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Abstract

We introduce a novel, computationally inexpensive approach for imaging with an active array of sensors, which probe an unknown medium with a pulse and measure the resulting waves. The imaging function is based on the principle of time reversal in non-attenuating media and uses a data driven estimate of the ‘internal wave’ originating from the vicinity of the imaging point and propagating to the sensors through the unknown medium. We explain how this estimate can be obtained using a reduced order model (ROM) for the wave propagation. We analyze the imaging function, connect it to the time reversal process and describe how its resolution depends on the aperture of the array, the bandwidth of the probing pulse and the medium through which the waves propagate. We also show how the internal wave can be used for selective focusing of waves at points in the imaging region. This can be implemented experimentally and can be used for pixel scanning imaging. We assess the performance of the imaging methods with numerical simulations and compare them to the conventional reverse-time migration method and the ‘backprojection’ method introduced recently as an application of the same ROM.

Keywords: imaging, waves, model reduction, data driven, time reversal, focusing

(Some figures may appear in colour only in the online journal)

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1. Introduction

This paper is concerned with an application of reduced order modeling to imaging reflective structures in a known, non-scattering and non-attenuating host medium, from data gathered by an active array of \( m \) sensors that emit probing pulses and measure the backscattered waves.

Model reduction is an important topic in computational science, which traditionally has been concerned with finding a low-dimensional reduced order model (ROM) that approximates the response (observables) of a given dynamical system for a set of inputs \([3, 4]\). In wave-based imaging the inputs and observables are controlled and measured by the sensors in the array, but the dynamical system is not given, as it is governed by the wave equation with unknown coefficients like the wave speed. Thus, we need a data-driven ROM.

Data driven reduced order modeling is a growing field which combines ideas from traditional model reduction and learning \([10]\). Much of it is concerned with studying dynamical systems using the Koopman operator theory \([21, \text{chapter 1}]\). Dynamical system identification based on the Koopman operator has been proposed for instance in \([11]\) and \([21, \text{chapter 13}]\). However, these approaches are difficult to use in imaging because they assume knowing the full state of the dynamical system, aka the snapshot of the wave, at a finite set of time instances. Since we only know the wave at the sensor locations, which are far from the imaging region, a new way of learning the wave propagation from the data is needed.

To our knowledge, the first sensor array data driven ROM for wave propagation was introduced in \([14]\) and was connected to a three point spatial finite difference scheme on special, so-called ‘optimal grids’. These grids are understood only in one dimension, and they were used in \([14]\) to estimate the acoustic impedance of layered media. The extension of the data driven ROM construction to higher dimensions and also to vector waves was obtained in \([8]\) and was analyzed in \([9]\). The latter study showed that wave propagation can be viewed as a discrete time dynamical system governed by a ‘propagator operator’, where the time step \( \tau \) is the data sampling interval. This propagator operator maps the wave from the states at instants \((j - 1)\tau \) and \( j\tau \) to the future state at time \((j + 1)\tau \), for any \( j \in \mathbb{N} \). The ROM in \([8, 9]\) is an algebraic analogue of the dynamical system. Its evolution is controlled by an \( nm \times nm \) propagator matrix, given by the Galerkin projection of the propagator operator on the function space spanned by the first \( n \) snapshots of the wave, assuming that the array records for the duration \((2n - 1)\tau \). Recall that \( m \) is the number of sensors in the array. What distinguishes the ROM construction from the many other Galerkin projections in the literature \([19, 20]\) is that it is obtained only from the measurements of the snapshots at the sensors in the array i.e., it does not require knowing the approximation space. Furthermore, the ROM approximates the wave field, as expected from results like \([18]\), but in addition it captures it exactly at times \((j\tau)_{j=0}^{n-1}\) and it also reproduces the array data at all the measurement time instants \((j\tau)_{j=0}^{2n-1}\).

The ROM introduced in \([8, 9]\) has been used so far to: (1) approximate the Fréchet derivative of the reflectivity to sensor array data map \([7]\). This gives the single scattering (Born) forward map used in conventional imaging in radar \([12, 13]\), seismic inversion \([5]\) and elsewhere. (2) Obtain a fast converging, iterative inverse scattering method for the acoustic impedance in a medium with known and smooth wave speed \([9]\). (3) Develop a non-iterative ‘backprojection’ imaging method that is free of multiple scattering artifacts \([15]\).

We propose yet another application of the ROM: estimate the ‘internal’ acoustic wave that originates from the vicinity of the imaging point and propagates through the unknown medium to the array of sensors. This idea has been tried before for Schrödinger’s equation in the spectral (frequency) domain \([6, 16]\), where the solutions are smooth functions that are easier to approximate than the internal waves in this paper. We use the internal waves for two novel
imaging methods: the first is a computationally inexpensive approach designed to sense rapid changes of the wave speed in the vicinity of the imaging point. Its imaging function is connected to the point spread function of the time reversal process, and we explain how the aperture of the array, the bandwidth of the probing pulse and the medium through which the waves propagate affect the resolution. The second method can be implemented experimentally. It controls the excitation from the array in order to focus waves at the imaging points, and then uses a matched field approach to image with the resulting backscattered wave in a pixel scanning manner.

The paper is organized as follows: we begin in section 2 with the mathematical formulation of the imaging problem and review briefly from [9] the relevant facts about the ROM, needed in the next sections. The estimation of the internal wave is described in section 3. The first imaging method based on this internal wave is introduced and analyzed in section 4. We also give there a comparison with the backprojection imaging method introduced in [15]. The second, pixel scanning imaging method is described in section 5. We use numerical simulations in section 6 to assess the performance of the imaging methods and to compare them with the backprojection approach [15] and with the conventional, reversed time migration method [5]. We end with a summary in section 7.

2. Formulation of the imaging problem and the ROM

We are interested in imaging reflective structures in a non-scattering and non-attenuating known host medium occupying the bounded domain \( \Omega \), using data gathered by an active array of sensors located at \( x_s \), for \( s = 1, \ldots, m \). We suppose that the aperture of the array is planar in three-dimensions or linear in two-dimensions, and call ‘range’ the spatial coordinate in the direction orthogonal to it. The coordinates in the plane (line in two-dimensions) parallel to the aperture are called ‘cross-range’.

The \( s \)th sensor is modeled by a source term in the form of a Dirac delta \( \delta_{x_s}(x) \). It probes the medium with a pulse \( f(t) \) and generates the wave \( u^{(s)}(t,x) \), the solution of the wave equation

\[
\partial_t^2 u^{(s)}(t,x) + A(c) u^{(s)}(t,x) = f'(t) \delta_{x_s}(x), \quad t \in \mathbb{R}, \ x \in \Omega,
\]

with quiescent initial condition

\[
u^{(s)}(t,x) \equiv 0, \quad t \ll 0, \ x \in \Omega.
\]

We assume henceforth a real valued pulse \( f(t) \), that is an even function supported in the short interval \((-t_f, t_f)\) and has non-negative Fourier transform\(^4\)

\[
\hat{f}(\omega) = \int_{\mathbb{R}} dt \ f(t)e^{i\omega t} = \int_{\mathbb{R}} dt \ f(t) \cos(\omega t) \geq 0,
\]

which is negligible in the complement of the set \((\omega_c - B, \omega_c + B) \cup (-\omega_c - B, -\omega_c + B)\), where \( \omega_c \) is the center (carrier) frequency and \( B = O(1/t_f) \) is the bandwidth.

The reflective structures are modeled in (2.1) by rough changes (jumps) of the wave speed \( c(x) \), with respect to the known and smooth reference wave speed \( c_0(x) \) of the host medium. These changes are supported in the imaging domain \( \Omega_{im} \), which is a subset of \( \Omega \) lying away

\(^4\)If the source emits an arbitrary pulse \( f(t) \), we can convolve the echos received at the array with \( f(-t) \). Mathematically, this is equivalent to having the even pulse \( f(t) = f(t) \ast f(-t) \), with Fourier transform \( \hat{f}(\omega) = |\hat{f}(\omega)|^2 \geq 0. \)
from the array. The unknown \( c(x) \) appears as a coefficient in the self-adjoint, second order elliptic operator
\[
A(c) = -c(x) \Delta [c(x)],
\]
with homogeneous boundary conditions at \( \partial \Omega \). The self-adjointness of \( A(c) \) is convenient for the operator calculus in [9] and the next sections, but we note that (2.1) can be written in the standard wave equation form for the acoustic pressure \( p^{(s)}(t, x) = c(x)w^{(s)}(t, x) \). Since \( c(x) \) is known and equal to \( c_0(x) \) at the sensor locations, the measurements of the pressure, sampled in time at interval \( \tau \), define the array data
\[
data = \{ w^{(s)}(t, x_r), \ s, r = 1, \ldots, m, \ t = j\tau, \ j = 0, \ldots, 2n - 1 \}.
\]
(2.5)

The domain \( \Omega \) may be physical or the truncation of an infinite domain, justified by hyperbolicity and the finite duration \( (2n - 1)\tau \) of the measurements. In either case, we divide the boundary in two parts: the ‘accessible’ boundary \( \partial \Omega_{ac} \), named so because it lies in the immediate vicinity of the array, and the ‘inaccessible’ boundary \( \partial \Omega_{inac} = \partial \Omega \setminus \partial \Omega_{ac} \). The accessible boundary is useful for the ROM construction because the waves propagate only on one side of the array5, as illustrated in figure 1. We model it as sound hard, using the homogeneous Neumann boundary condition
\[
\partial_n w^{(s)}(t, x) = 0, \ t \in \mathbb{R}, \ x \in \partial \Omega_{ac},
\]
(2.6)
where \( \partial_n \) denotes the normal derivative. The inaccessible boundary is modeled as sound soft,
\[
w^{(s)}(t, x) = 0, \ t \in \mathbb{R}, \ x \in \partial \Omega_{inac},
\]
(2.7)
and if it is due to the truncation of an infinite domain, it is sufficiently far away from the sensors to not affect the waves recorded by them over the duration of the measurements.

The imaging problem is to estimate the support of the large and localized variations \( c(x) - c_0(x) \) of the wave speed, from the data (2.5) collected by the array.

2.1. Review of the ROM for wave propagation

Here we review briefly from [9] the relevant facts about the ROM, needed to state the new results.

2.1.1. The dynamical system for wave propagation. We work with the even in time wave
\[
u^{(s)}_e(t, x) := w^{(s)}(t, x) + w^{(s)}(-t, x),
\]
(2.8)
which satisfies \( u^{(s)}_e(t, x) = u^{(s)}(t, x) \) for \( t > t_f \), due to causality and the initial condition (2.2). During the short duration \( t_f \) of the pulse, the wave senses only the vicinity of the sensor location \( x_s \), where the wave speed equals the known \( c_0(x) \). Thus, the second term in (2.8) can be calculated and we can work with the data matrices
\[
D_j = \left( D^{(s)}_j \right)_{s, r = 1 \ldots m}, \ D^{(s)}_j : = u^{(s)}_e(j\tau, x_r), \ j = 0, \ldots, 2n - 1.
\]
(2.9)

5 If such a boundary does not exist, the medium should be known and homogeneous on the other side of the array, so that the waves there can be removed with some additional processing.
Because \( f(t) \) is even, it is easy to obtain from (2.1), (2.6) and (2.7) that \( \psi(t,x) \) satisfies

\[
\frac{\partial^2}{\partial t^2} \psi(t,x) + A(c) \psi(t,x) = 0, \quad t > 0, \quad x \in \Omega,
\]

\[
\frac{\partial}{\partial n} \psi(t,x) = 0, \quad t > 0, \quad x \in \partial \Omega_{ac},
\]

\[
\psi(t,x) = 0, \quad t > 0, \quad x \in \partial \Omega_{inac},
\]

with initial conditions derived in [9, appendix A]

\[
\psi(0,x) = \hat{f} \left( \sqrt{A(c)} \right) \delta_{x,s}(x), \quad \frac{\partial}{\partial t} \psi(0,x) = 0, \quad x \in \Omega.
\]

We define throughout functions of the operator \( A(c) \) in the standard way, using its spectral decomposition deduced from [22, theorem 4.12]. Specifically, if we denote by \( \{\theta_l > 0, l \geq 1\} \) the eigenvalues, ordered like \( 0 < \theta_1 \leq \theta_2 \leq \ldots \) and satisfying \( \lim_{l \to \infty} \theta_l = \infty \), and by \( \{\psi_l(x), l \geq 1\} \) the eigenfunctions, which form an orthonormal basis of \( L^2(\Omega) \) with the appropriate boundary conditions, then we have

\[
\hat{f} \left( \sqrt{A(c)} \right) \delta_{x,s}(x) = \sum_{l=1}^{\infty} \hat{f} \left( \sqrt{\theta_l} \right) \psi_l(x) \psi_l(x_s).
\]

The pulse is band-limited, so the sum is for \( l \leq l_{\text{max}} \), where \( \sqrt{\theta_{l_{\text{max}} + 1}} > \omega_c + B \).

Note that the solution of (2.10)–(2.13) is

\[
w(t,x) = \cos \left( t \sqrt{A(c)} \right) \hat{f} \left( \sqrt{A(c)} \right) \delta_{x,s}(x) = \sum_{l=1}^{\infty} \cos \left( t \sqrt{\theta_l} \right) \hat{f} \left( \sqrt{\theta_l} \right) \psi_l(x) \psi_l(x_s)
\]

\[
= \hat{f} \left( \sqrt{A(c)} \right) \cos \left( t \sqrt{A(c)} \right) \hat{f} \left( \sqrt{A(c)} \right) \delta_{x,s}(x),
\]

and that the data matrices (2.9) can be written in symmetric inner product form as follows
Here we used that functions of $A(c)$ commute, and denoted by

$$
(\phi, \psi) := \int_{\Omega} dx \phi(x) \psi(x), \quad \forall \phi, \psi \in L^2(\Omega),
$$

the $L^2(\Omega)$ inner product. We also introduced the ‘sensor functions’

$$
\delta_{x_s}(x) := \hat{f} \left( \sqrt{A(c)} \right) \delta_{x_s}(x), \quad s = 1, \ldots, m, \tag{2.17}
$$

and used the assumption (2.3) to define the square root of $\hat{f}$.

The notation in (2.17) reminds us that $\delta_{x_s}(x)$ is a pulse dependent, blurry version of the Dirac $\delta_{x_s}(x)$. Indeed, comparing (2.17) with (2.13), we note that $\delta_{x_s}(x)/2$ is the initial state of the solution of (2.1), (2.2) and (2.6), (2.7), when the sensor emits the pulse

$$
\hat{f}^{\frac{1}{2}}(t) := \int_{\mathbb{R}} \frac{d\omega}{2\pi} \hat{f}^{\frac{1}{2}}(\omega) e^{-i\omega t} = \int_{\mathbb{R}} \frac{d\omega}{2\pi} \hat{f}^{\frac{1}{2}}(\omega) \cos(\omega t), \tag{2.18}
$$

which is real valued, even and satisfies $f(t) = \hat{f}^{\frac{1}{2}}(t) * \hat{f}^{\frac{1}{2}}(t)$. Because of causality and the finite wave speed, $\delta_{x_s}(x)$ is supported in a ball centered at $x_s$, with radius of order $c_o(x_s)T_f \ll \text{dist}(x_s, \Omega_{in})$ and it can be computed using the operator $A(c_o)$ in the host medium

$$
\delta_{x_s}(x) = \hat{f}^{\frac{1}{2}} \left( \sqrt{A(c)} \right) \delta_{x_s}(x) = \hat{f}^{\frac{1}{2}} \left( \sqrt{A(c_o)} \right) \delta_{x_s}(x). \tag{2.19}
$$

Let us group all the sensor functions in the $m$-dimensional row vector field

$$
\delta^f(x) = (\delta^f_{x_1}(x), \ldots, \delta^f_{x_m}(x)), \tag{2.20}
$$

and define the ‘snapshots’ as the $m$-dimensional row vector fields

$$
u_j(x) = (u^{(1)}_j(x), \ldots, u^{(m)}_j(x)) := \cos \left( j\tau \sqrt{A(c)} \right) \delta^f(x), \quad j \geq 0. \tag{2.21}
$$

These are the states of the discrete time dynamical system governed by the ‘propagator operator’

$$
P := \cos \left( \tau \sqrt{A(c)} \right). \tag{2.22}
$$

Indeed, using a trigonometric identity of the cosine, we get that the states evolve like

$$
u_{j+1}(x) = 2P \nu_j(x) - \nu_{j-1}(x), \quad j \geq 0, \quad x \in \Omega, \tag{2.23}
$$

starting from

$$
u_0(x) = \delta^f(x), \quad \nu_{-1}(x) = \nu_1(x) = P \delta^f(x), \quad x \in \Omega. \tag{2.24}
$$
2.1.2. Data driven ROM construction. The ROM is the algebraic analogue of the dynamical system (2.23),

\[ u_{j+1}^{\text{ROM}} = 2P^{\text{ROM}}u_j^{\text{ROM}} - u_{j-1}^{\text{ROM}}, \quad j \geq 0, \]  

(2.25)

with propagator matrix \( P^{\text{ROM}} \in \mathbb{R}^{nm \times nm} \) and states \( u_j^{\text{ROM}} \in \mathbb{R}^{nm \times m} \). It corresponds to the Galerkin projection of (2.23) on the \( nm \)-dimensional function space spanned by the first \( n \) snapshots (2.21). Using linear algebra notation, we write this space as

\[ \mathcal{S} := \text{range } U(x), \quad U(x) = (u_0(x), \ldots, u_{n-1}(x)), \]  

(2.26)

and note that \( U(x) \) is an \( nm \)-dimensional row vector field and that it is unknown. Nevertheless, the construction in [9, section 2] shows that it is possible to get the ROM (2.25) from what we know: the initial snapshot \( \delta^0(x) \) in (2.20) and the \( m \times m \) data matrices with entries (2.16),

\[ D_j = \int_{\Omega} dx \, \delta^j(x)^T u_j(x) := \langle \langle \delta^j, u_j \rangle \rangle, \quad j = 0, \ldots, 2n - 1. \]  

(2.27)

Here \( T \) denotes the transpose and we introduced the notation \( \langle \langle \cdot, \cdot \rangle \rangle \) for the integral of the outer product of \( m \)-dimensional row vector functions.

Key to the ROM construction are the data driven, symmetric, positive definite ‘mass’ matrix

\[ M := \int_{\Omega} dx \, U(x)^T U(x) \in \mathbb{R}^{nm \times nm} \]  

with \( m \times m \) blocks

\[ M_{jl} := \langle \langle u_j, u_l \rangle \rangle \]

\[ = \langle \langle \cos \left( j \tau \sqrt{A(c)} \right) \delta^j, \cos \left( l \tau \sqrt{A(c)} \right) \delta^l \rangle \rangle \]

\[ = \langle \langle \delta^j, \cos \left( j \tau \sqrt{A(c)} \right) \cos \left( l \tau \sqrt{A(c)} \right) \delta^l \rangle \rangle \]

\[ = \frac{1}{2} \left[ \langle \langle \delta^j, \cos \left( (j + l) \tau \sqrt{A(c)} \right) \delta^l \rangle \rangle + \langle \langle \delta^j, \cos \left( (j - l) \tau \sqrt{A(c)} \right) \delta^l \rangle \rangle \right] \]

\[ = \frac{1}{2} \left( D_{j+l} + D_{j-l} \right), \quad j, l = 0, \ldots, n - 1, \]  

(2.28)

and the ‘stiffness’ matrix \( S := \int_{\Omega} dx \, U(x)^T P U(x) \in \mathbb{R}^{nm \times nm} \) with \( m \times m \) blocks

\[ S_{jl} := \langle \langle u_j, P u_l \rangle \rangle \]

\[ = \frac{1}{2} \langle \langle u_j, u_{l+1} + u_{l-1} \rangle \rangle \]

\[ = \frac{1}{4} \left( D_{j+l+1} + D_{j-l+1} + D_{j+l-1} + D_{j-l-1} \right), \quad j, l = 0, \ldots, n - 1. \]  

(2.29)

Here we used definitions (2.21) and (2.27), the self-adjointness of \( A(c) \) and a trigonometric identity for the cosine.

The ROM propagator is defined in [9, section 2.2.1] as follows: let \( R \in \mathbb{R}^{nm \times nm} \) be the block upper triangular matrix obtained from the block Cholesky factorization of the mass matrix

\[ M = R^T R. \]  

(2.30)
This matrix $R$ can be used to write the Gram–Schmidt orthogonalization of $U(x)$

$$U(x) = V(x)R,$$  \hspace{1cm} (2.31)

which defines the orthonormal, causal basis of the approximation space (2.26), gathered in

$$V(x) = (v_0(x), \ldots, v_{n-1}(x)),$$  \hspace{1cm} (2.32)

with $m$-dimensional row vector field components $v_j(x), j = 0, \ldots, n - 1$, called the ‘orthonormal snapshots’. Then, we have

$$\mathcal{P}_{\text{ROM}} := R^{-T}S^{-1} = \int_\Omega V(x)^T P V(x) = (\langle\langle v_j, P v_l \rangle\rangle)_{j=0,\ldots,n-1},$$  \hspace{1cm} (2.33)

where the superscript $-T$ denotes the inverse and transpose. The first equality in this equation is used to compute $\mathcal{P}_{\text{ROM}}$ from the data, and the second equality shows that it is a projection of the operator $P$.

The first $n$ ROM states satisfy

$$R = \begin{pmatrix}
R_{0,0} & R_{0,1} & R_{0,2} & \cdots & R_{0,n-1} \\
0 & R_{1,1} & R_{1,2} & \cdots & R_{1,n-1} \\
0 & 0 & R_{2,2} & \cdots & R_{2,n-1} \\
& & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & R_{n-1,n-1}
\end{pmatrix} = (u_{0,\text{ROM}}, \ldots, u_{n-1,\text{ROM}})$$

$$= \int_\Omega V(x)^T U(x) = (\langle\langle v_j, u_l \rangle\rangle)_{j=0,\ldots,n-1},$$  \hspace{1cm} (2.34)

and we note how the algebraic structure of $R$ captures the causal wave propagation: the $nm \times m$ column blocks of $R$ are indexed using the time instants $j\tau$, for $j = 0, \ldots, n - 1$, while the $m \times nm$ row blocks of $R$ are indexed according to the range locations reached by the wavefront at these instants. The first column block of $R$, which equals $u_{0,\text{ROM}}$, has all but the first block equal to zero, because the true snapshot $u_0(x)$ is supported near the array. The second column block of $R$, which equals $u_{1,\text{ROM}}$, has an additional nonzero block because the true snapshot $u_1(x)$ reaches some range in the medium, and so on.

### 3. The internal wave

We now use the data driven ROM reviewed above to estimate an internal wave that originates from the vicinity of an arbitrary point $y \in \Omega_{\text{in}}$ and propagates to the array through the true, unknown medium. The best estimate that we could hope for would be

$$g_{\text{ideal}}(t,x;y) := \cos \left( t\sqrt{A(c)} \right) \delta^f(y),$$  \hspace{1cm} (3.1)

with the initial state
\[ g_{\text{ideal}}(0,x,y) = \delta_f(x) = \hat{f}^T \left( \sqrt{A(c)} \right) \delta_y(x) \approx \hat{f}^T \left( \sqrt{A(c)} \right) VV^T \delta_y(x), \quad (3.2) \]
given by the approximation
\[ VV^T \delta_y := \sum_{j=0}^{n-1} v_j(x)v_j^T(y) = V(x)V^T(y), \quad (3.3) \]
of \( \delta_y(x) \) in the space \((2.26)\), blurred a little by the pulse dependent operator \( \hat{f}^T \left( \sqrt{A(c)} \right) \). The same reasoning used for the sensor functions \((2.17)\) applies to \((3.2)\) and shows that it is supported in the ball centered at \( y \), with \( O(c(y)t_f) \) radius. We are interested in evaluating the wave \((3.1)\) at \( t > 0 \) and the sensor locations \( x_r \), for \( r = 1, \ldots, m \). However, we cannot get exactly \( g_{\text{ideal}}(t,x; y) \), because we do not know the approximation space \((2.26)\). Proposition 3.1 given below shows that we can compute instead
\[ g(t,x,y) := \cos \left( t \sqrt{A(c)} \right) \delta_y^{\text{ROM}}(x), \quad \delta_y^{\text{ROM}}(x) := \hat{f}^T \left( \sqrt{A(c)} \right) \delta_y^{\text{ROM}}(x), \quad (3.4) \]
at \( x = x_r \), for \( r = 1, \ldots, m \), where the approximation \((3.3)\) of \( \delta_y(x) \) is replaced by the ‘ROM point spread function’
\[ \delta_y^{\text{ROM}} := VV^T \delta_y(x) = \sum_{j=0}^{n-1} v_j(x)v_j^T(y) = V(x)V^T(y), \quad (3.5) \]
calculated with the orthonormal snapshots in the reference medium with known wave speed \( c_o(x) \)
\[ V_o(y) = \left( v_{o,0}(y), \ldots, v_{o,n-1}(y) \right). \quad (3.6) \]

We will see in the next section that the tighter the focus of \( \delta_y^{\text{ROM}}(x) \) around \( y \), the better the imaging using the internal wave \( g(t,x; y) \). So when can we expect such a result? The answer lies in how well we can approximate the snapshots in \( U(x) \) in the reference space calculated for the known \( c_o(x) \),
\[ \mathcal{U}_o := \text{range} \ U_o(x), \quad U_o(x) = \left( u_{o,0}(x), \ldots, u_{o,n-1}(x) \right). \quad (3.7) \]
If it is true that the approximation error is small, then the Gram–Schmidt orthogonalization, which is a stable procedure when the entries in \( U(x) \) are linearly independent, gives
\[ v_j(x) \approx v_{o,j}(x), \quad j = 0, \ldots, n-1, \quad (3.8) \]
and the ROM point spread function \((3.5)\) is an approximation of \((3.2)\).

We give in appendix \( A \) an explicit analysis of the approximation \((3.8)\) for two setups: in a layered medium and in a waveguide. The error in these two cases is controlled by the time step \( \tau \), the separation \( h \) between the sensors and the array aperture size. It is unknown how to extend the theoretical analysis of \((3.8)\) to more general settings, but we show with numerical simulations that if \( \tau \) and \( h \) are small enough and the aperture is large enough, then the ROM point spread function \( \delta_y^{\text{ROM}}(x) \) is peaked at \( y \). Note however that \( \tau \) and \( h \) cannot be too small, because the snapshots must be linearly independent to numerical precision. This is needed to carry out the Cholesky factorization of \( M \) and to compute \( V_o(x) \). The accuracy vs numerical stability
trade-off can be addressed in practice by setting $\tau$ close to the Nyquist sampling requirement for the probing pulse and letting $h$ be approximately half of the central wavelength.

**Proposition 3.1.** Let $R$ be the block upper triangular Cholesky factor of the data driven mass matrix $M$, with block entries given by (2.28). The estimated internal wave (3.4) evaluated at the sensor locations $x_r$ for $r = 1, \ldots, m$, and at the time instants $t_j = j\tau$, for $j = 0, \ldots, n - 1$, is given by the $m$-dimensional row vector field

$$
(g(t_j, x_1; y), \ldots, g(t_j, x_m; y)) = V_0(y)R e_j,
$$

where $e_j \in \mathbb{R}^{mn \times m}$ is the $(j + 1)^{th}$ column block of the $nm \times nm$ identity matrix $I_{nn}$.

**Proof.** Let us begin with the auxiliary $m$-dimensional column vectors

$$
\sigma_l(y) := e^T_l R^{-1} V_0^T(y), \quad l = 0, \ldots, n - 1,
$$

and note that

$$
\sum_{l=0}^{n-1} u_l(x)\sigma_l(y) = U(x) \sum_{l=0}^{n-1} e_l \sigma_l(y) = U(x) \left( \sum_{l=0}^{n-1} e_l e_l^T \right) R^{-1} V_0^T(y) = U(x) R^{-1} V_0^T(y) = V(x) V_0^T(y) = \delta_{y}^{\text{ROM}}(x).
$$

Here the first equality is by the definition (2.26) of $U(x)$, in the third equality we used that the sum over $l$ equals the identity matrix and the last equality is due to the Gram–Schmidt orthogonalization (2.31). Applying the operator $\cos \left( t_j \sqrt{A(c)} \right)$ to both sides of (3.11) and using definition (2.21) we get

$$
\cos \left( t_j \sqrt{A(c)} \right) \delta_{y}^{\text{ROM}}(x) = \cos \left( t_j \sqrt{A(c)} \right) \sum_{l=0}^{n-1} \cos \left( t_j \sqrt{A(c)} \right) \delta_0(x) \sigma_l(y)
$$

$$
= \frac{1}{2} \sum_{l=0}^{n-1} \left[ \cos \left( (j + l) \tau \sqrt{A(c)} \right) + \cos \left( (j - l) \tau \sqrt{A(c)} \right) \right] \delta_0(x) \sigma_l(y)
$$

$$
= \frac{1}{2} \sum_{l=0}^{n-1} \left[ u_{j+l}(x) + u_{j-l}(x) \right] \sigma_l(y).
$$

Furthermore, using definition (2.19) and the self-adjointness of $A(c)$, we have

$$
g(t_j, x_r; y) = \int_{\Omega} dx \, \delta_{x_r}(x) \sqrt{\sqrt{A(c)}} \cos \left( t_j \sqrt{A(c)} \right) \delta_{y}^{\text{ROM}}(x)
$$

$$
= \int_{\Omega} dx \, \delta_{x_r}(x) \frac{1}{2} \sum_{l=0}^{n-1} \left[ u_{j+l}(x) + u_{j-l}(x) \right] \sigma_l(y),
$$
for all \( r = 1, \ldots, m \). Gathering these results in an \( m \)-dimensional column vector and recalling definition (2.20) and the expression (2.27) of the data matrices, we obtain

\[
\begin{pmatrix}
g(t_1, x_1; y) \\
\vdots \\
g(t_m, x_m; y)
\end{pmatrix} = \left( \frac{1}{2} \sum_{l=0}^{n-1} (D_{j+l} + D_{j-l}) \right) \sigma_j(y) = \sum_{l=0}^{n-1} M_{j,l} \sigma_l(y),
\]

where the last equality is by (2.28). Finally, we substitute (3.10) into this equation and use the Cholesky factorization of the mass matrix to get the result

\[
\begin{pmatrix}
g(t_1, x_1; y) \\
\vdots \\
g(t_m, x_m; y)
\end{pmatrix} = \left( \frac{1}{2} \sum_{l=0}^{n-1} M_{j,l} \right) e^T R^{-1} V^T_o(y) = e^T (V_o(e) R e_j)^T.
\]

\[\square\]

4. Imaging with the internal wave

In this section we introduce a novel imaging approach based on the internal wave estimated in proposition 3.1. The imaging function is strikingly simple: it is the squared norm of this wave evaluated at the sensors

\[
I(y) = \sum_{r=1}^{m} \sum_{j=0}^{n-1} |g(j, \tau_r; y)|^2, \quad y \in \Omega_{im}.
\]

Moreover, \( I(y) \) is easy to compute and it does not even require the full ROM. It just uses the Cholesky factor \( R \) of the data driven mass matrix \( M \) with block entries (2.28), and the orthonormal snapshots (3.6) calculated in the reference medium with known wave speed \( c_o(x) \).

The analysis of (4.1) is given below, but let us describe in a few words the motivation behind it: our goal is to introduce an imaging function that is sensitive to the local variations of the wave speed. Denoting by \( \hat{G} \) the time-harmonic Green’s function in the unknown medium, it is known [2] that \( \text{Im}(\hat{G}(\omega, y; y)) \) is sensitive to the wave speed around \( y \). The imaging function \( I(y) \), which is computable from the array data, turns out to be roughly proportional to \( \int d\omega |\hat{G}(\omega, x; y)|^2 \hat{f}(\omega) \), which is itself approximately proportional to \( \int d\omega \text{Im}(\hat{G}(\omega, y; y)) \hat{f}(\omega)/\omega \) by the Helmholtz–Kirchhoff identity [2]. Hence, we anticipate that (4.1) should be sensitive to the local variations of the wave speed around \( y \). We also refer to section 4.2 for the connection between \( I(y) \) and the time reversal point spread function.

Our analysis of (4.1) is based on the continuum time approximation, where the sum over \( j \) is replaced by an integral over time. We also assume a long enough duration of the measurements and suppose, as is typical in applications, that the wave speed is constant near the sensors and therefore the accessible boundary,

\[
c(\tau) = c_o(\tau) = c_o, \quad \tau \text{ near } \partial \Omega_{ac}.
\]

We begin in section 4.1 with the connection between the internal wave and the Green’s function of the acoustic wave equation. This is useful for obtaining the main result in section 4.2, where we relate \( I(y) \) to the time reversal process and we discuss its resolution. We end in section 4.3 with a comparison of \( I(y) \) and the imaging function of the backprojection approach introduced in [15].
4.1. The Green’s function and its connection to the internal wave

The next lemma connects the internal wave (3.4) with the Green’s function $G(t, x; z)$ of the acoustic wave equation, satisfying

\[
\left[ \frac{1}{c^2(x)} \partial_t^2 - \Delta_x \right] G(t, x; z) = \delta(t)\delta_x(x), \quad t \in \mathbb{R}, \ x \in \Omega, \quad (4.3)
\]

\[
G(t, x; z) = 0, \quad t < 0, \ x \in \Omega, \quad (4.4)
\]

\[
\partial_t G(t, x; z) = 0, \quad t \in \mathbb{R}, \ x \in \partial\Omega_{\text{ac}}, \quad (4.5)
\]

\[
G(t, x; z) = 0, \quad t \in \mathbb{R}, \ x \in \partial\Omega_{\text{mac}}, \quad (4.6)
\]

where $z$ is an arbitrary point in $\Omega$ and $\Delta_x$ is the Laplace operator with respect to $x$.

**Lemma 4.1.** Let $y$ be any point in the imaging domain $\Omega_{\text{im}}$, which supports the sought after reflective structures. The internal wave (3.4) evaluated at the sensor locations satisfies

\[
g(t, x; y) = \partial_t \overline{f^\frac{1}{2}(t)} \ast \frac{1}{c(z)c_0} \int_{\Omega} G(t, x, z) \delta_y^{\text{ROM}}(z), \quad r = 1, \ldots, m, \quad (4.7)
\]

for any $t > 0$, where we recall that $\overline{f^\frac{1}{2}(t)}$ is defined in (2.18), $g$ is an even function in $t$, and $\ast$ denotes convolution in time $t$.

**Proof.** Let us begin with the even in time wave function

\[
G_r(t, x; z) = \cos \left( t \sqrt{A(c)} \right) \delta_x(x), \quad (4.8)
\]

and use linear superposition to write

\[
\cos \left( t \sqrt{A(c)} \right) \delta_y^{\text{ROM}}(x) = \int_{\Omega} \delta_y^{\text{ROM}}(z) G_r(t, x; z). \quad (4.9)
\]

The internal wave is

\[
g(t, x; y) = \hat{f}^\frac{1}{2} \left( \sqrt{A(c)} \right) \cos \left( t \sqrt{A(c)} \right) \delta_y^{\text{ROM}}(x) \times \delta_y^{\text{ROM}}(z)
\]

\[
= \sum_{l=1}^{\infty} j_l \left( \sqrt{\theta_l} \right) \cos \left( t \sqrt{\theta_l} \right) y_l(x) \langle y_l, \delta_y^{\text{ROM}} \rangle
\]

\[
= \sum_{l=1}^{\infty} \int_{-\infty}^{\infty} dt' \overline{f^\frac{1}{2}(t')} \cos \left( t' \sqrt{\theta_l} \right) \cos \left( t \sqrt{\theta_l} \right) y_l(x) \langle y_l, \delta_y^{\text{ROM}} \rangle
\]

\[
= \sum_{l=1}^{\infty} \int_{-\infty}^{\infty} dt' \overline{f^\frac{1}{2}(t')} \frac{1}{2} \left[ \cos \left( (t - t') \sqrt{\theta_l} \right) + \cos \left( (t + t') \sqrt{\theta_l} \right) \right] \times y_l(x) \langle y_l, \delta_y^{\text{ROM}} \rangle
\]

\[
= \sum_{l=1}^{\infty} \int_{-\infty}^{\infty} dt' \overline{f^\frac{1}{2}(t')} y_l(x) \langle y_l, \delta_y^{\text{ROM}} \rangle, \quad (4.10)
\]
where we used that operators of $A(c)$ commute, as well as the spectral decomposition of $A(c)$, definition (2.18) and that $\hat{f}^\sharp(t)$ is even. Since we have

$$\cos \left( \left( t - t' \right) \sqrt{A(c)} \right) \delta_y^{\text{ROM}}(x) = \int_\Omega dz \, g_x(t - t', x; z) \delta_y^{\text{ROM}}(z)$$

$$= \sum_{l=1}^{\infty} \cos \left( \left( t - t' \right) \sqrt{\theta_l} \right) y_l(x) \langle y_l, \delta_y^{\text{ROM}} \rangle,$$

(4.11)

pointwise in $t - t'$, and $\hat{f}^\sharp(t')$ has finite support, we can use the dominated convergence theorem to interchange the integral and sum in (4.10) and get

$$g(t, x; y) = \hat{f}^\sharp(t) * \int_\Omega dz \, g_x(t, x; z) \delta_y^{\text{ROM}}(z). \tag{4.12}$$

It remains to connect the Green’s function $G(t, x; z)$ to (4.8), which is the even extension in time

$${\mathcal G}_e(t, x; z) = G(t, x; z) + G(-t, x; z), \tag{4.13}$$

of the causal Green’s function $G(t, x; z)$, satisfying

$$[\partial_t^2 + A(c)] G(t, x; z) = \delta'(t) \delta_x(x), \quad t \in \mathbb{R}, \ x \in \Omega, \tag{4.14}$$

$$G(t, x; z) = 0, \quad t < 0, \ x \in \Omega, \tag{4.15}$$

$$\partial_t G(t, x; z) = 0, \quad t \in \mathbb{R}, \ x \in \partial \Omega_{\text{ac}}, \tag{4.16}$$

$$\partial_t G(t, x; z) = 0, \quad t \in \mathbb{R}, \ x \in \partial \Omega_{\text{mac}}. \tag{4.17}$$

We are interested in evaluating (4.12) at the sensor locations, which are far from $y$, where $\delta_y^{\text{ROM}}(x)$ peaks. By causality and the finite speed of propagation we should have $g(t, x; y) = 0$ for time $t = O(\tau_f)$. For larger time $t > 0$, we conclude from $G_e(t, x; z) = G(t, x; z)$ and the $\tau_f$ duration of the pulse that

$$g(t, x; y) = \hat{f}^\sharp(t) * \int_\Omega dz \, G(t, x; z) \delta_y^{\text{ROM}}(z), \quad t > O(\tau_f), \ r = 1, \ldots, m. \tag{4.18}$$

The solutions of (4.3)–(4.6) and (4.14)–(4.17) are related by

$$\partial_t G(t, x; z) = c(x) c(z) G(t, x; z), \tag{4.19}$$

where we used the expression (2.4) of the operator $A(c)$ and the assumption (4.2). The statement of the lemma follows from the identity

$$\hat{f}^\sharp(t) * \partial_t G(t, x; z) = \partial_t \hat{f}^\sharp(t) * G(t, x; z).$$

\[\square\]

4.2. Analysis of the imaging function

The expression of the imaging function (4.1) is given in the next proposition, obtained with the continuum time approximation and for a long duration of the measurements.
**Proposition 4.1.** The imaging function (4.1) is approximated by

\[
I(y) \approx \int_{\Omega} \int_{\Omega} \delta_y^\text{ROM}(z) \delta_y^\text{ROM}(z') \frac{\Gamma(z,z')}{c(z)c(z')}
\]

(4.20)

where

\[
\Gamma(z,z') := \frac{1}{\tau c^2} \sum_{j=1}^{n^T} \int_0^{\tau} dt \int_{\mathbb{R}^2} ds' \partial_s^2 \hat{f}_s^2(s) \partial_s \hat{f}_s^2(s') G(t-s,x,z) G(t-s',x',z').
\]

(4.21)

**Proof.** Approximating the sum over \( j \) in (4.1) by the integral in \( t \), we get

\[
I(y) \approx \frac{1}{\tau} \sum_{r=1}^{m} \int_0^{\tau} dt |g(t,x_r,y)|^2,
\]

and the result follows by substituting (4.7) into this expression.

In order to explain why the imaging function \( I(y) \) gives an image of the local changes of the velocity, we interpret its approximate expression (4.20) in terms of the result of the following time-reversal experiment:

- **First step.** Consider the source function

\[
n_1(t,x) = \frac{1}{\tau c(x) c_o} \partial_t \hat{f}_s^2(t) \delta_y^\text{ROM}(x),
\]

(4.22)

that is localized in space in the support of \( \delta_y^\text{ROM}(x) \), and in time in the support of \( \hat{f}_s^2(t) \).

The wave field generated by this source satisfies

\[
\left[ \frac{1}{c^2(x)} \partial_t^2 - \Delta_x \right] u_1(t,x) = n_1(t,x), \quad t \in \mathbb{R}, \ x \in \Omega,
\]

(4.23)

\[
u_1(t,x) = 0, \quad t \ll 0, \ x \in \Omega,
\]

(4.24)

\[
\partial_s u_1(t,x) = 0, \quad t \in \mathbb{R}, \ x \in \partial \Omega_{ac},
\]

(4.25)

\[
u_1(t,x) = 0, \quad t \in \mathbb{R}, \ x \in \partial \Omega_{mac}.
\]

(4.26)

We suppose that we record it at the sensor locations \( (x_r)_{r=1}^m \) for \( t \in [0,n^T] \). Using the Green’s function \( G(t,x; z) \), the solution of (4.3)–(4.6), we can write these recordings as

\[
u_1(t,x_r) = \int_{\mathbb{R}} ds \int_{\Omega} dz G(t-s,x_r,z)n_1(s,z).
\]

(4.27)

We then have from (4.20)–(4.21) that

\[
I(y) \approx \sum_{r=1}^{m} \int_0^{n^T} dt \int_{\Omega} dz \int_{\Omega} \frac{dz'}{c(z)c_o} G(t-s,x_r;z) \delta_y^\text{ROM}(z) \partial_s \hat{f}_s^2(s) u_1(t,x_r),
\]

(4.28)
and after the change of variables $t \mapsto -t$, $s \mapsto -s$ and using that $\partial_{y}j^{\frac{1}{2}}$ is odd, this equation becomes

$$
\mathcal{I}(y) = - \int_{\mathbb{R}} ds \int_{\Omega} \frac{dz}{c(z)c_o} \delta_{y}^{\text{ROM}}(z) \partial_{s}j^{\frac{1}{2}}(s) \times \int_{\mathbb{R}} dt \sum_{r=1}^{m} G(s - t, x_r; z) u_1(-t, x_r) 1_{[-\tau, 0]}(t).
$$

(4.29)

Here $1_{[-\tau, 0]}(t)$ is the indicator function of the interval $[-\tau, 0]$, equal to 1 when $t$ lies in this interval and 0 otherwise.

- **Second step.** Consider the source function

$$
n_2(t, x) = \sum_{r=1}^{m} u_1(-t, x_r) 1_{[-\tau, 0]}(t) \delta_{x_r}(x),
$$

(4.30)

that is localized in time in $[-\tau, 0]$ and in space at the sensor locations $(x_r)_{r=1}^{m}$. This source transmits the time-reversed recorded signals $u_1(-t, x_r)$, and the generated wave field $u_2(t, x)$ satisfies

$$
\left[ \frac{1}{c^2(x)} \frac{\partial^2}{\partial t^2} - \Delta_x \right] u_2(t, x) = n_2(t, x), \quad t \in \mathbb{R}, \; x \in \Omega,
$$

(4.31)

$$
u_2(t, x) = 0, \quad t \ll 0, \; x \in \Omega,
$$

(4.32)

$$\partial_{\nu}u_2(t, x) = 0, \quad t \in \mathbb{R}, \; x \in \partial \Omega_{ac},
$$

(4.33)

$$u_2(t, x) = 0, \quad t \in \mathbb{R}, \; x \in \partial \Omega_{mac}.
$$

(4.34)

We have, using again the Green’s function, that

$$u_2(s, z) = \int_{\mathbb{R}} dt \int_{\Omega} G(s - t, z; z') n_2(t, z')
$$

$$= \sum_{r=1}^{m} \int_{\mathbb{R}} dt \, G(s - t, z; x_r) u_1(-t, x_r) 1_{[-\tau, 0]}(t),
$$

(4.35)

and from (4.29) we find

$$\mathcal{I}(y) = - \int_{\mathbb{R}} ds \int_{\Omega} \frac{dz}{c(z)c_o} \delta_{y}^{\text{ROM}}(z) \partial_{s}j^{\frac{1}{2}}(s) u_2(s, z).
$$

(4.36)

If $\delta_{y}^{\text{ROM}}$ is localized around $y$, then this expression shows that we observe the time-reversed wave around time 0, in the support of $j^{\frac{1}{2}}$, and around $y$.

Suppose that both the recording time window $\tau$ and the aperture of the sensor array are large enough. Then, the theory of time reversal for waves [2] predicts that the refocused wave $u_2(s, z)$ in (4.36) should be close to the original source (4.22), but time-reversed, and therefore

$$\mathcal{I}(y) \approx - \int_{\mathbb{R}} ds \int_{\Omega} \frac{dz}{c(z)c_o} \delta_{y}^{\text{ROM}}(z) \partial_{s}j^{\frac{1}{2}}(s) n_1(-s, z)
$$

$$= \int_{\mathbb{R}} ds \left[ \partial_{s}j^{\frac{1}{2}}(s) \right]^{2} \left[ \sum_{r=1}^{m} \int_{\Omega} \frac{dz}{c(z)c_o} \delta_{y}^{\text{ROM}}(z)^{2} \right].
$$

(4.37)
This expression shows that the imaging function $\mathcal{I}(y)$ is related to the local velocity at $y$, provided $\delta_y^{\text{ROM}}$ is peaked at $y$. If there are sharp and significant changes $c(z) - c_o(z)$ around $y$, which correspond to reflective structures in the non-scattering host-medium, they appear in $\mathcal{I}(y)$ with a resolution that depends on $\delta_y^{\text{ROM}}$. The more focused this is at $y$, the better the resolution. The other resolution controlling factors are the pulse width (support of $f^T$), the recording time $n\tau$ and the array aperture, which determine how well the time reversal wave $\psi_2$ refocuses.

We display the ROM point spread function $\delta_y^{\text{ROM}}$ in the numerical results section 6 to show that it is indeed peaked at $y$ if the time sample interval $\tau$ and the sensor separation $h$ are chosen properly. The explicit analysis carried out in appendix A for layered media and in a waveguide with horizontal reflectors also gives that the focusing of $\delta_y^{\text{ROM}}$ is controlled by $\tau$ and $h$. However, if the medium contains reflectors that are steeply slanted, the generated echoes may be hard to approximate in the space of the snapshots in the reference medium and $\delta_y^{\text{ROM}}$ may display some spurious peaks. We introduce in section 6.3 a numerical approach for checking if $\delta_y^{\text{ROM}}$ is indeed focused around $y$ and therefore the image can be trusted there.

Note that the energy carried by $\delta_y^{\text{ROM}}(z)$ is insensitive to the variations $c(z) - c_o(z)$, so there is no cancellation of the wave speed in (4.36) and (4.37). To show this, we compute explicitly the $L^2(\Omega)$ norm of $\delta_y^{\text{ROM}}(z)$ using its definition (3.5) and the orthonormality of the components of $V(z)$.

$$v_j(z) = \left(v_j^{(1)}(z), \ldots, v_j^{(m)}(z)\right), \quad j = 0, \ldots, n - 1,$$

which gives

$$\int_\Omega dz \; v_j^{(s)}(z)v_j^{(s')}\left(z\right) = \delta_{j,j'}\delta_{s,s'}, \quad \forall \; j, j' = 0, \ldots, n - 1, \; s, s' = 1, \ldots, m,$$

where $\delta_{j,j'}$ is the Kronecker delta. We obtain that

$$\|\delta_y^{\text{ROM}}\|_{L^2(\Omega)}^2 = \int_\Omega dz \; \left[\sum_{j=0}^{n-1} \sum_{s=1}^{m} v_j^{(s)}(z)v_j^{(s)}(y)\right]^2 = \sum_{j=0}^{n-1} \sum_{s=1}^{m} \|v_j^{(s)}(y)\|^2. \quad (4.38)$$

where the right-hand side depends only on the orthonormal snapshots in the reference medium and is, therefore, insensitive to the variations $c(z) - c_o(z)$.

### 4.3. Comparison with backprojection imaging

The backprojection imaging function introduced in [15] is given by

$$\mathcal{I}^{\text{BP}}(y) = V_o(y) \left(\mathcal{P}^{\text{ROM}} - \mathcal{P}_o^{\text{ROM}}\right) V_o^T(y), \quad (4.39)$$

where $\mathcal{P}_o^{\text{ROM}}$ is the ROM propagator calculated in the reference medium with known wave speed $c_o(x)$. To compare it with our imaging function $\mathcal{I}(y)$, let us rewrite (4.39) using equation (2.33) for the ROM propagator and the analogue of (3.3) in the reference medium,

$$\delta_{o,y}(x) := V_o Y_o^{\text{T}}\delta_y(x) = \sum_{j=0}^{n-1} v_{o,j}(x)v_{o,j}^{\text{T}}(y) = V_o(x)V_o^T(y). \quad (4.40)$$
We obtain that
\[ I^{BP}(y) = V_oV^T P_VV_o^T(y) - V_oV^T P_{o}V_oV^T(y) \]
\[ = \sum_{j=0}^{n-1} v_{o,j}(y) \langle v_j, V\delta^{\text{ROM}} \rangle - \sum_{j=0}^{n-1} v_{o,j}(y) \langle v_{o,j}, P_{o}\delta_{o,y} \rangle, \tag{4.41} \]
where \( P_o \) is the wave propagator operator in the reference medium.

Let us explain the meaning of the two terms in the right-hand side of (4.41); the last term models the wave \( P_o\delta_{o,y}(x) \) with initial state \( \delta_{o,y}(x) := V_oV^T\delta_{y}(x) \) peaked around \( y \), and propagated in the reference medium for the duration \( \tau \). This wave is then projected onto the reference space \( \mathcal{S}_o \) using \( V_oV^T \), and the result is evaluated at \( y \). The first term in (4.41) involves the internal wave
\[ P_y\delta^{\text{ROM}}(y) = \cos(\tau\sqrt{\Lambda(c)}) \delta^{\text{ROM}}(x), \tag{4.42} \]
that is similar to our wave \( g(\tau \cdot x; y) \) given in (4.4). Ideally, this wave would be projected onto the space \( \mathcal{S} \), but since \( V(x) \) is unknown, the projector \( VV^T \) is replaced by \( V_oV^T \), based on the expectation that the approximation (3.8) holds.

By hyperbolicity and the short duration \( \tau \) of propagation of the waves involved in (4.41), both terms described above should be affected mostly by the medium in the vicinity of \( y \). Therefore, by taking the difference of the terms, the backprojection imaging function is designed to sense changes \( c(x) - c_o(x) \) in the vicinity of the imaging point, like \( I(y) \).

The numerical results in section 6 show that \( I(y) \) and \( I^{BP}(y) \) perform similarly, although \( I(y) \) has better cross-range resolution. They both outperform the reverse-time migration imaging approach, in the sense that they do not suffer from multiple scattering artifacts. We refer to section 6.1 for the computational benefits of using \( I(y) \) vs \( I^{BP}(y) \) and \( I^{RTM}(y) \).

5. Pixel scanning type imaging

In this section we use the internal wave estimated in proposition 3.1 as a steering control at the array, for focusing the wave at the imaging points \( y \in \Omega_{im} \). Such steering can be implemented experimentally and can be used for imaging in a pixel scanning manner.

The basic idea is the principle of time reversal: since we know the internal wave \( g(t, x; y) \) originating from the vicinity of \( y \in \Omega_{im} \), we can just time reverse it and re-emit it into the medium, where it refocuses near \( y \). If \( y \) lies near a reflector, the refocused wave will be reflected back towards the array, where it can be measured. The reflector location can then be estimated from the peaks of the ‘pixel scanning’ imaging function \( I^{PS}(y) \) defined below, which matches the reflected wave measured at \( x \), with \( g(t, x; y) \), for \( r = 1, \ldots, m \).

5.1. Imaging algorithm

The calculation of the imaging function \( I^{PS}(y) \) is carried out with the following steps:
\begin{enumerate}
  \item[(a)] Compute the mass matrix \( M \) from the data collected at the array, as given in equation (2.28).
  \item[(b)] Compute the Cholesky factorization (2.30) and store \( R \).
  \item[(c)] Compute \( V_o(x) \) by carrying out the Gram–Schmidt orthogonalization of \( U_o(x) \) computed by solving the wave equation in the reference medium with known wave speed \( c_o(x) \). This is especially easy to do if \( c_o(x) = \bar{c}_o \) for all \( x \in \Omega \).
\end{enumerate}
(d) For each \( y \in \Omega_{im} \) compute the internal wave \( g(t,x_r; y) \) using equation (3.9), for \( r = 1, \ldots, m \).

(e) Define the control at the array for focusing at \( y \)

\[
\tilde{g}(t,x_r; y) := 1_{[0,n\tau]}(t)g(n\tau - t,x_r; y), \quad s = 1, \ldots, m.
\] (5.1)

Measure the wave \( \gamma(t,x; y) \) at the sensors \( x = x_r \), after using the illumination (5.1).

(f) Calculate the imaging function

\[
I_{PS}(y) := \sum_{r=1}^{m} \int_{0}^{n\tau} dt \gamma(n\tau + t,x_r; y)g(t,x_r; y).
\] (5.2)

Note that equation (3.9) gives the internal wave at the discrete time instants \( t = j\tau \), for \( j = 0, \ldots, n-1 \). If \( \tau \) is small enough, we can use interpolation to get \( \tilde{g}(t,x_r; y) \) at \( t \in [0,n\tau] \). Note also that at step (d), the measurements should be for the acoustic pressure \( c(x)\gamma(t,x; y) \). Since the wave speed at the sensors equals the known constant \( \tilde{c}_r \), those measurements determine \( \gamma(t,x_r; y) \), for \( r = 1, \ldots, m \).

5.2. Expression of the refocusing and imaging functions

The mathematical model of the wave \( \gamma(t,x; y) \) measured at step (e) of the algorithm is the solution of the wave equation

\[
\partial_t^2 \gamma(t,x; y) + A(c)\gamma(t,x; y) = \partial_t \sum_{r=1}^{m} \tilde{g}(t,x_r; y), \quad t > 0, \ x \in \Omega,
\] (5.3)

\[
\gamma(t,x; y) = 0, \quad t < 0, \ x \in \Omega,
\] (5.4)

\[
\partial_t \gamma(t,x; y) = 0, \quad t > 0, \ x \in \partial\Omega_{ac},
\] (5.5)

\[
\gamma(t,x; y) = 0, \quad t > 0, \ x \in \partial\Omega_{imac}.
\] (5.6)

We now show that this wave focuses near \( y \) at time \( t = n\tau \).

Using the Green’s function \( \mathcal{G}(t,x;z) \) defined in equations (4.14)–(4.17), we can write using linear superposition that

\[
\gamma(t,x; y) = \sum_{s=1}^{m} \mathcal{F}(t,x_r; y) \ast \delta(t,x_s; y),
\] (5.7)

where \( \mathcal{F}(t,x_s; y) \) is given by (5.1) in terms of the internal wave (3.4). A calculation similar to that in the proof of lemma 4.1 gives that

\[
g(t,x_s; y) = \int_{\Omega} dz \mathcal{G}(t,x_s; z) \delta_{y}^{fROM}(z),
\] (5.8)

and substituting the result into (5.7) we get
\[
\gamma(t, x_t; y) = \int_{\Omega} dz \frac{\delta_{y}^{\text{ROM}}(z)}{\delta_{y}^{\text{ROM}}(z)} \int_{0}^{\tau} dt \sum_{s=1}^{m} G(n \tau - t', x_t; z) \mathcal{G}(t - t', x_t; x_s)
= \int_{\Omega} dz \frac{\delta_{y}^{\text{ROM}}(z)}{\delta_{y}^{\text{ROM}}(z)} \int_{0}^{\tau} dt \sum_{s=1}^{m} G(n \tau - t', z; x_s) \mathcal{G}(t - t', x_t; x_s),
\]

where we have used the reciprocity relation \(\mathcal{G}(t', x_t; z) = \mathcal{G}(t', z; x_t)\). This clearly peaks at the instant \(t = n \tau\), when the two Green’s functions are in sync, and at points \(x \approx z\) in the support of \(\delta_{y}^{\text{ROM}}(z)\) defined in (3.4). Similar reasoning to that used in section 2.1 for the sensor functions (2.17) gives that \(\delta_{y}^{\text{ROM}}(z)\) has a slightly larger support than \(\delta_{y}^{\text{ROM}}(x)\), by an \(O(\epsilon(y) t_{f})\) radius.

The expression of the imaging function follows once we use (5.8) and (5.9) in (5.2)
\[
\mathcal{I}^{\text{PS}}(y) = \sum_{r=1}^{m} \sum_{s=1}^{m} \int_{\Omega} dz \frac{\delta_{y}^{\text{ROM}}(z)}{\delta_{y}^{\text{ROM}}(z)} \int_{\Omega} dz' \frac{\delta_{y}^{\text{ROM}}(z')}{\delta_{y}^{\text{ROM}}(z')} \int_{0}^{\tau} dt \int_{0}^{\tau} dt' G(t, x_t; z) \mathcal{G}(t', z'; x_t) \mathcal{G}(t + t', x_t; x_s) 
\approx \sum_{r=1}^{m} \sum_{s=1}^{m} \int_{\Omega} dz \frac{\delta_{y}^{\text{ROM}}(z)}{\delta_{y}^{\text{ROM}}(z)} \int_{\Omega} dz' \frac{\delta_{y}^{\text{ROM}}(z')}{\delta_{y}^{\text{ROM}}(z')} \mathcal{G}(-t, x_t; z) \mathcal{G}(-t, z'; x_t) \mathcal{G}(t, x_t; x_s) |_{t=0},
\]

where the approximation is for large enough \(n \tau\). As was the case in the previous section, the ROM point spread function \(\delta_{y}^{\text{ROM}}(z)\) plays an important role in the imaging function. If \(\delta_{y}^{\text{ROM}}(z)\) is sharply peaked at \(y\), so is \(\delta_{y}^{\text{ROM}}(z)\) and we have a contribution to (5.10) from points \(z \approx z' \approx y\). Then, we can interpret the terms in (5.10) as follows: the first time convolution
\[
\mathcal{G}(t, x_t; z) \mathcal{G}(t, x_t; x_s) \approx \mathcal{G}(t, x_t; y) \mathcal{G}(t, y; x_s)
\]
models the wave propagating from the source at \(x_t\) to \(y\), where we suppose there is a reflector, it presumably scatters there and then propagates back to the receiver at \(x_t\) in the array. The second time convolution matches this wave with \(\mathcal{G}(t, x_t; x_s)\), which models the echoes received at \(x_s\), due to the illumination from \(x_t\). If indeed there is a scatterer at \(y\), then there should be an arrival in \(\mathcal{G}(t, x_t; x_s)\) that is synchronous to that in \(\mathcal{G}(t, x_t; y) \mathcal{G}(t, y; x_s)\), and we will get a large contribution to \(\mathcal{I}^{\text{PS}}(y)\).

**Remark 5.1.** The imaging function \(\mathcal{I}^{\text{PS}}(y)\) resembles that of the reverse-time migration approach, where the array data, modeled by \(f(t) * \mathcal{G}(t, x_t; x_s)\), are migrated to the imaging point \(y\) in the reference medium
\[
\mathcal{I}^{\text{RTM}}(y) \approx \sum_{r=1}^{m} \sum_{s=1}^{m} \mathcal{G}(-t, x_t; y) \mathcal{G}(-t, y; x_s) f(t) * \mathcal{G}(t, x_t; x_s) |_{t=0}.
\]

In (5.10) we use the Green’s function \(\mathcal{G}\) in the true medium and not the reference one, which should give a better result. However, we cannot obtain the ideal ‘time-reversal’ function
\[
\mathcal{I}^{\text{TR}}(y) \approx \sum_{r=1}^{m} \sum_{s=1}^{m} \mathcal{G}(-t, x_t; y) \mathcal{G}(-t, y; x_s) f(t) * \mathcal{G}(t, x_t; x_s) |_{t=0},
\]

Instead, we have the blurrer version (5.10), where we integrate over points in the support of \(\delta_{y}^{\text{ROM}}(x)\).
Remark 5.2. The imaging functions $I(y)$ and $I_{BP}(y)$ discussed in section 4 are quite different from $I_{PS}(y)$ and $I_{RTM}(y)$. They are designed to be sensitive only to changes of the wave speed in the vicinity of the imaging point $y$, and are not affected by the arrivals of the multiply scattered echoes in the medium. Such echoes are the cause of ghost reflectors present in the images formed with the functions (5.10) and (5.11), as we show with numerical simulations in section 6.

6. Numerical results

In this section we present numerical results in two-dimensions. The setup mimics that in figure 1, with a rectangular domain $\Omega$ and the accessible boundary near the array, modeled as sound hard. The inaccessible boundary is sound soft and consists of two side boundaries aligned with the range direction, and a remote boundary, parallel to the array, which does not affect the waves over the duration $(2n-1)\tau$ of the data gather. The side boundaries are close enough to each other to play a role in the simulations shown in sections 6.2–6.5 and thus cause a waveguide effect. We also present in section 6.6 simulations for well separated side boundaries, that have no effect for $t \in (0,(2n-1)\tau)$.

The reference (host) medium is homogeneous, with constant wave speed $\bar{c}_o$. The unknown wave speed $c(x)$ varies with the simulation and is displayed in the figures below. All length scales are in units of the central wavelength $\lambda_c = 2\pi\bar{c}_o/\omega_c$. The probing pulse is

$$f(t) = \sqrt{\frac{2\pi}{2}} \exp\left(-\frac{t^2B^2}{2}\right) \cos(\omega_c t), \quad B = 0.25\omega_c. \quad (6.1)$$

The data are generated by solving the wave equation for the acoustic pressure $(p^{\delta}(t,x))_{t=1,...,m}$ using a time domain, second order centered finite differences scheme, on a square mesh with size $\lambda_c/16$. The time steps in this scheme are chosen to satisfy the Courant Friedrichs Lewy (CFL) condition.

The presentation of the numerical results is organized as follows: we begin in section 6.1 with a discussion of the implementation and the cost of computing the internal wave and the imaging function (4.1). In section 6.2 we give imaging results in the waveguide setting and illustrate the effects of the time sampling interval $\tau$ and the sensor separation $h$ on the ROM point spread function $\delta_{\text{ROM}}^y$. In section 6.3 we introduce an approach for assessing the focusing of $\delta_{\text{ROM}}^y$ at the imaging points $y$ in the imaging domain. This could be used in practice for checking the reliability of the imaging function (4.1). The internal wave based control for focusing via time reversal is illustrated in section 6.4. The results in sections 6.2–6.4 are for noiseless data, but we assess the effect of noise in section 6.5. In section 6.6 we present results in the half space setting. We end in section 6.7 with some results that suggest possible improvements of imaging by either using strategically placed known reflectors, or by recomputing $\delta_{\text{ROM}}^y$ iteratively.

6.1. Computational cost and implementation

The imaging function $I(y)$ defined by (4.1) is slightly cheaper to compute than the backprojection function $I_{BP}(y)$ defined by (4.39), because the latter requires the computation of the ROM propagator $P_{\text{ROM}}$ given in (2.33). Both methods use the data driven mass matrix $M$ and its Cholesky factor $R$. The computation of $P_{\text{ROM}}$ involves, in addition, the formation of the stiffness matrix $S$ using (2.29), and the solution of two block triangular linear systems i.e., the
multiplication of $S$ on the left by $R^{-T}$ and on the right by $R^{-1}$. Such systems can be solved efficiently, so the added computational cost of $I^{BF}(y)$ vs $I(y)$ is not large. The real advantage of $I(y)$ is that it is more robust. Indeed, the mass matrix $M$ may become ill-conditioned and even indefinite due to noise, especially if the time sample interval $\tau$ and the sensor separation $h$ are small. Since the computation of $P_{ROM}$ and therefore $I^{BF}(y)$ involves the inverse $R^{-1}$, it needs proper regularization. The internal wave is computed without inverting $R$, so less regularization can be used, as shown in section 6.5.

Both $I(y)$ and $I^{BF}(y)$ involve $V_{o}(y)$, for $y$ in the imaging domain $\Omega_{im} \subset \Omega$, which can be computed as follows: first, we solve the wave equation in the reference medium with known wave speed $c_{r}(x)$ and use the values of the solution at the sensors to calculate the reference medium data matrices $(D_{s}(j\tau))_{j=0}^{2n-1}$. These determine the mass matrix $M_{o}$ by (2.28). The Cholesky factorization of $M_{o}$ gives $\tilde{R}_{o}$ and from the definition of $V_{o}(y)$ we have

$$V_{o}(y) = U_{o}(y)R_{o}^{-1} = \sum_{j=0}^{n-1} u_{o,j}(y)e_{j}^{T}R_{o}^{-1}.$$  

(6.2)

Thus, we solve the wave equation in the reference medium to get $u_{o,j}(y)$ at time steps $j = 0, \ldots, n-1$, for all $y \in \Omega_{im}$, and add the contributions, for each time step, to obtain $V_{o}(y)$. Note that in (6.2) the inverse $R_{o}^{-1}$ is less problematic than $R^{-1}$, because $R_{o}$ is not tainted by noise.

The reverse time migration imaging function $I^{RTM}(y)$ given in (5.11) involves the time convolution of the array data with two wave fields in the reference medium: the one that goes forward from the sources at $x_{s}$ to the imaging point $y$ and the one that goes backward from $y$ to the receivers at $x_{r}$, for $s, r = 1, \ldots, m$. A typical implementation of $I^{RTM}(y)$ stores in memory the backward going wave. This wave is then convolved with the forward wave that is computed incrementally, for each $y$. The repeated calls to memory and the large storage requirements make the computation of $I^{RTM}(y)$ significantly more expensive than that of $I(y)$.

While the computation of $I(y)$ involves the computation of the Cholesky factorizations of $M$ and $M_{o}$, these are less costly than solving the wave equation. As an illustration, for the results presented in section 6.2, it takes 317 seconds\(^6\) to solve the wave equation and 2.7 s to compute the Cholesky factorization.

6.2. Imaging in a waveguide setting

The numerical results in this section are for the setup illustrated in figure 2, where the side boundaries are close enough to play a role over the duration of the experiment, hence the name waveguide setting. We consider first a large aperture size $a = 30 \lambda_{c}$, with beginning and end at distance $\lambda_{c}$ from the side walls, and containing $m = 49$ equidistantly placed sensors. The time sample interval is $\tau = 0.4\sigma/\omega_{c}$, corresponding to 5 points per carrier period. However, we also test how the aperture size, the separation between the sensors and $\tau$ affect the results, so we give the values of $a$, $m$ and $\tau$ in the captions of the figures.

We display in the right plot of figure 2 the data $D^{y}_{j,r,s}$ for $j = 0, \ldots, n-1$, $r = 1, \ldots, m$ and $s = 25$, which indexes the center sensor in the array. We also show for comparison the data in the reference medium. Note the echoes from the side walls that are present in the true and the reference medium, and the echoes from the sought after reflectors that are emphasized in the data differences.

\(^6\)This is for a Matlab implementation running on an Intel(R) Xeon(R) E-2176G CPU 3.70 GHz.
Figure 2. Left: illustration of the setup for imaging in a waveguide: the array of $m = 49$ sensors (indicated with triangles) lying near the accessible boundary probes a medium with wave speed $c_0$, containing a few thin reflecting structures, modeled by the low velocity shown in the color bar. Right: the data corresponding to the illumination from the center element in the array. We show it for the medium with the reflectors, the reference medium and the difference between the two. Note the echoes from the side walls.

Figure 3. Imaging function $I(y)$ (left) and $I_{\text{ideal}}(y)$ (right) for the setup shown in figure 2. The aperture length is $a = 30\lambda_c$ and the array has $m = 49$ sensors. The data are sampled in time at interval $\tau = 0.4\pi/\omega_c$.

In figure 3 we display the imaging function $I(y)$ defined in (4.1) and the analogue function

$$I_{\text{ideal}}(y) = \sum_{r=1}^{m} \sum_{j=0}^{n-1} \left| g_{\text{ideal}}(j\tau, x_r; y) \right|^2, \quad y \in \Omega_{\text{im}}, \quad (6.3)$$

defined in terms of the ‘ideal’ internal wave (3.1) that cannot be computed from the data set. We can infer from proposition 4.1 that there is only one difference between these functions: the ROM point spread function $\delta^{\text{ROM}}_y$ in the expression (4.20) of $I(y)$ is replaced by the projection (3.3) of $\delta_y$ in the expression of $I_{\text{ideal}}(y)$. Due to the excellent focusing of (3.3), we see that $I_{\text{ideal}}(y)$ gives a very sharp (photo-like) estimate of the reflectors, whereas the image $I(y)$ is a blurrier estimate. Moreover, $I(y)$ captures only the top of the vertical reflector, and the unobstructed part of the bottom reflector.

Note that both plots in figure 3 display shadows of the reflectors and have larger values near the array, because of the energy trapped there. This is less visible in $I_{\text{ideal}}(y)$, because its peak
values are higher. To remove this effect, we display henceforth the derivative of the images in the range direction. This derivative is computed after smoothing the image in range by a convolution with a Gaussian function, with standard deviation 0.05λc.

The plots in figure 4 compare the four imaging functions discussed in the paper: \( I(y) \), \( I^{BP}(y) \), \( I^{PS}(y) \) and \( I^{RTM}(y) \). They all localize the reflectors, with the exception of the vertical one, whose top is the only visible part, and the obstructed part of the bottom one. However, the result given by the computationally inexpensive imaging function \( I(y) \) is the better one, because: (1) it gives a better separation of the two nearby horizontal reflectors; (2) it displays clearly the oblique reflectors; (3) it does not have the ghost reflector seen in \( I^{PS}(y) \) and especially \( I^{RTM}(y) \), due to the reverberation between the top reflector and the accessible boundary. The backprojection image is also free of the ghost, but its cross-range resolution is worse and it barely sees the oblique reflectors.

In figure 5 we illustrate the effect of the aperture size on the ROM point spread function \( \delta^R_{y} \) and the image \( I(y) \). The larger the aperture, the better the focusing of \( \delta^R_{y} \) in cross-range and the better the image.

Figure 6 shows the effect of the time sampling interval \( \tau \). The reference value is as in the previous experiments \( \tau = 0.4\pi/\omega_c \). For larger \( \tau \) the focus of the ROM point spread function deteriorates and the image becomes noisy. For smaller \( \tau \) the results are basically the same as in the bottom plots of figure 5. In practice \( \tau \) should not be reduced too much, because the snapshots become too close to each other and consequently, the Cholesky factorization and the Gram–Schmidt orthogonalization become ill conditioned.

Finally, we illustrate in figure 7 the effect of the separation between the sensors. The aperture is fixed at \( a = 30\lambda_c \) and we display results for \( m = 10, 20 \) and 60 equidistant sensors. We see that if the sensors are too far apart from each other, the focus of the ROM point spread

Figure 4. Top plots: range derivative of the imaging function \( I(y) \) (left), the backprojection image \( I^{BP}(y) \) (right). Bottom plots: the range derivative of the pixel scanning image \( I^{PS}(y) \) (left) and the reversed time migration image \( I^{RTM}(y) \) (right) for the setup shown in figure 2. The aperture length is \( a = 30\lambda_c \) and the array has \( m = 49 \) sensors. The time sample interval is \( \tau = 0.4\pi/\omega_c \).
Figure 5. Illustration of the effect of the aperture size. Left column: range derivative of the imaging function $I(y)$. Right column: the ROM point spread function $\delta_{y}^{\text{ROM}}(x)$ for the point $y$ between the two nearby horizontal reflectors. The aperture of the array is shown in blue at the top of the plots. Top row for 40% aperture, middle row for 60% aperture and bottom row for the full aperture $a = 30\lambda_c$ and $m = 49$ sensors. The separation between the sensors is kept the same, so the smaller the aperture, the fewer sensors. The time sample interval is $\tau = 0.4\pi/\omega_c$.

function deteriorates and the image becomes noisy. The bottom plots obtained with $m = 60$ are basically the same as those for $m = 49$ (shown in the bottom row of figure 5). In practice one should not take $m$ too large (i.e., sensors that are too close), because the Cholesky factorization of the mass matrix and the Gram–Schmidt orthogonalization become ill conditioned.

6.3. Numerical assessment of the focusing of the ROM point spread function

The results in figures 5–7 illustrate that when the reflected waves in the true medium are not well approximated in the span of the snapshots in the reference medium, the ROM point spread function $\delta_{y}^{\text{ROM}}$ displays spurious peaks. The largest such peaks lie above $y$, and their contribution to $g(t, x_r, y)$ consists of waves that arrive before the travel time from $y$ to $x_r$ (see the middle plot in figure 8). The quality of the image $I(y)$ depends mostly on the focusing of $\delta_{y}^{\text{ROM}}$, so one can use this fact to assess computationally the reliability of the image at $y$, as follows: let $T^{\text{eik}}(x_r, y)$ be the eikonal travel time from $y$ to $x_r$, calculated by solving the eikonal equation...
in the reference medium. Use this travel time to define the following measure of quality of the focusing at $y$

$$M(y) := \frac{1}{I(y)} \sum_{r=1}^{m} \sum_{j=0}^{m-1} 1_{[0,T]}(j\tau)|g(j\tau, x_r; y)|^2,$$

(6.4)

where $1_{[0,T]}(t)$ denotes the indicator function of the interval $[0, T]$, equal to 1 for $t \in [0, T)$ and 0 otherwise. We can normalize by $I(y)$ because it is strictly positive by definition (4.1).

The display of $M(y)$ in the right plot of figure 8 shows that it is small, less than 4% throughout the imaging domain. Its largest value is around the two slanted reflectors. The analysis in appendix A.2 shows that the waves scattered by horizontal reflectors in a waveguide can be approximated well in the span of the snapshots in the reference medium, so it is expected that the worse approximation is for the echoes from the slanted reflectors.
Figure 7. Illustration of the effect of $m$. Left column: range derivative of the imaging function $I(y)$. Right column: the ROM point spread function $\delta_{\text{ROM}}(x)$ for the point $y$ between the two nearby horizontal reflectors. The reference $\tau$ is $\tau_{\text{ref}} = 0.4\pi/\omega_c$. Top row for $m = 10$, middle row for $m = 20$ and bottom row for $m = 60$ (the case with $m = 49$ is shown in the bottom row of figure 5). The aperture is $a = 30\lambda_c$, so the smaller $m$ is, the larger the separation between the sensors. The time sample interval is $\tau = 0.4\pi/\omega_c$.

6.4. Focusing with the internal wave

In this section we illustrate the focusing of the wave $\gamma(t, x; y)$, the solution of (5.3)–(5.6) with the illumination (5.1) defined in terms of the internal wave computed as in proposition 3.1. The setup is as in figure 2 and we use the large aperture $a = 30\lambda_c$ with $m = 49$ sensors and the time sample $\tau = 0.4\pi/\omega_c$.

We display in the left column of figure 9 the wave $\gamma(t, x; y)$ at the time of focus, for three different points $y \in \Omega_{\text{im}}$: between the two nearby horizontal reflectors, on one of the oblique reflectors and near the hard to see vertical reflector. For comparison, we also display the waves given by the illumination calculated as in (5.1), with $g(t, x; y)$ replaced by: $g_{\text{c}}(t, x; y)$ calculated in the reference medium (middle column plots) and by $g^{\text{ideal}}(t, x; y)$ that cannot be computed in practice (right column plots). While the refocusing is not as good as the unattainable one obtained with $g^{\text{ideal}}(t, x; y)$, we see that using $g(t, x; y)$ is better than $g_{\text{c}}(t, x; y)$ for the two first points $y$. 

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Figure 8. Left: the ROM point spread function $\delta_{y}^{\text{ROM}}$ for a point between the two slanted reflectors in figure 2. Middle: the internal wave $g(j\tau, x_r; y)$ for this point. We draw with the black line the eikonal times $T_{eik}(x_r; y)$. Note the early arrivals due to the waves generated from the spurious peaks of $\delta_{y}^{\text{ROM}}$, above $y$. Right: our measure of the quality of the focusing of $\delta_{y}^{\text{ROM}}$. The setup is as in figure 4.

Figure 9. Illustration of focusing at three different points in the imaging domain. Left column: the refocused wave after the illumination computed with the internal wave $g(t, x; y)$. Middle column: the refocused wave after the illumination computed with $g_{r}(t, x; y)$ calculated in the reference medium. Right column: the refocused wave after the illumination computed with $g_{\text{ideal}}(t, x; y)$. The aperture is $a = 30\lambda$, the number of sensors is $m = 49$ and $\tau = 0.4\pi/\omega$. 

6.5. Simulations with noisy data

For noisy data, formula (2.28) gives a symmetric data driven mass matrix that may not be positive definite. Thus, to compute $I(y)$ or $I^{BP}(y)$ we use the following regularization. Let us call the data driven mass matrix $\tilde{M}$ and consider its eigenvalue decomposition $\tilde{M} = W \Lambda W^T$, 

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Figure 10. The range derivative imaging function \( I(y) \) (left) and the backprojection image \( I_{BP}(y) \) (right) when imaging with data that has 20% additive white Gaussian noise. The aperture is \( a = 30\lambda_c \), the number of sensors is \( m = 49 \) and \( \tau = 0.67\pi/\omega_c \).

Figure 11. Left: illustration of the setup: the array of \( m = 49 \) sensors (indicated with triangles) lying near the accessible boundary probes a medium with wave speed \( c_0 \), containing a few thin reflecting structures (cracks), modeled by the low velocity shown in the color bar. Right: the data corresponding to the illumination from the center element in the array. We show it for the medium with the reflectors, the reference medium and the difference between the two.

where \( \tilde{\Lambda} \) is the diagonal matrix of the eigenvalues \( \left( \tilde{\Lambda}_j \right)_{j=1}^{nm} \) and \( W \) is the orthogonal matrix of the eigenvectors. We transform \( \tilde{M} \) to a positive definite matrix \( M \) by using the eigenvalue decomposition of \( \tilde{M} \). Explicitly, we set a threshold of the lowest acceptable positive eigenvalue \( \Lambda_{\min} \) and define

\[
M = W\Lambda W^T, \quad (6.5)
\]

where \( \Lambda = \text{diag}(\Lambda_1, \ldots, \Lambda_{nm}) \) and \( \Lambda_j = \max\{\tilde{\Lambda}_j, \Lambda_{\min}\} \), for \( j = 1, \ldots, nm \). Then, we carry out the computation of the imaging functions using the Cholesky factor \( R \) of (6.5).

In figure 10 we show results for the same waveguide setting as above, and for data contaminated with white Gaussian, additive noise. The noisy data set is \( \{u^{(s)}(j, r, x) + \varepsilon_{s,r,j}, s, r = 1, \ldots, m, j = 0, \ldots, 2n-1\} \), where the \( \varepsilon_{s,r,j} \) are independent and identically distributed Gaussian random variables with zero-mean and variance \( \sigma^2_{\text{noise}} = 0.2^2 \max_{s,r,j}\{u^{(s)}(j, r, x)^2\} \). As we explained before, the conditioning of the noiseless
mass matrix depends on the time sampling interval $\tau$. The smaller $\tau$ is, the worse the conditioning. Thus, for noisy data it is beneficial to increase $\tau$ a little, as we do in the results in figure 10. Aside from increasing $\tau$, the mass matrix is regularized as described above. Because the computation of the backprojection image $I^{BP}(y)$ involves the unstable step of inverting the Cholesky factor $R$ of the mass matrix, the noise effect is much worse than in $I(y)$ when using the same regularization (6.5). A more involved stabilization of the backprojection image is needed, as explained in [8, 15]. We do not repeat that regularization strategy here, but note that it typically leads to ghost multiples at high noise levels.

### 6.6. Imaging in the half space

Here we present numerical results for the setup shown in figure 11, where the side boundaries are sufficiently far to have no effect on the data displayed in the right plots.

We show in figure 12 images obtained with the aperture size $a = 18\lambda_c$, containing $m = 49$ equidistantly spaced sensors. The time sampling interval is $\tau = 0.42 \pi/\omega_c$. Note that the multiple scattering artifacts in the reversed time migration and the pixel scanning images are more pronounced than in figure 4. The backprojection image $I^{BP}(y)$ and $I(y)$ do not have
Figure 13. Illustration of the effect of including known reflectors into the reference medium. Left: imaging configuration with two slanted reflectors and a vertical one, modeled by the low wave speed \(0.3c_0\). Middle: \(I(y)\) using a constant \(c_0\) to compute \(V_0\). Right: \(I(y)\) using the inhomogeneous reference medium that includes the two slanted reflectors. The aperture length is \(a = 30\lambda_c\) and the array has \(m = 49\) sensors. The time sample interval is \(\tau = 0.4\pi/\omega_c\).

Figure 14. Top left: the estimated reflectors obtained from the image \(I(y)\) using the Sobel edge detection algorithm. Top right: the ROM point spread function computed with the homogeneous reference medium. Bottom left: the ideal point spread function \(VV^T\delta_y\). Bottom right: the ROM point spread function computed with updated reference medium. The aperture length is \(a = 30\lambda_c\) and the array has \(m = 49\) sensors. The time sample interval is \(\tau = 0.4\pi/\omega_c\).

such artifacts and they both localize well the crack-like reflectors. Arguably, \(I(y)\) does a slightly better job at localizing the sloped part of the middle crack.

6.7 Possible improvements of the imaging process

We expect from the analysis in appendix A and from the numerical results displayed above that the reflectors that are the most difficult to image are those that are positioned almost vertically i.e., orthogonal or near orthogonal to the aperture of the array. If the application allows the
inclusion of known, strategically placed reflectors in the imaging region, then the imaging can be improved, as illustrated in figure 13. All three reflectors are difficult to image using a homogeneous reference medium, because the snapshots in such a medium do not give a rich enough space to approximate the echoes from the steeply slanted reflectors. This is seen in the middle plot of figure 13. However, if the slanted reflectors are known, and thus included in the reference medium, then the vertical reflector can be seen in the image displayed in the right plot of figure 13. Note that here we did not take the range derivative of $I(y)$ as in the other plots, because that derivative would cancel the image of the vertical reflector.

Another possible approach to image improvement is to update the reference medium using estimates of the locations of the reflectors from $I(y)$. Obviously, we cannot estimate the wave speed of the reflectors, because $I(y)$ is a qualitative imaging method. However, by assigning any values of the wave speed to the estimated reflectors, sufficiently different from $c_0$, we can create a new reference medium whose snapshots will contain echoes from these reflectors. The ROM point spread function $\delta_{y}^{\text{ROM}}$ recomputed using this medium will likely have an improved focus.

In figure 14, we illustrate this idea, by estimating the location of the reflectors from the image $I(y)$ displayed in the top left plot of figure 4. The estimation is done with the Sobel edge detection algorithm [23] and is displayed in the top plot of figure 14. We use it to update the reference medium, by setting the wave speed of the estimated reflectors to the value $0.5c_0$. In figure 14, we show the ROM point spread function $\delta_{y}^{\text{ROM}}$ for a point below the slanted reflectors, where it is harder to focus. The top right plot shows $\delta_{y}^{\text{ROM}}$ computed with the homogeneous reference medium, the bottom right plot shows $\delta_{y}^{\text{ROM}}$ computed with the new reference medium and the bottom left plot shows the ideal point spread function $V V^T \delta_y$, which cannot be computed from the data set. We note the improved focusing of $\delta_{y}^{\text{ROM}}$ (bottom right plot) which is not so different from the ideal one (bottom left).

7. Summary

We introduced and studied with analysis and numerical simulations a novel, computationally inexpensive approach for imaging reflectors in a host, non-scattering medium, with an active array of $m$ sensors, which probe the medium with a pulse $f(t)$ and measure the generated waves. The measurements are for a finite duration $2n\tau$, at instants spaced by $\tau$, chosen to satisfy the Nyquist sampling requirement for $f(t)$. The imaging is based on a data driven ROM of the wave propagator, the operator that maps the wave from one instant to the next. Specifically, it uses the ROM to estimate an ‘internal wave’ $g(t, y, x_r)$ originating from the vicinity of the imaging point $y$ and propagating through the unknown medium to the sensors at $x_r$, for $r = 1, \ldots, m$.

We introduced two kinds of imaging functions: the first, denoted by $I(y)$, has a very simple expression, given by the squared norm of the internal wave at the sensors. The second, denoted by $I_{PS}(y)$, can be implemented experimentally. It is a pixel scanning imaging approach which uses the internal wave to define a control of the illumination of the medium from the array, for improved focusing of a probing wave at the pixel (imaging point) $y$. It then uses a matched field approach to obtain $I_{PS}(y)$ from the resulting measured backscattered wave.

The functions $I(y)$ and $I_{PS}(y)$ use a different imaging principle: the first one is designed to be sensitive to variations of the wave speed locally, near the imaging point, so it is not affected by the arrivals of the multiply scattered echoes in the medium. The second one matches time arrivals at the array and is therefore affected by multiple scattering.
Both imaging functions use the time reversal refocusing principle. In particular, $I(y)$ is a blurry version of the function that models the refocusing of the wave in the time reversal experiment with a source at the imaging point $y$. The sharpness of the refocusing depends on the bandwidth of the probing pulse $f(t)$, the duration $2n\tau$ of the measurements and the aperture size of the array. These affect the resolution of $I(y)$, but the blur, quantified by the ROM point spread function $\delta_{ROM}(y)$, is the main factor. The better this peaks at the imaging point $y$, the better the image. We showed with analysis and numerical simulations that $\delta_{ROM}(y)$ is highly peaked at $y$ if: (1) the kinematics (the smooth wave speed in the host medium) is known accurately; (2) the time sampling interval $\tau$ and the sensor separation are small enough; and (3) the array has large enough aperture. Of all these requirements, knowing the kinematics may be harder to achieve in some applications.

It is well-known that attenuation breaks the time reversibility of the wave equation. As a consequence, it is difficult to implement modifications and corrections to reverse-time imaging in the presence of attenuation losses [1]. Therefore, our proposed imaging methods can only be used in non-attenuating media.

Since the imaging function $I(y)$ is easily computed and it is unfocused when the assumed kinematics is wrong, it could be possible to carry out an estimation of the smooth part of the wave speed based on an optimization of the sharpness of $I(y)$ quantified properly by some norm. This could be the subject of future work.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Appendix A. The approximation of the snapshots in the reference space

In this appendix we discuss two setups where we can analyze explicitly the approximation (3.8) of the orthonormal snapshots. Specifically, we quantify how well the snapshots $u_j^s(x)$, for $j = 0, \ldots, n-1$ and $s = 1, \ldots, m$, which span the projection space $\mathcal{S}$, can be approximated in the reference space $\mathcal{S}_r$. If the approximation error is small, since the Gram-Schmidt orthogonalization is a stable process, we have $V(x) \approx V_r(x)$.

The first setup is for a layered medium and it is discussed in appendix A.1. The second setup, discussed in appendix A.2, is for a waveguide.

A.1. Snapshots in a layered medium

The analysis is simplest in the one-dimensional case, where $m = 1$, so we consider it first. The higher dimensional case is discussed after that.

A.1.1. One-dimensional case. It is well known [17, chapter 3] that in one-dimension scattering occurs due to changes of the acoustic impedance, and that the wave speed $c(z)$ can be
eliminated from the wave equation by transforming to the travel time coordinate

$$T(z) := \int_0^z \frac{dz'}{c(z')}.$$  \hspace{1cm} (A.1)

Thus, we consider, only in this section, the more general acoustic wave equation corresponding to variable mass density $\rho(z)$ and bulk modulus $K(z)$, which define the wave speed $c(z) = \sqrt{K(z)/\rho(z)}$ and acoustic impedance $\zeta(z) = \sqrt{K(z)\rho(z)}$. The wave is modeled by the acoustic pressure $p(t, z)$, the solution of

$$\partial_t^2 p(t, z) - \zeta(z)c(z)\partial_z \left[ \frac{c(z)}{\zeta(z)} \partial_z p(t, z) \right] = f'(t)c(0)\delta(z), \quad t \in \mathbb{R}, \ z \in (0-, L),$$

\hspace{1cm} (A.2)

$$\partial_z p(t, 0-) = p(t, L) = 0, \quad t \in \mathbb{R},$$

$$p(t, z) = 0, \quad t \ll 0, \ z \in (0-, L),$$

\hspace{1cm} (A.4)

where $z = 0$ is the range coordinate of the accessible boundary, just above the sensor at $z = 0$, and $L > 0$ is the range of the inaccessible boundary, assumed large enough so it does not affect the wave over the duration of the measurements.

After the travel time coordinate transformation (A.1), we get that the even in time wave

$$P(t, T) := p(t, z(T)) + p(-t, z(T)),$$

\hspace{1cm} (A.5)

satisfies

$$\partial_t^2 P(t, T) - \zeta(z(T))\partial_T \left[ \frac{1}{\zeta(z(T))} \partial_T P(t, T) \right] = 0, \quad t > 0, \ T \in (0-, T(L)),$$

\hspace{1cm} (A.6)

$$\partial_T P(t, 0-) = P(t, T(L)) = 0, \quad t > 0,$$

\hspace{1cm} (A.7)

with initial conditions

$$P(0, T) = \varphi(T) \approx 2f(T), \quad \partial_T P(0, T) = 0, \quad T \in (0-, T(L)).$$

\hspace{1cm} (A.8)

The layered medium is modeled by the piecewise constant impedance

$$\zeta(z(T)) = \zeta_j, \quad T \in (T_{j-1}, T_j), \quad T_j := T(z_j), \quad j = 0, \ldots, \ell + 1,$$

\hspace{1cm} (A.9)

whose jumps at range coordinates $z_j$, ordered as $0- = z_{-1} < 0 < z_0 \ldots < z_{\ell + 1} = L$, give the reflection and transmission coefficients [17, chapter 3]

$$R_j := \frac{\zeta_j - \zeta_{j+1}}{\zeta_j + \zeta_{j+1}}, \quad T_j := \frac{2\sqrt{\zeta_j\zeta_{j+1}}}{\zeta_j + \zeta_{j+1}}, \quad j = 0, \ldots, \ell.$$

\hspace{1cm} (A.10)

In the reference medium with constant impedance $\zeta_o = \zeta(0)$, the wave is given by d’Alembert’s solution

$$P_o(t, T) = \frac{1}{2} [\varphi(T - t) + \varphi(T + t)].$$

\hspace{1cm} (A.11)

It is the sum of a forward and a backward wave, due to the accessible boundary.
If there is a single scattering layer ($\ell = 0$) in the medium, we obtain after a standard calculation as described in [17, chapter 3] that

\[ P(t, T) = \sum_{q=0}^{\infty} (-\mathcal{R}_0)^q \left\{ \frac{1}{2} \varphi(T - t + 2qT_0) + \varphi(T + t - 2qT_0) \right\} \]

\[ = \sum_{q=0}^{\infty} (-\mathcal{R}_0)^q P_0(t - 2qT_0, T), \quad (A.12) \]

if $T \in (0-, T_0)$, whereas for $T > T_0$ we have

\[ P(t, T) = \frac{\mathcal{S}_0 \sqrt{\mathcal{C}_0}}{\sqrt{\mathcal{G}_0}} \sum_{q=0}^{\infty} (-\mathcal{R}_0)^q \left\{ \frac{1}{2} \varphi(T - t + 2qT_0) \right\} \]

\[ = \frac{\mathcal{S}_0 \sqrt{\mathcal{C}_0}}{\sqrt{\mathcal{G}_0}} \sum_{q=0}^{\infty} (-\mathcal{R}_0)^q P_0(t - 2qT_0, T). \quad (A.13) \]

In the last equation we used that $\varphi(T + t - 2qT_0) = 0$ for $T > T_0 \gg t_f$ and time $t > 2qT_0$ at which the $q$th transmitted wave can be observed.

The series over $q$ in equations (A.12) and (A.13) account for the multiple reflections at the interface $T = T_0$. We have a train of waves that look just like the wave in the reference medium, with delays $2qT_0$ corresponding to the number of roundtrips between the accessible boundary and the interface. Using causality, we conclude that

\[ P(j\tau, T) \in \mathcal{F}_{o,j} := \text{span} \{ P_0(j'\tau, T), \quad j' = 0, \ldots, j \}, \quad \text{if } \frac{2T_0}{\tau} \in \mathbb{N}. \quad (A.14) \]

Otherwise, $P(j\tau, T)$ is approximated in $\mathcal{F}_{o,j}$ with some error, which is small if $\tau$ is small with respect to the scale of variation of $\varphi(t)$ and therefore $f(t)$.

If the medium has multiple layers ($\ell \geq 1$), the expression of $P(t, T)$ is given by a more complicated series, with each term corresponding to a sequence of scattering events [17, chapter 3]. Nevertheless, the conclusion is similar to the above: if the travel time between the interfaces is an integer multiple of $\tau$, which corresponds to a ‘Goupillaud medium’ [17, section 3.5.4], then the snapshots $P(j\tau, T)$ are represented exactly in the span of the snapshots in the reference medium. Otherwise, we have an error that is small if $\tau$ is small with respect to the scale of variation of $f(t)$.

In conclusion, in the one-dimensional case, as long as $\tau$ is small enough, the orthonormal snapshots are approximately the same as in the reference medium, in the travel time coordinate. Furthermore, if we have an accurate estimate of the smooth part of the wave speed, called $c_0(z)$, we can transform to the range coordinate and obtain (3.8).

### A.1.2. Higher dimensions.

Here we suppose that the waves generated by a source at range $z = 0$ propagate in the half space $z > 0$ filled with a layered medium with wave speed $c(z)$ and impedance $\zeta(z)$. Consider the system of coordinates $x = (x^+, z)$, with cross-range $x^+ \in \mathbb{R}^d$, for $d = 1$ or 2, and let the source be $f(t)\delta(x^+)\delta(z)$, with cross-range profile $S(x^+)$. Then, if we Fourier transform the acoustic wave equation for the pressure $p(t, x)$ with respect to $t$ and $x^+$, we obtain a family of one-dimensional Helmholtz equations.
we return to the wave equation in a medium with constant density, and assume for simplicity a two-dimensional waveguide $x = (x^+, z) \in (0, D) \times (0, \infty)$, with sound hard wall at $z = 0$, representing the accessible boundary $\partial \Omega_{ac}$, and sound soft side walls at $x^+ \in \{0, D\}$ that are part of the inaccessible boundary $\partial \Omega_{inc}$. The remaining part of $\partial \Omega_{inc}$ is an artificial sound soft boundary at $z = L$, for large enough $L$ so that the waves do not reach it over the duration of the measurements. Note that this is the setup for the numerical simulations in section 6.2.

The waveguide is filled with a homogeneous medium with wave speed $c_0$, and contains a thin reflector localized for simplicity at the range $z = z_0$, modeled by the reflectivity $r(x^+)\delta_{in}(z)$.
as follows
\[
\frac{1}{c^2(x)} = \frac{1}{\bar{c}_o} \left[ 1 + r(x^\perp) \delta_{z_0}(z) \right].
\] (A.21)

We analyze the acoustic pressure \(p^{(o)}(t, x)\) in the waveguide, related to the wave \(u^{(o)}(t, x)\) as explained in section 2. The excitation is as in (2.1), and the pulse \(f(t)\) is given by an even envelope function \(F\) supported in the interval \((-1, 1)\) and modulated at the central frequency \(\omega_c\),
\[
f(t) := F \left( \frac{t}{t_f} \right) \cos(\omega_c t).
\] (A.22)

The snapshots at \(z \neq z_0\) are defined by the even extension in time of the pressure, divided by the constant speed \(\bar{c}_o\),
\[
u^{(o)}(t, x) := \frac{p^{(o)}(t, x) + p^{(o)}(-t, x)}{\bar{c}_o}.
\] (A.23)

The analysis uses the mode decomposition of \(u^{(o)}(t, x)\), based on its expansion in the \(L^2(0, D)\) orthonormal basis \(\{\psi_j(x^\perp), j \geq 1\}\), where
\[
\psi_j(x^\perp) = \sqrt{\frac{2}{D}} \sin(\alpha_j x^\perp), \quad \alpha_j := \frac{\pi j}{D}, \quad j \geq 1,
\] (A.24)

are the eigenfunctions of the operator \(\partial^2_{x^\perp}\) acting on functions of \(x^\perp \in (0, D)\), with homogeneous Dirichlet boundary conditions. We are interested in the propagating modes, indexed by \(j = 1, \ldots, N = \lfloor k_c D/\pi \rfloor\), because the evanescent modes generated by the reflectivity at \(z = z_0\) are negligible by the time they reach the array. Here \(k_c = \omega_c / \bar{c}_o\) is the wave number at the central frequency, and we assume that the bandwidth \(B = O(1/t_f)\) of the probing pulse is small enough, so that
\[
\left| \frac{\omega D}{\pi \bar{c}_o} \right| \approx \left| \frac{k_c D}{\pi} \right|, \quad \forall \omega \in (\omega_c - B, \omega_c + B).
\] (A.25)

The expression of the snapshots in the empty (reference) waveguide is obtained after a standard calculation, as explained for example in [17, chapter 20],
\[
u^{(o)}(t, x) \approx \sum_{j=1}^{N} \psi_j(x^\perp) u^{(o)}_{\omega_j}(t, z) + \text{evanescent}.
\] (A.26)

It is a superposition of one-dimensional propagating waves (modes)
\[
u^{(o)}_{\omega_j}(t, z) := \frac{2k_c \psi_j(x^\perp)}{c_o \beta_j(\omega_c)} \left\{ F \left( \frac{t - z/c_o}{t_f} \right) \cos \left[ \beta_j(\omega_c) z - \omega_c t \right] + F \left( \frac{t + z/c_o}{t_f} \right) \cos \left[ \beta_j(\omega_c) z + \omega_c t \right] \right\},
\] (A.27)

with wave numbers
\[
\beta_j(\omega) := \text{sign}(\omega) \sqrt{\frac{\omega^2}{c_o^2} - \alpha_j^2}, \quad j = 1, \ldots, N, \tag{A.28}
\]

and the approximation in (A.26) is due to the small bandwidth assumption that allows us to write

\[
\beta_j(\omega) z \approx \beta_j(\omega_c) z + (\omega - \omega_c) \beta_j'(\omega_c) z, \quad \forall \omega \in (\omega_c - B, \omega_c + B). \tag{A.29}
\]

Again, we see that due to the accessible boundary, we have both forward and backward going waves in (A.27). The backward waves are observed only at small \(z\) and time \(t = O(t_f)\). The waveguide is dispersive, so the propagation is at mode dependent group speed

\[
c_{\omega,j} := \frac{1}{\beta_j'(\omega_c)} = \frac{\epsilon_{\omega,j}(\omega_c)}{k_c}, \tag{A.30}
\]

which is different than the phase speed \(\omega_c/\beta_j(\omega_c)\), for \(j = 1, \ldots, N\).

The expression of the snapshots in the waveguide with the reflectivity given in (A.21) involves a series of multiple scattering events at the reflector. For our purposes it suffices to look at the first two terms in this series, corresponding to the single scattering, Born approximation. The analysis of the higher order terms is similar and does not bring anything new. A standard calculation that uses approximations like (A.29) gives that the snapshots are

\[
u^{(s)}(t, x) \approx \sum_{j=1}^{N} \psi_j(x^+) \left[ u^{(s)}_{\omega,j}(t, z) + u^{(s)}_{\text{Born},j}(t, z) + O(\tau^2) \right] \quad \text{evanescent,} \tag{A.31}
\]

where for \(z \in (0-, z_0)\) we have

\[
u^{(s)}_{\text{Born},j}(t, z) \approx \frac{k_c^2}{2 \epsilon_{\omega,j} \beta_j} \sum_{l=1}^{N} \tau_{jl} \frac{\psi_l(x)}{\beta_l'} \frac{\partial}{\partial t} \left\{ F \left[ t - \beta_l' z - \omega_0 t + \omega_0 \beta_l \right] \right\} \tag{A.32}
\]

and for \(z > z_0\) we have

\[
u^{(s)}_{\text{Born},j}(t, z) \approx \frac{k_c^2}{2 \epsilon_{\omega,j} \beta_j} \sum_{l=1}^{N} \tau_{jl} \frac{\psi_l(x)}{\beta_l'} \frac{\partial}{\partial t} \left\{ F \left[ t - \beta_l' z - \omega_0 t + \omega_0 \beta_l \right] \right\} \tag{A.33}
\]

In these equations we simplified the notation by dropping the \(\omega_c\) arguments of \(\beta_j, \beta_l\) and their derivatives, and we introduced the reflectivity matrix

\[
\tau_{jl} := \int_0^B dx^+ \tau(x^+) \psi_j(x^+) \psi_l(x^+). \tag{A.34}
\]

Note that the terms in (A.32) model two kinds of waves: the first kind strikes the reflector as mode \(l\), it is converted to mode \(j\), travels to the accessible boundary, it is reflected there and then travels forward. The second kind strikes the reflector as mode \(l\), it is converted to mode
and then travels backward. Similarly, the first term in (A.33) models the wave that starts as mode \( l \), it is converted to mode \( j \), travels to the accessible boundary where it reflects and then propagates forward. The second term models the wave that strikes the reflector as mode \( l \), it is converted to mode \( j \) and then propagates forward. We now show that these waves can be approximated in the span of the time delayed reference waveguide modes (A.27).

Let us introduce the travel times

\[
t_{lj} := \frac{z_0 (\beta_j + \beta_l)}{c_{o,j}} = \frac{z_0}{c_{o,j}},
\]

(A.35)
corresponding to the propagation of the envelope of the wave at group speeds (A.30), and

\[
T_{lj} := \frac{z_0 (\beta_j + \beta_l)}{c, k_c},
\]

(A.36)
corresponding to the propagation of the phase. Then, expanding the cosine in (A.32) we get

\[
\psi(t, z) \approx \frac{k^2}{2 c_o^2} \sum_{j=1}^{N} \psi_{lj} e^{i k c_{o,j} t_{lj}} \left\{ \cos \left[ \omega_c (T_{lj} - t_{lj}) \right] \partial_t \left\{ F \left[ \frac{t - t_{lj} - z/c_{o,j}}{t_f} \right] \cos \left[ \beta_j - \omega_c (t - t_{lj}) \right] \right\} \\
+ F \left[ \frac{t - t_{lj} + z/c_{o,j}}{t_f} \right] \cos \left[ \beta_z - \omega_c (t - t_{lj}) \right] \} \\
- \sin \left[ \omega_c (T_{lj} - t_{lj}) \right] \partial_t \left\{ F \left[ \frac{t - t_{lj} - z/c_{o,j}}{t_f} \right] \sin \left[ \beta_j - \omega_c (t - t_{lj}) \right] \} \\
- F \left[ \frac{t - t_{lj} + z/c_{o,j}}{t_f} \right] \sin \left[ \beta_z - \omega_c (t - t_{lj}) \right] \right\} \quad (A.37)
\]

for \( z \in (0, z_0) \). Recalling equation (A.27), we note that the first curly bracket is proportional to \( \partial_t \psi(t_{lj}, z) \). The second curly bracket is approximately proportional to \( \partial_t^2 \psi(t_{lj}, z) \), because

\[
\partial_t \left\{ F \left[ \frac{t - t_{lj} - z/c_{o,j}}{t_f} \right] \cos \left[ \beta_j - \omega_c (t - t_{lj}) \right] \right\} \\
= \omega_c \left[ F \left[ \frac{t - t_{lj} - z/c_{o,j}}{t_f} \right] \sin \left[ \beta_j - \omega_c (t - t_{lj}) \right] + O \left( \frac{1}{\omega_c t_f} \right) \right] \\
\partial_t \left\{ F \left[ \frac{t - t_{lj} + z/c_{o,j}}{t_f} \right] \cos \left[ \beta_j + \omega_c (t - t_{lj}) \right] \right\} \\
= -\omega_c \left[ F \left[ \frac{t - t_{lj} + z/c_{o,j}}{t_f} \right] \sin \left[ \beta_j + \omega_c (t - t_{lj}) \right] + O \left( \frac{1}{\omega_c t_f} \right) \right]
\]

and we have assumed

\[
\frac{1}{\omega_c t_f} = O \left( \frac{B}{\omega_c} \right) \ll 1.
\]

For a small enough time sample interval \( \tau \), the time derivatives of \( \psi(t_{lj}, z) \) can be approximated with finite differences, so we conclude that the snapshots (A.32) evaluated at range \( z \in (0, z_0) \) can be approximated by linear combinations of the time delayed reference snapshots (A.27). Similarly, it follows that the result also holds for the snapshots (A.33) evaluated at \( z > z_0 \).
In the ROM construction we do not use the mode decomposition. However, if the sensors are closely spaced in the array, so that we can approximate the sum over them by an integral over the array aperture $A \subseteq (0, D)$, we can write

$$\int_A dx^\perp u_0^m(t, x)\psi_j(x^\perp) \approx \sum_{l=1}^N Q_{jl} u_l^0(t, z), \quad (A.38)$$

where $Q = (Q_{jl})_{j, l = 1, \ldots, N}$ is the mode coupling matrix

$$Q_{jl} := \int_A dx^\perp \psi_j(x^\perp)\psi_l(x^\perp), \quad j, l = 1, \ldots, N. \quad (A.39)$$

If $A$ is large enough, $Q$ is invertible, so the snapshots at the array carry the same information as the modes (A.27). This is what we need for our approximation, in addition to the small $\tau$ required to deal with the discrete time samples of the wave, as in the previous section.

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