

## NETWORK APPROXIMATION FOR EFFECTIVE VISCOSITY OF CONCENTRATED SUSPENSIONS WITH COMPLEX GEOMETRY\*

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**Abstract.** We study suspensions of rigid particles (inclusions) in a viscous incompressible fluid. The particles are close to touching one another, so that the suspension is near the packing limit, and the flow at small Reynolds number is modeled by the Stokes equations. The objective is to determine the dependence of the effective viscosity  $\langle \mu \rangle$  on the geometric properties of the particle array. We study spatially irregular arrays, for which the volume fraction alone is not sufficient to estimate the effective viscosity. We use the notion of the interparticle distance parameter  $\delta$ , based on the Voronoi tessellation, and we obtain a discrete network approximation of  $\langle \mu \rangle$ , as  $\delta \rightarrow 0$ . The asymptotic formulas for  $\langle \mu \rangle$ , derived in dimensions two and three, take into account translational and rotational motions of the particles. The leading term in the asymptotics is rigorously justified in two dimensions by constructing matching upper and lower variational bounds on  $\langle \mu \rangle$ . While the upper bound is obtained by patching together local approximate solutions, the construction of the lower bound cannot be obtained by a similar local analysis because the boundary conditions at fluid-solid interfaces must be resolved for all particles simultaneously. We observe that satisfying these boundary conditions, as well as the incompressibility condition, amounts to solving a certain algebraic system. The matrix of this system is determined by the total number of particles and their coordination numbers (number of neighbors of each particle). We show that the solvability of this system is determined by the properties of the network graph (which is uniquely defined by the array of particles) as well as by the conditions imposed at the external boundary.

**Key words.** effective viscosity, discrete network, variational bounds, concentrated suspension

**AMS subject classifications.** 74Q, 35Q72, 74F10, 76T20

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**1. Introduction.** In this paper, we obtain and justify approximate formulas for the effective viscosity  $\langle \mu \rangle$  of a highly concentrated suspension of solid particles in a viscous incompressible fluid. We study generic, nonperiodic spatial distributions of particles and focus on a particular type of highly concentrated suspension, which can be approximately modeled on the macroscale by a single phase fluid, called the effective fluid. The effective viscosity is determined from the equality of the viscous dissipation rates in the suspension and the effective fluid. This is a classical approach that goes back to Einstein [14], who approximated the effective viscosity in the limit of an infinitely small particle concentration (the so-called dilute limit). Further results for dilute suspensions can be found in [3] and the references therein.

While in the dilute limit the interactions between the particles are negligible, the case of finite (non-small) concentrations is much harder to analyze because these interactions must be taken into account. In [25], an asymptotic expansion of the effective viscosity was constructed assuming a periodic distribution of particles. In

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[25], the formal two-scale homogenization was carried out under the assumption that the number of particles tends to infinity while their total volume remains constant. In this case, the distances between the particles are of the order of their sizes, which is the key feature of the so-called finite (moderate) concentration regime.

By contrast, our interest lies in the high concentration regime, where the particles are close to touching one another, so the typical interparticle distances are much smaller than their sizes. In this case, the hydrodynamic interactions lead to the blow-up of the dissipation rate in the thin gaps between the closely spaced particles. Note that the effective fluid can be either Newtonian [15] or non-Newtonian [1, 2, 4, 24]. We consider only the former case, following [15, 16, 27] (see also [19] for a review of physical data). Also, we consider only noncolloidal suspensions, which means that hydrodynamic interactions are much stronger than Brownian interactions, so the latter can be neglected. For effective rheology of colloidal suspensions, one may consult [12].

For periodic arrays of particles [15, 16, 27], the estimation of the effective viscosity reduces to solving the flow problem locally, in a thin gap between two neighboring particles. In [15], this is done by a formal asymptotic method, similar to the well-known lubrication approximation, which takes into account only the translational motions of particles along the lines of their centers. The contributions of rotations and shear-type translations are neglected in [15]. In [16], a more general formula for the effective viscosity is obtained, which combines the results in [15] and [14] for dilute and high concentration regimes, respectively. In [27], the definition of the effective viscosity involves the traction exerted by the fluid on a single sphere. This traction satisfies an integral equation derived and solved (for a cubic periodic lattice) in [27]. Note in particular that the periodicity assumption in [27] reduces the boundary conditions on the surface of the particles to just a rigid body rotation (no translations).

In this paper, we consider generic, nonperiodic arrays, where different particles have different translational and rotational body motions. Since the rigid motions of the particles are not known a priori, the effective viscosity cannot be obtained simply by solving a local problem in the gap between adjacent particles. The motion of one particle influences the motion of all the particles in the array and, to find the effective viscosity, we must solve the global problem. A key ingredient in our method of solution is the so-called discrete network approximation.

Discrete network models have been used in the engineering and physics literature [21, 17, 28, 29], although the relation between the continuum problem and the discrete network has not been established. The first rigorous mathematical characterization of high-contrast media, in terms of discrete networks, was obtained for electromagnetic transport problems in [8, 9, 10, 11], where the electrical conductivity (and permittivity) were modeled as exponentials of the form  $e^{S(x)/\epsilon}$ . This continuum high-contrast model is due to Kozlov [22], where  $S$  is a smooth, Morse function and where  $\epsilon \ll 1$ , such that small variations of  $S$  are highly amplified by the exponential, thus giving the high contrast. Kozlov's model is especially useful in the context of imaging [10], where the medium is not known and it is approximated by a generic, high-contrast continuum. The high-contrast continuum model leads to an explicit characterization of two-dimensional flow of DC (AC) electric current in the material, in terms of a network of resistors (and capacitors), which is uniquely defined by the distribution of critical points of  $S$ . Explicitly, in the DC case, the nodes of the network are the local maxima of the electrical conductivity function (i.e., of  $S$ ) and the branches of the network connect adjacent nodes through the saddle points of  $S$ . The resistor associated with each branch is determined by the conductivity and by the curvatures of  $S$  at

the saddle point, respectively. The boundary currents and voltages of the asymptotic network are also uniquely defined by  $S$  and by the boundary conditions specified for the continuum problem, so the asymptotic results in [9, 10, 11] give more than the homogenized electrical properties of the high contrast continuum. They give that the Neumann-to-Dirichlet map of the continuum problem is asymptotically equivalent to the discrete Neumann-to-Dirichlet map of the asymptotic network, in the limit  $\epsilon \rightarrow 0$  [9, 11]. All the results in [8, 9, 10, 11] apply to the two-dimensional case for all smooth functions  $S$  with isolated, nondegenerate critical points. Extensions to three dimensions are straightforward for a special class of functions  $S$ , but for a general  $S$ , the network approximation may not apply.

In [5], another network approximation has been developed for a scalar, DC conductivity problem which models dispersive high contrast composites. In this case,  $S(x)$  is the characteristic function of the particles, the high-contrast parameter is  $\epsilon = 0$  (perfectly conducting particles), and the asymptotic analysis is carried out in the limit of the interparticle distance parameter  $\delta$  tending to zero. The particle radii are not treated as small parameters, and the number of the particles is sufficiently large but bounded from above by  $N_{\max}$ , where  $N_{\max}$  is the maximal close packing number. In [5], the connectivity patterns and the interparticle distance parameter for irregular spatial arrays of particles are rigorously defined using Voronoi tessellation. It is demonstrated that the network approximation is an efficient numerical tool, capable of capturing various percolation effects as well as effects due to the polydispersity of particles. This approach also allows for analytical error estimates, subsequently obtained in [6].

In [6, 8, 9, 10, 11, 5], the network approximation was rigorously justified by employing variational duality. The key point is the construction of trial functions, the electric potential and current density for the direct and dual variational problems, respectively. The choice of trial functions depends on both the mathematical and the physical features of the problem. For example, the construction of trial functions in [5] is essentially different from those in [8, 9, 10, 11], and it requires the development of new mathematical tools. While the upper bound can be obtained by patching together the appropriate test functions based on the local analysis of [20], such a straightforward approach does not work for the lower bound. The difficulty in obtaining the latter lies in the construction of trial functions for the dual problem, when the boundary conditions on the surfaces of the particles cannot be satisfied independently for each particle, and one must deal with all inclusions simultaneously. The dual (lower) bound was obtained in [5] by constructing an approximate, divergence-free trial electric current density in the gap between adjacent particles and extending it to zero elsewhere in the domain. Then, the network equations are used to choose the unknown parameters in the dual trial field, so that the boundary conditions on the surface of the particles are satisfied exactly. Note, however, that this construction is specialized to the scalar, electrical conductivity problem, and does not admit a generalization to vectorial problems.

In this work, we study the vectorial problem described by Stokes's flow in a closely packed suspension with rigid particles. Since the array of particles is irregular, our construction uses the interparticle distance parameter introduced in [5], based on the Voronoi tessellation. Due to the high concentration of particles of finite size, in a fixed volume, the particles are close to touching. Thus, we assume that distances  $\delta^{ij}$  between adjacent particles  $D^{(i)}$  and  $D^{(j)}$  become infinitesimally small, but positive. More precisely, we say that  $c\delta \leq \delta^{ij} \leq \delta$  for all pairs  $D^{(i)}, D^{(j)}$  of neighboring particles, where  $0 < c < 1$  is fixed and where  $\delta$  is the small parameter of the problem.

We are interested in the asymptotics of the effective viscosity as  $\delta \rightarrow 0$ , while the particle radii  $a_i$  are kept fixed and the number of particles  $N$  approaches  $N_{\max}$ , from below.

The goal of this paper is twofold. The first objective is to obtain a method for estimating the effective viscosity which captures explicitly the effects of the complex geometry (the irregular distribution of the location and size of the particles). This is done in both two and three dimensions, and our derivation is based on the generalization of the lubrication approximation technique. We take into account all possible translations and rotations of the rigid particles in the suspension, which we quantify by constant vectors  $\mathbf{T}^{(p)}$  and  $\boldsymbol{\omega}^{(p)}$ , respectively, for  $1 \leq p \leq N$ . Using the linearity of the problem, we approximate first the velocity, pressure, and stress in the gaps (necks) between the particles for translational and rotational motions, separately, and then we superpose the results. The lubrication analysis is local for each gap, and by summing the contribution of all the gaps, we obtain the discrete approximation of  $\langle \mu \rangle$ , parameterized in terms of the rigid body translational and rotational velocities  $\mathbf{T}^{(p)}$  and  $\boldsymbol{\omega}^{(p)}$ , respectively, for  $1 \leq p \leq N$ . These rigid body motions are not arbitrary, but they are calculated by solving a system of linear equations, which corresponds to the conditions of mechanical equilibrium for all particles in the suspension.

For the reader interested mainly in numerical estimation of the effective viscosity, we describe our approach in Remark 3.1. (See also the forthcoming paper [7], where the effective viscosity was computed for several boundary conditions and various particle arrays by adapting the approach developed in this paper.)

The second objective of the paper is to provide a rigorous mathematical justification of the asymptotic approximation of the effective viscosity. The rigorous justification of the leading order term in the asymptotic approximation is done here in two dimensions. The most subtle part of this justification is the construction of the dual trial function for the lower bound on the effective viscosity. None of the techniques developed previously in [5, 6, 8, 9, 10, 11] for constructing trial functions for the dual problem work here. There are two main difficulties in the construction of the bounds on the effective viscosity. The first difficulty is that the trial functions must be divergence free in the fluid domain. The second difficulty is raised by the boundary conditions on fluid-solid interfaces. While these issues can be handled in the upper bound construction with an approach inspired by the work in [5, 6, 8, 9, 10, 11], the dual problem is significantly more challenging because the trial fields are tensors. In the dual problem, neither of the above two difficulties can be resolved by doing local analysis, that is, by choosing approximate solutions in each gap followed by patching these solution together. First, we must consider the global problem to ensure that the boundary conditions are satisfied for all inclusions at once. Second, we show that the divergence-free requirement on the stress trial fields is also global, analogous to the interface conditions. Then, we observe that the solvability of a certain algebraic system is sufficient to ensure that these two global requirements are satisfied. The size of the matrix of the linear system is determined by  $N$ , the total number of particles, and by their coordination number (number of neighbors). The solvability of the system, in turn, is determined by the connectivity and the coordination numbers of the network graph corresponding to the particle array, as well as by the conditions at the external boundary. We present geometric conditions for the network graph (topology) so that this linear system is solvable. In particular, we point out that these conditions are satisfied by network graphs which model typical close packing configurations.

The paper deals with irregular spatial arrays of particles. In this case, the total

volume fraction of particles (the only parameter in the formulas from [14, 15]) is not sufficient for estimating the effective viscosity. Instead of a formula, we give an algorithm, which essentially reduces computation of the effective viscosity to solving a linear algebraic system for translational and angular velocities of particles. The gain here is that we obtain an accurate yet computationally inexpensive approximation for the effective viscosity, which, unlike the above mentioned formulas, takes into account variable distances between neighboring particles. Note that variability in these distances for a fixed total volume fraction of particles may result in significant changes in the effective properties due to percolation effects (see [6]).

The focus of this paper is on derivation and, particularly, analytical justification of this algorithm, while its implementation will be investigated elsewhere. (See, for example, the forthcoming [7], where both shear and compression boundary conditions for various arrays of particles are investigated.) This paper, however, contains results of immediate practical interest, such as determination of the order of magnitude of the effective viscosity in the interparticle distance parameter  $\delta$ . An interesting feature of the vectorial problem that distinguishes it from the scalar case considered in [5, 6, 8, 9, 10, 11] is that the order of magnitude of the effective viscosity depends crucially on the geometry of the particle array and on the boundary conditions. For the scalar problem, the order of magnitude is the same for all networks satisfying a natural connectedness assumption [5], which is not the case in our vectorial problem. In section 6.2.6 we give a sufficient condition on the particle array such that the effective viscosity blows up at the rate  $\delta^{-3/2}$ , in two dimensions, and the leading term in the asymptotics of  $\langle\mu\rangle$  is given by the so-called spring network approximation, in which only the translational motions of adjacent particles, along the axis of their centers, are taken into account. In this case, the rotations of the particles do not contribute to the leading term of the asymptotics of  $\langle\mu\rangle$ . If an array does not satisfy this condition, the blow up rate may be a weaker ( $\delta^{-1/2}$ ), in which case rotational contributions cannot be ignored. A detailed study of this phenomenon is presented in the forthcoming [7], where we also use network approximation to explain the discrepancy, observed in [31], between the effective shear viscosity formulas for periodic arrays and estimates obtained from experimental results and numerical simulations.

In this paper, we give the rigorous justification of the spring network approximation. An important physical problem is to calculate the second order term in  $\langle\mu\rangle$ , which depends on the rotational motions of particles. The two-term (formal) asymptotics obtained here provide physical insight and the quantitative estimate of the contributions of rotations, as well as the effects of variable size distribution. Rigorous justification of these formulas requires a more careful lower bound construction than we attempt here, and it remains an interesting and challenging open problem.

Our study is motivated by the problem of transport of highly concentrated slurries, which arises in numerous industrial applications ranging from construction engineering to combustion processes [30, 32]. It is often necessary to use slurries with high solid content (highly packed). The transport of such slurries is impeded by the fact that their effective viscosity is very high. Thus the goal is to find an optimal balance between the effective viscosity and the concentration of the solid phase. The first step in achieving this goal is to obtain relatively simple formulas which show how the effective viscosity depends on the control parameters (e.g., geometrical parameters, such as the particle size distributions, particle locations, and shapes). The network approximation we propose here can be used in the prediction of optimal properties of such slurries.

The paper is organized as follows. In section 2, we give the mathematical formu-

lation of the problem. Section 3 deals with the discrete network approximation of the effective viscosity. We also give here, and in section 4, the lubrication approximation of  $\langle \mu \rangle$ , in two and three dimensions. In section 5, we construct the upper bound on the effective viscosity, which accounts for both translational and rotational motions of the inclusions. In section 6, we give the rigorous justification of the spring network approximation of the effective viscosity. This accounts just for the leading order term in the asymptotics of the effective viscosity of the high contrast, closely packed suspension of particles. Finally, in section 7, we give a summary and conclusions.

## 2. Formulation of the problem.

**2.1. The Stokes flow problem.** Consider a cube

$$(2.1) \quad \Omega = \left\{ \mathbf{x} = \sum_{j=1}^n x_j \mathbf{e}_j, \quad -L \leq x_j \leq L, \quad 1 \leq j \leq n \right\}$$

of volume  $|\Omega| = (2L)^n$ , where  $n = 2$  or  $3$  and where  $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$  is an orthonormal basis. We suppose that  $\Omega$  is filled with  $N$ , nonoverlapping, rigid balls (particles)  $D^{(j)}$  of radius  $a_j$ , suspended in an incompressible fluid of viscosity  $\mu$ . We study the Stokes flow of this suspension, where the fluid occupies the perforated, connected domain

$$(2.2) \quad \Omega_F = \Omega \setminus \bigcup_{j=1}^N D^{(j)}.$$

We are particularly interested in concentrated suspensions with volume fraction

$$(2.3) \quad \alpha = 1 - \frac{|\Omega_F|}{|\Omega|},$$

close to maximal packing (neighboring particles are close to touching).

Let  $\mathbf{u}(\mathbf{x})$  be the velocity field at point  $\mathbf{x} \in \Omega_F$  and let  $\mathcal{E}(\mathbf{x})$  be the rate of strain tensor

$$(2.4) \quad \mathcal{E}(\mathbf{x}) = \frac{1}{2} [\nabla \mathbf{u}(\mathbf{x}) + (\nabla \mathbf{u}(\mathbf{x}))^T],$$

which satisfies

$$(2.5) \quad \text{trace } \mathcal{E}(\mathbf{x}) = \text{div } \mathbf{u}(\mathbf{x}) = 0$$

by incompressibility. The stress in the fluid is

$$(2.6) \quad \mathcal{S}(\mathbf{x}) = -P(\mathbf{x})I + 2\mu\mathcal{E}(\mathbf{x}),$$

where  $\mu$  is the viscosity,  $P$  is the hydrostatic pressure, and  $I$  denotes the unit tensor. In the rigid balls,  $\mathcal{E} = 0$ . In the absence of external forces, the velocity field  $\mathbf{u}(\mathbf{x})$  in the fluid satisfies Stokes's equation

$$(2.7) \quad \text{div } \mathcal{S}(\mathbf{x}) = \mu \Delta \mathbf{u}(\mathbf{x}) - \nabla P(\mathbf{x}) = \mathbf{0}$$

and the incompressibility constraint (2.5).

Let us denote by  $\partial\Omega^+$  and  $\partial\Omega^-$  the top and bottom parts of the external boundary  $\partial\Omega$ , respectively,

$$(2.8) \quad \partial\Omega^+ = \{\mathbf{x} \in \partial\Omega : x_n = L\} \quad \text{and} \quad \partial\Omega^- = \{\mathbf{x} \in \partial\Omega : x_n = -L\}.$$

In this paper, we work with the model boundary conditions prescribed as follows. On  $\partial\Omega^+ \cup \partial\Omega^-$ , the velocity satisfies

$$(2.9) \quad \mathbf{u}(\mathbf{x}) = \mathbf{g}(x), \quad \text{where } \mathbf{g}(\mathbf{x}) = \begin{cases} -\frac{\mathbf{e}_n}{2L} & \text{on } \partial\Omega^-, \\ \frac{\mathbf{e}_n}{2L} & \text{on } \partial\Omega^+, \end{cases}$$

and the remaining part of  $\partial\Omega$  is traction free,

$$(2.10) \quad \mathcal{S}(\mathbf{x}) \mathbf{n}(\mathbf{x}) = \mathbf{0} \quad \text{for } \mathbf{x} \in \partial\Omega \setminus \{\partial\Omega^+ \cup \partial\Omega^-\}.$$

At the surface of each rigid ball  $D^{(j)}$ , the velocity satisfies

$$(2.11) \quad \mathbf{u}(\mathbf{x}) = \boldsymbol{\omega}^{(j)} \times (a_j \mathbf{n}^{(j)})(\mathbf{x}) + \mathbf{T}^{(j)} \quad \text{on } \partial D^{(j)}, \quad j = 1, 2, \dots, N,$$

where  $\boldsymbol{\omega}^{(j)}$ ,  $\mathbf{T}^{(j)}$  are constant but unknown rotational and translational velocities of  $D^{(j)}$  and where  $\mathbf{n}^{(j)}(\mathbf{x})$  is the outer normal at  $\partial D^{(j)}$ . Finally, since each rigid ball is in equilibrium, the total force and torque exerted on  $D^{(j)}$  by the fluid must be zero,

$$(2.12) \quad \int_{\partial D^{(j)}} \mathcal{S} \mathbf{n}^{(j)} ds = \mathbf{0} \quad \text{and} \quad \int_{\partial D^{(j)}} \mathbf{n}^{(j)} \times \mathcal{S} \mathbf{n}^{(j)} ds = \mathbf{0} \quad \text{for } j = 1, 2, \dots, N.$$

It is known that (2.7) and (2.5), with boundary conditions (2.9), (2.11), and (2.12), have a unique solution  $\mathbf{u}(\mathbf{x})$ , at least in the weak sense, with components in  $H^1(\Omega_F)$ .

**2.2. The effective viscosity.** The rate of viscous dissipation of the energy is given by [23]

$$(2.13) \quad E = \frac{1}{2} \int_{\Omega_F} (\mathcal{S}(\mathbf{x}), \mathcal{E}(\mathbf{x})) d\mathbf{x},$$

where  $(\cdot, \cdot)$  denotes the Frobenius tensor inner product

$$(2.14) \quad (\mathcal{S}(\mathbf{x}), \mathcal{E}(\mathbf{x})) = \sum_{i,j=1}^n \mathcal{S}_{ij}(\mathbf{x}) \mathcal{E}_{ij}(\mathbf{x}).$$

Integrating by parts and using (2.5), (2.6), (2.9), (2.10), (2.11), and the identity

$$(2.15) \quad (\mathcal{S}, \mathcal{E}) = -P \text{trace } \mathcal{E} + 2\mu(\mathcal{E}, \mathcal{E}) = \frac{\mu}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T, \nabla \mathbf{u} + (\nabla \mathbf{u})^T),$$

we obtain

$$(2.16) \quad E = \frac{1}{2} \int_{\partial\Omega^+ \cup \partial\Omega^-} \frac{\mathbf{e}_n}{2L} \cdot \mathcal{S}(\mathbf{x}) \mathbf{e}_n ds - \frac{1}{2} \sum_{j=1}^N \int_{\partial D^{(j)}} (\boldsymbol{\omega}^{(j)} \times \mathbf{n}^{(j)}(\mathbf{x}) + \mathbf{T}^{(j)}) \cdot \mathcal{S}(\mathbf{x}) \mathbf{n}^{(j)} ds.$$

Furthermore, due to the balance equations (2.12), the integrals at the surface of the particles vanish and (2.13) can be rewritten as

$$(2.17) \quad E = \frac{1}{4L} \int_{\partial\Omega^+ \cup \partial\Omega^-} \mathbf{e}_n \cdot \mathcal{S}(\mathbf{x}) \mathbf{e}_n ds.$$

The effective viscosity  $\langle \mu \rangle$  is defined by the equation

$$(2.18) \quad \frac{\langle \mu \rangle}{\mu} = \frac{E}{E^0} = \frac{\int_{\partial\Omega^+ \cup \partial\Omega^-} \mathbf{e}_n \cdot \mathcal{S}(\mathbf{x}) \mathbf{e}_n \, ds}{\int_{\partial\Omega^+ \cup \partial\Omega^-} \mathbf{e}_n \cdot \mathcal{S}^0(\mathbf{x}) \mathbf{e}_n \, ds},$$

where  $\mathcal{S}^0(\mathbf{x})$  is the stress tensor that would occur in  $\Omega$ , in the absence of all the particles, under the same external boundary conditions (2.9), (2.10), and where  $E^0$  is the corresponding rate of dissipation (see, for example, [15]). An equivalent definition of  $\langle \mu \rangle$  can be obtained directly from (2.13) and (2.15) by equating the viscous dissipation rates

$$(2.19) \quad \langle \mu \rangle \int_{\Omega} (\mathcal{E}^0, \mathcal{E}^0) \, d\mathbf{x} = \mu \int_{\Omega_F} (\mathcal{E}, \mathcal{E}) \, d\mathbf{x}.$$

We note that the definition of the effective viscosity, via the dissipation rate, is introduced in [3] for dilute suspensions, where the energy of the particulate phase is negligible. However, since the particles are rigid and condition (2.12) holds, the total mechanical energy of the particles is conserved. Thus, definitions (2.18) and (2.19) can be used as well for the suspensions considered in this paper.

**2.3. The variational principles.** The dissipation rate (2.13) or, equivalently, the effective viscosity (2.18), have a primal and dual variational formulation. The primal variational principle is widely known (see, for example, [13]),

$$(2.20) \quad E = \min_{\mathbf{u} \in \mathcal{U}} W_{\Omega_F}(\mathbf{u}), \text{ where } W_{\Omega_F}(\mathbf{u}) = \frac{\mu}{4} \sum_{i,j=1}^n \int_{\Omega_F} \left( \frac{\partial u_i(\mathbf{x})}{\partial x_j} + \frac{\partial u_j(\mathbf{x})}{\partial x_i} \right)^2 \, d\mathbf{x},$$

and where the function space  $\mathcal{U}$  of admissible velocity fields is

$$(2.21) \quad \mathcal{U} = \left\{ \mathbf{u} = \sum_{j=1}^n u_j \mathbf{e}_j, \, u_j \in H^1(\Omega_F), \, j = 1 \dots n, \, \text{div } \mathbf{u} = 0, \, (2.9) \text{ and } (2.11) \text{ hold} \right\}.$$

Note that the minimizer in (2.20) is the solution of the Stokes flow equation (2.7), where  $P(\mathbf{x})$  is the Lagrange multiplier for the incompressibility constraint  $\text{div } \mathbf{u}(\mathbf{x})=0$ .

The dual variational principle<sup>1</sup> is

$$(2.22) \quad E = \max_{\mathcal{S} \in \mathcal{F}} \left\{ \frac{1}{2L} \int_{\partial\Omega^+ \cup \partial\Omega^-} \mathbf{e}_n \cdot \mathcal{S}(\mathbf{x}) \mathbf{e}_n \, ds - \frac{1}{4\mu} \int_{\Omega_F} \left[ (\mathcal{S}(\mathbf{x}), \mathcal{S}(\mathbf{x})) - \frac{(\text{trace } \mathcal{S}(\mathbf{x}))^2}{n} \right] \, d\mathbf{x} \right\},$$

where we maximize over the space  $\mathcal{F}$  of admissible stress fields

$$(2.23) \quad \mathcal{F} = \{ \mathcal{S} \in \mathbb{R}^{n \times n}, \, \mathcal{S} = \mathcal{S}^T, \, \text{div } \mathcal{S} = \mathbf{0}, \, \mathcal{S}_{ij} \in L^2(\Omega_F), \, i, j = 1, \dots, n, \, (2.10) \text{ and } (2.12) \text{ hold} \}.$$

The maximizer in (2.22) is the stress field  $\mathcal{S}(\mathbf{x})$ , which determines the minimizing velocity field  $\mathbf{u}(\mathbf{x})$  in (2.20) by Newton's law (2.6), where

$$(2.24) \quad P(\mathbf{x}) = -\frac{\text{trace } \mathcal{S}(\mathbf{x})}{n}.$$

<sup>1</sup>For the derivation, see Appendix A in the preprint version of this article, available at <http://www.math.wsu.edu/math/faculty/panchenko/welcome.html>.



**3. The discrete approximation of the effective viscosity.** Intuitively, in highly packed suspensions, we expect that most energy is dissipated in the thin gaps between the rigid particles. Let us then define the local dissipation rate in a gap  $\Pi$ , between two adjacent particles in  $\Omega_F$ , by

$$(3.1) \quad W_\Pi(\mathbf{u}) = \mu \int_\Pi (\mathcal{E}(\mathbf{u}), \mathcal{E}(\mathbf{u})) \, d\mathbf{x}.$$

In this paper, we show that, in the asymptotic limit of infinitesimally small gap thickness

$$(3.2) \quad \frac{\delta}{a} = \max_{j,k} \frac{\delta^{jk}}{a} \rightarrow 0,$$

the effective viscosity is determined by the sum of local dissipation rates (3.1) over all the gaps in  $\Omega_F$ . In a highly packed suspension, the rate of dissipation of the energy can be written as an asymptotic series, in the limit (3.2), with the first and second terms blowing up at different rates (as powers or at least logarithmically in  $a/\delta$ ). The remainder of the series is  $O(1)$ .

**3.1. Connectivity patterns for densely packed suspensions.** In the case of regular (cubic, hexagonal, etc.) arrays of particles in  $\Omega$ , the volume fraction is sufficient to describe the distance between the particles and therefore the effective behavior of the suspension. However, for general distributions of particles in highly packed suspensions, one has to consider irregular connectivity patterns.

Let us consider an arbitrary distribution of particles  $D^{(i)}$ , centered at  $\mathbf{x}^{(i)} \in \Omega$ , for  $i = 1, 2, \dots, N$ . We suppose that  $N$  is close to  $N_{\max}$  such that particles can get close to touching one another. The concept of adjacent particles is essential to the analysis, and, to make it rigorous, we use Voronoi tessellations.

DEFINITION 3.1. *The Voronoi cell  $V_i$ , corresponding to  $\mathbf{x}^{(i)}$ , is the polyhedron*

$$V_i = \{\mathbf{x} \in \bar{\Omega} \text{ such that } |\mathbf{x} - \mathbf{x}^{(i)}| \leq |\mathbf{x} - \mathbf{x}^{(j)}| \text{ for all } j = 1, 2, \dots, N, j \neq i\}.$$

*The plane faces of  $V_i$  can lie either on  $\partial\Omega$  or in the interior of  $\Omega$ . On each face of  $V_i$  that lies inside  $\Omega$ ,*

$$|\mathbf{x} - \mathbf{x}^{(i)}| = |\mathbf{x} - \mathbf{x}^{(j)}| \text{ for some } i \neq j.$$

In Figure 1, we illustrate a Voronoi tessellation in two dimensions.

DEFINITION 3.2. *Given the Voronoi tessellation and an arbitrary  $D^{(i)}$ , for  $i = 1, 2, \dots, N$ , we define the set of indices of its neighbors as  $\mathcal{N}_i = \{j \in \mathbb{N}, j \neq i, \text{ such that } V_i \text{ and } V_j \text{ have a common face}\}$ . The coordination number of  $D^{(i)}$  is equal to the cardinal number of  $\mathcal{N}_i$ .*

Neighboring particles  $D^{(i)}$  and  $D^{(j)}$  are separated by a gap (neck)  $\Pi^{ij}$  (see Figure 3) of minimum thickness,

$$(3.3) \quad \delta^{ij} = |\mathbf{x}^{(i)} - \mathbf{x}^{(j)}| - (a_i + a_j),$$

and width  $R^{ij} = O(a^{ij})$ , where

$$(3.4) \quad a^{ij} = \frac{2a_i a_j}{a_i + a_j}.$$

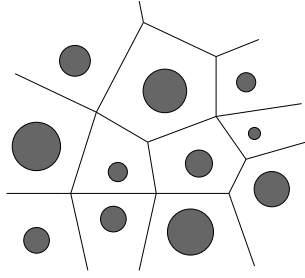


FIG. 1. Two dimensional Voronoi tessellation.

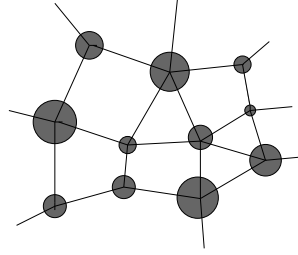


FIG. 2. The Delaunay graph.

Then the topology of network  $\Gamma$ , needed in the asymptotic approximation of  $\langle \mu \rangle$ , in the limit  $\delta^{ij}/a^{ij} \rightarrow 0$ , for  $i = 1, \dots, N$  and  $j \in \mathcal{N}_i$ , is uniquely defined, as follows.

**DEFINITION 3.3.** *The interior vertices of the network (graph)  $\Gamma$  are given by  $\mathbf{x}^{(i)}$ , the locations of the centers of particles  $D^{(i)}$  in  $\Omega$  for  $i = 1, 2, \dots, N$ . The interior branches (edges)  $b^{ij}$  of the network connect vertices  $\mathbf{x}^{(i)}$  and  $\mathbf{x}^{(j)}$  ( $j \in \mathcal{N}_i$ ) through the gaps (necks)  $\Pi^{ij}$ . For Voronoi cells  $V_i$  with faces belonging to  $\partial\Omega^+ \cup \partial\Omega^-$ , we join  $\mathbf{x}^{(i)}$  with  $\partial\Omega^\pm$  through a normal segment  $\tilde{b}^i$  (exterior branch or edge) and we call the intersection  $\tilde{\mathbf{x}}^{(i)}$  an exterior vertex. Finally, we let  $\mathcal{B}$  be the set of indices  $i$  corresponding to the boundary Voronoi cells, that is, the cells at least one face of which belongs to  $\partial\Omega^+ \cup \partial\Omega^-$ .*

**ASSUMPTION 3.1.** *We assume that the distances between the neighboring balls are bounded below by  $c\delta$ , where  $c > 0$  is fixed and  $\delta$  is the small parameter of the problem. Thus the length of each edge in the graph is larger than  $2A + c\delta$ , where  $A$  is the smallest ball radius.*

Note that  $\Gamma$  is the Delaunay graph, which is dual to the Voronoi tessellation. The Delaunay graph for the two-dimensional tessellation of Figure 1 is shown in Figure 2. Note also the following properties of  $\Gamma$ ,<sup>2</sup> which we use in the analysis.

*Property 3.1.*  $\Gamma$  is connected in the sense that each pair of interior vertices can be connected by a path consisting entirely of interior edges.

*Property 3.2.* Suppose there exists a Voronoi cell contained strictly inside  $\Omega$ . Then there exists a closed path consisting entirely of interior edges.

*Property 3.3.* At least two edges originate at each interior vertex of  $\Gamma$ .

**3.2. The two-term discrete asymptotic approximation.** The asymptotic approximation of the viscous dissipation rate in the high-contrast suspension is obtained by summing the local dissipation rates  $W_{\Pi^{ij}}$  in the gaps  $\Pi^{ij}$  between  $D^{(i)}$  and  $D^{(j)}$  for  $i = 1, \dots, N$  and  $j \in \mathcal{N}_i$ . Then, focusing attention on one such gap (see Figure 3), we introduce a local system of coordinates  $(x_1, \dots, x_n)$  in  $\Pi^{ij}$ , with the origin at  $(\mathbf{x}^{(i)} + \mathbf{x}^{(j)})/2$  and coordinate  $x_n$  measured along the axis of the centers, pointing from  $\mathbf{x}^{(j)}$  toward  $\mathbf{x}^{(i)}$ . The width of the gap is  $R^{ij} = O(a^{ij})$  and the height (thickness) is

$$(3.5) \quad h(r) = \delta^{ij} + a_i \left( 1 - \sqrt{1 - \frac{r^2}{a_i^2}} \right) + a_j \left( 1 - \sqrt{1 - \frac{r^2}{a_j^2}} \right), \quad r = \sqrt{\sum_{k=1}^{n-1} x_k^2} \leq R^{ij}.$$

<sup>2</sup>For the proof, see Appendix B in the preprint version of this article, available at <http://www.math.wsu.edu/math/faculty/panchenko/welcome.html>.

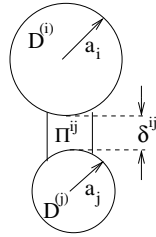


FIG. 3. Two nearby particles  $D^{(i)}$  and  $D^{(j)}$  of radii  $a_i$  and  $a_j$ , respectively, separated by a gap  $\delta^{ij}$ .

The dissipation rate density  $\mu(\mathcal{E}, \mathcal{E})$  is expected to be highest at radial distances  $r \ll \min(a_i, a_j)$ , so, in the calculation of  $W_{\Pi^{ij}} = \mu \int_{\Pi^{ij}} (\mathcal{E}, \mathcal{E}) d\mathbf{x}$ , we can approximate the spherical surfaces by paraboloids and the thickness of the gap by

$$(3.6) \quad h(r) \approx \delta^{ij} + \frac{r^2}{a^{ij}}.$$

**3.2.1. The two-dimensional discrete approximation of  $\langle \mu \rangle$ .** Clearly,  $W_{\Pi^{ij}}$  depends on the velocity at the top and bottom surfaces of the gap, where  $x_2 = \pm h(x_1)/2$ , in two dimensions. Using boundary conditions (2.11) at  $\partial D^{(i)}$  and  $\partial D^{(j)}$ , and approximating the outer normals as  $\mathbf{n}^{(i)} \approx \frac{x_1}{a_i} \mathbf{e}_1 - \mathbf{e}_2$  and  $\mathbf{n}^{(j)} \approx \frac{x_1}{a_j} \mathbf{e}_1 + \mathbf{e}_2$ , by the normal vectors to the parabolas touching the disks (see (3.6)), we have

$$(3.7) \quad \mathbf{u} \left( x_1, \pm \frac{h(x_1)}{2} \right) \approx \pm (T_2^{(i)} - T_2^{(j)}) \frac{\mathbf{e}_2}{2} \pm (T_1^{(i)} - T_1^{(j)} + a_i \omega^{(i)} + a_j \omega^{(j)}) \frac{\mathbf{e}_1}{2} \pm (\omega^{(i)} - \omega^{(j)}) x_1 \mathbf{e}_2 + \mathcal{R},$$

where

$$(3.8) \quad \mathcal{R} = [T_2^{(i)} + T_2^{(j)} + (\omega^{(i)} + \omega^{(j)}) x_1] \frac{\mathbf{e}_2}{2} + (T_1^{(i)} + T_1^{(j)} + a_i \omega^{(i)} - a_j \omega^{(j)}) \frac{\mathbf{e}_1}{2}.$$

Equation (3.7) can be viewed as a decomposition of  $\mathbf{u}$  in the following elementary velocity fields:

1. The first elementary velocity field in (3.7) is  $\mathbf{u}_{\text{sp}}$ , and it solves the Stokes equations in  $\Pi^{ij}$  with boundary conditions

$$(3.9) \quad \mathbf{u}_{\text{sp}} \left( x_1, \pm \frac{h(x_1)}{2} \right) = \pm (T_2^{(i)} - T_2^{(j)}) \frac{\mathbf{e}_2}{2}.$$

We can associate  $\mathbf{u}_{\text{sp}}$  with the oscillatory motion, along  $\mathbf{e}_2$ , of two particles joined by a spring, with elastic constant  $C_{\text{sp}}^{ij} = O((a^{ij}/\delta^{ij})^{3/2})$  (see sections 4, 5, and 6). The velocity  $(T_2^{(i)} - T_2^{(j)})/2$  of the particles is constant and unknown, so far. It is to be determined later from the global conditions of mechanical equilibrium of all inclusions in the suspension.

2. The second term in (3.7), denoted by  $\mathbf{u}_{\text{sh}}$ , satisfies the Stokes equations in  $\Pi^{ij}$  with boundary conditions

$$(3.10) \quad \mathbf{u}_{\text{sh}} \left( x_1, \pm \frac{h(x_1)}{2} \right) = \pm (T_1^{(i)} - T_1^{(j)} + a_i \omega^{(i)} + a_j \omega^{(j)}) \frac{\mathbf{e}_1}{2}.$$

This accounts for a shear strain in the gap, where the fluid moves to the right and left, at the top and bottom surfaces of  $\Pi^{ij}$ , respectively, at the constant, unknown velocity  $(T_1^{(i)} - T_1^{(j)} + a_i\omega^{(i)} + a_j\omega^{(j)})/2$ . The contribution of this term to the dissipation rate is  $C_{\text{sh}}^{ij} = O(\sqrt{a^{ij}/\delta^{ij}})$  (see sections 4, 5).

3. The third term in (3.7) corresponds to a shear strain in the gap due to rotations. The boundary conditions are given by

$$(3.11) \quad \mathbf{u}_{\text{rot}} \left( x_1, \pm \frac{h(x_1)}{2} \right) = \pm(\omega^{(i)} - \omega^{(j)})x_1\mathbf{e}_2,$$

as if the fluid were pushed and pulled, in direction  $\mathbf{e}_2$ , on the left and right sides of  $\Pi^{ij}$ , respectively (see Figure 6). The contribution of this term to the dissipation rate is  $C_{\text{rot}}^{ij} = O(\sqrt{a^{ij}/\delta^{ij}})$  (see sections 4, 5).

4. Finally, the remainder  $\mathcal{R}$  corresponds to a constant,  $O(1)$  shear strain in the gap and, as such, it gives an  $O(1)$  contribution to  $W_{\Pi^{ij}}$  (see sections 4, 5).

In section 4, we obtain the formal asymptotic approximation

$$(3.12) \quad W_{\Pi^{ij}} \approx C_{\text{sp}}^{ij}(T_2^{(i)} - T_2^{(j)})^2 + C_{\text{sh}}^{ij}(T_1^{(i)} - T_1^{(j)} + a_i\omega^{(i)} + a_j\omega^{(j)})^2 + C_{\text{rot}}^{ij}(\omega^{(i)} - \omega^{(j)})^2 + O(1),$$

where

$$(3.13) \quad C_{\text{sp}}^{ij} = \frac{3\pi\mu}{4} \left( \frac{a^{ij}}{\delta^{ij}} \right)^{\frac{3}{2}} + \frac{12\pi\mu}{5} \sqrt{\frac{a^{ij}}{\delta^{ij}}}, \quad C_{\text{sh}}^{ij} = \frac{\pi\mu}{2} \sqrt{\frac{a^{ij}}{\delta^{ij}}}, \quad \text{and} \quad C_{\text{rot}}^{ij} = \frac{9\pi\mu}{16} \sqrt{\frac{a^{ij}}{\delta^{ij}}}.$$

The approximation (3.12) applies to interior inclusions  $D^{(i)}$ . For  $i \in \mathcal{B}$ , we have  $D^{(i)}$  joined to a fictitious disk of infinite radius (i.e.,  $\partial\Omega^+$  or  $\partial\Omega^-$ ) and the harmonic average of the radii is  $a^i = 2a_i$ . Given boundary conditions (2.9) at  $\partial\Omega^\pm$ , we have, similar to (3.12),

$$(3.14) \quad W_{\Pi^i} \approx C_{\text{sp}}^i(T_2^{(i)} - \mathbf{g} \cdot \mathbf{e}_2)^2 + C_{\text{sh}}^i(T_1^{(i)} - \mathbf{g} \cdot \mathbf{e}_1 + a_i\omega^{(i)})^2 + C_{\text{rot}}^i(2\omega^{(i)})^2 + O(1),$$

where  $C_{\text{sp}}^i$ ,  $C_{\text{sh}}^i$ , and  $C_{\text{rot}}^i$  are given by (3.13), with  $a^{ij}$  replaced by  $a^i = 2a_i$  and  $\delta^{ij}$  replaced by  $\delta^i$ , the distance between  $\partial D^{(i)}$  and the upper or lower boundary  $\partial\Omega^\pm$ .

Next, we approximate  $E$  by summing the local dissipation rates in all gaps  $\Pi^{ij}$  for  $i = 1, \dots, N$ ,  $i \notin \mathcal{B}$ ,  $j \in \mathcal{N}_i$ , and  $\Pi^i$  for  $i \in \mathcal{B}$ . For this purpose, let us rename the orthonormal basis vectors in each gap  $\Pi^{ij}$  as

$$\mathbf{q}^{ij} = \frac{\mathbf{x}^{(i)} - \mathbf{x}^{(j)}}{|\mathbf{x}^{(i)} - \mathbf{x}^{(j)}|}$$

and  $\mathbf{p}^{ij} =$  the rotated  $\mathbf{q}^{ij}$ , clockwise, by  $\pi/2$ , in the two-dimensional plane. In the boundary gaps  $\Pi^i$ , joining a particle  $D^{(i)}$  with  $\partial\Omega^\pm$ , these vectors are called  $\mathbf{q}^i$  and  $\mathbf{p}^i$ , respectively. The discrete approximation of  $\langle \mu \rangle$  is given by (2.19), with the right-hand side

$$(3.15) \quad E \approx \min_{\mathbf{T}, \omega} \sum_{i=1}^N \sum_{\substack{j \in \mathcal{N}_i \\ j < i}} \{C_{\text{sp}}^{ij}[(\mathbf{T}^{(i)} - \mathbf{T}^{(j)}) \cdot \mathbf{q}^{ij}]^2 + C_{\text{sh}}^{ij}[(\mathbf{T}^{(i)} - \mathbf{T}^{(j)}) \cdot \mathbf{p}^{ij} + a_i\omega^{(i)} + a_j\omega^{(j)}]^2 + C_{\text{rot}}^{ij}(\omega^{(i)} - \omega^{(j)})^2\} + \sum_{i \in \mathcal{B}} \{C_{\text{sp}}^i[(\mathbf{T}^{(i)} - \mathbf{g}) \cdot \mathbf{q}^i]^2 + C_{\text{rot}}^i(2\omega^{(i)})^2 + C_{\text{sh}}^i[(\mathbf{T}^{(i)} - \mathbf{g}) \cdot \mathbf{p}^i + a_i\omega^{(i)}]^2\}.$$

Note that in (3.15) we minimize a quadratic functional, over translational and rotational velocities  $\mathbf{T}^{(i)}$  and  $\omega^{(i)}$ , for  $i = 1, \dots, N$ , respectively. This is equivalent to solving the Euler–Lagrange equations

$$(3.16) \quad \sum_{j \in \mathcal{N}_i} \{C_{\text{sp}}^{ij} [(\mathbf{T}^{(i)} - \mathbf{T}^{(j)}) \cdot \mathbf{q}^{ij}] \mathbf{q}^{ij} + C_{\text{sh}}^{ij} [(\mathbf{T}^{(i)} - \mathbf{T}^{(j)}) \cdot \mathbf{p}^{ij} + a_i \omega^{(i)} + a_j \omega^{(j)}] \mathbf{p}^{ij}\} + \mathbf{F}_{\mathcal{B}}(\mathbf{T}^{(i)}, \omega^{(i)}) = \mathbf{0},$$

$$(3.17) \quad \sum_{j \in \mathcal{N}_i} \{C_{\text{sh}}^{ij} [(\mathbf{T}^{(i)} - \mathbf{T}^{(j)}) \cdot \mathbf{p}^{ij} + \omega^{(i)} + \omega^{(j)}] + C_{\text{rot}}^{ij} (\omega^{(i)} - \omega^{(j)})\} + \mathcal{M}_{\mathcal{B}}(\mathbf{T}^{(i)}, \omega^{(i)}) = 0$$

for all  $i = 1, \dots, N$ , where

$$(3.18) \quad \mathbf{F}_{\mathcal{B}}(\mathbf{T}^{(i)}, \omega^{(i)}) = \begin{cases} C_{\text{sp}}^i [(\mathbf{T}^{(i)} - \mathbf{g}) \cdot \mathbf{q}^i] \mathbf{q}^i + C_{\text{sh}}^i [(\mathbf{T}^{(i)} - \mathbf{g}) \cdot \mathbf{p}^i + a_i \omega^{(i)}] \mathbf{p}^i & \text{if } i \in \mathcal{B}, \\ \mathbf{0} & \text{otherwise,} \end{cases}$$

$$(3.19) \quad \mathcal{M}_{\mathcal{B}}(\mathbf{T}^{(i)}, \omega^{(i)}) = \begin{cases} C_{\text{sh}}^i [(\mathbf{T}^{(i)} - \mathbf{g}) \cdot \mathbf{p}^i + a_i \omega^{(i)}] + 4C_{\text{rot}}^i \omega^{(i)} & \text{if } i \in \mathcal{B}, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

These are the equations of force and torque balance of the inclusions, and the minimization in (3.15) ensures that the rigid body translational and rotational velocities are chosen in such a way that the suspension is in mechanical equilibrium.

**3.2.2. The three-dimensional discrete approximation of  $\langle \mu \rangle$ .** Consider the local system of coordinates described at the beginning of section 3.2 in three dimensions. Similar to our two-dimensional calculation, we write

$$(3.20) \quad \mathbf{u} \left( x_1, x_2, \pm \frac{h(x_1, x_2)}{2} \right) \approx \pm (T_3^{(i)} - T_3^{(j)}) \frac{\mathbf{e}_3}{2} \pm (T_1^{(i)} - T_1^{(j)} - a_i \omega_2^{(i)} - a_j \omega_2^{(j)}) \frac{\mathbf{e}_1}{2} \\ \pm (T_2^{(i)} - T_2^{(j)} + a_i \omega_1^{(i)} + a_j \omega_1^{(j)}) \frac{\mathbf{e}_2}{2} \pm (\omega_1^{(i)} - \omega_1^{(j)}) \frac{x_2 \mathbf{e}_3}{2} \\ \mp (\omega_2^{(i)} - \omega_2^{(j)}) \frac{x_1 \mathbf{e}_3}{2} + \mathcal{R},$$

where the remainder is

$$(3.21) \quad \mathcal{R} = (T_3^{(i)} + T_3^{(j)}) \frac{\mathbf{e}_3}{2} + (T_1^{(i)} + T_1^{(j)} - a_i \omega_2^{(i)} + a_j \omega_2^{(j)}) \frac{\mathbf{e}_1}{2} \\ + (T_2^{(i)} + T_2^{(j)} + a_i \omega_1^{(i)} - a_j \omega_1^{(j)}) \frac{\mathbf{e}_2}{2} + (\omega_1^{(i)} + \omega_1^{(j)}) \frac{x_2 \mathbf{e}_3}{2} \\ - (\omega_2^{(i)} - \omega_2^{(j)}) \frac{x_1 \mathbf{e}_3}{2}.$$

As in two dimensions, we associate the first term in (3.20), due to the motion of the inclusions along the axis of their centers, with the oscillatory motion of two particles

joined by a spring of elastic constant  $C_{\text{sp}}^{ij} = O(a^{ij}/\delta^{ij})$ . The next two terms in (3.20) correspond to shear strains in the gap, where the fluid is pulled in the positive and negative directions of  $\mathbf{e}_1$  and  $\mathbf{e}_2$  at the top and bottom surfaces of  $\Pi^{ij}$ , respectively. The contribution of these terms to the dissipation rate is  $C_{\text{sh}}^{ij} = O(\ln a^{ij}/\delta^{ij})$ . The fourth and fifth terms in (3.20) correspond to a shear strain in the gap as well, but now the fluid is pushed and pulled, in direction  $\mathbf{e}_3$ , on opposite sides of the axis of  $\Pi^{ij}$ , respectively. The contribution of these terms to the dissipation rate is  $C_{\text{rot}}^{ij} = O(\ln a^{ij}/\delta^{ij})$ . Finally, the remainder  $\mathcal{R}$  gives an  $O(1)$  contribution to  $W_{\Pi^{ij}}$ .

A formal asymptotic analysis, which is very similar to the two-dimensional one in section 4 and, as such, is not detailed here, gives

$$(3.22) \quad \begin{aligned} W_{\Pi^{ij}} \approx & C_{\text{sp}}^{ij}(T_3^{(i)} - T_3^{(j)})^2 + C_{\text{rot}}^{ij}[(\omega_1^{(i)} - \omega_1^{(j)})^2 + (\omega_2^{(i)} - \omega_2^{(j)})^2] \\ & + C_{\text{sh}}^{ij}[(T_1^{(i)} - T_1^{(j)} - a_i\omega_2^{(i)} - a_j\omega_2^{(j)})^2 \\ & + (T_2^{(i)} - T_2^{(j)} + a_i\omega_1^{(i)} + a_j\omega_1^{(j)})^2] + O(1), \end{aligned}$$

where

$$(3.23) \quad C_{\text{sp}}^{ij} = \frac{3\pi\mu a^{ij}}{4} \left( \frac{a^{ij}}{\delta^{ij}} \right) + \frac{9\pi\mu a^{ij}}{5} \ln \frac{a^{ij}}{\delta^{ij}}, \quad C_{\text{sh}}^{ij} = \frac{\pi\mu a^{ij}}{2} \ln \frac{a^{ij}}{\delta^{ij}}, \quad \text{and} \quad C_{\text{rot}}^{ij} = \frac{9\pi\mu a^{ij}}{16} \ln \frac{a^{ij}}{\delta^{ij}} (a^{ij})^2.$$

If  $i \in \mathcal{B}$ ,  $D^{(i)}$  is joined to  $\partial\Omega^\pm$  through gap  $\Pi^i$  and we obtain (see section 3.2.1)

$$(3.24) \quad \begin{aligned} W_{\Pi^i} \approx & C_{\text{sp}}^i(T_3^{(i)} - \mathbf{g} \cdot \mathbf{e}_3)^2 + C_{\text{rot}}^i[(2\omega_1^{(i)})^2 + (2\omega_2^{(i)})^2] \\ & + C_{\text{sh}}^i[(T_1^{(i)} - \mathbf{g} \cdot \mathbf{e}_1 - a_i\omega_2^{(i)})^2 + (T_2^{(i)} - \mathbf{g} \cdot \mathbf{e}_2 + a_i\omega_1^{(i)})^2] + O(1) \end{aligned}$$

with constants  $C_{\text{sp}}^i$ ,  $C_{\text{sh}}^i$ , and  $C_{\text{rot}}^i$  given by (3.23), where  $a^{ij}$  is replaced by  $a^i = 2a_i$  and  $\delta^{ij}$  is replaced by  $\delta^i$ , the distance between  $\mathbf{x}^{(i)}$  and the upper or lower boundary.

The discrete approximation of the viscous dissipation rate  $E$  in the suspension is given by the sum of the local dissipation rates in all gaps  $\Pi^{ij}$  for  $i = 1, \dots, N$ ,  $i \notin \mathcal{B}$ ,  $j \in \mathcal{N}_i$ , and  $\Pi^i$ , for  $i \in \mathcal{B}$ . Let us then introduce, in gap  $\Pi^{ij}$ , the orthonormal vectors

$$\mathbf{q}^{ij} = \frac{\mathbf{x}^{(i)} - \mathbf{x}^{(j)}}{|\mathbf{x}^{(i)} - \mathbf{x}^{(j)}|}, \quad \mathbf{r}^{ij} = \text{the rotated } \mathbf{q}^{ij},$$

$$\text{in the plane } \mathcal{P}^{ij}, \text{ clockwise, by } \frac{\pi}{2}, \text{ and } \mathbf{p}^{ij} = \mathbf{r}^{ij} \times \mathbf{q}^{ij},$$

where  $\mathbf{p}^{ij}$ ,  $\mathbf{r}^{ij}$ , and  $\mathbf{q}^{ij}$  play the role of  $\mathbf{e}_1$ ,  $\mathbf{e}_2$ , and  $\mathbf{e}_3$ , respectively, in the above calculation of  $W_{\Pi^{ij}}$ . Since plane  $\mathcal{P}^{ij}$  is not uniquely defined by  $\mathbf{q}^{ij}$ , there are infinitely many choices of  $\mathbf{r}^{ij}$ , although they give the same dissipation rate  $W_{\Pi^{ij}}$ . Let us then pick  $\mathcal{P}^{ij}$  arbitrarily but ensure, at the same time, that when indices  $i$  and  $j$  are interchanged, we have

$$\mathcal{P}^{ij} = \text{span}\{\mathbf{q}^{ij}, \mathbf{r}^{ij}\} = \text{span}\{\mathbf{q}^{ji}, \mathbf{r}^{ji}\} = \mathcal{P}^{ji}$$

or, equivalently,

$$\mathbf{q}^{ij} = -\mathbf{q}^{ji}, \quad \mathbf{r}^{ij} = -\mathbf{r}^{ji}, \quad \text{and} \quad \mathbf{p}^{ij} = \mathbf{p}^{ji} \quad \text{for all } i = 1, \dots, N, \quad i \neq \mathcal{B}, \quad j \in \mathcal{N}_i.$$

For  $i \in \mathcal{B}$ ,  $D^{(i)}$  is joined with  $\partial\Omega^\pm$  by gap  $\Pi^i$ , and the unit vectors are denoted by  $\mathbf{p}^i$ ,  $\mathbf{r}^i$ , and  $\mathbf{q}^i$ , respectively. Then, the discrete approximation of the effective viscosity is given by (2.19), with the right-hand side

$$\begin{aligned}
 E \approx \min_{\mathbf{T}, \boldsymbol{\omega}} & \sum_{i=1}^N \sum_{\substack{j \in \mathcal{N}_i \\ j < i}} \{ C_{\text{sp}}^{ij} [(\mathbf{T}^{(i)} - \mathbf{T}^{(j)}) \cdot \mathbf{q}^{ij}]^2 + C_{\text{rot}}^{ij} [(\boldsymbol{\omega}^{(i)} - \boldsymbol{\omega}^{(j)}) \cdot \mathbf{p}^{ij}]^2 \\
 & + C_{\text{rot}}^{ij} [(\boldsymbol{\omega}^{(i)} - \boldsymbol{\omega}^{(j)}) \cdot \mathbf{r}^{ij}]^2 + C_{\text{sh}}^{ij} [(\mathbf{T}^{(i)} - \mathbf{T}^{(j)}) \cdot \mathbf{p}^{ij} \\
 & - (a_i \boldsymbol{\omega}^{(i)} + a_j \boldsymbol{\omega}^{(j)}) \cdot \mathbf{r}^{ij}]^2 \\
 (3.25) \quad & + C_{\text{sh}}^{ij} [(\mathbf{T}^{(i)} - \mathbf{T}^{(j)}) \cdot \mathbf{r}^{ij} + (a_i \boldsymbol{\omega}^{(i)} + a_j \boldsymbol{\omega}^{(j)}) \cdot \mathbf{p}^{ij}]^2 \} \\
 & + \sum_{i \in \mathcal{B}} \{ C_{\text{sp}}^i [(\mathbf{T}^{(i)} - \mathbf{g}) \cdot \mathbf{q}^i]^2 \\
 & + C_{\text{rot}}^i [(2\boldsymbol{\omega}^{(i)} \cdot \mathbf{p}^i)^2 + (2\boldsymbol{\omega}^{(i)} \cdot \mathbf{r}^i)^2] \\
 & + C_{\text{sh}}^i [(\mathbf{T}^{(i)} - \mathbf{g}) \cdot \mathbf{p}^i - a_i \boldsymbol{\omega}^{(i)} \cdot \mathbf{r}^i]^2 \\
 & + C_{\text{sh}}^i [(\mathbf{T}^{(i)} - \mathbf{g}) \cdot \mathbf{r}^i + a_i \boldsymbol{\omega}^{(i)} \cdot \mathbf{p}^i]^2 \} + O(1).
 \end{aligned}$$

Finally, as in section 3.2.1, the minimization in (3.25), over translational and rotational velocities  $\mathbf{T}^{(i)}$  and  $\boldsymbol{\omega}^{(i)}$  for  $i = 1, \dots, N$ , ensures that all the inclusions in the suspension are in mechanical equilibrium.

*Remark 3.1* (computation of the effective viscosity). We now summarize the steps necessary to compute the effective viscosity in problem (2.7)–(2.12). First, compute the approximate dissipation rate  $E$  by minimizing the quadratic functional (3.15) (in two dimensions) or (3.25) (in three dimensions). Next, solve the Stokes equations in the domain  $\Omega$  (see (2.1)) with viscosity equal to one and boundary conditions given by (2.9), (2.10). Then compute the corresponding strain rate  $\mathcal{E}^0 = 1/2(\nabla \mathbf{u}^0 + \nabla^T \mathbf{u}^0)$  and the normalized dissipation rate  $\int_{\Omega} \mathcal{E}_{ij}^0 \mathcal{E}_{ij}^0 d\mathbf{x}$ . Finally, compute the approximate value of the effective viscosity by the formula

$$(3.26) \quad \langle \mu \rangle = \frac{E}{\int_{\Omega} \mathcal{E}_{ij}^0 \mathcal{E}_{ij}^0 d\mathbf{x}}.$$

When the contributions of rotations can be neglected, the leading term in (3.26) is given by the leading term in the formula (6.85). Note that this term corresponds to the spring network approximation, which takes into account only motions of particles along the line of their centers. Detailed analysis of computational formulas for  $\langle \mu \rangle$ , based on the approach developed in this paper, for various boundary conditions and different arrays of particles, is presented in [7].

**4. The local dissipation rate in a gap between two adjacent particles. Formal asymptotics in two dimensions.** We begin our estimation of  $E$  with a formal asymptotic analysis which extends the lubrication approximations in [15, 16, 27] beyond the leading term by accounting for all possible rigid body motions of the inclusions in the suspension. To find  $E$ , we construct a velocity field in  $\Omega_F$  which satisfies boundary conditions (2.11) but solves the Stokes equations approximately in the following sense: Since the density  $\mu(\mathcal{E}, \mathcal{E})$  of the viscous dissipation rate is very high near the axis of the centers of adjacent inclusions  $D^{(i)}$  and  $D^{(j)}$ , we approximate  $\mathbf{u}$  in each gap  $\Pi^{ij}$  by the solution of the Stokes problem between two parallel plates, at distance  $h$  (which we pretend is a constant) apart, and we calculate the corresponding

rate of strain  $\mathcal{E}$ . Then we integrate over the gap to obtain the local dissipation rate

$$W_{\Pi^{ij}} \approx \int_{-a^{ij}}^{a^{ij}} dx_1 \int_{-\frac{h(x_1)}{2}}^{\frac{h(x_1)}{2}} dx_2 \mu(\mathcal{E}, \mathcal{E}).$$

Since most energy is dissipated in the gaps, we expect that the contribution to  $E$  from the region outside the gaps remains uniformly bounded in the limit  $\delta \rightarrow 0$ .

Let us denote by  $E^{\text{a}}$  the approximation of the dissipation rate, obtained with the formal asymptotic, lubrication type, approach. Since  $E^{\text{a}}$  is a heuristic estimate, it requires rigorous justification, which we give in sections 5 and 6, where we calculate upper and lower variational bounds on  $E$  that match  $E^{\text{a}}$  to leading order. Nevertheless, both bounds are inspired to some extent by the calculation of  $E^{\text{a}}$ , so we describe next, in detail, our formal asymptotic analysis.

We begin by recalling the local system of coordinates  $(x_1, x_2)$  in gap  $\Pi^{ij}$ , as defined in section 3.2. At the surface of  $D^{(i)}$ , the velocity is given by

$$\mathbf{u}|_{\partial D^{(i)}} = (T_1^{(i)} + a_i \omega^{(i)}) \mathbf{e}_1 + (T_2^{(i)} + a_i \omega^{(i)} n_1^{(i)}) \mathbf{e}_2 - a_i \omega^{(i)} (n_2^{(i)} + 1) \mathbf{e}_1,$$

and the two components of the outer normal at  $\partial D^{(i)}$  are  $n_1^{(i)} = \frac{x_1}{a_i}$  and  $n_2^{(i)} = -\sqrt{1 - \frac{x_1^2}{a_i^2}}$ . Similarly,

$$\mathbf{u}|_{\partial D^{(j)}} = (T_1^{(j)} - a_j \omega^{(j)}) \mathbf{e}_1 + (T_2^{(j)} + a_j \omega^{(j)} n_1^{(j)}) \mathbf{e}_2 - a_j \omega^{(j)} (n_2^{(j)} - 1) \mathbf{e}_1,$$

where  $n_1^{(j)} = \frac{x_1}{a_j}$  and  $n_2^{(j)} = \sqrt{1 - \frac{x_1^2}{a_j^2}}$ . Equivalently, we rewrite the boundary conditions on  $\mathbf{u}$  as

$$\begin{aligned} (4.1) \quad \mathbf{u} \left( x_1, \pm \frac{h}{2} \right) &= \pm \left( T_1^{(i)} - T_1^{(j)} + \omega^{(i)} \sqrt{a_i^2 - x_1^2} + \omega^{(j)} \sqrt{a_j^2 - x_1^2} \right) \frac{\mathbf{e}_1}{2} \pm (T_2^{(i)} - T_2^{(j)}) \frac{\mathbf{e}_2}{2} \\ &\quad \pm (\omega^{(i)} - \omega^{(j)}) \frac{x_1 \mathbf{e}_2}{2} + \left( T_1^{(i)} + T_1^{(j)} + \omega^{(i)} \sqrt{a_i^2 - x_1^2} - \omega^{(j)} \sqrt{a_j^2 - x_1^2} \right) \frac{\mathbf{e}_1}{2} \\ &\quad + (T_2^{(i)} + T_2^{(j)}) \frac{\mathbf{e}_2}{2} + (\omega^{(i)} + \omega^{(j)}) \frac{x_1 \mathbf{e}_2}{2} \approx \mathbf{u}^{\text{a}} \left( x_1, \pm \frac{h}{2} \right), \end{aligned}$$

where  $\mathbf{u}^{\text{a}}$  is an approximation of the velocity field near the axis of the gap (i.e., for  $x_1/a^{ij} \ll 1$ , where the density of the dissipation rate is highest). The boundary conditions on  $\mathbf{u}^{\text{a}}$  are

$$\begin{aligned} \mathbf{u}^{\text{a}} \left( x_1, \pm \frac{h}{2} \right) &= \pm (T_1^{(i)} - T_1^{(j)} + a_i \omega^{(i)} + a_j \omega^{(j)}) \frac{\mathbf{e}_1}{2} \\ &\quad \pm (T_2^{(i)} - T_2^{(j)}) \frac{\mathbf{e}_2}{2} \pm (\omega^{(i)} - \omega^{(j)}) \frac{x_1 \mathbf{e}_2}{2} \\ &\quad + (T_1^{(i)} + T_1^{(j)} + a_i \omega^{(i)} - a_j \omega^{(j)}) \frac{\mathbf{e}_1}{2} \\ &\quad + (T_2^{(i)} + T_2^{(j)}) \frac{\mathbf{e}_2}{2} + (\omega^{(i)} + \omega^{(j)}) \frac{x_1 \mathbf{e}_2}{2}, \end{aligned}$$



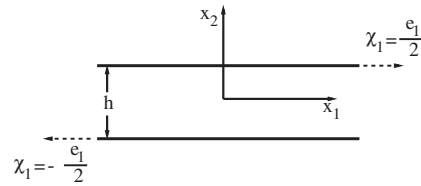


FIG. 4. The setup for the calculation of  $\chi_1$ .

and, using the linearity of the problem, we decompose  $\mathbf{u}^a$  as

$$(4.2) \quad \mathbf{u}^a = (T_1^{(i)} - T_1^{(j)} + a_i\omega^{(i)} + a_j\omega^{(j)})\chi_1 + (T_2^{(i)} - T_2^{(j)})\chi_2 + (\omega^{(i)} - \omega^{(j)})\lambda + \mathcal{R},$$

where  $\chi_1, \chi_2, \lambda$ , and  $\mathcal{R}$  are elementary velocity fields satisfying

$$(4.3) \quad \chi_k \left( x_1, \frac{h}{2} \right) = -\chi_k \left( x_1, -\frac{h}{2} \right) = \frac{1}{2}\mathbf{e}_k, \quad k = 1, 2,$$

$$(4.4) \quad \lambda \left( x_1, \frac{h}{2} \right) = -\lambda \left( x_1, -\frac{h}{2} \right) = \frac{x_1}{2}\mathbf{e}_2,$$

$$(4.5) \quad \begin{aligned} \mathcal{R} \left( x_1, \pm \frac{h}{2} \right) &= [T_2^{(i)} + T_2^{(j)} + (\omega^{(i)} + \omega^{(j)})x_1] \frac{\mathbf{e}_2}{2} \\ &+ (T_1^{(i)} + T_1^{(j)} + a_i\omega^{(i)} - a_j\omega^{(j)}) \frac{\mathbf{e}_1}{2}. \end{aligned}$$

We approximate all elementary velocity fields in the decomposition (4.2) by solving the simplified Stokes flow problem between two parallel plates at distance  $h$  (treated as constant) apart.

Clearly, velocity field

$$(4.6) \quad \mathcal{R}(x_1, x_2) = [T_2^{(i)} + T_2^{(j)} + (\omega^{(i)} + \omega^{(j)})x_1] \frac{\mathbf{e}_2}{2} + (T_1^{(i)} + T_1^{(j)} + a_i\omega^{(i)} - a_j\omega^{(j)}) \frac{\mathbf{e}_1}{2}$$

is divergence free and satisfies (2.7) for a constant pressure field. Moreover, its rate of strain is uniformly bounded in the asymptotic limit  $\delta^{ij}/a^{ij} \rightarrow 0$  and the contribution of  $\mathcal{R}$  to the local dissipation rate in  $\Pi^{ij}$  is negligible.

**Velocity field  $\chi_1$ .** As we zoom in near the axis of the centers  $\mathbf{x}^{(i)}$  and  $\mathbf{x}^{(j)}$ , the top and bottom boundaries of  $\Pi^{ij}$  which belong to  $\partial D_i$  and  $\partial D^{(j)}$ , respectively, are approximated by parallel planes which move in opposite directions, as shown in Figure 4. Using separation of variables, we find

$$(4.7) \quad \chi_1(x_1, x_2) = \left[ \frac{x_2}{h} + \frac{C}{2\mu} \left( x_2^2 - \frac{h^2}{4} \right) \right] \mathbf{e}_1.$$

Integrating,<sup>3</sup> we obtain

$$(4.8) \quad W_{\Pi^{ij}}^{\chi_1} = \frac{\mu}{4} \int_{-a^{ij}}^{a^{ij}} dx_1 \int_{-\frac{h(x_1)}{2}}^{\frac{h(x_1)}{2}} dx_2 (\nabla\chi_1 + \nabla\chi_1^T, \nabla\chi_1 + \nabla\chi_1^T) \approx \frac{\pi\mu}{2} \sqrt{\frac{a^{ij}}{\delta^{ij}}} + O(1).$$

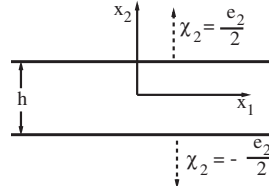


FIG. 5. The setup for the calculation of  $\chi_2$ .

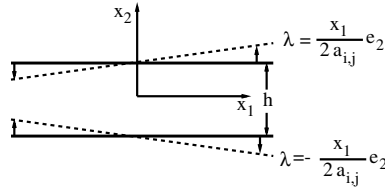


FIG. 6. The setup for the calculation of  $\lambda$ .

**Velocity field  $\chi_2$ .** We approximate  $\chi_2$  by the velocity of an incompressible fluid between two parallel plates which move at constant speed away from each other along the axis  $\mathbf{e}_2$  (see Figure 5). Separating variables, we obtain

$$(4.9) \quad \chi_2(x_1, x_2) \approx \frac{6x_1}{h} \left( \frac{x_2^2}{h^2} - \frac{1}{4} \right) \mathbf{e}_1 + \left[ \frac{3x_2}{2h} - 2 \left( \frac{x_2}{h} \right)^3 \right] \mathbf{e}_2,$$

$$(4.10) \quad p(x_1, x_2) \approx \frac{6\mu x_1^2}{h^3} - \frac{6\mu}{h} \left( \frac{x_2^2}{h^2} - \frac{1}{4} \right) + C.$$

Then,

$$(4.11) \quad \begin{aligned} W_{\Pi^{ij}}^{\chi_2} &= \frac{\mu}{4} \int_{-a^{ij}}^{a^{ij}} dx_1 \int_{-\frac{h(x_1)}{2}}^{\frac{h(x_1)}{2}} dx_2 (\nabla \chi_2 + \nabla \chi_2^T, \nabla \chi_2 + \nabla \chi_2^T) \\ &\approx \frac{3\pi\mu}{4} \left( \frac{a^{ij}}{\delta^{ij}} \right)^{\frac{3}{2}} + \frac{12\pi\mu}{5} \sqrt{\frac{a^{ij}}{\delta^{ij}}} + O(1). \end{aligned}$$

**Velocity field  $\lambda$ .** The setup for the calculation of  $\lambda$  is shown in Figure 6. In this case, the approximate solution is

$$(4.12) \quad \lambda(x_1, x_2) \approx \frac{3x_1^2}{h} \left( \frac{x_2^2}{h^2} - \frac{1}{4} \right) \mathbf{e}_1 + \left( \frac{3x_1}{2} \frac{x_2}{h} - 2 \frac{x_1 x_2^3}{h^3} \right) \mathbf{e}_2,$$

$$(4.13) \quad p(x_1, x_2) \approx \frac{2\mu x_1^3}{h^3} - \frac{6\mu x_1}{h} \left( \frac{x_2^2}{h^2} - \frac{1}{4} \right),$$

and

$$(4.14) \quad W_{\Pi^{ij}}^\lambda = \frac{\mu}{4} \int_{x_1=-a^{ij}}^{a^{ij}} dx_1 \int_{x_2=-\frac{h(x_1)}{2}}^{\frac{h(x_1)}{2}} dx_2 (\nabla \lambda + \nabla \lambda^T, \nabla \lambda + \nabla \lambda^T) \approx \frac{9\pi\mu}{16} \sqrt{\frac{a^{ij}}{\delta}} (a^{ij})^2 + O(1).$$

<sup>3</sup>The integration is done in MAPLE.

Finally, using straightforward MAPLE calculations, we find that the contributions to  $W_{\Pi^{ij}}$  of all the cross terms such as  $(\nabla\chi_1 + \nabla\chi_1^T, \nabla\chi_2 + \nabla\chi_2^T)$  is  $O(1)$ . Gathering all the results, we have

$$\begin{aligned}
 W_{\Pi^{ij}} \approx & \left[ \frac{3\pi\mu}{4} \left( \frac{a^{ij}}{\delta^{ij}} \right)^{\frac{3}{2}} + \frac{12\pi\mu}{5} \sqrt{\frac{a^{ij}}{\delta^{ij}}} \right] (T_2^{(i)} - T_2^{(j)})^2 \\
 (4.15) \quad & + \frac{\pi\mu}{2} \sqrt{\frac{a^{ij}}{\delta^{ij}}} (T_1^{(i)} - T_1^{(j)} + a_i\omega^{(i)} + a_j\omega^{(j)})^2 \\
 & + \frac{9\pi\mu}{16} \sqrt{\frac{a^{ij}}{\delta^{ij}}} (a^{ij})^2 (\omega^{(i)} - \omega^{(j)})^2 + O(1).
 \end{aligned}$$

This is precisely the result (3.12), and the approximation  $E^a$  of the viscous dissipation rate in the suspension is obtained by summing contributions (4.15) of all the gaps, as explained in section 3.2.

**5. The upper bound.** Any test velocity field  $\mathbf{u} \in \mathcal{U}$  gives an upper bound on the viscous dissipation rate  $E$  when used in variational principle (2.20). However, of all choices of  $\mathbf{u}$ , we are interested in those that give tight, correct-to-leading-orders bounds on  $E$ . In this section, we give the construction of such a velocity field in two dimensions. We begin with the construction of  $\mathbf{u}$  in the gap  $\Pi^{ij}$  between two adjacent particles  $D^{(i)}$  and  $D^{(j)}$  (see section 5.1), and, to capture the important features of the flow, we use the formal asymptotic analysis of section 4 as a guide. Then in section 5.2, we extend  $\mathbf{u}$  to the remainder of the domain, where the flow is diffuse and, as such, contributes to  $O(1)$  terms in  $E$ .

**5.1. Definition of the trial velocity field  $\mathbf{u}$  in a gap  $\Pi^{ij}$ .** The local construction of section 4 captures the important features of the flow in the gap  $\Pi^{ij}$  between adjacent particles  $D^{(i)}$  and  $D^{(j)}$ . However, since the gap thickness  $h(x_1)$  is not a constant (as it is treated in section 4),  $\mathbf{u}^a$  derived in section 4 is not divergence free and, therefore, it is not an admissible trial field in variational principle (2.20). In this section, we modify the velocity field calculated in section 4 in such a way that the incompressibility condition is satisfied and yet the effect of the corrections on  $E$  is minimal.

Using (4.1), (4.2), and the linearity of the problem, we decompose trial velocity field  $\mathbf{u}$  as

$$\begin{aligned}
 (5.1) \quad \mathbf{u}(\mathbf{x}) = & (T_1^{(i)} - T_1^{(j)} + a_i\omega^{(i)} + a_j\omega^{(j)})\chi_1(\mathbf{x}) + (T_2^{(i)} - T_2^{(j)})\chi_2(\mathbf{x}) \\
 & + (\omega^{(i)} - \omega^{(j)})\lambda(2) + \mathcal{R}(\mathbf{x}) + \mathcal{C}(\mathbf{x}),
 \end{aligned}$$

where  $\mathcal{R}$  is given by (4.6),  $\chi_1, \chi_2$ , and  $\lambda$  satisfy boundary conditions (4.3) and (4.4), and

$$\begin{aligned}
 (5.2) \quad \mathcal{C} \left( x_1, \pm \frac{h(x_1)}{2} \right) = & \left( \sqrt{1 - \frac{x_1^2}{a_i^2}} + \sqrt{1 - \frac{x_1^2}{a_j^2}} - 2 \right) \left( \pm \frac{\omega^{(i)} + \omega^{(j)}}{2} + \frac{\omega^{(i)} - \omega^{(j)}}{2} \right) \frac{\mathbf{e}_1}{2} \\
 & + \left( \sqrt{1 - \frac{x_1^2}{a_i^2}} - \sqrt{1 - \frac{x_1^2}{a_j^2}} \right) \left( \pm \frac{\omega^{(i)} - \omega^{(j)}}{2} + \frac{\omega^{(i)} + \omega^{(j)}}{2} \right) \frac{\mathbf{e}_1}{2}.
 \end{aligned}$$

Elementary velocity fields  $\chi_1, \chi_2$ , and  $\lambda$  have been approximated in section 4. Here, we modify their expression to ensure that they are divergence free in the gap of variable thickness  $h(x_1)$ . Velocity field  $\mathbf{C}$  accounts for the curvature of the gap and it has been omitted in section 4. In this section, we calculate an admissible field  $\mathbf{C}$  and we show that its influence on  $E$  is  $O(1)$ .

**Velocity field  $\chi_1$ .** Using the formal asymptotic analysis of section 4, we have that

$$(5.3) \quad \chi_1(x_1, x_2) \sim \frac{x_2}{h(x_1)} \mathbf{e}_1.$$

However, the right-hand side in (5.3) is not divergence free, so we correct (5.3) as

$$(5.4) \quad \chi_1(x_1, x_2) = \nabla^\perp F(x_1, x_2), \quad \text{where } F(x_1, x_2) = -\frac{x_2^2}{2h(x_1)} - \frac{h(x_1)}{8}$$

and  $\nabla^\perp = (-\partial/\partial x_2, \partial/\partial x_1)$ . Then,

$$(5.5) \quad \chi_1(x_1, x_2) = \frac{x_2}{h(x_1)} \mathbf{e}_1 + \frac{1}{2} \frac{dh(x_1)}{dx_1} \left( \frac{x_2^2}{h^2(x_1)} - \frac{1}{4} \right) \mathbf{e}_2, \quad \text{div} \chi_1(x_1, x_2) = 0$$

and, on the top/bottom parts of boundary  $\partial \Pi^{ij}$ ,  $\chi_1(x_1, x_2 = \pm h(x_1)/2) = \pm \mathbf{e}_1/2$ . The calculation of local rate of dissipation  $W_{\Pi^{ij}}^{\chi_1}$  is now straightforward and the result coincides with (4.8).

**Velocity field  $\chi_2$ .** The formal asymptotic analysis of section 4 gives

$$(5.6) \quad \chi_2(x_1, x_2) \sim \frac{6x_1}{h(x_1)} \left( \frac{x_2^2}{h^2(x_1)} - \frac{1}{4} \right) \mathbf{e}_1 + \left[ \frac{3x_2}{2h(x_1)} - 2 \left( \frac{x_2}{h(x_1)} \right)^3 \right] \mathbf{e}_2,$$

but, since  $h$  is, in truth, a function of  $x_1$ , (5.6) is not divergence free and cannot be used as such in the upper bound calculation. Instead, we define the trial field

$$(5.7) \quad \chi_2(x_1, x_2) = \nabla^\perp F(x_1, x_2), \quad \text{where } F(x_1, x_2) = -\frac{2x_1x_2^3}{h^3(x_1)} + \frac{3x_1x_2}{2h(x_1)}.$$

The corresponding local dissipation rate  $W_{\Pi^{ij}}^{\chi_2}$  is

$$(5.8) \quad W_{\Pi^{ij}}^{\chi_2} = \frac{3\pi\mu}{4} \left( \frac{a^{ij}}{\delta^{ij}} \right)^{\frac{3}{2}} + \frac{27\pi\mu}{10} \sqrt{\frac{a^{ij}}{\delta^{ij}}} + O(1)$$

and we note that it coincides, with leading order, with (4.11).

**Velocity field  $\lambda$ .** We define a divergence free trial field  $\lambda$ , which is approximately equal to (4.12), as

$$(5.9) \quad \lambda(x_1, x_2) = \nabla^\perp F(x_1, x_2), \quad \text{where } F(x_1, x_2) = \left( \frac{3x_1^2x_2}{4h(x_1)} - \frac{x_1^2x_2^3}{h^3(x_1)} \right),$$

Then  $\lambda(x_1, x_2 = \pm h(x_1)/2) = \pm \frac{x_1}{2} \mathbf{e}_2$ , and the corresponding local rate of dissipation  $W_{\Pi^{ij}}^\lambda$  is given by (4.14).

**Velocity field  $\mathbf{C}(\mathbf{x})$ .** We define trial field  $\mathbf{C}(\mathbf{x})$  as

$$(5.10) \quad \mathbf{C}(x_1, x_2) = (\omega^{(i)} + \omega^{(j)}) \nabla^\perp F(x_1, x_2) + (\omega^{(i)} - \omega^{(j)}) \nabla^\perp G(x_1, x_2),$$

where

$$\begin{aligned}
F(x_1, x_2) = & \left(1 - \frac{1}{2}\sqrt{1 - \frac{x^2}{a_i^2}} - \frac{1}{2}\sqrt{1 - \frac{x^2}{a_j^2}}\right) \left(\frac{x_2^2}{2h(x_1)} + \frac{h(x_1)}{8}\right) \\
& + \int_0^{x_1} \frac{h(s)}{8} \frac{d}{ds} \left(\sqrt{1 - \frac{s^2}{a_i^2}} + \sqrt{1 - \frac{s^2}{a_j^2}}\right) ds \\
& + \left(\frac{1}{2}\sqrt{1 - \frac{x_1^2}{a_i^2}} - \frac{1}{2}\sqrt{1 - \frac{x_1^2}{a_j^2}}\right) \left(-\frac{x_2}{2} + \frac{3h(x_1)x_2}{2} - \frac{2x_2^3}{h(x_1)}\right) \\
& - \int_0^{x_1} \left(\frac{3h(s)x_2}{2h(x_1)} - \frac{2h(s)x_2^3}{h^3(x_1)}\right) \times \left[\left(\frac{1}{4} - \frac{h(s)}{2}\right) \left(\frac{s/a_i^2}{\sqrt{1 - s^2/a_i^2}} - \frac{s/a_j^2}{\sqrt{1 - s^2/a_j^2}}\right) \right. \\
& \quad \left. + \frac{dh(s)}{ds} \left(\sqrt{1 - s^2/a_i^2} - \sqrt{1 - s^2/a_j^2}\right)\right] ds
\end{aligned}$$

and

$$\begin{aligned}
G(x_1, x_2) = & - \left(\frac{1}{2}\sqrt{1 - \frac{x_1^2}{a_i^2}} - \frac{1}{2}\sqrt{1 - \frac{x_1^2}{a_j^2}}\right) \left(\frac{x_2^2}{2h(x_1)} - \frac{h(x_1)}{4}\right) \\
& - \frac{1}{4} \int_0^{x_1} \frac{dh(s)}{ds} \frac{\sqrt{1 - s^2/a_i^2} - \sqrt{1 - s^2/a_j^2}}{2} ds \\
& + \frac{x_2}{2} + \frac{\sqrt{1 - x_1^2/a_i^2} + \sqrt{1 - x_1^2/a_j^2}}{2} \left(-\frac{x_2}{2} + \frac{3h(x_1)x_2}{2} - \frac{2x_2^3}{h(x_1)}\right) \\
& - \int_0^{x_1} \left(\frac{3h(s)x_2}{2h(x_1)} - \frac{2h(s)x_2^3}{h^3(x_1)}\right) \times \left[\left(\frac{1}{4} - \frac{h(s)}{2}\right) \left(\frac{s/a_i^2}{\sqrt{1 - s^2/a_i^2}} + \frac{s/a_j^2}{\sqrt{1 - s^2/a_j^2}}\right) \right. \\
& \quad \left. + \frac{dh(s)}{ds} \left(\sqrt{1 - s^2/a_i^2} + \sqrt{1 - s^2/a_j^2}\right)\right] ds.
\end{aligned}$$

Although the expression (5.10) is rather complicated, it can be checked with straightforward calculations (which we have done in MAPLE) that it satisfies boundary conditions (5.2) and, as such, it is an admissible trial field, which gives a local rate of dissipation  $W_{\Pi^{ij}}^C = O(1)$ .

Finally, we find through MAPLE calculations that the contribution of the cross terms to  $W_{\Pi^{ij}}$  is  $O(1)$ .

We have now defined a trial velocity field that satisfies the exact boundary conditions on the top and bottom boundaries of the gap  $\Pi^{ij}$ , is divergence free, and, most important, gives an upper bound on the gap dissipation rate which agrees, with leading order, with the formal asymptotic result of section 4.

**5.2. Extension of the trial velocity field  $\mathbf{u}$  outside the gaps between the particles in suspension.** Let us denote the union of all gaps by  $U_\Pi$  and define the complement in  $\Omega_F$  of the union of all gaps

$$(5.11) \quad U_E = \Omega_F \setminus U_\Pi.$$

We wish to extend the trial velocity field  $\mathbf{u}$  from the gaps  $\Pi^{ij}$  to  $U_E$ , so that the leading order terms of the dissipation rate are not affected. Clearly, when there are many particles in the suspension, the set  $U_E$  is the union of many disjoint, connected components, which we denote by  $C_j$ . Let us then focus attention on one such component and drop the subscript  $j$ . To avoid boundary corners in the connected component  $C$ , we take a slightly larger domain  $\tilde{C} \subset \Omega_F$  such that  $C \subset \tilde{C}$  and  $\partial\tilde{C}$  is smooth.<sup>4</sup> Note that the construction of section 5.1 gives a trial velocity of the form  $\mathbf{u} = \nabla^\perp F$  and, since the gap thickness is  $h = O(a) \gg \delta^{ij}$  at  $\partial\Pi^{ij} \cap \partial\tilde{C}$ , the first and second derivatives of  $F$  are uniformly bounded on  $\partial\tilde{C}$ , as  $\delta \rightarrow 0$ . We now wish to extend  $\mathbf{u}$  to the interior of  $\tilde{C}$ .

Let us take a  $\gamma > 0$ , independent of  $\delta$ , and define the boundary layer

$$(5.12) \quad C_\gamma = \{\mathbf{x} \in \tilde{C} \text{ such that } \text{dist}(\mathbf{x}, \partial\tilde{C}) < \gamma\}.$$

Since the arcs in  $\partial\tilde{C}$  are independent of  $\delta$ , we can choose a cover  $\mathcal{P}_j$ ,  $j = 1, 2, \dots, J$  independent of  $\delta$ , and a subordinate partition of unity  $\phi_j$  with support  $\phi_j = \tilde{\mathcal{P}}_j \subset \mathcal{P}_j$  such that  $\tilde{\mathcal{P}}_j \cap \tilde{C} \subset C_\gamma$ . Then let us extend  $\mathbf{u}$  in  $\tilde{\mathcal{P}}_j$  and, for simplicity of notation, drop the index  $j$ .

In  $\tilde{\mathcal{P}}$ , define local coordinates  $\mathbf{y} = (y_1, y_2)$ , such that  $y_2 = 0$  at  $\partial\tilde{\mathcal{P}} \cap \partial\tilde{C}$ , and  $\tilde{\mathcal{P}} \cap \tilde{C}$  is mapped into a tensor product of intervals  $I_1(y_1) \times I_2(y_2)$ , for  $y_2 > 0$ . Take then a smooth function  $g(y_2)$ , which vanishes outside interval  $I_2(y_2)$  and, at  $y_2 = 0$ ,  $g(0) = 1$ , and define the extension of  $F$ , from  $\tilde{\mathcal{P}} \cap \partial\tilde{C}$  to  $\tilde{\mathcal{P}} \cap \tilde{C}$ , as<sup>5</sup>

$$(5.13) \quad F(y_1, y_2) = g(y_2) \left[ F(y_1, 0) + y_2 \frac{\partial F(y_1, 0)}{\partial y_2} \right].$$

Clearly, the extended  $F$  is smooth and its first derivatives are equal to the previously specified values on  $\tilde{\mathcal{P}} \cap \partial\tilde{C}$ . We also have

$$(5.14) \quad \|F(y_1, y_2)\|_{H^2(I_1 \times I_2)} \leq A$$

with  $A$ , independent of  $\delta$ . Repeating the same procedure, we extend  $F$  to all  $\tilde{\mathcal{P}}_j \cap \tilde{C}$ ,  $j = 1, \dots, J$ , or, equivalently, to  $\tilde{C}$ . Then, taking  $\mathbf{u} = \nabla^\perp F$ , we have  $\text{div } \mathbf{u} = 0$ , and the strain tensor  $\mathcal{E}(\mathbf{u})$  with components in  $L^2(\tilde{C})$  and a corresponding viscous dissipation rate

$$(5.15) \quad \int_{\tilde{C}} \mu (\mathcal{E}(\mathbf{u}), \mathcal{E}(\mathbf{u})) \, d\mathbf{x} \leq A|\tilde{C}|.$$

We end this section with the remark that it is not necessary that  $\tilde{C}$  lie inside  $\Omega_F$  for estimate (5.15) to hold (see Figure 7). Indeed, even if the connected component  $C$

<sup>4</sup> $\partial\tilde{C}$  is the union of arcs which lie either inside a gap  $\Pi^{ij}$  or on the boundary of a surrounding particle.

<sup>5</sup>Note that (5.13) is a simplified version of the classic Borel construction in [18, Theorem 1.2.6].

intersects the exterior boundary  $\partial\Omega$ , we can always extend  $F$  to a smooth  $H^2$  function supported away from the corners of  $\partial\Omega$ , and (5.15) follows.

Gathering all the results in this section, we have, in the notation of section 3.2.1, the upper bound

$$\begin{aligned}
 (5.16) \quad E \leq & \min_{\mathbf{T}, \omega} \sum_{i=1}^N \sum_{\substack{j \in \mathcal{N}_i \\ j < i}} \left\{ \left[ \frac{3\pi\mu}{4} \left( \frac{a^{ij}}{\delta^{ij}} \right)^{\frac{3}{2}} + \frac{27\pi\mu}{10} \sqrt{\frac{a^{ij}}{\delta^{ij}}} \right] [(\mathbf{T}^{(i)} - \mathbf{T}^{(j)}) \cdot \mathbf{q}^{ij}]^2 \right. \\
 & + \frac{\pi\mu}{2} \sqrt{\frac{a^{ij}}{\delta^{ij}}} [(\mathbf{T}^{(i)} - \mathbf{T}^{(j)}) \cdot \mathbf{p}^{ij} + a_i\omega^{(i)} + a_j\omega^{(j)}]^2 \\
 & \left. + \frac{9\pi\mu}{16} \sqrt{\frac{a^{ij}}{\delta^{ij}}} (\omega^{(i)} a^{ij} - \omega^{(j)} a^{ij})^2 \right\} \\
 & + \sum_{i \in \mathcal{B}} \left\{ \left[ \frac{3\pi\mu}{4} \left( \frac{2a_i}{\delta^i} \right)^{\frac{3}{2}} + \frac{27\pi\mu}{10} \sqrt{\frac{2a_i}{\delta^i}} \right] [(\mathbf{T}^{(i)} - \mathbf{g}) \cdot \mathbf{q}^i]^2 \right. \\
 & + \frac{9\pi\mu}{16} \sqrt{\frac{2a_i}{\delta^i}} (2\omega^{(i)})^2 (a^{ij})^2 \\
 & \left. + \frac{\pi\mu}{2} \sqrt{\frac{2a_i}{\delta^i}} [(\mathbf{T}^{(i)} - \mathbf{g}) \cdot \mathbf{p}^i + a_i\omega^{(i)}]^2 \right\} + O(1),
 \end{aligned}$$

where, for the boundary nodes  $i \in \mathcal{B}$ ,  $\delta^i$  is the distance between  $\partial D^{(i)}$  and the upper or lower boundary  $\partial\Omega^\pm$ .

**6. Rigorous justification of the leading-order spring network approximation.** In this section, we derive and justify rigorously the spring network approximation in two dimensions (recall section 3.2.1) by constructing a lower bound on  $E$ , which agrees with (5.16), to  $O((\frac{a}{\delta})^{\frac{3}{2}})$ .

**6.1. A simplified upper bound.** Since the leading order term is not affected by the rotations of the particles, we set in (5.16)  $\omega^{(i)} = 0$  for all  $i = 1, \dots, N$ , and we obtain a less precise but simplified upper bound

$$\begin{aligned}
 (6.1) \quad E \leq & W_{\Omega_F}(\mathbf{u}) = \min_{\mathbf{T}} \sum_{i=1}^N \sum_{\substack{j \in \mathcal{N}_i \\ j < i}} \left\{ \left[ \frac{3\pi\mu}{4} \left( \frac{a^{ij}}{\delta^{ij}} \right)^{\frac{3}{2}} + \frac{27\pi\mu}{10} \sqrt{\frac{a^{ij}}{\delta^{ij}}} \right] [(\mathbf{T}^{(i)} - \mathbf{T}^{(j)}) \cdot \mathbf{q}^{ij}]^2 \right. \\
 & + \frac{\pi\mu}{2} \sqrt{\frac{a^{ij}}{\delta^{ij}}} [(\mathbf{T}^{(i)} - \mathbf{T}^{(j)}) \cdot \mathbf{p}^{ij}]^2 \left. \right\} + \sum_{i \in \mathcal{B}} \left\{ \left[ \frac{3\pi\mu}{4} \left( \frac{2a_i}{\delta^i} \right)^{\frac{3}{2}} + \frac{27\pi\mu}{10} \sqrt{\frac{2a_i}{\delta^i}} \right] [(\mathbf{T}^{(i)} - \mathbf{g}) \cdot \mathbf{q}^i]^2 \right. \\
 & \left. + \frac{\pi\mu}{2} \sqrt{\frac{2a_i}{\delta^i}} [(\mathbf{T}^{(i)} - \mathbf{g}) \cdot \mathbf{p}^i]^2 \right\} + O(1).
 \end{aligned}$$

Except for the  $O(1)$  term, the right-hand side of (6.1) involves the minimization of a quadratic form in the translation velocities  $\mathbf{T}^{(i)}$  for  $i = 1, \dots, N$ , and the minimum

is achieved by the solution of the linear system of equations,

$$(6.2) \quad \sum_{j \in \mathcal{N}_i} \left\{ \left[ \frac{3\pi\mu}{4} \left( \frac{a^{ij}}{\delta^{ij}} \right)^{\frac{3}{2}} + \frac{27\pi\mu}{10} \sqrt{\frac{a^{ij}}{\delta^{ij}}} \right] [(\mathbf{T}^{(i)} - \mathbf{T}^{(j)}) \cdot \mathbf{q}^{ij}] \mathbf{q}^{ij} \right. \\ \left. + \frac{\pi\mu}{2} \sqrt{\frac{a^{ij}}{\delta^{ij}}} [(\mathbf{T}^{(i)} - \mathbf{T}^{(j)}) \cdot \mathbf{p}^{ij}] \mathbf{p}^{ij} \right\} + \mathbf{F}_{\mathcal{B}}(\mathbf{T}^{(i)}) = \mathbf{0} \quad \text{for } 1 \leq i \leq N,$$

where

$$(6.3) \quad \mathbf{F}_{\mathcal{B}}(\mathbf{T}^{(i)}) = \begin{cases} \left[ \frac{3\pi\mu}{4} \left( \frac{2a_i}{\delta^i} \right)^{\frac{3}{2}} + \frac{27\pi\mu}{10} \sqrt{\frac{2a_i}{\delta^i}} \right] [(\mathbf{T}^{(i)} - \mathbf{g}) \cdot \mathbf{q}^i] \mathbf{q}^i + \frac{\pi\mu}{2} \sqrt{\frac{2a_i}{\delta^i}} [(\mathbf{T}^{(i)} - \mathbf{g}) \cdot \mathbf{p}^i] \mathbf{p}^i & \text{if } i \in \mathcal{B}, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Next, we prove the unique solvability of these equations.

PROPOSITION 6.1. *The linear system of (6.2) has a unique solution,*

$$\boldsymbol{\tau} = (T_1^{(1)}, T_2^{(1)}, \dots, T_1^{(N)}, \dots, T_2^{(N)})^T \in \mathbb{R}^{2N},$$

where superscript  $T$  stands for transpose.

*Proof.* Let us write the upper bound (6.1) in compact form as

$$(6.4) \quad E \leq \min_{\boldsymbol{\tau}} (\boldsymbol{\tau} \cdot A\boldsymbol{\tau} - 2\boldsymbol{\tau} \cdot \mathbf{f}) + r + O(1),$$

where matrix  $A \in \mathbb{R}^{2N \times 2N}$  is symmetric,  $\mathbf{f} \in \mathbb{R}^{2N}$ , and  $r \in \mathbb{R}$ . We prove the unique solvability of (6.2) (i.e., of  $A\boldsymbol{\tau} = \mathbf{f}$ ) by showing that  $A$  is positive definite. Since we take the limit  $\delta \rightarrow 0$ , we have from (6.1) and (6.4) that

$$(6.5) \quad \boldsymbol{\tau} \cdot A\boldsymbol{\tau} = \sum_{i=1}^N \sum_{\substack{j \in \mathcal{N}_i \\ j < i}} \left\{ \left[ \frac{3\pi\mu}{4} \left( \frac{a^{ij}}{\delta^{ij}} \right)^{\frac{3}{2}} + \frac{27\pi\mu}{10} \sqrt{\frac{a^{ij}}{\delta^{ij}}} \right] [(\mathbf{T}^{(i)} - \mathbf{T}^{(j)}) \cdot \mathbf{q}^{ij}]^2 \right. \\ \left. + \frac{\pi\mu}{2} \sqrt{\frac{a^{ij}}{\delta^{ij}}} [(\mathbf{T}^{(i)} - \mathbf{T}^{(j)}) \cdot \mathbf{p}^{ij}]^2 \right\} \\ + \sum_{i \in \mathcal{B}} \left\{ \left[ \frac{3\pi\mu}{4} \left( \frac{2a_i}{\delta^i} \right)^{\frac{3}{2}} + \frac{27\pi\mu}{10} \sqrt{\frac{2a_i}{\delta^i}} \right] (\mathbf{T}^{(i)} \cdot \mathbf{q}^i)^2 + \frac{\pi\mu}{2} \sqrt{\frac{2a_i}{\delta^i}} (\mathbf{T}^{(i)} \cdot \mathbf{p}^i)^2 \right\} \\ \geq C\delta^{-\frac{1}{2}} \sum_{i=1}^N \sum_{\substack{j \in \mathcal{N}_i \\ j < i}} [(\mathbf{T}^{(i)} - \mathbf{T}^{(j)}) \cdot \mathbf{q}^{ij}]^2 + [(\mathbf{T}^{(i)} - \mathbf{T}^{(j)}) \cdot \mathbf{p}^{ij}]^2 \\ + C\delta^{-\frac{1}{2}} \sum_{i \in \mathcal{B}} [(\mathbf{T}^{(i)} \cdot \mathbf{q}^i)^2 + (\mathbf{T}^{(i)} \cdot \mathbf{p}^i)^2] \\ = C\delta^{-\frac{1}{2}} \left( \sum_{i=1}^N \sum_{\substack{j \in \mathcal{N}_i \\ j < i}} |\mathbf{T}^{(i)} - \mathbf{T}^{(j)}|^2 + \sum_{i \in \mathcal{B}} |\mathbf{T}^{(i)}|^2 \right) = \boldsymbol{\tau} \cdot \tilde{A}\boldsymbol{\tau},$$



where  $C$  is independent of  $\delta$  and matrix  $\tilde{A}$  is clearly symmetric, nonnegative definite. To show that  $\tilde{A}$  is, in fact, positive definite, let us suppose that there exists a nontrivial  $\boldsymbol{\tau}$  in the null space of  $\tilde{A}$ . Then, by (6.5), we have  $\mathbf{T}^{(i)} - \mathbf{T}^{(j)} = 0$  for  $i = 1, \dots, N$ ,  $j \in \mathcal{N}_i$ , and  $\mathbf{T}^{(i)} = 0$  for  $i \in \mathcal{B}$ . Since the graph  $\Gamma$  is connected (Property 3.1), this implies  $\mathbf{T}^{(i)} = 0$  for all  $i = 1, \dots, N$  or, equivalently,  $\boldsymbol{\tau} = \mathbf{0}$ . However, this contradicts our initial assumption on  $\boldsymbol{\tau}$ , so the null space of  $\tilde{A}$  must be trivial. This implies, in turn, that  $A$  is positive definite and that the linear system of (6.2) is uniquely solvable.  $\square$

*Remark 6.1.* In the remainder of this paper, we denote by  $\mathbf{u}$  the trial velocity field constructed in section 5, where all rotational velocities  $\omega^{(i)}$  are set to zero and where translational velocities  $\mathbf{T}^{(i)}$  solve linear system of equations (6.2) for  $1 \leq i \leq N$ . In particular, in gap  $\Pi^{ij}$ , connecting adjacent disks  $D^{(i)}$  and  $D^{(j)}$ , the trial field is

$$(6.6) \quad \mathbf{u}(\mathbf{x}) = [(\mathbf{T}^{(i)} - \mathbf{T}^{(j)}) \cdot \mathbf{e}_1] \boldsymbol{\chi}_1(\mathbf{x}) + [(\mathbf{T}^{(i)} - \mathbf{T}^{(j)}) \cdot \mathbf{e}_2] \boldsymbol{\chi}_2(\mathbf{x}) + \frac{1}{2}(\mathbf{T}^{(i)} + \mathbf{T}^{(j)}),$$

where  $\boldsymbol{\chi}_1$  and  $\boldsymbol{\chi}_2$  are given by (5.5) and (5.7), respectively. Note that this trial field yields upper bound (see (6.1))

$$(6.7) \quad E \leq W_{\Omega_F}(\mathbf{u})$$

when used in variational principle (2.20).

**6.2. Lower Bound.**

**6.2.1. Outline of the construction.** Given a subdomain  $M$  of  $\Omega_F$ , define a functional

$$(6.8) \quad W_M^*(\mathcal{S}) = \int_{\partial\Omega \cap \bar{M}} \mathbf{g} \cdot \mathcal{S} \mathbf{n} ds - \int_M F(\mathcal{S}) dx,$$

where  $\bar{M}$  is the closure of  $M$ ,  $\mathbf{g}$  is defined by (2.9),

$$(6.9) \quad F(\mathcal{S}) = \frac{1}{4\mu} \left[ (\mathcal{S}, \mathcal{S}) - \frac{1}{2}(\text{trace } \mathcal{S})^2 \right],$$

and  $\mathcal{S}$  is a symmetric (stress) tensor in  $\mathcal{F}$ . In the context of this paper, subdomain  $M$  stands for either a gap  $\Pi^{ij}$  between adjacent particles or a connected component  $\mathcal{C}$  in  $U_E$ , where the flow is diffuse (see section 5.2). Then  $W_{\Omega_F}^*$  is given by the sum of  $W_M^*(\mathcal{S})$  for all such disjoint subdomains in  $\Omega_F$ . For any  $\mathcal{S} \in \mathcal{F}$ , we have by dual variational principle (2.22)

$$(6.10) \quad W_{\Omega_F}^*(\mathcal{S}) \leq E \leq W_{\Omega_F}(\mathbf{u}).$$

Our goal in this section is to construct a trial tensor  $\mathcal{S} \in \mathcal{F}$  such that  $W_{\Omega_F}^*(\mathcal{S})$  matches leading order upper bound  $W_{\Omega_F}(\mathbf{u})$ .

The construction of the trial tensor  $\mathcal{S}$  proceeds as follows.

**Step 1.** In an ideal case, where  $\hat{\mathbf{u}}$  and  $\hat{\mathcal{S}}$ , the minimizer and maximizer of the direct and dual problems (2.20) and (2.22), respectively, would be known, the constitutive equations for the incompressible fluid would give

$$(6.11) \quad \int_M F(\hat{\mathcal{S}}) dx = W_M(\hat{\mathbf{u}}) = \mu \int_M (\mathcal{E}(\mathbf{u}), \mathcal{E}(\mathbf{u})) dx,$$

and, by integration by parts,

$$(6.12) \quad \int_{\partial\Omega \cap \bar{M}} \mathbf{g} \cdot \hat{\mathbf{S}} \mathbf{n} \, ds = 2W_M(\hat{\mathbf{u}}).$$

However, we don't know  $\hat{\mathbf{u}}$ , so we use instead trial velocity field  $\mathbf{u}$  described in Remark 6.1. With this  $\mathbf{u}$ , we find, as a first step in our construction, an approximate pressure  $P$  and the corresponding approximate stress tensor  $\mathcal{S}_0 = 2\mu\mathcal{E}(\mathbf{u}) - P\mathcal{I}$ .

For this purpose, let us focus on a gap  $\Pi^{ij}$ , where  $\mathcal{S}_0$  satisfies

$$(6.13) \quad \int_{\Pi^{ij}} F(\mathcal{S}_0) d\mathbf{x} = W_{\Pi^{ij}}(\mathbf{u}) = O\left(\left(\frac{a^{ij}}{\delta^{ij}}\right)^{\frac{3}{2}}\right).$$

Note, however, that  $\mathcal{S}_0 \notin \mathcal{F}$  because  $\operatorname{div}\mathcal{S}_0 \neq \mathbf{0}$ , so we define the trial tensor in  $\Pi^{ij}$  as

$$\mathcal{S} = \mathcal{S}_0 - \mathcal{K},$$

where  $\mathcal{K}$  is a compensating tensor chosen such that  $\operatorname{div}(\mathcal{S}_0 - \mathcal{K}) = 0$  in  $\Pi$ , and

$$(6.14) \quad \int_{\Pi^{ij}} F(\mathcal{S}) d\mathbf{x} = W_{\Pi^{ij}}(\mathbf{u}) + O\left(\sqrt{\frac{a^{ij}}{\delta^{ij}}}\right).$$

**Step 2.** This is the crucial step in the construction of the lower bound. In Step 1, we obtained tensor  $\mathcal{S}(\mathbf{x})$  in  $\Pi^{ij}$  and, in particular, on the portion of  $\partial D^{(j)}$  which belongs to the neck  $\Pi^{ij}$ . In the second step, we extend  $\mathcal{S}$  to the remaining parts of  $\partial D^{(j)}$ , so that the net force and torque conditions (2.12) hold. Such an extension cannot be constructed for each  $\partial D^{(j)}$  individually. Recall that  $U_\Pi$  is the union of all gaps. For each connected component  $\mathcal{C}$  of the set  $U_E = \Omega_F \setminus U_\Pi$ , where the flow is diffuse, we must have

$$(6.15) \quad \int_{\partial\mathcal{C}} \mathcal{S} \mathbf{n} \, ds = \mathbf{0}$$

for any divergence free extension of  $\mathcal{S}$ , from  $\Pi$  to  $\mathcal{C}$ . But, since each  $\partial\mathcal{C}$  contains parts of the boundaries of several neighboring disks, the extensions of  $\mathcal{S}$  to the boundaries of these disks must be coupled. An attempt to satisfy the balance of forces and torques (2.12) for an individual disk  $D^{(j)}$  influences the balance on all neighboring disks. Since these disks have other neighbors as well (recall that the graph  $\Gamma$  is connected), the extension of  $\mathcal{S}$  from the necks  $\Pi^{ij}$  to the remaining parts of  $\partial D^{(j)}$ , for  $1 \leq j \leq N$ , is a global problem.

Note that a similar difficulty arises in the scalar problem of electrical conduction [5], where a simple construction of the dual trial field (which is a vector flux) is given as follows. In a gap  $\Pi^{ij}$ , the dual trial field is taken as the vector  $\mathbf{j} = (0, \zeta(x_1))$ , where  $\zeta$  is a smooth function of  $x_1$ , the local coordinate in the direction orthogonal to the axis of symmetry of the gap. Outside the union of all gaps, the dual trial field is extended to  $\mathbf{0}$ . While this choice satisfies the divergence-free condition locally in each subdomain of  $\Omega_F$ , one must ensure that the total flux through  $\partial D^{(j)}$  intersected with the union of all the gaps connected with  $D^{(j)}$  is zero for all  $1 \leq j \leq N$  as well. The latter condition is satisfied in [5] by setting

$$\int_{\partial D^{(i)} \cap \Pi^{ij}} \mathbf{j} \cdot \mathbf{n}^{(i)} \, ds = J_{\Pi^{ij}},$$

where  $J_\Pi$  is the net current flowing through the corresponding branch of the asymptotic network (graph  $\Gamma$ ). More explicitly, the condition of flux balance at  $\partial D^{(j)}$  is formulated as Kirchhoff’s current law at the node  $\mathbf{x}^{(j)}$  in the asymptotic network.

While in the scalar problem, the two conditions (divergence free and flux balance) on the dual trial field can be dealt with separately in the vectorial problem that we consider here, they appear to be coupled, and one cannot simply generalize the construction in [5] to find an admissible  $\mathcal{S} \in \mathcal{F}$ . We introduce in section 6.2.3 our novel construction of the extension of  $\mathcal{S}$ , which is divergence free and satisfies the momentum balance equations for all disks.

**Step 3.** Extend the tensor  $\mathcal{S}$ , defined so far in the gaps and at  $\partial D^{(j)}$  for  $1 \leq j \leq N$ , to the whole  $\Omega_F$ . The main point of this step is to control the energy of the extension in such a way that

$$(6.16) \quad W_{\Omega_F \setminus U_\Pi}^*(\mathcal{S}) \ll O(\delta^{-\frac{3}{2}}).$$

**Step 4.** In this step we gather all the results of the previous steps and show that  $W_{\Omega_F}(\mathbf{u})$  and  $W_{\Omega_F}^*(\mathcal{S})$  are the same leading order.

**6.2.2. The trial field  $\mathcal{S}$  in a gap.** We begin our construction of  $\mathcal{S}$  in a gap  $\Pi^{ij}$ , with the help of velocity field (6.6). Recall from sections 4 and 5 that (6.6) is divergence free and, furthermore, it is an approximate solution of Stokes’s equations in the following sense: if the gap thickness  $h$  were a constant, we would have  $\text{curl } \Delta \mathbf{u} = 0$ , the pressure would be well defined by  $\mu \Delta u = \nabla P$ , and the stress

$$\mathcal{S}_0 = 2\mu \mathcal{E}(\mathbf{u}) - PT$$

would be divergence free. However, in truth, gap  $\Pi^{ij}$  is not flat and the condition  $\text{div} \mathcal{S} = 0$  that any dual trial field  $\mathcal{S}$  must satisfy needs to be ensured for the variable thickness  $h(x_1)$ . In that case,  $\Delta u$  is not a gradient of a scalar function, so we introduce an approximate pressure  $P$  and a compensating symmetric tensor  $\mathcal{K}$  such that

$$(6.17) \quad \mathcal{S} = \mathcal{S}_0 - \mathcal{K}$$

is divergence free. Because  $\text{div } \mathbf{u} = 0$ , we have

$$\begin{aligned} F(\mathcal{S}_0) &= \frac{1}{4\mu} \left[ (\mathcal{S}_0, \mathcal{S}_0) - \frac{1}{2}(\text{trace } \mathcal{S}_0)^2 \right] \\ &= \frac{1}{4\mu} [(2\mu \mathcal{E}(\mathbf{u}) - PT, 2\mu \mathcal{E}(\mathbf{u}) - PT) - 2P^2] = \mu (\mathcal{E}(\mathbf{u}), \mathcal{E}(\mathbf{u})) \end{aligned}$$

and

$$(6.18) \quad \int_{\Pi^{ij}} F(\mathcal{S}_0) d\mathbf{x} = W_{\Pi^{ij}}(\mathbf{u}) = O\left(\frac{a^{ij}}{\delta^{ij}}\right)^{\frac{3}{2}},$$

so to get a lower bound that matches the upper one to leading order, we wish that

$$(6.19) \quad \int_{\Pi^{ij}} [F(\mathcal{S}) - F(\mathcal{S}_0)] d\mathbf{x} = O\left(\sqrt{\frac{a^{ij}}{\delta^{ij}}}\right).$$

This can be accomplished, for example, by choosing  $P$  and  $\mathcal{K}$  to satisfy

$$(6.20) \quad \int_{\Pi^{ij}} (\mathcal{S}_0, \mathcal{K}) d\mathbf{x} = O\left(\sqrt{\frac{a^{ij}}{\delta^{ij}}}\right) \text{ and } \int_{\Pi^{ij}} (\mathcal{K}, \mathcal{K}) d\mathbf{x} = O\left(\sqrt{\frac{a^{ij}}{\delta^{ij}}}\right)$$

since

$$F(\mathcal{S}) - F(\mathcal{S}_0) = -2(\mathcal{S}_0, \mathcal{K}) + (\mathcal{K}, \mathcal{K}) + \text{trace } \mathcal{S}_0 \text{ trace } \mathcal{K} - \frac{1}{2}(\text{trace } \mathcal{K})^2.$$

Let us then begin our search for  $\mathcal{K}$  by rewriting equation  $\text{div} \mathcal{S} = \mathbf{0}$ , in terms of the components of  $\mathcal{K}$ , as

$$(6.21) \quad \begin{aligned} \partial_{x_1} \mathcal{K}_{11} + \partial_{x_2} \mathcal{K}_{12} &= R_1, \\ \partial_{x_1} \mathcal{K}_{12} + \partial_{x_2} \mathcal{K}_{22} &= R_2, \end{aligned}$$

where the discrepancy vector

$$(6.22) \quad \mathbf{R} = \text{div } \mathcal{S}_0 = \mu \Delta \mathbf{u} - \nabla P$$

depends on the choice of  $P$ . We define the approximate pressure by

$$(6.23) \quad P(\mathbf{x}) = \mu \int_{-h/2}^{x_2} \Delta u_2(s_1, s_2) ds_2 + \mu \int_{-R^{ij}}^{x_1} r_1(s_1) ds_1,$$

where  $r_1(x_1)$  is given in terms of the Laplacian of the first component of  $\mathbf{u}$  as

$$(6.24) \quad \Delta u_1(x_1, x_2) = r_1(x_1) + r_2(x_1, x_2).$$

Then we set the first entry  $\mathcal{K}_{11}$  in the compensating tensor to zero, and we find from (6.21) that

$$(6.25) \quad \mathcal{K}_{12}(\mathbf{x}) = \int_{-h/2}^{x_2} R_1(s_1, s_2) ds_2, \quad \mathcal{K}_{22}(\mathbf{x}) = - \int_{-R^{ij}}^{x_1} R_1(s_1, s_2) ds_2$$

for discrepancy vector

$$(6.26) \quad \mathbf{R}(\mathbf{x}) = \mu \Delta \mathbf{u}(\mathbf{x}) - \nabla P(\mathbf{x}) = \begin{pmatrix} \mu \Delta u_1(\mathbf{x}) - \partial_{x_1} P(\mathbf{x}) \\ 0 \end{pmatrix}.$$

Now, to verify that estimates (6.20) hold, we note that the components of  $\Delta \mathbf{u}$  are sums of terms of the form

$$(6.27) \quad \text{const } \frac{x_1^k x_2^l}{h(x_1)^m}$$

for some nonnegative integers  $k, l, m$ , and that we have the following estimate.

LEMMA 6.1. *For  $k$  even, there exists a positive constant  $c$  such that*

$$(6.28) \quad \int_{\Pi^{ij}} \frac{x_1^k x_2^l}{h^m} d\mathbf{x} \leq c \int_{-R^{ij}}^{R^{ij}} h^{\frac{k}{2} + l + 1 - m} dx_1.$$

If  $k$  is odd, then

$$(6.29) \quad \int_{\Pi^{ij}} \frac{x_1^k x_2^l}{h^m} d\mathbf{x} = 0.$$

Moreover, for any positive integer  $p$ , we have

$$(6.30) \quad \int_{-R^{ij}}^{R^{ij}} \frac{dx_1}{(\delta^{ij} + x_1^2/a^{ij})^p} = O \left( \left( \frac{a^{ij}}{\delta^{ij}} \right)^{p - \frac{1}{2}} \right).$$

*Proof.* To prove (6.30), we write  $\int_{-R^{ij}}^{R^{ij}} (\delta^{ij} + x_1^2/a^{ij})^{-p} dx_1 = \mathcal{I}_1 + \mathcal{I}_2$ , where

$$\mathcal{I}_1 = \int_{-\sqrt{\delta^{ij}}}^{\sqrt{\delta^{ij}}} \frac{dx_1}{(\delta^{ij} + x_1^2/a^{ij})^p}.$$

Scaling  $x_1$  by  $\sqrt{\delta^{ij}}$ , we get

$$\mathcal{I}_1 = (\delta^{ij})^{1/2-p} \int_{-1}^1 \frac{dt}{(1 + t^2/a^{ij})^p} = c_1(p, a^{ij})(\delta^{ij})^{1/2-p},$$

where  $c_1$  is independent of  $\delta^{ij}$ . For  $\mathcal{I}_2$ , we have

$$\begin{aligned} \mathcal{I}_2 &= 2 \int_{\sqrt{\delta^{ij}}}^{R^{ij}} \frac{dx_1}{(\delta^{ij} + x_1^2/a^{ij})^p} \leq 2 \int_{\sqrt{\delta^{ij}}}^{R^{ij}} \left(\frac{a^{ij}}{x_1^2}\right)^p dx_1 \\ &= \frac{2}{2p-1} [(\delta^{ij})^{\frac{1}{2}-p} - (R^{ij})^{1-2p}] \leq c_2(p)(\delta^{ij})^{1/2-p} \end{aligned}$$

and the proof of (6.30) is complete. Identity (6.29) follows immediately because the integrand is an odd function of  $x_1$ . Finally, (6.29) and  $x_1^2/a^{ij} < h(x_1)$  imply (6.28).  $\square$

In light of Lemma 6.1, we obtain with explicit calculations that (6.20) holds and, therefore,

$$(6.31) \quad \int_{\Pi^{ij}} F(\mathcal{S}) d\mathbf{x} = W_{\Pi^{ij}}(\mathbf{u}) + O\left(\sqrt{\frac{a^{ij}}{\delta^{ij}}}\right).$$

In the next section, we extend  $\mathcal{S}$  from  $\Pi^{ij}$  to  $\partial D^{(i)}$  and  $\partial D^{(j)}$  in such a way that the net force and torque on  $D^{(i)}$  and  $D^{(j)}$  vanish. For that purpose, we need to examine the integrals of  $\mathbf{S}\mathbf{n}$  over various parts of  $\partial\Pi^{ij}$ . We show that, roughly speaking, the integrals of  $\mathbf{S}\mathbf{n}$  over opposite sides of  $\partial\Pi^{ij}$  cancel each other. To make this precise, let us denote the lateral parts of  $\partial\Pi^{ij}$  by

$$L_{\pm} = \left\{ (x_1, x_2) : x_1 = \pm R^{ij}, -\frac{1}{2}h(R^{ij}) < x_2 < \frac{1}{2}h(R^{ij}) \right\}.$$

PROPOSITION 6.2.

$$\int_{L_+} \mathbf{S}\mathbf{n} ds + \int_{L_-} \mathbf{S}\mathbf{n} ds = 0.$$

*Proof.* Since  $\mathbf{S}\mathbf{n} = (\mathcal{S}_{11}(\pm R^{ij}, x_2), \mathcal{S}_{12}(\pm R^{ij}, x_2))^T$  on  $L_{\pm}$ , we must show that

$$(6.32) \quad \int_{-\frac{h(R^{ij})}{2}}^{\frac{h(R^{ij})}{2}} \mathcal{S}_{1k}(-R^{ij}, x_2) dx_2 = \int_{-\frac{h(R^{ij})}{2}}^{\frac{h(R^{ij})}{2}} \mathcal{S}_{1k}(R^{ij}, x_2) dx_2 \quad \text{for } k = 1, 2.$$

This, in turn, follows by direct calculation from the expression of trial stress field  $\mathcal{S}$  constructed above.  $\square$

*Remark 6.2.* Since  $\text{div } \mathcal{S} = 0$  in  $\Pi^{ij}$ ,  $\int_{\partial\Pi^{ij}} \mathbf{S}\mathbf{n} ds = 0$  and, by Proposition 6.2, we have for the top and bottom parts of  $\partial\Pi^{ij}$ ,

$$(6.33) \quad \int_{\partial D^{(i)} \cap \overline{\Pi^{ij}}} \mathbf{S}\mathbf{n}^{(i)} ds = - \int_{\partial D^{(j)} \cap \overline{\Pi^{ij}}} \mathbf{S}\mathbf{n}^{(j)} ds.$$

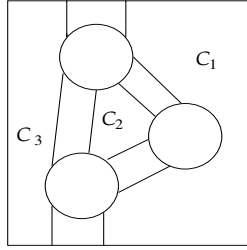


FIG. 7. Three-disk network. Connected components.

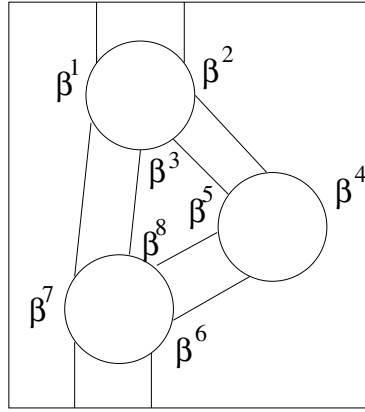


FIG. 8. Three-disk network. Designation of the vector integrals  $\beta^j$ .

**6.2.3. Extension of  $\mathcal{S}$  to the boundaries of the disks.** In section 6.2.2, we defined  $\mathcal{S}$  in  $\Pi^{ij}$  and, in particular, on  $\partial D^{(j)} \cap \overline{\Pi^{ij}}$ . Here, we wish to extend  $\mathcal{S}$  to the whole boundary  $\partial D^{(j)}$  in such a way that

$$(6.34) \quad \int_{\partial D^{(j)}} \mathcal{S} \mathbf{n}^{(j)} ds = \mathbf{0} \quad \text{and} \quad \int_{\partial \mathcal{C}} \mathcal{S} \mathbf{n} ds = \mathbf{0}$$

for any connected component  $\mathcal{C}$  of diffuse flow in  $U_E = \Omega_F \setminus U_\Pi$  and for all  $j = 1, \dots, N$ . We note that  $\partial D^{(j)} \cap \overline{U_E}$  is a union of circular, complementary arcs, and we let vectors  $\beta^k$  denote the unknown integrals of  $\mathcal{S} \mathbf{n}^{(j)}$  over various parts of  $\partial D^{(j)} \cap \overline{U_E}$  for  $1 \leq j \leq N$ . We begin by showing that there exist vectors  $\beta^k$  consistent with (6.34). This is done first for a simple, three-disk network and is generalized later to  $N$  disks. Then we construct  $\mathcal{S}$  on  $\partial D^{(j)} \cap \overline{U_E}$  for  $1 \leq j \leq N$  so the integral of  $\mathcal{S} \mathbf{n}^{(j)}$  over the  $k$ th complementary arc is equal to  $\beta^k$  for all  $k$ .

**Part I: A simple, three-disk network.** To simplify the presentation, let us begin by considering a simple three-disk network, as shown in Figure 7, where there are three connected regions  $\mathcal{C}_1$ ,  $\mathcal{C}_2$ , and  $\mathcal{C}_3$  of diffuse flow.

The unknown integrals of  $\mathcal{S} \mathbf{n}$  over the complementary arcs in  $\partial D^{(j)} \cap \overline{U_E}$  for  $j = 1, 2, 3$  are denoted by  $\beta^k$ ,  $1 \leq k \leq 8$  (see Figure 8). We also let  $\mathbf{F}^k$ , and  $\mathbf{B}^k$  for  $1 \leq k \leq 5$  be the known integrals of  $\mathcal{S} \mathbf{n}$  over the parts of  $\partial D^{(j)} \cap \overline{U_\Pi}$  and the lateral segments of the gaps, respectively (see Figure 9 and recall Proposition 6.2). Finally, for connected components  $\mathcal{C}_1$  and  $\mathcal{C}_3$ , we need  $\mathcal{S} \mathbf{n}$  on the exterior boundaries  $\partial \Omega \setminus \overline{U_\Pi}$  of the domain. On the vertical segments of the external boundary, we set  $\mathcal{S} = 0$ ,

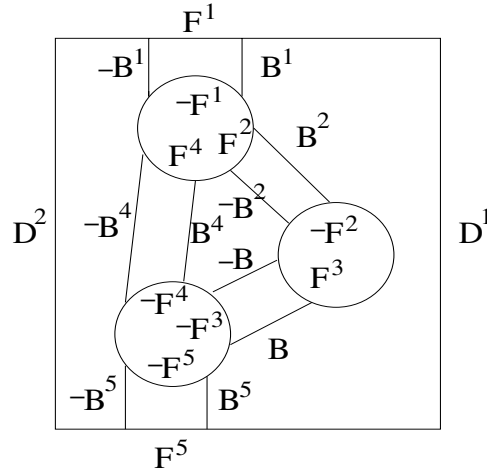


FIG. 9. Three-disk network. Right-hand sides.

and on the horizontal segments, we let  $\mathcal{S}$  be constant. Letting  $\mathbf{D}^1$  and  $\mathbf{D}^2$  be the net traction over  $\partial\mathcal{C}_3 \cap \partial\Omega$  and  $\partial\mathcal{C}_1 \cap \partial\Omega$ , respectively, we can now write (6.34) as

$$(6.35) \quad \mathbf{F}^1 + \mathbf{D}^1 + \mathbf{F}^5 + \mathbf{D}^2 = 0,$$

$$(6.36) \quad \begin{matrix} \beta^1 + & \beta^2 + & \beta^3 = & \mathbf{Q}^1, \\ \beta^4 + & & \beta^5 = & \mathbf{Q}^2, \\ \beta^6 + & \beta^7 + & \beta^8 = & \mathbf{Q}^3, \end{matrix} \quad \text{where} \quad \begin{matrix} \mathbf{Q}^1 = & -\mathbf{F}^2 & -\mathbf{F}^4 & +\mathbf{F}^1, \\ \mathbf{Q}^2 = & -\mathbf{F}^3 & +\mathbf{F}^2, \\ \mathbf{Q}^3 = & \mathbf{F}^3 & +\mathbf{F}^4 & +\mathbf{F}^5, \end{matrix}$$

and

$$(6.37) \quad \begin{matrix} \beta^2 + & \beta^4 + & \beta^6 = & \mathbf{Q}^4, \\ \beta^3 + & \beta^5 + & \beta^8 = & \mathbf{Q}^5, \\ \beta^1 + & & \beta^7 = & \mathbf{Q}^6, \end{matrix} \quad \text{where} \quad \begin{matrix} \mathbf{Q}^4 = & -\mathbf{B}^1 & -\mathbf{B}^2 & -\mathbf{B}^3 & -\mathbf{B}^5 & -\mathbf{D}^1, \\ \mathbf{Q}^5 = & -\mathbf{B}^4 & +\mathbf{B}^2 & +\mathbf{B}^3, \\ \mathbf{Q}^6 = & \mathbf{B}^1 & +\mathbf{B}^4 & +\mathbf{B}^5 & -\mathbf{D}^2. \end{matrix}$$

We now have an undetermined system of six vectorial equations (6.36), (6.37), with eight unknown vectors  $\beta^k$  for  $1 \leq k \leq 8$  with right-hand sides satisfying constraint (6.35).

PROPOSITION 6.3. *There exist solutions of the linear system of (6.36), (6.37).*

*Proof.* Note that the vector system (6.36), (6.37) is equivalent to two scalar systems with the same matrix for the components of  $\beta^j$ . It is therefore sufficient to prove the proposition for any one of the two scalar systems. The matrix  $A$  is

$$(6.38) \quad A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

and we denote its rows by  $\mathbf{a}_j$  for  $1 \leq j \leq 6$ . We call the first three rows in  $A$  disk-rows or simply  $d$ -rows, and the last three rows  $c$ -rows (in reference to the connected

components  $\mathcal{C}_j$ ). We observe that each arc in  $\partial D^{(j)} \cap \overline{U_E}$  belongs to exactly one disk and one connected component and thus exactly two entries in each column are equal to 1. Moreover, matrix  $A$  possesses the following two-ones property: one of these unit entries appears in a  $d$ -row and another appears in a  $c$ -row.

Next, we show that each equation in (6.36), (6.37) is a linear combination of the other five. Indeed, summing up first the equations in (6.36) and then those in (6.37), we obtain

$$\sum_{j=1}^8 \beta^j = \mathbf{F}_1 + \mathbf{F}_5 \quad \text{and} \quad \sum_{j=1}^8 \beta^j = -(\mathbf{D}_1 + \mathbf{D}_2),$$

respectively. These equations are consistent by (6.35), which gives  $\mathbf{F}_1 + \mathbf{F}_5 = -(\mathbf{D}_1 + \mathbf{D}_2)$ , and the rows of  $A$  are clearly linearly dependent.

Let us then eliminate from the original system of equations one equation, say, the first one, and show that the reduced system is solvable. Let  $A_R$  be the matrix of the reduced system. The rows of  $A_R$  are  $\mathbf{a}_j$  with  $2 \leq j \leq 6$ . We show next that the rows of  $A_R$  are linearly independent (i.e.,  $\text{rank} A_R = 5$ ), and then that the existence of solutions follows from standard linear algebra.

Arguing by contradiction, suppose that there exists a  $k$ , between 2 and 6, such that  $\mathbf{a}_k$  is a linear combination of  $\mathbf{a}_p$  for  $2 \leq p \leq 6$ ,  $p \neq k$ . Explicitly, we have

$$(6.39) \quad \mathbf{a}_k = \sum_{m=2, m \neq k}^6 \lambda_m \mathbf{a}_m,$$

where not all  $\lambda_m$  are zero. By the two-ones property, three rows  $\mathbf{a}_4, \mathbf{a}_5, \mathbf{a}_6$  have a unit entry at a column where all other rows of  $A_R$  have zeros. Take, for example, row  $\mathbf{a}_4$ . It has a unit entry in column 2, whereas the other four remaining rows  $\mathbf{a}_j, 2 \leq j \leq 6, j \neq 4$ , have zeros in this column. Hence, (6.39) implies that  $k \neq 4$ . The same argument shows that  $k \neq 5, k \neq 6$ . When  $k = 2$  or  $3$ , direct inspection of the first column of  $A_R$  shows that  $\lambda_6$  from (6.39) is zero. Similarly we obtain  $\lambda_4 = \lambda_5 = 0$ . Then (6.39) reduces to  $\mathbf{a}_2 = \lambda \mathbf{a}_3$ , which is impossible since  $\mathbf{a}_2$  and  $\mathbf{a}_3$  are linearly independent.  $\square$

**Part II: A general,  $N$  disk network with  $M$  connected components.**

Analogous to (6.36), we write the momentum balance equations on the boundary of each disk  $D^{(j)}$  for  $1 \leq j \leq N$ . We call these equations  $d$ -equations. Furthermore, analogous to (6.37), we write the equations for each connected component  $\mathcal{C}_p, 1 \leq p \leq M$ , where  $M \geq N$ . These are referred to as  $c$ -equations. As above, we consider the scalar system of  $N + M$  equations for the components of unknown vectors  $\beta^p, 1 \leq p \leq P$  ( $P = 8$  in the example with three disks). This linear system is referred to as the  $d$ - $c$ -system. Similar to the case of three disks, the right-hand side of the system involves integrals of  $\mathcal{S}\mathbf{n}$  over parts of  $\partial\Omega$  which do not belong to gaps. We assume that  $\mathcal{S}$  is extended to the external boundary  $\partial\Omega$  so that condition  $\int_{\partial\Omega} \mathcal{S}\mathbf{n} ds = \mathbf{0}$  holds.

The solvability of the  $d$ - $c$ -system is determined by matrix  $A$ , which has  $M + N$  rows  $\mathbf{a}_i, i = 1, \dots, M + N$ , and  $P$  columns. The rows of  $A$  that correspond to  $d$ -equations are called  $d$ -rows, and those remaining are called  $c$ -rows. The entries of  $A$  are again either 0 or 1. Since each complementary arc in  $\partial D^{(j)} \cap \overline{U_E}$  belongs to exactly one disk and one connected component, we observe that in each column of  $A$ , exactly two entries are equal to 1. One of these entries appears in a  $d$ -row and the other in a  $c$ -row. ( $A$  has the two-ones property.)



In what follows, we recall from section 3.1 that network  $\Gamma$  is a Delaunay graph corresponding to a Voronoi tessellation of  $\Omega$ . We restrict our attention to the case of large  $N$  (for technical reasons it is sufficient to have  $N \geq 3$ ) and consider only Voronoi tessellations with at least one Voronoi cell being strictly inside  $\Omega$ . We also make use of Properties 3.1 to 3.3 of  $\Gamma$ .

**THEOREM 6.1.** *The  $d$ - $c$ -system has a solution.*

*Proof.* First, we show that the  $d$ - $c$ -system is underdetermined (i.e.,  $P > M + N$ ). Indeed, by Property 3.2, at least two edges of  $\Gamma$  originate from each interior vertex (which is the center of some disk  $D^{(i)}$ ). Then  $P \geq 2N$ . Next, by Property 3.3, there exists a closed path which consists of interior edges. Therefore, there exists a connected component  $\mathcal{C}_j$  with its closure disjoint from  $\partial\Omega$  and, as such, there are at least three edges and three arcs in  $\partial\mathcal{C}_j$ . If the connected component would contain parts of  $\partial\Omega$ , there would be at least two arcs in its boundary. Thus<sup>6</sup>  $2M < P$  and, since  $P \geq 2N$ ,  $P > M + N$ .  $\square$

Next, we show that matrix  $A$  of the  $d$ - $c$ -system has linearly dependent rows. Indeed, similar to the case of three disks, we have that the sum of the  $d$ -equations is equal to the sum of  $k$ -equations. Then we eliminate the first equation in the  $d$ - $c$ -system and we denote by  $A_R$  the reduced  $(M + N - 1) \times P$  matrix. To finish the proof of the theorem, we now show that the reduced system is full rank.

**LEMMA 6.2.** *The rank of  $A_R$  is  $M + N - 1$ .*

*Proof.* We argue by contradiction. Assume that the rows of  $A_R$  are linearly dependent, that is, for some  $k > 1$ ,

$$(6.40) \quad \mathbf{a}_k = \sum_{m \neq k, m=2}^{M+N} \lambda_m \mathbf{a}_m,$$

where at least one  $\lambda_m$  is nonzero. The strategy of the proof is as follows. We introduce a multistep procedure where on each consecutive step  $l$  we have a set  $X_l$  of  $d$ -rows and a set  $Y_l$  of  $c$ -rows. We show that the rows from  $X_l \cup Y_l$  cannot appear on the left-hand side of (6.40). Furthermore, we show that if either of these rows are present in the right-hand side of (6.40), then the coefficients  $\lambda_m$  in front of these rows in (6.40) must be zero. The process is stopped after  $L$  steps, when either all  $d$ -rows are included in  $\cup_{l=1}^L X_l$  or all  $c$ -rows belong to  $\cup_{l=1}^L Y_l$ . At that point, (6.40) contains only  $d$ -rows (or only  $c$ -rows). Then the lemma follows from the linear independence of the  $d$ -rows and ( $c$ -rows), respectively.

Before giving the multistep procedure, let us introduce some notation. Given a collection of disks  $S = \{D^{(i_1)}, D^{(i_2)}, \dots, D^{(i_k)}\}$ , denote by  $\mathcal{C}(S)$  the set of all connected components of  $\Omega_F \setminus U_\Pi$  adjacent to a disk in  $S$ . Also, given a collection  $Q$  of connected components  $\mathcal{C}_j$ , denote by  $D(Q)$  the set of all disks having an arc in common with the boundary of an element of  $Q$ . Moreover, since there is a one-to-one correspondence between a disk and a  $d$ -row, use  $X_l$  to denote both the sets of disks and the corresponding sets of  $d$ -rows. Similarly, use the same notation for the set  $Y_l$  of connected components and the corresponding set of  $c$ -rows.

The multistep procedure is as follows.

*Step 1.* Set  $X_1 = D^{(1)}$  and  $Y_1 = \mathcal{C}(X_1)$ . The set  $Y_1$  consists of all connected components adjacent to  $D^{(1)}$ . We also identify  $X_1$  with the  $d$ -row  $\mathbf{a}_1$ . Recall that  $X_1$  is eliminated in the above reduction. The two-ones property implies that for each  $\mathbf{a}_j \in Y_1$ , there is a column of  $A_R$  with the only nonzero entry belonging to row  $\mathbf{a}_j$ .

<sup>6</sup>Note that, in fact, for large  $N$  and  $M$ , we have  $P > 3M - O(1)$  as  $M \rightarrow \infty$ .

This is the single-one property and it follows from the two-ones property after the elimination of  $X_1$ . This shows that if  $\mathbf{a}_k \in Y_1$ , it cannot appear in the left-hand side of (6.40) and so it appears in the right-hand side of (6.40) with coefficient  $\lambda^k = 0$ .

*Step 2.* Let  $X_2 = D(Y_1) \setminus X_1$  and observe that  $X_2$  consists of all disks except  $D^{(1)}$ , which have a part of the boundary in common with one of the connected components in  $Y_1$ . Then define  $Y_2 = \mathcal{C}(X_2) \setminus Y_1$ . The elements of  $Y_2$  are connected components which do not belong to  $Y_1$  and whose boundary intersects the boundary of some disk from  $X_2$ . Again, none of the vectors in  $X_2 \cup Y_2$  can be in the left-hand side of (6.40) and so they must be in the right-hand side of (6.40), with corresponding coefficients  $\lambda_m$  equal to zero.

*Step 3.* Define  $Y_l, X_l$  recursively by

$$X_l = D(Y_{l-1}) \setminus X_{l-1}, \quad Y_l = \mathcal{C}(X_l) \setminus Y_{l-1}.$$

The elements of  $X_l$  are disks that do not belong to  $X_{l-1}$  and whose boundary intersects the boundary of some connected component in  $Y_{l-1}$ . The set  $Y_l$  consists of the connected components which do not belong to  $Y_{l-1}$  and whose boundary intersects the boundary of some disk in  $X_l$ . Repeating the argument used in the previous step, we show that all corresponding  $\lambda_m$  must be zero.

By Property 3.2 of graph  $\Gamma$ , sets  $Y_l$  and  $X_l$  are nonempty, unless for some  $L$ ,  $Y_{L-1} = Y_L = \Omega_F \setminus U_\Pi$ , or  $X_L = X_{L-1} = \{D^{(1)}, \dots, D^{(N)}\}$ . Then we stop the process and note that the rows remaining in (6.40) are either all  $d$ -rows or all  $c$ -rows. By the two-ones property, we obtain that if a  $d$ -row has a unit entry in some column, the other  $d$ -rows have zeros in the same column. Hence all  $d$ -rows are linearly independent. The same reasoning yields linear independence of all  $c$ -rows. This means that by the time we stop the process, all vectors possibly remaining in (6.40) are linearly independent and that the coefficients  $\lambda_m$  in front of these rows must be zero. This finishes the proof of Lemma 6.2 and of Theorem 6.1.  $\square$

*Remark 6.3.* The above iterative procedure can be illustrated as follows. Remove a disk  $D^{(1)}$  from  $\Omega$ . This disk has adjacent connected components, say, three of them, if there are three edges originating from  $\mathbf{x}^{(1)}$ . Remove these connected components. Now, the just-removed connected components were adjacent to three disks (second generation of disks) which are neighbors of  $D^{(1)}$ . Remove the second generation of disks and consider the remaining connected components (second generation of connected components) adjacent to them. Remove the second generation of connected components. The remaining neighbors of second generation disks are called third generation disks. Remove them and consider the remaining connected components adjacent to third-generation disks. Continue removing objects from  $\Omega$  until there is nothing left. Due to the connectedness of the graph, the process does not stop until all the disks and all the connected components are removed.

**Part III. Extending  $\mathcal{S}$  to  $\partial D^{(j)} \cap \overline{U_E}$ .** We wish to define a trial tensor  $\mathcal{S}$  along the  $p$ th complementary arc in  $\partial D^{(j)} \cap \overline{U_E}$  such that its integral is equal to  $\beta^p$  for some  $1 \leq j \leq N$  and for  $1 \leq p \leq P$ . This ensures that conditions (6.34) hold and the existence of vectors  $\beta^p$  has been proved in Parts I and II. However, the trial stress tensor must also satisfy the balance of angular momentum

$$(6.41) \quad \int_{\partial D^{(j)}} \mathbf{n}^{(j)} \times \mathcal{S} \mathbf{n}^{(j)} ds = \mathbf{0}$$

for all  $1 \leq j \leq N$ . Let us then focus attention on one disk, say,  $D^{(j)}$ , of radius  $a_j$  centered at  $\mathbf{x}^{(j)}$ . At  $\partial D^{(j)}$ ,  $\mathbf{x} = \mathbf{x}^{(j)} + a_j \mathbf{n}^{(j)}$ , so we rewrite (6.41) as

$$\mathbf{x}_0 \times \int_{\partial D^{(j)}} \mathcal{S} \mathbf{n}^{(j)} ds + a_j \int_{\partial D^{(j)}} \mathbf{n}^{(j)} \times \mathcal{S} \mathbf{n}^{(j)} ds = \mathbf{0}.$$

Due to (6.34), the first integral is zero, and (6.41) reduces to

$$\int_{\partial D} \mathbf{n}^{(j)} \times \mathcal{S} \mathbf{n}^{(j)} ds = \mathbf{0}.$$

Let  $\boldsymbol{\tau}$  be the tangent unit vector at  $\partial D^{(j)}$ , pointing in the clockwise direction. Since  $\mathcal{S} \mathbf{n}^{(j)} = (\mathcal{S} \mathbf{n}^{(j)} \cdot \boldsymbol{\tau}) \boldsymbol{\tau} + (\mathcal{S} \mathbf{n}^{(j)} \cdot \mathbf{n}^{(j)}) \mathbf{n}^{(j)}$  and  $\boldsymbol{\tau} \cdot \mathbf{n}^{(j)} = 0$ , we have

$$0 = \int_{\partial D^{(j)}} \mathbf{n}^{(j)} \times \mathcal{S} \mathbf{n}^{(j)} ds = \mathbf{k} \int_{\partial D^{(j)}} \mathcal{S} \mathbf{n}^{(j)} \cdot \boldsymbol{\tau} ds,$$

where  $\mathbf{k} = \mathbf{n}^{(j)}(\mathbf{x}) \times \boldsymbol{\tau}(\mathbf{x})$  is a constant (independent of  $\mathbf{x}$ ) unit vector, orthogonal to the two-dimensional plane and pointing into it. Therefore, any tensor  $\mathcal{S}$  obeying the balance of angular momentum (6.41) satisfies

$$(6.42) \quad \int_{\partial D^{(j)} \cap \overline{U_E}} \mathcal{S} \mathbf{n}^{(j)} \cdot \boldsymbol{\tau} ds = - \int_{\partial D^{(j)} \cap \overline{U_\Pi}} \mathcal{S} \mathbf{n}^{(j)} \cdot \boldsymbol{\tau} ds.$$

Now, since  $\mathcal{S}$  is already defined in  $U_\Pi$ , we estimate the integral in the right-hand side of (6.42). In the local coordinates of gap  $\Pi^{ij}$ , a complementary arc in  $\partial D^{(j)} \cap \overline{\Pi^{ij}}$  is given by equation  $f(x_1, x_2) = x_2 - \delta^{ij}/2 - a_j + \sqrt{a_j^2 - x_1^2} = 0$ . Then

$$(6.43) \quad \mathbf{n}^{(j)} = \frac{1}{a_j} \begin{pmatrix} x_1 \\ -(a_j^2 - x_1^2)^{1/2} \end{pmatrix}, \quad \boldsymbol{\tau} = \frac{1}{a_j} \begin{pmatrix} (a_j^2 - x_1^2)^{1/2} \\ x_1 \end{pmatrix}.$$

Using the explicit expression of  $\mathcal{S}$  from section 6.2.2 and Lemma 6.1, we obtain

$$(6.44) \quad \int_{\partial D^{(j)} \cap \overline{\Pi^{ij}}} \mu \mathcal{E}(\mathbf{u}) \mathbf{n}^{(j)} \cdot \boldsymbol{\tau} ds = O\left(\sqrt{\frac{a^{ij}}{\delta^{ij}}}\right)$$

and

$$(6.45) \quad \int_{\partial D^{(j)} \cap \overline{\Pi^{ij}}} \mathcal{K} \mathbf{n}^{(j)} \cdot \boldsymbol{\tau} ds = O(1).$$

Let  $\mathcal{A}_{\Pi^{ij}}$  be a complementary arc from  $\partial D^{(j)} \cap \overline{U_E}$  adjacent to gap  $\Pi^{ij}$  and oriented in the clockwise direction. We wish to construct tensor  $\mathcal{S}$  on  $\mathcal{A}_{\Pi^{ij}}$  so that

$$(6.46) \quad \int_{\mathcal{A}_{\Pi^{ij}}} \mathcal{S} \mathbf{n}^{(j)} ds = \boldsymbol{\beta}$$

and

$$(6.47) \quad \int_{\mathcal{A}_{\Pi^{ij}}} \mathcal{S} \mathbf{n}^{(j)} \cdot \boldsymbol{\tau} ds = -\rho.$$

Here,  $\boldsymbol{\beta}$  is found by solving the  $d$ -c-system, and  $\rho$  stands for the sum of the integrals in (6.44), (6.45). Parameterize  $\mathcal{A}_{\Pi^{ij}}$  as follows:

$$(6.48) \quad \mathcal{A}_{\Pi^{ij}} = \{(x_1, x_2) \in \partial D^{(j)} : x_1 = a_j \cos t, x_2 = a_j \sin t, t \in [0, \alpha]\}.$$

Then we rewrite (6.46), (6.47) as

$$(6.49) \quad \begin{aligned} a \int_0^\alpha (\mathcal{S}_{11}(t) \cos t - \mathcal{S}_{12} \sin t) dt &= \beta_1, \\ a_j \int_0^\alpha (\mathcal{S}_{12}(t) \cos t - \mathcal{S}_{22} \sin t) dt &= \beta_2, \\ a_j \int_0^\alpha (\mathcal{S}_{11}(t) \cos t \sin t - \mathcal{S}_{22} \sin t \cos t) dt &= -\rho. \end{aligned}$$

To accomplish the task of this section, we now set the components  $\mathcal{S}_{kl}$  of our trial tensor on  $\mathcal{A}_{\Pi^{ij}}$  to constant values satisfying (6.49). This implies that

$$(6.50) \quad M \begin{pmatrix} \mathcal{S}_{11} \\ \mathcal{S}_{12} \\ \mathcal{S}_{13} \end{pmatrix} = \begin{pmatrix} \beta_1/a_j \\ \beta_2/a_j \\ -\rho/a_j \end{pmatrix},$$

where

$$M = \begin{pmatrix} \sin \alpha & \cos \alpha - 1 & 0 \\ 0 & \sin \alpha & \cos \alpha - 1 \\ 1/2 \sin^2 \alpha & 0 & -1/2 \sin^2 \alpha \end{pmatrix}$$

and  $\det(M) = \sin^2 \alpha \cos \alpha (1 - \cos \alpha)$ . Thus, unless  $\alpha = 0, \pi/2$  or  $\pi$ , (6.50) is uniquely solvable, and our extension of  $\mathcal{S}$  to the complementary arc  $\mathcal{A}_{\Pi^{ij}}$  is complete. Finally, all cases of  $\alpha$  that make  $M$  singular can be eliminated. Indeed,  $\alpha = 0$  is discarded by the observation that it implies an empty  $\mathcal{A}_{\Pi^{ij}}$ . The other cases,  $\alpha = \pi/2$  or  $\pi$ , can also be avoided by modifying the length of  $\mathcal{A}_{\Pi^{ij}}$ , i.e., by changing slightly the widths of gaps  $\Pi^{ij}$  adjacent to  $\mathcal{A}_{\Pi^{ij}}$ .

**6.2.4. Extension of  $\mathcal{S}$  in the set  $U_E$  of connected components.** The goal of this section is to extend  $\mathcal{S}$  outside the gaps in such a way that the dual dissipation rate in  $U_E$  is much smaller than  $O(\delta^{-3/2})$ . First, we show that the components of the extended  $\mathcal{S}$  at complementary arcs  $\mathcal{A}_{\Pi^{ij}} \in \partial D^{(j)}$  (see section 6.2.3) for  $1 \leq j \leq N$  and  $i \in \mathcal{N}_j$  are bounded, pointwise by  $C_{kl} \delta^{-1/2}$ . Then we consider the extension of  $\mathcal{S}$  from  $\mathcal{A}^+ = \overline{U_\Pi} \cap \partial \Omega^+$  and  $\mathcal{A}^- = \overline{U_\Pi} \cap \partial \Omega^-$  (the parts of  $\partial \Omega^\pm$  included in gaps) to the whole  $\partial \Omega$  and we prove the pointwise estimate  $|\mathcal{S}_{kl}| \leq C_{kl} \delta^{-1/2}$  for  $k, l = 1, 2$ . Finally, we extend  $\mathcal{S}$  in the interior of  $U_E$  and show that the dissipation rate there is at most  $O(\delta^{-1})$ . Once this is done, the first three steps in the outline of section 6.2.1 would be completed. The fourth step in section 6.2.1 is accomplished in section 6.2.6, where we give the main theorem of the paper.

**Part I: Estimates on the boundaries of the disks.** To prove the desired pointwise estimates of the components of  $\mathcal{S}$  at  $\partial D^{(i)}$  for  $1 \leq i \leq N$ , we need the following proposition.

PROPOSITION 6.4. *For each disk  $D^{(i)}$ , we have*

$$\sum_{j \in \mathcal{N}_i} \int_{\partial D^{(i)} \cap \overline{\Pi^{ij}}} \mathbf{S} \mathbf{n}^{(i)} ds = O(\delta^{-1/2}) \quad \text{for } i = 1, \dots, N.$$

This proposition states that if we fix a disk  $D^{(i)}$  and consider the forces that act on each arc in  $\partial D^{(i)} \cap \overline{U_\Pi}$ , then the sum of these forces over all the arcs is  $O(\delta^{-1/2})$ ,

whereas the force on each disk may be of order  $\delta^{-3/2}$ . Thus, we have a cancellation of terms due to the fact that the forces depend on translation velocities  $\mathbf{T}^{(i)}$ , the solutions of network equations (6.2). This is yet another manifestation of the global nature of the lower bound construction, which cannot be obtained by simply patching together trial functions obtained in each gaps  $\Pi^{ij}$ .

*Proof of Proposition 6.4.* Fix a gap  $\Pi^{ij}$  which joins disks  $D^{(i)}$  and  $D^{(j)}$  and use (6.22) to calculate

$$\int_{\Pi^{ij}} (\mu\Delta\mathbf{u} - \nabla P) \cdot \mathbf{u} d\mathbf{x} = \int_{\Pi^{ij}} \operatorname{div} (2\mu\mathcal{E}(\mathbf{u}) - P\mathcal{I}) \cdot \mathbf{u} d\mathbf{x},$$

where  $\mathbf{u}$  is defined by (6.6). The approximate pressure  $P$  is defined by (6.23). Integrating by parts, using the symmetry of  $\mathcal{E}$  and the incompressibility of  $\mathbf{u}$ , we have

$$(6.51) \quad \int_{\Pi^{ij}} (\mu\Delta\mathbf{u} - \nabla P) \cdot \mathbf{u} d\mathbf{x} = -2\mu \int_{\Pi^{ij}} \mathcal{E}(\mathbf{u}) \cdot \mathcal{E}(\mathbf{u}) d\mathbf{x} + \int_{\partial\Pi^{ij}} \mathcal{S}_0 \mathbf{n} \cdot \mathbf{u} ds.$$

Recall that  $\mu\Delta\mathbf{u} - \nabla P = \operatorname{div} \mathcal{S}_0 = \operatorname{div} \mathcal{K}$ , where  $\mathcal{K}$  is the compensating tensor defined in (6.25). Write  $\mathcal{S}_0 = \mathcal{S} + \mathcal{K}$  and integrate by parts to obtain

$$(6.52) \quad 2\mu \int_{\Pi^{ij}} \mathcal{E}(\mathbf{u}) \cdot \mathcal{E}(\mathbf{u}) d\mathbf{x} = \int_{\partial\Pi^{ij}} \mathcal{S} \mathbf{n} \cdot \mathbf{u} d\Gamma + \int_{\Pi^{ij}} \mathcal{K} \cdot \mathcal{E}(\mathbf{u}) d\mathbf{x}.$$

In section 6.2.2 we showed that  $\int_{\Pi} \mathcal{K} \cdot \mathcal{E}(\mathbf{u}) d\mathbf{x} = O(\delta^{-1/2})$ . Then rewrite the first integral in the right-hand side of (6.52) as

$$\begin{aligned} \int_{\partial\Pi^{ij}} \mathcal{S} \mathbf{n} \cdot \mathbf{u} ds &= \int_{\partial D^{(i)} \cap \partial\Pi^{ij}} \mathcal{S} \mathbf{n}^{(i)} \cdot \mathbf{u} ds + \int_{\partial D^{(j)} \cap \partial\Pi^{ij}} \mathcal{S} \mathbf{n}^{(j)} \cdot \mathbf{u} ds + \int_{\partial U_E \cap \partial\Pi^{ij}} \mathcal{S} \mathbf{n} \cdot \mathbf{u} ds \\ &= \int_{\partial D^{(i)} \cap \partial\Pi^{ij}} \mathcal{S} \mathbf{n}^{(i)} \cdot \mathbf{T}^{(i)} ds + \int_{\partial D^{(j)} \cap \partial\Pi^{ij}} \mathcal{S} \mathbf{n}^{(j)} \cdot \mathbf{T}^{(j)} ds + \int_{\partial U_E \cap \partial\Pi^{ij}} \mathcal{S} \mathbf{n} \cdot \mathbf{u} ds \\ &\quad + \int_{\partial D^{(i)} \cap \partial\Pi^{ij}} \mathcal{S} \mathbf{n}^{(i)} \cdot (\mathbf{u} - \mathbf{T}^{(i)}) ds + \int_{\partial D^{(j)} \cap \partial\Pi^{ij}} \mathcal{S} \mathbf{n}^{(j)} \cdot (\mathbf{u} - \mathbf{T}^{(j)}) ds. \end{aligned}$$

However, by Proposition 6.2, we have

$$\int_{\partial D^{(i)} \cap \partial\Pi^{ij}} \mathcal{S} \mathbf{n}^{(i)} \cdot \mathbf{u} ds = - \int_{\partial D^{(j)} \cap \partial\Pi^{ij}} \mathcal{S} \mathbf{n}^{(j)} \cdot \mathbf{u} ds,$$

and using the constructed  $\mathcal{S}$  and  $\mathbf{u}$  (see sections 6.1 and 6.2.2) and Lemma 6.1 gives

$$\int_{\partial\Pi^{ij}} \mathcal{S} \mathbf{n} \cdot \mathbf{u} ds = (\mathbf{T}^{(i)} - \mathbf{T}^{(j)}) \cdot \int_{\partial D^{(i)} \cap \partial\Pi^{ij}} \mathcal{S} \mathbf{n}^{(i)} ds + O(\delta^{-1/2}).$$

Finally, combining this with (6.52) yields

$$(6.53) \quad W_{\Pi^{ij}}(\mathbf{u}) = \frac{1}{2}(\mathbf{T}^{(i)} - \mathbf{T}^{(j)}) \cdot \int_{\partial D^{(i)} \cap \partial\Pi^{ij}} \mathcal{S} \mathbf{n}^{(i)} ds + O(\delta^{-1/2}).$$

Next, recall that  $W_{\Pi^{ij}}(\mathbf{u})$  in the left-hand side of (6.53) is a quadratic form in  $\mathbf{T}^{(i)} - \mathbf{T}^{(j)}$  for  $i = 1, \dots, N$  and  $j \in \mathcal{N}_i$  (see section 6.1). Denote the matrix of this

quadratic form by  $A(\delta)$ . From the definition (6.6) of  $\mathbf{u}$ , it follows that  $\mathcal{S}$  is a linear function of  $\mathbf{T}^i - \mathbf{T}^j$ , so we write

$$(6.54) \quad \frac{1}{2} \int_{\partial D^{(i)} \cap \partial \Pi^{ij}} \mathbf{S} \mathbf{n}^{(i)} ds = B(\delta)(\mathbf{T}^{(i)} - \mathbf{T}^{(j)}),$$

where the matrix  $B(\delta)$  is independent of  $\mathbf{T}^i, \mathbf{T}^j$ . Then the first term in the right-hand side of (6.53) is a quadratic form in  $\mathbf{T}^{(i)} - \mathbf{T}^{(j)}$ . Replacing the terms in (6.53) with the corresponding quadratic forms, we obtain

$$(6.55) \quad (\mathbf{T}^{(i)} - \mathbf{T}^{(j)}) \cdot A(\delta)(\mathbf{T}^{(i)} - \mathbf{T}^{(j)}) = (\mathbf{T}^{(i)} - \mathbf{T}^{(j)}) \cdot B(\delta)(\mathbf{T}^{(i)} - \mathbf{T}^{(j)}) + O(\delta^{-1/2}).$$

Summing up over all disks  $D^{(i)}, i = 1, \dots, N$ , and then differentiating with respect to the components of a fixed vector  $\mathbf{T}^{(i)}$ , we have

$$(6.56) \quad \sum_{j \in \mathcal{N}_i} A(\delta)(\mathbf{T}^{(i)} - \mathbf{T}^{(j)}) = \sum_{j \in \mathcal{N}_i} B(\delta)(\mathbf{T}^{(i)} - \mathbf{T}^{(j)}) + O(\delta^{-1/2})$$

for each disk  $D^{(i)}, i = 1, \dots, N$ . However, by the network equations (6.2), the left-hand side in (6.56) is zero and so

$$0 = \sum_{j=1}^{J_i} B(\delta)(\mathbf{T}^i - \mathbf{T}^j) + O(\delta^{-1/2}).$$

This completes the proof of Proposition 6.4.  $\square$

**Part II: Controlled extension of  $\mathcal{S}$  to  $\partial\Omega$ .** In this step, we deal with the extension of  $\mathcal{S}$  from  $\mathcal{A}^\pm$  to  $\partial\Omega$ .

PROPOSITION 6.5.

$$\int_{\mathcal{A}^+} \mathbf{S} \mathbf{n} ds + \int_{\mathcal{A}^-} \mathbf{S} \mathbf{n} ds = O(\delta^{-1/2}).$$

*Proof.* Since  $\mathcal{S}$  is divergence free in each gap  $\Pi^{ij}$ , we have

$$(6.57) \quad \sum_{ij} \int_{\partial \Pi^{ij}} \mathbf{S} \mathbf{n} ds = \mathbf{0},$$

where the sum is taken over all gaps. If a gap is connected to  $\partial\Omega$ , its boundary consists of a segment from  $\mathcal{A}^+$  or  $\mathcal{A}^-$ , an arc which belongs to one of the disks, and two lateral segments. The boundary of an interior gap contains two disk arcs and two lateral segments. By Proposition 6.2, the sum of the integrals over the lateral parts of  $\partial \Pi^{ij}$  is zero. Thus (6.57) reduces to

$$\int_{\mathcal{A}^+ \cup \mathcal{A}^-} \mathbf{S} \mathbf{n} ds + \sum_{j=1}^N \int_{\partial D^{(j)} \cap \bar{U}_\Pi} \mathbf{S} \mathbf{n} ds = 0,$$

where the sum is taken over all disks. By Proposition 6.4,

$$\sum_{j=1}^N \int_{\partial D^{(j)} \cap \bar{U}_\Pi} \mathbf{S} \mathbf{n} ds = O(\delta^{-1/2}),$$

and Proposition 6.5 follows.  $\square$

To obtain pointwise estimates on the complementary arcs, we restrict our attention to the example of a three-disk network from Figure 7 and to the corresponding algebraic system (6.36), (6.37). Generalizing our arguments to the general case of  $N$  disks is straightforward.

Let us denote the vector  $\mathbf{F}^1 + \mathbf{F}^5$  by  $-\mathbf{P}$ . By Proposition 6.5,  $\mathbf{P} = O(\delta^{-1/2})$ . Define  $\mathcal{S} = 0$  on the lateral part of  $\partial\Omega$ . On  $\partial\Omega^+ \setminus \mathcal{A}^+$  (or  $\partial\Omega^- \setminus \mathcal{A}^-$ ), we choose the constant components  $\mathcal{S}_{11} = 0$ ,

$$\mathcal{S}_{12} = \pm \frac{1}{2} \frac{\mathbf{P}_1}{|\partial\Omega^+ \setminus \mathcal{A}^+|}, \quad \mathcal{S}_{22} = \pm \frac{1}{2} \frac{\mathbf{P}_2}{|\partial\Omega^+ \setminus \mathcal{A}^+|},$$

where  $|\cdot|$  denotes the length of a curve. Then

$$\int_{\partial\Omega^+ \setminus \mathcal{A}^+} \mathcal{S} \mathbf{n} \, ds = \int_{\partial\Omega^- \setminus \mathcal{A}^-} \mathcal{S} \mathbf{n} \, ds = \frac{1}{2} \mathbf{P}, \quad \int_{\partial\Omega} \mathcal{S} \mathbf{n} \, ds = 0,$$

and

$$(6.58) \quad \sup_{\partial\Omega \setminus (\mathcal{A}^+ \cup \mathcal{A}^-)} |\mathcal{S}_{kl}| \leq c_{kl} \delta^{-1/2}, \quad k, l = 1, 2,$$

with  $c_{kl}$  independent of  $\delta$ . Thus, we can define  $\mathcal{S}$  on  $\partial\Omega \setminus (\mathcal{A}^+ \cup \mathcal{A}^-)$  so that (6.35) is satisfied and

$$(6.59) \quad \mathbf{D}^1 = O(\delta^{-1/2}), \quad \mathbf{D}^1 = O(\delta^{-1/2}).$$

From the definitions of  $\mathbf{u}, P$ , and  $\mathcal{S}$ , it follows that  $|\mathcal{S}_{kl}|$  are pointwise bounded independent of  $\delta$  on the lateral parts of the gap boundaries. Hence,

$$(6.60) \quad \mathbf{B}^j = O(1) \quad \text{for } 1 \leq j \leq 4.$$

Moreover, by Proposition 6.4,

$$(6.61) \quad -\mathbf{F}^1 + \mathbf{F}^3 + \mathbf{F}^4 = O(\delta^{-1/2}), \quad -\mathbf{F}^2 + \mathbf{F}^3 = O(\delta^{-1/2}), \quad -\mathbf{F}^4 - \mathbf{F}^3 - \mathbf{F}^5 = O(\delta^{-1/2}),$$

and, combining (6.59)–(6.61), we see that the components of the right-hand side of the algebraic system (6.36), (6.37) are bounded by  $c\delta^{-1/2}$  with  $c$  independent of  $\delta$ . Since the matrix of this algebraic system is independent of  $\delta$  as well, we can choose a solution of (6.36), (6.37) so that all its components are bounded by  $c\delta^{-1/2}$  with  $c$  independent of  $\delta$ . This means that for each complementary arc in  $\partial D^{(j)} \cap \partial U_E$ ,  $1 \leq j \leq 8$ ,  $\int_{\partial D^{(j)} \cap \partial U_E} \mathcal{S} \mathbf{n} \, ds = O(\delta^{-1/2})$ . The latter implies that the right-hand side of algebraic system (6.50) is bounded by  $c\delta^{-1/2}$  and, since matrix  $M$  is independent of  $\delta$  and invertible, we can find a stress field  $\mathcal{S}$  at the boundaries of the disks which is bounded by  $c\delta^{-1/2}$  with  $c$  independent of  $\delta$ . Then for each complementary arc, we have

$$(6.62) \quad \sup_{\partial D^{(j)} \cap \partial U_E} |\mathcal{S}_{kl}| \leq c_{ij} \delta^{-1/2} \quad \text{for } 1 \leq j \leq N$$

with  $c_{ij}$  independent of  $\delta$ .

The boundary of each connected component  $\mathcal{C}_j$  of  $U_E$  consists of complementary arcs, pieces of the external boundary  $\partial\Omega$ , and lateral parts of gap boundaries. Thus

the estimates (6.58), (6.62) and the uniform estimates on the lateral segments yield the following.

PROPOSITION 6.6. *For each connected component  $\mathcal{C}_j$  of  $U_E$ , we have*

$$\sup_{\partial\mathcal{C}_j} |\mathcal{S}_{kl}| \leq c_j \delta^{-1/2}, \quad k, l = 1, 2,$$

with  $c_j$  independent of  $\delta$ .

**Part III: Extension from the boundary, to the connected components.**

Now,  $\mathcal{S}$  is defined on the boundary of each connected component  $\mathcal{C}_j$  of  $U_E = \Omega_F \setminus U_\Pi$ . Moreover, we have

$$(6.63) \quad \int_{\partial\mathcal{C}_j} \mathcal{S} \mathbf{n} \, ds = 0$$

and

$$(6.64) \quad \sup_{\partial\mathcal{C}_j} |\mathcal{S}(\mathbf{x})| \leq c \delta^{-1/2}$$

with  $c$  independent of  $\delta$ . Next, we construct a divergence-free extension of  $\mathcal{S}$  from  $\partial U_E$  to  $U_E$ .

PROPOSITION 6.7. *Let  $\mathcal{C}_j \subset U_E$  be a connected component, and let  $\mathcal{S}$  be the trial tensor defined on  $\partial\mathcal{C}_j$  satisfying (6.63), (6.64). Then there exists an extension  $\hat{\mathcal{S}} \in L^2(\mathcal{C}_j)$  in  $\mathcal{C}_j$  such that  $\operatorname{div} \hat{\mathcal{S}} = 0$  in  $\mathcal{C}_j$ ,  $\hat{\mathcal{S}} = \mathcal{S}$  on  $\partial\mathcal{C}_j$  and*

$$\int_{\mathcal{C}_j} \hat{\mathcal{S}} \cdot \hat{\mathcal{S}} \, ds \leq c \delta^{-1}$$

with  $c$  independent of  $\delta$ .

This proposition is proved using the same techniques as those in section 5.2.<sup>7</sup>

**6.2.5. Estimates of the dual dissipation functional in the gaps.** In our proof of the main theorem of the paper (see section 6.2.6), we use the following estimate.

PROPOSITION 6.8. *Let  $\mathbf{u}$  be defined by (6.6) and let  $\mathcal{S}$  be the trial tensor defined by (6.17) and (6.23)–(6.25). Then*

$$(6.65) \quad W_{U_\Pi}^*(\mathcal{S}) = W_{U_\Pi}(\mathbf{u}) + O(\delta^{-1/2}).$$

*Proof.* By (6.8), we have  $W_{U_\Pi}^* = \int_{\partial\Omega \cap \bar{U}_\Pi} \mathbf{g} \cdot \mathcal{S} \mathbf{n} \, ds - \int_{U_\Pi} F(\mathcal{S}) \, d\mathbf{x}$ . Let us then estimate first the boundary integral. Fix a gap  $\Pi$  and consider  $\int_{\partial\Pi} \mathcal{S} \mathbf{n} \cdot \mathbf{u} \, ds$ . Integrating by parts using the definition of  $\mathcal{S}$ , the incompressibility of  $\mathbf{u}$ , and (6.18), we get

$$(6.66) \quad \int_{\partial\Pi} \mathcal{S} \mathbf{n} \cdot \mathbf{u} \, ds = 2W_\Pi(\mathbf{u}) - \int_\Pi \mathcal{K} \cdot \mathcal{E}(\mathbf{u}) \, d\mathbf{x}.$$

The integral  $\int_\Pi \mathcal{K} \cdot \mathcal{E}(\mathbf{u}) \, d\mathbf{x}$  is estimated using the explicit expressions of  $\mathcal{K}$  and  $\mathcal{E}(\mathbf{u})$  and then by applying Lemma 6.1. These calculations, already carried out in section

<sup>7</sup>See also Appendix C in the preprint version of this article, available at <http://www.math.wsu.edu/math/faculty/panchenko/welcome.html>.



6.2.2 show that the integral in the right-hand side of (6.66) is  $O(\delta^{-1/2})$ . Summing up over all gaps, we have

$$(6.67) \quad \int_{\partial U_{\Pi}} \mathbf{S}\mathbf{n} \cdot \mathbf{u} ds = 2W_{U_{\Pi}}(\mathbf{u}) + O(\delta^{-1/2}).$$

Since  $\partial U_{\Pi}$  is a union of circular arcs, lateral segments that belong to the gap boundaries, and a set  $(\partial\Omega \cap \bar{U}_{\Pi}) \subset \partial\Omega$ , we write

$$(6.68) \quad \begin{aligned} \int_{\partial\Omega \cap \bar{U}_{\Pi}} \mathbf{g} \cdot \mathbf{S}\mathbf{n} ds &= \int_{\partial U_{\Pi}} \mathbf{u} \cdot \mathbf{S}\mathbf{n} ds \\ &- \sum_{j=1}^N \mathbf{T}^{(j)} \cdot \int_{\partial D^{(j)} \cap \bar{U}_{\Pi}} \mathbf{S}\mathbf{n} ds \\ &- \int_{\partial U_{\Pi} \setminus (\cup_j \partial D^{(j)})} \mathbf{u} \cdot \mathbf{S}\mathbf{n} ds + O(\delta^{-1/2}). \end{aligned}$$

Here, we use the same technique that gave (6.53) from (6.52). By Proposition 6.4, the sum of the second and third terms in the right-hand side of (6.68) is  $O(\delta^{-1/2})$ . Hence,

$$(6.69) \quad \int_{\partial\Omega \cap \bar{U}_{\Pi}} \mathbf{g} \cdot \mathbf{S}\mathbf{n} ds = \int_{\partial U_{\Pi}} \mathbf{u} \cdot \mathbf{S}\mathbf{n} ds + O(\delta^{-1/2}).$$

Combining (6.67) and (6.69), we obtain

$$(6.70) \quad \int_{\partial\Omega \cap \bar{U}_{\Pi}} \mathbf{g} \cdot \mathbf{S}\mathbf{n} ds = 2W_{U_{\Pi}}(\mathbf{u}) + O(\delta^{-1/2}).$$

Finally, to estimate the second integral in the definition of  $W_{U_{\Pi}}^*$ , we apply section 6.14 and sum up over all gaps:

$$(6.71) \quad \int_{U_{\Pi}} F(\mathcal{S}) d\mathbf{x} = W_{U_{\Pi}}(\mathbf{u}) + O(\delta^{-1/2}).$$

The estimate (6.65) follows from (6.70) and (6.71).  $\square$

We remark here that construction of the lower bound which accounts for rotations (with the error term  $O(1)$ ) requires developing more sophisticated techniques even for periodic densely packed arrays, and this will be addressed elsewhere.

**6.2.6. The main theorems.** The trial field for the upper bound (6.7) is constructed by patching up the local approximate solutions (6.6), which depend on the translational particle velocities  $\mathbf{T}^{(i)}$ ,  $i = 1, \dots, N$ , minimizing the quadratic functional

$$(6.72) \quad \begin{aligned} Q = \sum_{i=1}^N \sum_{\substack{j \in \mathcal{N}_i \\ j < i}} \left\{ \left[ \frac{3\pi\mu}{4} \left( \frac{a}{\delta^{ij}} \right)^{\frac{3}{2}} + \frac{27\pi\mu}{10} \sqrt{\frac{a}{\delta^{ij}}} \right] [(\mathbf{T}^{(i)} - \mathbf{T}^{(j)}) \cdot \mathbf{q}^{ij}]^2 \right. \\ \left. + \frac{\pi\mu}{2} \sqrt{\frac{a}{\delta^{ij}}} [(\mathbf{T}^{(i)} - \mathbf{T}^{(j)}) \cdot \mathbf{p}^{ij}]^2 \right\} \\ + \sum_{i \in \mathcal{B}} \left\{ \left[ \frac{3\pi\mu}{4} \left( \frac{2a}{\delta^i} \right)^{\frac{3}{2}} + \frac{27\pi\mu}{10} \sqrt{\frac{2a}{\delta^i}} \right] [(\mathbf{T}^{(i)} - \mathbf{g}) \cdot \mathbf{q}^i]^2 \right. \\ \left. + \frac{\pi\mu}{2} \sqrt{\frac{2a}{\delta^i}} [(\mathbf{T}^{(i)} - \mathbf{g}) \cdot \mathbf{p}^i]^2 \right\}. \end{aligned}$$

In this section we introduce the corresponding dissipation rate

$$(6.73) \quad E_2 = \min_{\mathbf{T}^{(i)}, i=1, \dots, N} Q = Q(\mathbf{T}_{min}^{(i)}, i=1, \dots, N).$$

By Proposition 6.1, the minimizing collection of vectors  $\mathbf{T}_{min}^{(i)}$  (solving the system (6.2)) is unique.

Since the error terms appearing in the construction of the lower bound are of order  $\delta^{-1}$  (recall Proposition 6.7), we need to make sure that  $E_2 \geq c\delta^{-3/2}$  with  $c$  independent of  $\delta$ . So far, we know from the upper bound in section 6.1 that if  $(\mathbf{T}^{(i)} - \mathbf{T}^{(j)})_2 \neq 0$ , the local dissipation rate in each gap  $\Pi^{ij}$  blows up as  $\delta^{-3/2}$ . Otherwise, the rate of growth is at most  $\delta^{-1/2}$ . The vectors  $\mathbf{T}_{min}^{(i)}$  are solutions of a (large) system of network equations (6.2) and, until these are solved, we cannot say whether the quantities  $((\mathbf{T}_{min}^{(i)} - \mathbf{T}_{min}^{(j)}) \cdot \mathbf{q}^{ij})^2$  vanish as  $\delta \rightarrow 0$ . That is, we cannot determine the global rate of blow up of  $E_2$  as  $\delta \rightarrow 0$ . In the scalar case of electrical conduction, it has been shown in [5] that for all connected graphs, the total energy blows up at the same rate as the energy in each gap. In the vectorial case considered here, connectivity is not sufficient to ensure the analogous property. The global rate of blow up depends on other geometrical characteristics of a connected graph (e.g., the coordination number; see [7] for details).

The functional  $Q$  depends on the interparticle distances  $\delta^{ij} = \delta d^{ij}$ , where the rescaled distances  $d^{ij}$  do not depend on  $\delta$ , and  $0 < c \leq d^{ij} \leq 1$  for all pairs of neighboring disks. To study asymptotic behavior of  $Q$  as  $\delta \rightarrow 0$ , we factor out the powers of  $\delta$  and write

$$(6.74) \quad Q(\mathbf{T}^{(1)}, \dots, \mathbf{T}^{(N)}) = \delta^{-3/2} \widehat{Q}(\mathbf{T}^{(1)}, \dots, \mathbf{T}^{(N)}) + \delta^{-1/2} Q'(\mathbf{T}^{(1)}, \dots, \mathbf{T}^{(N)}),$$

where the coefficients of  $Q$  and  $Q'$  do not depend on  $\delta$ , and

$$(6.75) \quad \widehat{Q} = \sum_{i=1}^N \sum_{\substack{j \in \mathcal{N}_i \\ j < i}} A^{ij} [(\mathbf{T}^{(i)} - \mathbf{T}^{(j)}) \cdot \mathbf{q}^{ij}]^2 + \sum_{i \in \mathcal{B}} A^i [(\mathbf{T}^{(i)} - \mathbf{g}) \cdot \mathbf{q}^i]^2$$

with

$$(6.76) \quad A^{ij} = \frac{3\pi\mu}{4} \left( \frac{a}{d^{ij}} \right)^{\frac{3}{2}}, \quad A^i = \frac{3\pi\mu}{4} \left( \frac{2a}{d^i} \right)^{\frac{3}{2}}.$$

Note that our boundary conditions (2.9) correspond to  $\mathbf{g} = \pm \mathbf{e}_2$  on  $\partial\Omega^\pm$  and by Definition 3.3,  $\mathbf{q}^i = \pm \mathbf{e}_2$  on  $\partial\Omega^\pm$ . We keep the general notation in (6.75) because it may be applied to more general boundary conditions. In this section we use the rescaled dissipation rate

$$(6.77) \quad \widehat{E} = \min_{\mathbf{T}^{(i)}, i=1, \dots, N} \widehat{Q},$$

which does not depend on  $\delta$ , and the corresponding minimizers  $\widehat{\mathbf{T}}^{(i)}, i=1, \dots, N$ . From (6.72) and (6.74),

$$(6.78) \quad \delta^{-3/2} \widehat{E} \leq \delta^{-3/2} \widehat{Q}(\mathbf{T}_{min}^{(i)}) \leq E_2 = Q(\mathbf{T}_{min}^{(i)}) \leq Q(\widehat{\mathbf{T}}^{(i)}) = \delta^{-3/2} \widehat{E} + \delta^{-1/2} Q'(\widehat{\mathbf{T}}^{(i)}).$$

Since  $Q'$  and  $\widehat{\mathbf{T}}^{(i)}$  are independent of  $\delta$ , (6.78) and (6.73) yield  $\delta^{-3/2}\widehat{E} \leq E_2 \leq \delta^{-3/2}\widehat{E} + O(\delta^{-1/2})$  and thus

$$(6.79) \quad E_2 = \delta^{-3/2}\widehat{E} + O(\delta^{-1/2}).$$

Therefore, the inequality

$$(6.80) \quad \widehat{E} > 0$$

would imply  $E_2 = O(\delta^{-3/2})$  as  $\delta \rightarrow 0$ , and the leading term of the asymptotics of  $E_2$  would be determined by minimizing the  $\delta$ -independent functional  $\widehat{Q}$ .

In this paper, we consider a mathematical model for uniformly closely packed suspensions. For geometrical arrays of particles which correspond to such suspensions, the inequality (6.80) does hold. The detailed investigation of geometric properties of arrays for which (6.80) does or does not hold under various external boundary conditions is a subject of a separate investigation carried out in [7]. Here, we describe only one sufficient condition for validity of (6.80), discuss its physical relevance, and present an example which illustrates this condition.

Uniform, closely packed geometries can be modeled by the so-called densely packed quasi-triangular graphs. Roughly speaking, these are graphs such that each particle in the corresponding array has six neighbors, and the interparticle distances are uniformly small. More precisely, a quasi-triangular graph  $\Gamma$  is defined as follows. We start with a graph  $\Gamma'$  in  $\Omega$  such that the interior vertices of  $\Gamma'$  are points of the triangular periodic lattice. Then  $\Gamma$  is obtained by perturbing the locations of the vertices of  $\Gamma'$  in such a way that if two vertices were neighbors, they would remain neighbors. Moreover, a vertex of  $\Gamma'$  is connected to  $\partial\Omega$  if and only if the corresponding vertex of  $\Gamma$  is connected to  $\partial\Omega$ . More precisely, let  $\Gamma$  denote a network graph, and let  $\Gamma'$  be a graph corresponding to a periodic triangular lattice restricted to  $\Omega$ . We also define  $K$  and  $K'$  to be (topological) complexes associated with  $\Gamma$  and  $\Gamma'$ , respectively. We say that the graph  $\Gamma$  is quasi-triangular if  $K$  and  $K'$  are combinatorially equivalent. (The definition of combinatorial equivalence can be found, for instance, in [26, p. 4].)

To define the close packing condition for such graphs, recall that the interior vertices of  $\Gamma'$  are the centers of disks of radius  $a$  and that the corresponding periodic lattice is closely packed if the interparticle distance  $\delta = l - 2a \ll 1$ , where  $l$  denotes the length of an interior edge. For a densely packed quasi-triangular graph, we require that

$$(6.81) \quad \max_{ij} \delta^{ij} = \max_{ij} (l^{ij} - 2a) \ll 1,$$

where the maximum is taken over all pairs of neighbors and  $l^{ij}$  is the length of the corresponding interior edge of  $\Gamma$ .

In [7], we prove that (6.80) holds for a quasi-triangular graph under the close packing condition. An example of a network satisfying (6.80) is presented in the appendix.

We now formulate the main theorems.

**THEOREM 6.2.** *Let  $E$  be the dissipation rate (2.20) equal to the effective viscosity  $\langle \mu \rangle$  up to a constant normalizing factor (see (2.19)). Then as  $\delta \rightarrow 0$ ,*

$$(6.82) \quad E \leq E_2 + O(1)$$

and

$$(6.83) \quad E_2 + O(\delta^{-1}) \leq E,$$

where  $E_2$  is the minimum of the quadratic form (6.72).

THEOREM 6.3. For uniform, closely packed geometries such that condition (6.80) holds, the rescaled effective viscosity (dissipation rate)  $E$  defined by (2.20) has the asymptotic representation

$$(6.84) \quad E = \delta^{-3/2} \widehat{E} + O(\delta^{-1}) \quad \text{as } \delta \rightarrow 0,$$

where  $\widehat{E}$  is a minimum of the quadratic form  $\widehat{Q}$  defined by (6.75).

COROLLARY 6.1. Let  $\langle \mu \rangle$  be the effective viscosity defined by (2.19), and let the conditions of Theorem 6.3 hold. Then

$$(6.85) \quad \langle \mu \rangle = \frac{\widehat{E}}{\int_{\Omega} (\mathcal{E}(\mathbf{u}^0), \mathcal{E}(\mathbf{u}^0)) d\mathbf{x}} \delta^{-3/2} + O(\delta^{-1}) \quad \text{as } \delta \rightarrow 0,$$

where  $\mathbf{u}^0$  solves the Stokes equation  $\Delta \mathbf{u}^0 - \nabla P^0 = 0$  in  $\Omega$  with the boundary conditions (2.9) and (2.10).

*Proof of the Theorem 6.2.* Let us define the trial tensor  $\mathcal{S}$  in  $\Omega_F$  as follows. In each gap  $\Pi^{ij}$ , we use formula (6.17), and in each connected component  $\mathcal{C}_l$  of  $U_E = \Omega_F \setminus U_{\Pi}$ , we let  $\mathcal{S}$  be an extension from  $\partial \mathcal{C}_l$ , as given in Proposition 6.7. Note that, through our construction, we have ensured that  $\mathcal{S} \in \mathcal{F}$  and, as such, it is an admissible trial field for the dual variational problem (2.22). Furthermore, let  $\mathbf{u} \in \mathcal{U}$ , defined by (6.6), be the trial function for primal variational problem (2.20).

Let us evaluate the dual functional  $W_{\Omega_F}^*(\mathcal{S})$  defined in (6.8) and (6.9). Since  $\mathbf{S}\mathbf{n} = 0$  on  $\partial \Omega \setminus (\partial \Omega^+ \cup \partial \Omega^-)$ ,

$$W_{\Omega_F}^*(\mathcal{S}) = \int_{\partial \Omega^+ \cup \partial \Omega^-} \mathbf{g} \cdot \mathbf{S}\mathbf{n} ds - \int_{\Omega_F} F(\mathcal{S}) d\mathbf{x}.$$

First, we estimate the boundary integral. By Proposition 6.6,  $|\mathcal{S}| \leq c\delta^{-1/2}$  on  $(\partial \Omega^+ \cup \partial \Omega^-) \setminus \partial U_{\Pi}$ , and  $\mathbf{g}$  is independent of  $\delta$ . Hence,

$$(6.86) \quad \int_{\partial \Omega^+ \cup \partial \Omega^-} \mathbf{g} \cdot \mathbf{S}\mathbf{n} ds = \int_{(\partial \Omega^+ \cup \partial \Omega^-) \cap \partial U_{\Pi}} \mathbf{g} \cdot \mathbf{S}\mathbf{n} ds + O(\delta^{-1/2}).$$

Next, we use the notation from (5.11) to write

$$(6.87) \quad \int_{\Omega_F} F(\mathcal{S}) d\mathbf{x} = \int_{U_{\Pi}} F(\mathcal{S}) d\mathbf{x} + \int_{U_E} F(\mathcal{S}) d\mathbf{x}.$$

The second integral in the right-hand side of (6.87) is  $O(\delta^{-1})$  by Proposition 6.7. Using (6.8) with  $M = U_{\Pi}$  and taking into account the boundary conditions (2.10), we write

$$(6.88) \quad \int_{(\partial \Omega^+ \cup \partial \Omega^-) \cap \partial U_{\Pi}} \mathbf{g} \cdot \mathbf{S}\mathbf{n} ds - \int_{U_{\Pi}} F(\mathcal{S}) d\mathbf{x} = W_{U_{\Pi}}^*(\mathcal{S}),$$

and, combining (6.88) with (6.86) and (6.87), we obtain

$$(6.89) \quad W_{\Omega_F}^*(\mathcal{S}) = W_{U_{\Pi}}^*(\mathcal{S}) + O(\delta^{-1}).$$

Now Proposition 6.8 and (6.1), (6.72), and (6.73) imply

$$(6.90) \quad W_{\Omega_F}^*(\mathcal{S}) = W_{U_{\Pi}}(\mathbf{u}) + O(\delta^{-1}) = E_2 + O(\delta^{-1}).$$

Applying the direct and dual variational principles (2.20)–(2.21), (2.22)–(2.23) with the trial fields  $\mathbf{u}$  defined in (6.6) and  $\mathcal{S}$  defined in the beginning of the proof, we obtain

$$(6.91) \quad W_{\Omega_F}^*(\mathcal{S}) \leq E \leq W_{\Omega_F}(\mathbf{u}),$$

and the estimates (6.82) and (6.83) follow.  $\square$

*Proof of Theorem 6.3.* The inequalities (6.82) and (6.83) imply  $E = E_2 + O(\delta^{-1})$ . Together with (6.79), this yields  $E = \widehat{E}\delta^{-3/2} + O(\delta^{-1})$ , which gives the representation (6.84) provided (6.80) holds.  $\square$

**7. Summary.** In this paper we obtain and rigorously justify an asymptotic formula for the effective viscosity of a suspension of closely packed solid particles in a viscous Newtonian fluid. This formula accounts for variable distances between particles which form a nonperiodic (e.g., random) array.

The rigorous justification is presented in two dimensions. It is based on a construction of matching to the leading order lower and upper bounds by means of two, dual to each other, variational principles for the effective viscosity. The key point here is the construction of the lower bound, which accounts for all pairwise interactions between neighboring particles as well as for the incompressibility condition in the fluid domain. These interactions influence each other over the entire domain, leading to considerable difficulties in the construction of the corresponding trial function.

In both three and two dimensions, we obtain formal asymptotics formulas for the effective viscosity for nonperiodic arrays of particles of different sizes. For a particular case of a periodic array when identical particles move toward each other (along the line which joins their centers), the leading term in our formulas recovers the formal asymptotics previously obtained by [15, 16]. Our formulas also contain lower order terms which take into account the rotations and movements of adjacent particles in directions orthogonal to the axis of their centers. In our formal asymptotic analysis, we develop the corresponding generalization of the lubrication approximation.

While the previously obtained asymptotic formulas [15, 16, 27] capture the dependence of the effective viscosity on the volume fraction in a periodic array of closely packed particles, the network approximation proposed in this work also accounts for other geometrical characteristics such as variable distances between particles and the coordination number (the number of neighboring particles).

**Appendix. Proof of estimate (6.80).** Here we prove that (6.80) holds for the spring network corresponding to the graph in Figure 10.

**PROPOSITION 7.1.** *Let  $\widehat{Q}$  be the rescaled dissipation rate (6.75) corresponding to the network in Figure 10. Then  $\min \widehat{Q} > 0$ .*

*Proof.* The functional  $\widehat{Q}$  is of the form

$$(7.1) \quad \begin{aligned} \widehat{Q} = & A^{12}((\mathbf{T}^{(1)} - \mathbf{T}^{(2)}) \cdot \mathbf{q}^{12})^2 + A^{13}((\mathbf{T}^{(1)} - \mathbf{T}^{(3)}) \cdot \mathbf{q}^{13})^2 + A^{23}((\mathbf{T}^{(2)} - \mathbf{T}^{(3)}) \cdot \mathbf{q}^{23})^2 \\ & + A^{24}((\mathbf{T}^{(2)} - \mathbf{T}^{(4)}) \cdot \mathbf{q}^{24})^2 + A^1((\mathbf{T}^{(1)} - \frac{1}{2}\mathbf{e}_2) \cdot \mathbf{e}_2)^2 + A^2((\mathbf{T}^{(2)} - \frac{1}{2}\mathbf{e}_2) \cdot \mathbf{e}_2)^2 \\ & + A^3((\mathbf{T}^{(3)} + \frac{1}{2}\mathbf{e}_2) \cdot \mathbf{e}_2)^2 + A^4((\mathbf{T}^{(4)} + \frac{1}{2}\mathbf{e}_2) \cdot \mathbf{e}_2)^2, \end{aligned}$$

where  $A^{ij}, A^i, i, j = 1, 2, \dots, 4$ , are given by (6.76). We show that  $\min_{\mathbf{T}^{(i)}, i=1,2,\dots,4} \widehat{Q} > 0$ . Arguing by contradiction, assume that  $\min \widehat{Q} = 0$ . This is possible only if the minimizing set of vectors  $\mathbf{T}^{(i)}$  satisfies the system of equations

$$(7.2) \quad (\mathbf{T}^{(1)} - \mathbf{T}^{(2)}) \cdot \mathbf{q}^{12} = 0, \quad (\mathbf{T}^{(1)} - \mathbf{T}^{(3)}) \cdot \mathbf{q}^{13} = 0, \quad (\mathbf{T}^{(2)} - \mathbf{T}^{(3)}) \cdot \mathbf{q}^{23} = 0,$$

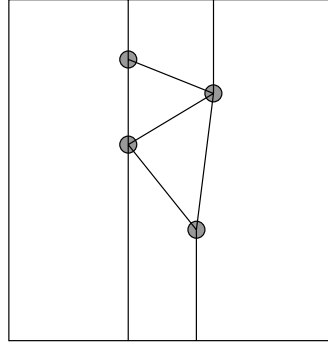


FIG. 10. A four-disk network.

$$(7.3) \quad (\mathbf{T}^{(2)} - \mathbf{T}^{(4)}) \cdot \mathbf{q}^{24} = 0, \quad (\mathbf{T}^{(3)} - \mathbf{T}^{(4)}) \cdot \mathbf{q}^{34} = 0,$$

$$(7.4) \quad \mathbf{T}^{(1)} \cdot \mathbf{e}_2 = \frac{1}{2}, \quad \mathbf{T}^{(2)} \cdot \mathbf{e}_2 = \frac{1}{2}, \quad \mathbf{T}^{(3)} \cdot \mathbf{e}_2 = -\frac{1}{2}, \quad \mathbf{T}^{(4)} \cdot \mathbf{e}_2 = -\frac{1}{2}.$$

To write this system of nine equations in a more compact form  $A\mathbf{z} = \mathbf{b}$ , introduce a  $1 \times 8$  vector of unknowns,

$$\mathbf{z} = (\mathbf{T}^{(1)}, \mathbf{T}^{(2)}, \mathbf{T}^{(3)}, \mathbf{T}^{(4)})^T.$$

The right-hand side of (7.2)–(7.4) is a  $1 \times 9$  vector  $\mathbf{b}$  that has the entries  $b_i = 0, i = 1, 2, \dots, 5, b_6 = b_7 = \frac{1}{2}, b_8 = b_9 = -\frac{1}{2}$ . Performing (partial) Gaussian elimination on the transpose of the augmented matrix  $(A | \mathbf{b})^T$ , we find that it is similar to the matrix

$$(7.5) \quad C = \begin{pmatrix} \hat{q}^{12} & \hat{q}^{13} & \hat{o} & \hat{o} & \hat{o} & \hat{e}_2 & \hat{o} & \hat{o} & \hat{o} \\ \hat{o} & \hat{q}^{13} & \hat{q}^{23} & \hat{q}^{24} & \hat{o} & \hat{e}_2 & \hat{e}_2 & \hat{o} & \hat{o} \\ \hat{o} & \hat{o} & \hat{o} & \hat{q}^{24} & \hat{q}^{34} & \hat{e}_2 & \hat{e}_2 & \hat{e}_2 & \hat{o} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \end{pmatrix}.$$

Here

$$(\hat{q}^{12} \quad \hat{q}^{13} \quad \hat{o} \quad \hat{o} \quad \hat{o} \quad \hat{e}_2 \quad \hat{o} \quad \hat{o} \quad \hat{o})$$

is a shorthand notation for two rows:

$$\mathbf{c}_1 = (q_1^{12} \quad q_1^{13} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0)$$

and

$$\mathbf{c}_2 = (q_2^{12} \quad q_2^{13} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0),$$

and, similarly, the other two boldfaced rows in (7.5) are the shorthand notation for the four rows  $\mathbf{c}_3, \dots, \mathbf{c}_6$ . The last two rows of  $C$  in (7.5) are denoted by  $\mathbf{c}_7, \mathbf{c}_8$ .

To show that  $\mathbf{c}_j, j = 1, 2, \dots, 8$ , are linearly independent, argue by contradiction. When  $\mathbf{c}_j$  are linearly dependent, there exist  $\lambda_j, j = 1, 2, \dots, 8$ , not all zero, such that

$$(7.6) \quad \sum_{j=1}^8 \lambda_j \mathbf{c}_j = 0.$$

Let us take the sixth and seventh columns in the vector equation (7.6). Then (7.6) yields the equations

$$(7.7) \quad \lambda_2 + \lambda_4 + \lambda_6 + \lambda_7 = 0 \quad \text{and} \quad \lambda_4 + \lambda_6 + \lambda_7 = 0.$$

Thus  $\lambda_2 = 0$ . Next, consider the first column in (7.6) to obtain  $\lambda_1 q_1^{12} + \lambda_2 q_2^{12} = 0$ . Note that  $q_1^{12}$  cannot be zero since the edge  $e^{12}$  connects the boundary vertices 1, 2, and thus cannot be vertical. Hence  $\lambda_1 = 0$ . Next, consider columns 2 and 3 in (7.6). Since  $\lambda_1 = \lambda_2 = 0$ , from (7.6) we obtain two equations for  $\lambda_3, \lambda_4$ :

$$(7.8) \quad \begin{aligned} \lambda_3 q_1^{13} + \lambda_4 q_2^{13} &= 0, \\ \lambda_3 q_1^{23} + \lambda_4 q_2^{23} &= 0. \end{aligned}$$

Since  $\mathbf{q}^{13}, \mathbf{q}^{23}$  are linearly independent,  $\lambda_3 = \lambda_4 = 0$ .

Consider columns four and five in (7.6). Since  $\lambda_3, \lambda_4$  are zero, we obtain two equations for  $\lambda_5, \lambda_6$ :

$$(7.9) \quad \begin{aligned} \lambda_5 q_1^{24} + \lambda_4 q_2^{24} &= 0, \\ \lambda_3 q_1^{34} + \lambda_4 q_2^{34} &= 0. \end{aligned}$$

Linear independence of  $\mathbf{q}^{24}, \mathbf{q}^{34}$  implies  $\lambda_5 = \lambda_6 = 0$ . Returning to (7.7), we see that  $\lambda_7 = 0$ . Finally, considering column 9, we obtain the equation for  $\lambda_8$ ,

$$(7.10) \quad \lambda_7 - \lambda_8 = 0,$$

which yields  $\lambda_8 = 0$ . Thus, all  $\lambda_j$  must be zero, and we arrive at a contradiction, which yields  $\text{rank}(A \mid \mathbf{b}) = 8$ . Applying the same Gaussian elimination to  $A^T$ , we see that  $\text{rank}(A) = 7$  and thus  $\text{rank}(A \mid \mathbf{b}) > \text{rank}(A)$ . This means that the system (7.2)–(7.4) has no solutions, and the minimum of  $\widehat{Q}$  must be positive.  $\square$

*Remark 7.1.* A quasi-triangular structure of the graph is sufficient for positivity of  $\widehat{Q}$ . The proof of the Proposition 7.1 shows that if a graph contains a triangulated path (see Figure 10), then  $\widehat{Q} > 0$  for the external boundary conditions (2.9), (2.10). We now explain heuristically why triangulation ensures positivity of  $\widehat{Q}$ . Start from vertices 1, 2 in Figure 10. They are connected by the nonvertical edge  $e^{12}$ , which implies  $\lambda_1 = 0$ . We next add vertex 3 and observe that it is connected to vertices 1, 2 by noncollinear edges  $e^{12}, e^{23}$ , which are adjacent sides of a triangle 123. The noncollinearity of these edges implies  $\lambda_3 = \lambda_4 = 0$ . Next, add vertex 4 to obtain triangle 234 and, as before, noncollinearity of the edges  $e^{34}, e^{42}$  implies  $\lambda_5 = \lambda_6 = 0$ . Finally, since  $e^{34}$  is nonvertical,  $\lambda_7 = \lambda_8 = 0$ . This argument also admits a straightforward generalization to a triangulated path of  $n$  vertices.

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