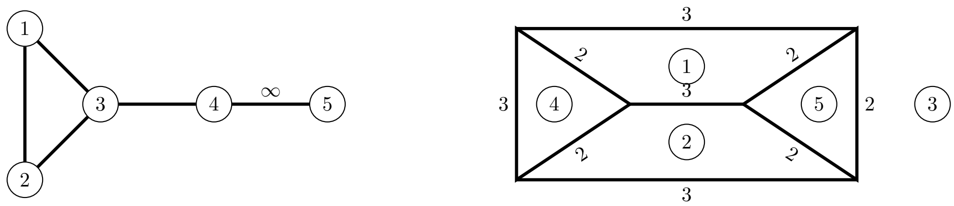

Figure 11.1: Le diagrame de Γ_Y et son polytope

11.4.1 L'exemple de Benoist

L'exemple qui va suivre est dû à Benoist [Ben06a, Sec. 4.3] (ce fut le premier exemple de convexe divisible irréductible non symétrique et non strictement convexe). Soit Γ_Y le groupe de Coxeter donné par le diagrame à gauche dans la figure 11.1 ; On note r_1, r_2, r_3, r_4 and r_5 les 5 réflexions associées au sommet du graphe. Le groupe de Coxeter abstrait Γ_Y est infini, non affine est irréductible. On définit pour tout $t > 1$ une representation ρ_t de Γ_Y dans $\mathrm{GL}(\mathbb{R}^4)$ en donnant l'image des générateurs $r_1, \dots, r_5$.

$$\rho_t(r_1) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ \rho_t(r_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ \rho_t(r_3) = \begin{pmatrix} 0 & 0 & 1/t & 0 \\ 0 & 1 & 0 & 0 \\ t & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\rho_t(r_4) = \begin{pmatrix} 1+\mu & \mu & \mu & -\mu \\ \mu & 1+\mu & \mu & -\mu \\ \mu & \mu & 1+\mu & -\mu \\ \nu & \nu & \nu & 1-\nu \end{pmatrix}, \ \rho_t(r_5) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

où $\mu = t/(t-1)^2$ et $\nu = 2 + 3\mu$; le polytope P_t est donné par

$$P_t = \{[x_1, x_2, x_3, x_4] \in \mathrm{P}(\mathbb{R}^4) : x_1 \leq x_2 \leq x_3 \leq tx_1 \text{ et } 0 \leq x_4 \leq x_1 + x_2 + x_3\},$$

et l'on note $\Omega_t = \rho_t(\Gamma_Y) \cdot P_t$.

Pour tout $t > 1$, on vérifie à l'aide du fait 11.3.1 que ρ_t est injective d'image discrète, définit un groupe de Coxeter linéaire dont le cône polyèdral fondamental se projette sur P_t et dont le cône de Vinberg est proprement convexe, et enfin que $\rho_t(\Gamma)$ divise la projection dans $\mathrm{P}(\mathbf{V})$ de l'intérieur du cône de Vinberg.

Lorsque t tend vers l'infini, $\ell(\rho_t(r_4 r_5))$ tend vers zéro. En effet,

$$\rho_t(r_4)\rho_t(r_5) = \begin{pmatrix} 1+\mu & \mu & \mu & \mu \\ \mu & 1+\mu & \mu & \mu \\ \mu & \mu & 1+\mu & \mu \\ \nu & \nu & \nu & \nu-1 \end{pmatrix} \underset{t\to\infty}{\longrightarrow} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & 2 & 1 \end{pmatrix}.$$

Lorsque t tend vers 1, c'est $\ell(\rho_t(r_2 r_1 r_2 r_3))$ qui tend vers zéro. En effet,

$$\rho_t(r_2)\rho_t(r_1)\rho_t(r_2)\rho_t(r_3) = \begin{pmatrix} t & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/t & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \underset{t\to 1}{\longrightarrow} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

D'après le corollaire 11.2.2 et le fait, dû à Crampon [Cra11, Prop. 5.4.1], que $t \mapsto \delta_{\rho_t}$ est continue, on déduit que $\inf_{t>1} \delta_{\rho_t} > 0$.

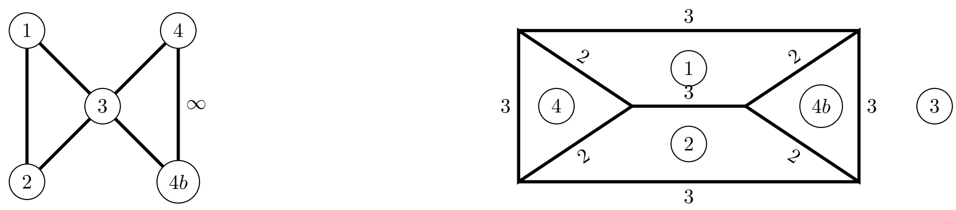

Figure 11.2: Le diagrame de Γ_L et son polytope

11.4.2 Plions l'exemple de Benoist

Considérons à présent le sous-groupe d'indice 2 de Γ_Y engendré par les réflexions r_1, r_2, r_3, r_4 et $r_{4b} := r_5 r_4 r_5$; c'est lui aussi un groupe de Coxeter, dont le diagrame est donné dans la figure 11.2 à gauche. On déforme par pliage la restriction à Γ_L des représentations $(\rho_t)_{t>1}$ présentées à la section précédente, et cela donne une famille de représentations à deux paramètres $(\rho_{t,b})_{t>1,b\geq 1}$, où b est le paramètre de pliage (bending). Plus précisément, pour $t>1$ et $b \geq 1$ on définit $\rho_{t,b}$ en posant $\rho_{t,b}(r_1) = \rho_t(r_1)$, $\rho_{t,b}(r_2) = \rho_t(r_2)$, $\rho_{t,b}(r_3) = \rho_t(r_3)$, $\rho_{t,b}(r_4) = \rho_t(r_4)$, et

$$\rho_{t,b}(r_{4b}) = \beta_b \rho_t(r_5)\rho_t(r_4)\rho_t(r_5)\beta_b^{-1},$$

où

$$\beta_b = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & b \end{pmatrix} ;$$

on pose de plus $P_{t,b} = P_t \cup \beta_b \rho_t(r_5) P_t$ et $\Omega_{t,b} = \rho_{t,b}(\Gamma_L) \cdot P_{t,b}$.

Comme avant, pour tous $t > 1$ et $b \geq 1$, on vérifie à l'aide du fait 11.3.1 que $\rho_{t,b}$ est injective d'image discrète, définit un groupe de Coxeter linéaire dont le cône polyèdral fondamental se projette sur $P_{t,b}$ et dont le cône de Vinberg est proprement convexe, et enfin que $\rho_{t,b}(\Gamma)$ divise la projection dans $\mathrm{P}(\mathbf{V})$ de l'intérieur du cône de Vinberg.

Lorsque t tend vers 1, $\ell(\rho_{t,b}(r_2 r_1 r_2 r_3))$ tend vers zéro indépendamment de b. Le cas qui nous intéresse est lorsque t tend vers l'infini. Nous allons démontrer, à l'aide de l'observation 11.1.2, que la grosseur de $(P_{t,t^2}, \Omega_{t,t^2})$ tend vers l'infini avec t. Étant donnée la combinatoire de $P_{t,b}$, on se convainc facilement que pour toute paire de facettes disjointes F et F', on peut trouver une face F'_1 disjointe de F et qui contient F'. Minorons au cas par cas l'écart entre les paires de facettes disjointes F et F' telles que F' est de dimension 2 et F est maximale pour l'inclusion.

- Soit F la face fixée par r_4 et F' la face fixée par r_{4b}. Alors

$$\beta_t \rho_{t,t^2}(r_4) \beta_t^{-1} \underset{t\to\infty}{\longrightarrow} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

  tandis que

$$\beta_t \rho_{t,t^2}(r_{4b}) \beta_t^{-1} \underset{t\to\infty}{\longrightarrow} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & -2 & -2 & 0 \end{pmatrix},$$

d'où

$$\beta_t \rho_{t,t^2}(r_4 r_{4b})\beta_t^{-1} \underset{t\to\infty}{\longrightarrow} \begin{pmatrix} 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \end{pmatrix}.$$

Cette dernière matrice a une valeur propre nulle et une autre non nulle. En conséquence, $\ell(\rho_{t,t^2}(r_4 r_{4b}))$ tend vers l'infini avec t.

- De manière symétrique, si F est la face fixée par r_{4b} et F' est la face fixée par r_4, alors $r_{4b} r_4 = (r_4 r_{4b})^{-1}$, donc $\ell(\rho_{t,t^2}(r_{4b} r_4))$ tend vers l'infini avec t.

- Soit F l'intersection des faces fixées par respectivement r_1 et r_2, et soit F' la face fixée par r_3. Alors

$$\rho_{t,t^2}(r_2 r_1 r_2 r_3) = \rho_t(r_2 r_1 r_2 r_3) = \begin{pmatrix} t & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/t & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Ainsi $\ell(\rho_{t,t^2}(r_2 r_1 r_2 r_3))$ tend vers l'infini avec t.

- Soit F l'intersection des faces fixées respectivement par r_1 et r_3, et soit F' la face fixée par r_2. Alors $r_1 r_3 r_1 r_2$ égale $r_2 r_1 r_1 r_2 r_1 r_3 r_1 r_2$, qui est conjugué à $r_1 r_2 r_1 r_3 = r_2 r_1 r_2 r_3$. Par voie de conséquence, $\ell(\rho_{t,t^2}(r_1 r_3 r_1 r_2))$ tend vers l'infini avec t.

- Soit F l'intersection des faces fixées respectivement par r_2 et r_3, et soit F' la face fixée par r_1. Alors $r_2 r_3 r_2 r_1$ égale $r_1 r_2 r_2 r_1 r_2 r_3 r_2 r_1$, qui est conjugué à $r_2 r_1 r_2 r_3$. Par voie de conséquence, $\ell(\rho_{t,t^2}(r_2 r_3 r_2 r_1))$ tend vers l'infini avec t.

11.5 Polytopes à pointes

11.5.1 Définition des polytopes à pointes

On s'attache à présent à définir rigoureusement les polytopes de Coxeter hilbertiens à pointes paraboliques de type $\tilde{A}_{\mathtt{d}-1}$ (ou polytopes à pointes) et leur grosseur non cuspidale. On commence par définir un polytope de Coxeter particulier, qu'on pourrait qualifier de *polytope de Coxeter parabolique de type $\tilde{A}_{\mathtt{d}-1}$ standard.* Munissons $\mathbb{R}^{\mathtt{d}-1}$ de sa norme euclidienne usuelle $\|\cdot\|$, et considérons le morphisme

$$\begin{array}{rccc} \tau_A & : & \mathrm{Isom}(\mathbb{R}^{\mathtt{d}-1}) \simeq \mathrm{O}(\mathtt{d}-1) \ltimes \mathbb{R}^{\mathtt{d}-1} & \longrightarrow & \mathrm{GL}(\mathbf{V}) \\ & & (M, v) & \longmapsto & \begin{pmatrix} 1 & {}^t v & \frac{\|v\|^2}{2} \\ 0 & M & v \\ 0 & 0 & 1 \end{pmatrix} \end{array}.$$

On considère également le sous-groupe à un paramètre $U = \{u_t\}_{t\in\mathbb{R}} \subset \mathrm{SL}(\mathbf{V})$, où, pour $t \in \mathbb{R}$, on pose $u_t(e_i) = e_i$ pour $i \leq \mathtt{d}$ et $u_t(e_{\mathtt{d}+1}) = e_{\mathtt{d}+1} + te_1$; notons que U est dans le centralisateur de $\tau_A(\mathrm{Isom}(\mathbb{R}^{\mathtt{d}-1}))$.

On fixe un simplexe régulier Δ_A de $\mathbb{R}^{\mathtt{d}-1}$ centré en 0, on note $S_A \subset \mathrm{GL}(\mathbf{V})$ l'image par τ_A de l'ensemble des réflexions le long des facettes du simplexe ainsi que $\Gamma_A \subset \mathrm{GL}(\mathbf{V})$ le sous-groupe engendré par S_A. On pose $\Omega_A = u_{]0,\infty[} \cdot \tau_A(\mathbb{R}^{\mathtt{d}-1}) \cdot [e_{\mathtt{d}+1}]$, et l'on considère les polytopes $\hat{P}_A = U \cdot \tau_A(\Delta_A) \cdot [e_{\mathtt{d}+1}]$ et $P_A = \hat{P}_A \cap \Omega_A = u_{]0,\infty[} \cdot \tau_A(\Delta_A) \cdot [e_{\mathtt{d}+1}]$. La paire (P_A, Ω_A) est le *polytope de Coxeter parabolique de type $\tilde{A}_{\mathtt{d}-1}$ standard.* On va en fait s'interesser à toutes les paires de la forme $(\hat{P}_A \cap \Omega, \Omega)$ où Ω est un ouvert proprement convexe Γ_A-invariant.

Un polytope de Coxeter hilbertien (P,Ω) de groupe (Γ,S) est dit *parabolique de type* $\tilde{A}_{\mathtt{d}-1}$ s'il existe $g \in \mathrm{GL}(\mathbf{V})$ tel que $gSg^{-1} = S_A$, auquel cas on peut vérifier que $gP = P_A \cap g\Omega$. Un polytope de Coxeter hilbertien (P,Ω) est dit *à pointes paraboliques de type* $\tilde{A}_{\mathtt{d}-1}$ (on dira *polytope à pointes* pour faire court) si $\overline{P} \cap \partial\Omega$ est fini et non vide, et si pour tout $p \in \overline{P} \cap \partial\Omega$ (*sommet à l'infini* de P), la paire $(P_{|p},\Omega)$ est un polytope de Coxeter hilbertien parabolique de type $\tilde{A}_{\mathtt{d}-1}$, où $P_{|p}$ est l'adhérence de la composante connexe de Ω privé des hyperplans induits par P adjacents à p ; autrement dit s'il existe $g \in \mathrm{GL}(\mathbf{V})$ tel que $gS'g^{-1} = S_A$, où S' est l'ensemble des réflexions le long des faces de P qui fixent p.

11.5.2 Voisinage des sommets à l'infini et grosseur non-cuspidale

Pour définir la grosseur non cuspidale d'un polytope à pointes (P,Ω), on a besoin de choisir de façon canonique un voisinage dans Ω de chaque sommet à l'infini. Pour ce faire on utilise le lemme suivant, qui est en fait un lemme-clé dans la démonstration du théorème. Précisons que l'idée derrière ce lemme est classique, on peut par exemple la trouver dans [Mar12b, Mar12a, CM14a] ; on a simplement un peu renforcé l'énoncé en inversant deux quantificateurs, car l'ouvert proprement convexe Ω sera appelé à varier, et aura besoin que le nombre t_A du lemme reste constant.

Lemme 11.5.1. *Il existe $t_A > 0$ tel que pour tout ouvert proprement convexe Γ_A-invariant Ω, il existe $t \in \mathbb{R}$ tel que $u_{t+t_A}\Omega_A \subset \Omega \subset u_t\Omega_A$.*

Preuve. Soit Ω un ouvert proprement convexe Γ_A-invariant. D'après le fait 2.2.3, l'ouvert Ω n'intersecte pas $\mathrm{P}(\mathbb{R}^{\mathtt{d}} \times \{0\})$, et il est inclus dans la carte affine $\mathrm{P}(\mathbf{V}) \smallsetminus \mathrm{P}(\mathbb{R}^{\mathtt{d}} \times \{0\})$, égale à $U\tau_A(\mathbb{R}^{\mathtt{d}-1})[e_{\mathtt{d}+1}]$, qu'on utilise dans la suite de la preuve quand on veut parler d'enveloppe convexe. Notons $\tilde{\Gamma}_A \subset \mathrm{Isom}(\mathbb{R}^{\mathtt{d}-1})$ le sous-groupe engendré par les réflexions le long des faces de Δ_A.

Remarquons à présent que $u_t\overline{\Omega} \subset \overline{\Omega}$ pour tout $x \geq 0$, et donc que $[e_1] \in \partial\Omega$. En effet, si $x \in \overline{\Omega}$ est dans la carte affine, alors $[x,\gamma x] \subset \overline{\Omega}$ pour tout $\gamma \in \Gamma_A$ (le segment est pris au sens de l'enveloppe convexe dans la carte affine qu'on a fixée), et ces segments s'accumulent sur $u_{[0,\infty[} \cup \{[e_1]\}$, qui est donc aussi inclus dans $\overline{\Omega}$.

Vérifier que $\partial\Omega$ contient $u_t[e_{\mathtt{d}+1}]$ pour un certain $t \in \mathbb{R}$. En effet, soit $t_1 \in \mathbb{R}$ et $v_1 \in \mathbb{R}^{\mathtt{d}-1}$ tels que $u_{t_1}\tau_A(v_1) \in \overline{\Omega}$. Comme $\tilde{\Gamma}_A$ agit cocompactement sur $\mathbb{R}^{\mathtt{d}-1}$, le point 0 appartient à l'enveloppe convexe de $\tilde{\Gamma}_A \cdot v_1$, et donc l'enveloppe convexe dans la carte affine de $\Gamma_A \cdot u_{t_1}\tau_A(v_1)$ contient un point de $U[e_{\mathtt{d}+1}]$, donc on peut trouver $t \in \mathbb{R}$ tel que $u_t[e_{\mathtt{d}+1}] \in \partial\Omega$.

En utilisant encore la cocompacité de $\tilde{\Gamma}_A$, on peut trouver un sous-ensemble fini $F \subset \tilde{\Gamma}_A$ tel que l'enveloppe convexe de $F \cdot 0$ contient Δ_A. Notons C l'enveloppe convexe dans la carte affine de $\{\tau_A(\gamma)[e_{\mathtt{d}+1}] : \gamma \in F\}$. Pour $t' > 0$ assez grand (qui ne dépend que de $\mathtt{d}$), $u_{[-t',0]}C$ contient $\tau_A(\Delta_A)$. Comme U et $\tau_A(\mathrm{Isom}(\mathbb{R}^{\mathtt{d}-1}))$ commutent, il s'ensuit que $\overline{\Omega}$ contient $u_{t+t'}\tau_A(\Delta_A)$, donc aussi $u_{t+t'}\tau_A(\mathbb{R}^{\mathtt{d}-1})$ et même $u_{t+t'}\Omega_A$ puisque $u_s\overline{\Omega} \subset \overline{\Omega}$ pour tout $s \geq 0$.

Soit $\mathcal{O}$ un voisinage ouvert de $u_{t+t'+1}[e_{\mathtt{d}+1}]$ dans Ω. Par convexité, pour tout point y de la carte affine, s'il existe $z \in \mathcal{O}$ tel que $u_t[e_{\mathtt{d}+1}] \in [y,z]$, alors $y \notin \overline{\Omega}$; or l'ensemble des points y qui satisfont ceci est un cône ouvert basé en $u_t[e_{\mathtt{d}+1}]$ qui contient $u_{t-t''}\tau_A(\Delta_A)$ pour un certain $t'' \in \mathbb{R}$ qui ne dépend que de $\mathtt{d}$, si bien que $\overline{\Omega} \subset u_{t-t''}\Omega_A$. Ceci conlut la preuve. ☐

Pour tout ouvert proprement convexe Γ_A-invariant Ω, on pose $\mathcal{H}_\Omega = u_{t_A+1}\Omega$, qu'on appelle *voisinage du sommet à l'infini* ; c'est une horoboule de Ω ; pour tout $s \in \mathbb{R}$ on note $\mathcal{H}_\Omega^s = u_{(t_A+1)e^{-2s}}\Omega$ l'horoboule à distance (algébrique) s, qu'on appelle *s-voisinage du sommet à l'infini.* Soit (P, Ω) un polytope de Coxeter hilbertien parabolique de type $\tilde{A}_{\mathtt{d}-1}$, de groupe (Γ, S) ; le voisinage canonique du sommet à l'infini est $g^{-1}\mathcal{H}_{g\Omega}$, où $g \in \mathrm{GL}(\mathbf{V})$ est tel que $gSg^{-1} = S_A$. Pour s'assurer que cette définition a un sens, il convient de vérifier que $g^{-1}\mathcal{H}_{g\Omega} = \mathcal{H}_\Omega$ pour tout ouvert proprement convexe Γ_A-invariant Ω et tout élément g du normalisateur de S_A dans $\mathrm{GL}(\mathbf{V})$; cela provient du lemme qui suit.

Lemme 11.5.2. *Le normalisateur de S_A dans $\mathrm{GL}(\mathbf{V})$ est l'ensemble des éléments de la forme $\lambda\tau_A(\sigma)u_t$, où $\lambda \in \mathbb{R}^*$, $t \in \mathbb{R}$ et σ est dans le groupe de symétrie de Δ_A.*

Preuve. Soit g dans le normalisateur de S_A. Alors g permute les hyperplans fixés par les éléments de S_A, qui bordent les faces de $\hat{P}_A$; il permute donc également les faces de Δ_A. On peut trouver σ dans le groupe de symétrie de Δ_A qui induit la même permutation, de sorte que $g' = \tau_A(\sigma)^{-1}g$ préserve chaque face de $\hat{P}_A$, et ainsi commute avec chaque élément de S_A. Mais alors g' est dans le centralisateur de Γ_A, donc dans celui de $\Gamma_A \cap \tau_A(\mathbb{R}^{\mathtt{d}-1})$, qui est cocompact, donc Zariski-dense, dans $\tau_A(\mathbb{R}^{\mathtt{d}-1})$. On a démontré que g' est dans le centralisateur de $\tau_A(\mathbb{R}^{\mathtt{d}-1})$, et il n'est pas dur de voir que cela entraîne que g' s'écrit sous la forme

$$\begin{pmatrix} \lambda & {}^t v & \frac{\|v\|^2}{2} + t \\ 0 & \lambda\,\mathrm{id} & v \\ 0 & 0 & \lambda \end{pmatrix},$$

pour un certain $\lambda \in \smallsetminus\{0\}$, un certain $t \in \mathbb{R}$ et un certain $v \in \mathbb{R}^{\mathtt{d}-1}$.

Par ailleurs, les vecteurs propres $v_1, \ldots, v_{\mathtt{d}} \in \mathrm{Vect}(e_1, e_2, \ldots, e_{\mathtt{d}+1})$, associés à la valeur propre -1, des éléments de S_A, sont tous des vecteurs propres de g', et engendrent $\mathrm{Vect}(e_1, \ldots, e_{\mathtt{d}})$. Ainsi la restriction de g' à $\mathrm{Vect}(e_1, \ldots, e_{\mathtt{d}})$ est diagonalisable, ce qui entraîne que $v = 0$, et conclut la démonstration. □

Soit (P, Ω) un polytope à pointes. On définit la *grosseur non cuspidale* de (P, Ω) comme le plus petit écart possible entre deux sous-ensembles A et $B \subset \Omega$ qui sont ou bien deux facettes de P disjointes et non adjacentes à un même sommet à l'infini, ou bien deux voisinages de sommets à l'infini différents, ou bien le voisinage d'un sommet à l'infini et une facette de P non adjacente au sommet à l'infini.

11.5.3 Un mot sur la preuve du théorème 11.1.3

La démonstration du théorème 11.1.3 est un mélange de celle du théorème 11.1.1 et du chapitre précédent. Donnons-nous un polytope à pointes (P, Ω) de groupe (Γ, S), un élément $\gamma \in \Gamma$ et un point $x \in P$ qui n'est dans aucun voisinage de sommet à l'infini. On va établir une minoration de $d_\Omega(x, \gamma x)$ (c'est (11.5.8)), similaire à celle du théorème 11.1.1 et à (10.3.1), qui fait intervenir la grosseur non cuspidale, des constantes universelles, des constantes qui dépendent de (Γ, S), et enfin des quantités qui dépendent de $\gamma \in \Gamma$ (l'équivalent longueur pour la métrique des mots dans le théorème 11.1.1 mais en plus compliqué).

11.5.4 Saisir le taureau par les cornes

Dans cette section on rassemble quelques estimées utiles sur les polytopes de Coxeter hilbertiens de type $\tilde{A}_{\mathtt{d}-1}$, ainsi que plus généralement sur les polytopes à pointes. Le lemme suivant une conséquence du lemme 11.5.1.

Lemme 11.5.3. *Il existe des constantes $C_{A,1}, C_{A,2}, C_{A,3} \geq 1$ qui vérifient les propriétés suivantes. Soit Ω un ouvert proprement convexe Γ_A-invariant, soit $h \in \Gamma_A$, soient $x \in \hat{P}_A \cap \Omega \smallsetminus \mathcal{H}_\Omega$ et $y \in h\hat{P}_A \cap \Omega \smallsetminus \mathcal{H}_\Omega$. On note x' (resp. y') est le point d'intersection de $U \cdot x$ (resp. $U \cdot y$) avec $\partial\mathcal{H}_\Omega$. Alors*

1. *Si $d_{\Omega_A}(x_A, hx_A) \geq C_{A,1}$, alors $d_\Omega(x,x') + d_\Omega(x',y') + d_\Omega(y',y) \leq d_\Omega(x,y) + 2$;*
2. *si $h = \mathrm{id}$ et les facettes de x et y sont disjointes, alors*
$$d_\Omega(x,x') + d_\Omega(x',y') + d_\Omega(y',y) \leq C_{A,2} d_\Omega(x,y);$$
3. *$d_\Omega(x',y') - C_{A,3} \leq d_{\Omega_A}(x_A, hx_A) \leq d_\Omega(x,y) + C_{A,3}$.*

Preuve. Démontrons le point 1. En fait on va trouver une constante $C'_{A,1}$ qui ne dépend que de t_A et qui vérifie la propriété suivante. Fixons la carte affine $(t_1, \ldots, t_d) \mapsto [t_1 e_1 + \cdots + t_d e_d + e_{\mathtt{d}+1}] \in \mathrm{P}(\mathbf{V})$. Soit Ω un ouvert proprement convexe (pas forcément Γ_A-invariant) tel que $\Omega_A \subset \Omega \subset u_{-t_A}\Omega_A$. Soient $x, y \in \Omega$ dont la différence a une projection orthogonale sur $\mathrm{Vect}(e_2, \ldots, e_d)$ de norme plus grande que $C'_{A,1}$, soit x' (resp. y') le point d'intersection de $U \cdot x = x + e_1\mathbb{R}$ (resp. $U \cdot y$) avec $u_{t_A+1}\partial\Omega$, et soit z le milieu de $[x,y]$, alors la distance de x' (resp. y') à $[x,z]$ (resp. $[z,y]$) dans Ω est plus petite que 1 (ceci implique que $d_\Omega(x,x') + d_\Omega(x',y') + d_\Omega(y',y) \leq d_\Omega(x,y) + 2$).

Pour démontrer ceci, on considère le point d'intersection x_1 (resp. y_1) de $U \cdot x$ (resp. $U \cdot y$) avec $u_{-t_A}\partial\Omega_A$, z_1 le milieu de $[x_1, y_1]$, et x'_1 le point d'intersection de $U \cdot x$ avec $u_{t_A+1}\partial\Omega_A$. Par convexité, car $d_\Omega \leq d_{\Omega_A}$, et par le lemme 2.1.4,
$$d_\Omega(x', [x,z]) \leq d_\Omega(x', \Omega \cap [x_1, z_1]) \leq d_{\Omega_A}(x', \Omega_A \cap [x_1, z_1]) \leq d_{\Omega_A}(x'_1, \Omega_A \cap [x_1, z_1]).$$

Le problème ne dépend à présent plus de Ω, et l'on voit rapidement qu'on s'est ramené à démontrer que pour tout $t > 0$, la distance entre $(0, t+1)$ et le segment de $(0, -t)$ à $(x/2, x^2/4 - t)$ tend vers zéro quand x tend vers l'infini, ce qui est vrai.

Démontrons le point 3. D'après le lemme 11.5.1, on peut supposer que $\Omega_A \subset \Omega \subset u_{-t_A}\Omega_A$. Alors
$$\begin{aligned} d_\Omega(x',y') &\geq d_{u_{-t_A}\Omega_A}(x',y') \\ &\geq d_{u_{-t_A}\Omega_A}(x_A, hx_A) - d_{u_{-t_A}\Omega_A}(x', x_A) - d_{u_{-t_A}\Omega_A}(h^{-1}y', x_A) \\ &\geq d_{u_{-t_A}\Omega_A}(x_A, hx_A) - 2\,\mathrm{diam}_{\Omega_A}(\hat{P}_A \cap u_{[1,1+t_A]}\partial\Omega_A). \end{aligned}$$

Or $|d_{\Omega_A}(x_A, hx_A) - d_{u_{-t_A}\Omega_A}(x_A, hx_A)|$ est bornée par une constante indépendante de h, d'où
$$d_\Omega(x',y') \geq d_{\Omega_A}(x_A, hx_A) - C'_{A,3}.$$

Pour une certaine constante $C'_{A,3}$. L'autre inégalité est similaire, sinon plus simple :
$$d_\Omega(x',y') \leq d_{\Omega_A}(x_A, hx_A) + 2\,\mathrm{diam}_{\Omega_A}(\hat{P}_A \cap u_{[1,1+t_A]}\partial\Omega_A).$$

Montrons pour finir qu'il existe une constante $C_{A,3}$ telle que $d_\Omega(x,y) \geq d_{\Omega_A}(x_A, hx_A) - C_{A,3}$. Si $d_{\Omega_A}(x_A, hx_A) \leq C_{A,1}$ alors on peut prendre $C_{A,3} = C_{A,1}$. Si au contraire $d_{\Omega_A}(x_A, hx_A)$, alors on peut utiliser le point 1, et l'on a
$$d_\Omega(x,y) \geq d_\Omega(x',y') - 2 \geq d_{\Omega_A}(x_A, hx_A) - 2 - C'_{A,3}.$$

Démontrons le point 2. On fixe $h_A \in \Gamma_A$ tel que $d_{\Omega_A}(x_A, h_A x_A) \geq C_{A,1} + C_{A,3} + 2$, que l'on écrit sous la forme $h_A = s_1 s_2 \cdots s_n$ où $s_i \in S_A$ pour $1 \leq i \leq n$ et $n = |h_A|$. Comme les facettes de x et y sont disjointe, tout élément de S_A fixe x ou y, donc

$$[s_1 \cdots s_k x, s_1 \cdots s_k y] \cup [s_1 \cdots s_{k+1} x, s_1 \cdots s_{k+1} y] = s_1 \ldots s_k \left([x, y] \cup [s_{k+1} x, s_{k+1} y]\right)$$

est connexe pour tout $1 \leq k < n$. Ainsi $\bigcup_{k=1}^{n} [s_1 \cdots s_k x, s_1 \ldots s_k y]$ est connexe, et cela implique que

$$d_\Omega(x, h_A y) \leq \sum_{k=1}^{n} d_\Omega(s_1 \cdots s_k x, s_1 \ldots s_k y) = |h_A| d_\Omega(x, y).$$

D'après les points 1 et 3, $d_\Omega(x, h_A y) \geq 2$ et

$$d_\Omega(x, h_A y) \geq \frac{1}{2}(d_\Omega(x, x') + d_\Omega(x', h_A y') + d_\Omega(h_A y', h_A y)).$$

Or d'après 3, on a $d_\Omega(x', y') \leq C_{A,3}$ et $d_\Omega(x', h_A y') \geq 2$, si bien que

$$\begin{aligned} d_\Omega(x, y) &\geq \frac{|h_A|}{2} \left(d_\Omega(x, x') + \frac{2}{C_{A,3}} d_\Omega(x', y') + d_\Omega(y', y) \right) \\ &\geq \frac{|h_A|}{2C_{A,3}} (d_\Omega(x, x') + d_\Omega(x', y') + d_\Omega(y', y)) \end{aligned}$$ □

Nous aurons aussi besoin d'estimées similaires à celles du fait 11.3.6, mais valable pour un système de facettes tronquées, c'est-à-dire auquelles on a enlevé des voisinages de sommets à l'infini. Soit (P, Ω) un polytope à pointes, de groupe (Γ, S) ; pour chaque sommet à l'infini $p \in \overline{P} \cap \partial\Omega$, notons $\mathcal{H}_p$ son voisinage. Supposons que la grosseur non cuspidale R de (P, Ω) soit non nulle ; en particulier, pour tout $0 \leq s < R/2$, les ensembles $\overline{\mathcal{H}}_p^s$ et $\overline{\mathcal{H}}_{p'}^s$ sont disjoints pour toute paire de sommet à l'infini distincts $p, p' \in \overline{P} \cap \partial\Omega$, et $\overline{\mathcal{H}}_p^s$ et F sont disjoints pour toute facette F de P et $p \in \overline{P} \cap \partial\Omega \smallsetminus \overline{F}$.

Pour tout $s \in \mathbb{R}$, on définit le *polytope tronqué* $\mathrm{tro}_s P$ comme P privé des s-voisinages de ses sommets à l'infini, et l'on définit les facettes du polytope tronqué comme les ensembles de la forme $F \cap \mathrm{tro}_s P$ ou bien $\partial\mathcal{H}_p^s \cap F$ (de dimension $\dim(F) - 1$) ou bien $\partial\mathcal{H}_p^s \cap P$ (de dimension $\mathtt{d} - 1$), où F est une facette de P et p un sommet à l'infini de P. On pose $\mathrm{tro}_s\Omega = \Gamma \cdot \mathrm{tro}_s P$; les chambres tronquées sont les sous-ensembles de la forme $\gamma \mathrm{tro}_s P$ pour $\gamma \in \Gamma$.

Lemme 11.5.4. *Soit (P, Ω) un polytope à pointes de groupe (Γ, S) et de grosseur non cuspidale $R > 0$, et soit $0 \leq s < R/2$. Alors pour tout $x, y \in \mathrm{tro}_s P$ dont les facettes tronquées s'intersectent, il existe dans leur intersection un point z tel que*

$$d_\Omega(x, z) + d_\Omega(z, y) \leq 6\alpha_{\Gamma,S} d_\Omega(x, y).$$

Preuve. Les facettes non tronquées F et F' de x et y s'intersectent, donc d'après le fait 11.3.6, il existe $z \in F \cap F'$ tel que

$$d_\Omega(x, z) + d_\Omega(z, y) \leq 3\alpha_{\Gamma,S} d_\Omega(x, y).$$

Si z n'est pas dans l'intersection des facettes tronquées, alors il est dans le s-voisinage $\mathcal{H}_p^s$ d'un sommet à l'infini $p \in \overline{P} \cap \partial\Omega$ qui appartient à $\overline{F}$ et $\overline{F'}$. Soit $z' \in \partial\mathcal{H}_p^s$ tel que $z \in [p, z']$. Il suffit de démontrer que $d_\Omega(x, z') \leq 2d_\Omega(x, z)$ (la preuve de $d_\Omega(y, z') \leq 2d_\Omega(y, z)$ est similaire). Soit $z'' \in \mathrm{Vect}(p, z)$ tel que $\mathtt{b}_p(x, z'') = 0$, alors $z'' \in \mathrm{tro}_s P$ puisque $x \in \mathrm{tro}_s P$, donc $z' \in [z, z'']$. Comme les boules pour la métrique de Hilbert sont convexes, il suffit de démontrer que $d_\Omega(x, z'') \leq 2d_\Omega(x, z)$:

$$d_\Omega(x, z'') \leq d_\Omega(x, z) + d_\Omega(z, z'') = d_\Omega(x, z) - \mathtt{b}_p(z, z'') = d_\Omega(x, z) - \mathtt{b}_p(z, x) \leq 2d_\Omega(x, z).$$ □

11.5.5 Démonstration du théorème 11.1.3

Soit (P, Ω) un polytope à pointes de groupe (Γ, S) et de grosseur non cuspidale R supposée non nulle ; notons $s = R/3$. Par le lemme 11.5.3.2, l'écart entre deux facettes disjointes du polytope tronqué $\mathrm{tro}_s P$ est plus grand que $\frac{R}{3C_{A,2}}$.

Pour chaque sommet à l'infini $p \in \overline{P} \cap \partial\Omega$, on note (Γ_p, S_p) le groupe de $(P_{|p}, \Omega)$ et l'on se donne un isomorphisme $\tau_p : \Gamma_A \to \Gamma_p$ qui provient d'une identification entre S_A et S_p. Fixons enfin δ dans $](d-1)/2, d-1]$, et démontrons que $\sum_{\gamma\in\Gamma} e^{-\delta d_\Omega(x,\gamma x)}$ est finie lorsque R assez grand, quel que soit $x \in P$. Soient x dans l'intérieur de $\mathrm{tro}_s P$ et $\gamma \in \Gamma$.

On se donne $k \geq 0$ et

$$x = x_0^+,\ x_1^-,\ x_1^+,\ x_2^-,\ x_2^+,\ \ldots,\ x_k^-,\ x_k^+,\ x_{k+1}^- = \gamma x \in [x, \gamma x]$$

alignés dans cet ordre tels que $[x_i^+, x_{i+1}^-] \subset \mathrm{tro}_s\Omega$ pour tout $0 \leq i \leq k$, et $[x_i^-, x_i^+]$ est contenu dans le s-voisinage fermé d'un sommet à l'infini pour tout $1 \leq i \leq k$.

Pour tout $0 \leq i \leq k$, puisque $[x_i^+, x_{i+1}^+]$ est contenu dans $\mathrm{tro}_s\Omega$, on peut trouver un entier $k_i \geq 0$ et une suite de points

$$x_i^+ = x_{i,0},\ x_{i,1},\ \ldots,\ x_{i,k_i},\ x_{i,k_i+1} = x_{i+1}^- \in [x_i^+, x_{i+1}^-]$$

dans cet ordre tels que $x_{i,j}$ et $x_{i,j+1}$ sont dans une même chambre tronquée pour tout $0 \leq j \leq k_i$. D'après les lemmes 11.3.7 et 11.5.4, on peut trouver pour chaque $0 \leq i \leq k$ un entier $m_i \geq 0$ et une suite

$$x_i^+ = y_{i,0},\ y_{i,1},\ \ldots,\ y_{i,m_i},\ y_{i,m_i+1} = x_{i+1}^- \in \mathrm{tro}_s\Omega$$

telle que pour tout $0 \leq j \leq m_i$, les points $y_{i,j}$ et $y_{i,j+1}$ sont dans une même chambre tronquée, de la forme $\gamma_{i,j}\mathrm{tro}_s P$ pour un certain $\gamma_{i,j} \in \Gamma$, et ont des facettes tronquées disjointes si $0 < j < m_i$, et de plus telle que, en posant $C_1 = 6\alpha_{\Gamma,S}$,

$$d_\Omega(y_{i,1}, y_{i,2}) + \cdots + d_\Omega(y_{i,m_i-1}, y_{i,m_i}) \leq C_1^d d_\Omega(x_i^+, x_i^-), \tag{11.5.1}$$

Observons que $\gamma_{0,0} = \mathrm{id}$, tandis que $\gamma_{k,m_k} = \gamma$. Pour tout $0 \leq i \leq k$, on pose $g_i = \gamma_{i,0}^{-1}\gamma_{i,m_i}$ et, si $i > 0$, on se donne un sommet à l'infini p_i de P, et $h_i \in \Gamma_A$ tels que $\tau_{p_i}(h_i) = \gamma_{i-1,m_i}^{-1}\gamma_{i,0}$. Alors

$$\gamma = g_0 \cdot \tau_{p_1}(h_1) \cdot g_1 \cdot \tau_{p_2}(h_2) \cdots g_{k-1} \cdot \tau_{p_k}(h_k) \cdot g_k. \tag{11.5.2}$$

Soit $0 \leq i \leq k$. Pour chaque $1 \leq j \leq m_i$, l'élément $k_{i,j} = \gamma_{i,j-1}^{-1}\gamma_{i,j}$ appartient à un sous-groupe standard sphérique de Γ. Comme $g_i = k_{i,1} \cdots k_{i,m_i}$, et $\alpha_{\Gamma,S} \leq C_1$, on en déduit que

$$|g_i| \leq C_1 m_i. \tag{11.5.3}$$

Par ailleurs, pour tout $1 \leq j < m_j$, les facettes tronquées de $y_{i,j}$ et $y_{i,j+1}$ étant disjointes, on a $d_\Omega(y_{i,j}, y_{i,j+1}) \geq \frac{R}{3C_{A,2}}$, si bien que

$$d_\Omega(y_{i,1}, y_{i,2}) + \cdots + d_\Omega(y_{i,m_i-1}, y_{i,m_i}) \geq \frac{R}{3C_{A,2}}(m_i - 1). \tag{11.5.4}$$

Il ressort de (11.5.1), (11.5.3) et (11.5.4) l'estimation suivante.

$$d_\Omega(x_i^+, x_{i+1}^-) \geq \frac{R}{C_1'}|g_i| \quad \text{si} \quad |g_i| \geq 2C_1. \tag{11.5.5}$$

Si $i > 0$, alors on peut remarquer en outre que

$$d_\Omega(x_i^+, x_{i+1}^-) \geq \frac{R}{3C_{A,2}C_1^{d+1}}. \tag{11.5.6}$$

En effet, cela découle de (11.5.1) et (11.5.4) lorsque $m_i \geq 2$, tandis que si $m_i = 1$, alors x_i^+ et x_{i+1}^- sont sur deux s-voisinages de sommets à l'infini différents, si bien que $d_\Omega(x_i^+, x_{i+1}^-) \geq \frac{R}{3C_{A,2}}$.

D'après le lemme 11.5.3.3, pour tout $1 \leq i \leq k$, on a

$$d_\Omega(x_i^-, x_i^+) \geq d_{\Omega_A}(x_A, h_i x_A) - C_{A,3}. \tag{11.5.7}$$

En combinant (11.5.5), (11.5.6) et (11.5.7), on obtient, en notant $C_1' = 12C_{A,2}C_1^{d+1}$:

$$d_\Omega(x, \gamma x) \geq \sum_{i=0}^{k} \frac{R}{C_1'}|g_i|1_{|g_i|\geq 2C_1} + \sum_{i=1}^{k} d_{\Omega_A}(x_A, h_i x_A) + k\left(\frac{R}{C_1'} - C_{A,3}\right). \tag{11.5.8}$$

Estimons maintenant $\beta = \sum_{\gamma\in\Gamma} e^{-\delta d_\Omega(x,\gamma x)}$:

$$\beta \leq \sum_{k\geq 0} \sum_{\substack{g_0,\dots,g_k\in\Gamma \\ h_1,\dots,h_k\in\Gamma_A \\ p_1,\dots,p_k\in\overline{P}\cap\partial\Omega}} \prod_{i=0}^{k} e^{-\frac{R\delta}{C_1'}|g_i|1_{|g_i|\geq 2C_1}} \cdot \prod_{i=1}^{k} e^{-\delta d_{\Omega_A}(x_A, h_i x_A)} \cdot e^{-k\delta\left(\frac{R}{C_1'} - C_{A,3}\right)}$$

$$\leq \sum_{k\geq 0} \left(\#\{\gamma : |\gamma| \leq 2C_1\} + \chi_\Gamma\left(-\frac{R\delta}{C_1'}\right)\right)^{k+1} \cdot \chi_A(\delta)^k \cdot \left(\#\overline{P}\cap\partial\Omega\right)^k \cdot \left(e^{-\delta\left(\frac{R}{C_1'} - C_{A,3}\right)}\right)^k,$$

où $\chi_\Gamma(t) = \sum_{\gamma\in\Gamma} e^{-t|\gamma|}$ et $\chi_A(t) = \sum_{\gamma\in\Gamma_A} e^{-t d_{\Omega_A}(x_A,\gamma x_A)}$ pour tout $t \geq 0$. Ainsi, pour que β soit fini, il suffit que

$$\left(\#\{\gamma : |\gamma| \leq 2C_1\} + \chi_\Gamma\left(-\frac{R\delta}{C_1'}\right)\right) \cdot \chi_A(\delta) \cdot \left(\#\overline{P}\cap\partial\Omega\right) \cdot e^{-\delta\left(\frac{R}{C_1'} - C_{A,3}\right)} < 1.$$

Or il existe une constante $C_2 \geq 1$ telle que $\chi_A(t) \leq \frac{C_2}{t - \frac{d-1}{2}}$ pour tout $t > (d-1)/2$, et de plus $\chi_\Gamma(-\log\#S - 1) < \infty$. Puisque $(d-1)/2 < \delta \leq (d-1)$, pour que β soit fini il suffit que

$$\frac{R(d-1)}{2C_1'} \geq \log\#S + 1, \quad \text{et}$$

$$\left(\#\{\gamma : |\gamma| \leq 2C_1\} + \chi_\Gamma\left(-\log\#S - 1\right)\right) \cdot \frac{C_2}{\delta - \frac{d-1}{2}} \cdot \left(\#\overline{P}\cap\partial\Omega\right) \cdot e^{-\frac{R(d-1)}{C_1'} + (d-1)C_{A,2}} < 1.$$

En résumé, on a démontré que pour tout système de Coxeter (Γ, S), il existe une constante $C \geq (d-1)e \geq 1$ qui satisfait la propriété suivante. Soit (P, Ω) un polytope à pointes de groupe (Γ, S). Si sa grosseur R est plus grande que C, si $\frac{d-1}{2} < \delta \leq d-1$, et si

$$\frac{Ce^{-\frac{R}{C}}}{\delta - \frac{d-1}{2}} < 1, \quad \text{i.e. } \delta > \frac{d-1}{2} + Ce^{-\frac{R}{C}}$$

alors pour tout $x \in \Omega$,

$$\sum_{\gamma\in\Gamma} e^{-\delta d_\Omega(x,\gamma x)} < \infty.$$

D'après [Tho17, Th. 2], l'exposant critique de (P, Ω) est dans tous les cas inférieur à $d-1$. Donc L'exposant critique de (P, Ω) est inférieur à $\frac{d-1}{2} + Ce^{-\frac{R}{C}}$.

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ÉCOLE DOCTORALE
de mathématiques
Hadamard (EDMH)

Titre: Aspects dynamiques des structures projectives convexes

Mots clés: Ouvert proprement convexe, Convexe divisible, Flot géodésique, Densité conforme, Groupe de Coxeter, Exposant critique

Résumé: Cette thèse est consacrée à l'étude de la dynamique du flot géodésique des variétés projectives convexes, et fait suite aux travaux de Benoist, Bray, Crampon, Marquis et F. Zhu sur le sujet. Ces variétés sont des quotients d'ouverts projectifs proprement convexes, munis de la métrique de Hilbert, par des groupes linéaires discrets sans torsion, et le flot géodésique y suit les lignes droites projectives. Elles présentent des similitudes avec les variétés riemanniennes à courbure négative ou nulle, ainsi on s'inspirera des nombreux travaux, notamment ceux de Knieper et Roblin, portant sur le flot géodésique de ces dernières. En particulier, la majeure partie des résultats présentés ici portent sur les variétés projectives convexes de rang un, introduites par M.Islam et A.Zimmer, et analogues aux variétés riemanniennes de rang un.

Étant donnée une variété projective convexe de rang un, nous introduisons un fermé du fibré unitaire tangent invariant par le flot géodésique, appelé fibré unitaire tangent biproximal, dont nous montrons qu'il est le fibré unitaire tangent tout entier dans la cas compact, et que le flot géodésique y est topologiquement mélangeant en général. (Un résultat de mélange est aussi obtenu dans le cas compact de rang supérieur.) Nous développons la théorie des densités de Patterson–Sullivan pour les variétés projectives convexes de rang un, qui est utilisée pour obtenir une dichotomie de Hopf–Tsuji–Sullivan–Roblin et l'existence et unicité de la mesure d'entropie maximale sur le fibré unitaire tangent biproximal lorsque le cœur convexe (introduit par Danciger–Guéritaud–Kassel) est compact. Cette théorie nous permet aussi d'obtenir sous certaines conditions plusieurs résultats d'équidistribution : les géodésiques périodiques s'équidistribuent dans le fibré unitaire tangent biproximal, tandis que chaque orbite du groupe fondamental dans le revêtement universel de la variété s'accumule sur le bord projectif de ce dernier comme prescrit par la densité de Patterson–Sullivan.

En parallèle, faisant suite aux travaux de X.Nie et T.Zhang, nous étudions l'exposant critique des variétés projectives convexes. Plus précisément, nous nous intéressons à l'ensemble des valeurs prises par l'exposant critique lorsque l'on fait varier la structure projective convexe sur une variété topologique fixée. Par exemple, nous déterminons la borne inférieure de l'exposant critique pour certaines familles de telles structures, comme l'ensemble des structures de volume fini sur une surface.

Title: Dynamical aspects of convex projective structures

Keywords: Properly convex open set, Divisible convex set, Geodesic flow, Conformal density, Coxeter group, Critical exponent

Abstract: In this thesis we study the dynamics of the geodesic flow on convex projective manifolds, following work of Benoist, Bray, Crampon, Marquis and F. Zhu. These manifolds are quotients of properly convex, open, projective sets, equipped with the Hilbert metric, by discrete linear groups; the geodesic flow parametrises straight projective lines. These manifolds show similarities with non-positively curved Riemannian manifolds, hence we take inspiration from the literature on the geodesic flow of the latter, especially from work of Knieper and Roblin. In particular, most results presented here concern rank-one convex projective manifolds, which were introduced by M. Islam and A. Zimmer, and are analogues of rank-one Riemannian manifolds.

Given a rank-one convex projective manifold, we introduce a flow-invariant closed subset of the unit tangent bundle, called the biproximal unit tangent bundle, which is proved to be the full unit tangent bundle if the manifold is compact, and on which the geodesic flow is proved to be topologically mixing in general. (A mixing result is also obtained in the higher-rank compact case.) We develop the theory of Patterson–Sullivan densities for rank-one convex projective manifolds, which is used to establish a Hopf–Tsuji–Sullivan–Roblin dichotomy and, if the convex core (introduced by Danciger–Guéritaud–Kassel) is compact, existence and uniqueness of the measure of maximal entropy on the biproximal unit tangent bundle. This theory also allows us to establish, under certain conditions, equidistribution results: closed geodesics equidistribute in the biproximal unit tangent bundle, while orbits of the fundamental group in the universal cover accumulate on the projective boundary of the latter as prescribed by the Patterson–Sullivan density.

We also study the critical exponent of convex projective manifolds, building on work of X. Nie and T. Zhang. More precisely, we look at the values taken by the critical exponent when letting the convex projective structure vary on a fixed topological manifold. For instance, we give the infimum of the critical exponents of certain families of convex projective structures, such as the family of finite-volume structures on a surface.

Université Paris-Saclay
Espace Technologique / Immeuble Discovery
Route de l'Orme aux Merisiers RD 128 / 91190 Saint-Aubin, France