ON LOOP DELIGNE–LUSZTIG VARIETIES OF COXETER-TYPE FOR INNER FORMS OF GL_n

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Abstract. For a reductive group G over a local non-archimedean field K one can mimic the construction from classical Deligne–Lusztig theory by using the loop space functor. We study this construction in the special case that G is an inner form of GL_n and the "loop Deligne–Lusztig variety" is of Coxeter type. After simplifying the proof of its representability, our main result is that its ℓ-adic cohomology realizes many irreducible supercuspidal representations of G, notably almost all among those whose L-parameter factors through an unramified elliptic maximal torus of G. This gives a purely local, purely geometric and – in a sense – quite explicit realization of special cases of the local Langlands and Jacquet–Langlands correspondences.

1. Introduction

Let $\mathbf{G}$ be an inner form of $\mathbf{GL}_n$ ($n \geq 2$) over a local non-archimedean field K and let $G = \mathbf{G}(K)$ be the group of its K-points. Let $\mathbf{T} \subseteq \mathbf{G}$ be a maximal elliptic unramified torus. Then $\mathbf{T}$ is uniquely determined up to G-conjugation and $T = \mathbf{T}(K) \cong L^\times$ where L/K is the unramified extension of degree n. In [CI18] we constructed a scheme X over $\overline{\mathbb{F}}_q$ with an action by $G \times T$, which can be seen as an analogue over K of a Deligne–Lusztig variety attached to $\mathbf{T} \subseteq \mathbf{G}$. As in the classical Deligne–Lusztig theory [DL76], this allows to attach to a smooth character $\theta \colon T \to \overline{\mathbb{Q}}_\ell^\times$ the θ-isotypic component $R_T^G(\theta)$ of the ℓ-adic Euler characteristic of X, which is a smooth virtual G-representation. If θ is primitive (i.e., the Howe decomposition of θ has at most one member), we showed that $R_T^G(\theta)$ is irreducible supercuspidal and isomorphic to the representation attached to $(L/K, \theta)$ by Howe [How77], and hence provides a geometric and purely local realization of the local Langlands and Jacquet–Langlands correspondences.

These results indicate that X and more generally, another schemes obtained by similar Deligne–Lusztig-type constructions for other reductive groups over K allow a quite *explicit*, *purely local* and *purely geometric* way to realize the local Langlands correspondence and/or some instances of automorphic induction for at least those irreducible representations of $\mathbf{G}$, whose L-parameter factors through an unramified torus. This is highly desirable, as the existing local proofs of the local Langlands correspondence are purely algebraic (e.g. via Bushnell–Kutzko types), and the existing geometric proofs tend to be very inexplicit and/or use global arguments (except for [BW16], which – similar to [CI18] – only deals with primitive θ). Moreover, an exact analogue of the classical Deligne–Lusztig theory over non-archimedean local fields is highly interesting in its own right.

The first goal of the present article is to give a more satisfactory definition of X and simplify the proof of its representability. The second goal is to show that $R_T^G(\theta)$ is irreducible supercuspidal and realizes the local Langlands and Jacquet–Langlands correspondences for a much wider class of irreducible supercuspidal representations of $G(K)$ (almost all among those, whose L-parameter factors through $T \subseteq G$), thus going far beyond the corresponding results of [BW16] and [CI18]. As the methods from [CI18] for primitive θ do not apply anymore, our main concern here will be to develop new geometric methods to study the cohomology of Deligne–Lusztig constructions of Coxeter type over local fields, in particular generalizing results of [Lus04] away from the case when θ is regular, and providing nice description for the quotient of (subschemes of) X by unipotent radicals of rational parabolic subgroups of G, which generalizes (in the special case for $\mathbf{G}, \mathbf{T}$) to the situation over K particular results of [Lus76]. Some of these methods immediately work for all reductive groups, and some rely on $\mathbf{G}$ being an inner form of $\mathbf{GL}_n$.

To describe our result, we need more notation. First of all there is an unique integer $\kappa \in \{0, 1, \dots, n-1\}$, such that if $n' = \gcd(n, \kappa)$, $n = n'n_0$, $\kappa = n'k_0$, we have $G \cong \mathrm{GL}_{n'}(D_{k_0/n_0})$, where D_{k_0/n_0} denotes the central division algebra over K with Hasse-invariant k_0/n_0.

Let ε be any character of $K^\times$ with $\ker(\varepsilon) = \mathrm{N}_{L/K}(L^\times)$, the image of the norm map of L/K. Denote by

- $\mathscr{X}$ the set of smooth characters of $L^\times$ with trivial stabilizer in $\mathrm{Gal}(L/K)$,
- $\mathscr{G}_K^\varepsilon(n)$ the set of isomorphism classes of smooth n-dimensional representations σ of the Weil group $\mathcal{W}_K$ of K satisfying $\sigma \cong \sigma \otimes (\varepsilon \circ \mathrm{rec}_K)$,
- $\mathscr{A}_K^\varepsilon(n, \kappa)$ the set of smooth irreducible supercuspidal representations π of G ($= \mathbf{G}(K)$ with $\mathbf{G}$ corresponding to κ) such that $\pi \cong \pi \otimes (\varepsilon \circ \mathrm{Nrd}_G)$.

There are natural *bijections*

$$\begin{array}{ccccccc} \mathscr{X}\,/\,\mathrm{Gal}(L/K) & \longrightarrow & \mathscr{G}_K^\varepsilon(n) & \xrightarrow{\quad \mathrm{LL} \quad} & \mathscr{A}_K^\varepsilon(n,0) & \xrightarrow{\quad \mathrm{JL} \quad} & \mathscr{A}_K^\varepsilon(n,\kappa) \\ \theta & \longmapsto & \sigma_\theta & \longmapsto & \mathrm{LL}(\sigma_\theta) =: \pi_\theta^{\mathrm{GL}_n} & \longmapsto & \mathrm{JL}(\pi_\theta^{\mathrm{GL}_n}) =: \pi_\theta. \end{array}$$

The latter two maps are the local Langlands and the Jacquet–Langlands correspondences respectively. Here $\sigma_\theta := \mathrm{Ind}_{\mathcal{W}_L}^{\mathcal{W}_K}(\theta \cdot \mu)$ is the induction to $\mathcal{W}_K$ of the character $\mathcal{W}_L \to \mathcal{W}_L^{\mathrm{ab}} \stackrel{\mathrm{rec}_L}{\to} L^\times \stackrel{\theta\cdot\mu}{\to} \overline{\mathbb{Q}}_\ell^\times$, where μ is the *rectifier*, i.e. the unramified character of $L^\times$ defined by $\mu(\varpi) = (-1)^{n-1}$ (here ϖ uniformizer of L).

Our main result is the following theorem.

Theorem A. *Assume that $p > n$. Let $\theta \colon T \cong L^\times \to \overline{\mathbb{Q}}_\ell^\times$ be a smooth character such that $\theta|_{U_L^1}$ has trivial stabilizer in $\mathrm{Gal}(L/K)$. Then $\pm R_T^G(\theta)$ is a genuine G-representation and*

$$\pm R_T^G(\theta) \cong \pi_\theta.$$

In particular, $\pm R_T^G(\theta)$ is irreducible supercuspidal and $\sigma_\theta \leftrightarrow \pm R_T^G(\theta)$ is a realization of the local Langlands and Jacquet–Langlands correspondences.

For θ trivial on U_L^1 (and with trivial $\mathrm{Gal}(L/K)$-stabilizer), as well as for θ primitive, Theorem A is shown in [CI18] for all p, n. When G is the group of units of a central division algebra over K, Theorem A essentially follows (for all p, n and all θ with trivial $\mathrm{Gal}(L/K)$-stabilizer) from Lusztig's original work [Lus79] along with a result of Henniart [Hen92, 3.1 Théorème], see [Cha19]. The case $G = \mathrm{GL}_2$ was first studied in [Iva16].

In the rest of this introduction we explain the strategy of the proof of Theorem A and discuss the geometric methods used in it. To begin with, $\mathbf{G}$ has a (unique up to conjugacy) smooth affine model $\mathbf{G}_\mathcal{O}$ over the integers $\mathcal{O}_K$ of K, whose $\mathcal{O}_K$-points are the maximal compact subgroup $G_\mathcal{O} \cong \mathrm{GL}_{n'}(\mathcal{O}_{D_{k_0/n_0}})$, where $\mathcal{O}_{D_{k_0/n_0}}$ is the ring of integers of D_{k_0/n_0}. Moreover, $G_\mathcal{O}$ can be chosen compatibly with $\mathbf{T}$ so that $T_\mathcal{O} := T \cap G_\mathcal{O} \cong U_L$ is the maximal compact subgroup of T. As is shown in [CI18] (see also Proposition 2.6 below), X admits a scheme-theoretically disjoint decomposition,

$$X = \coprod_{g \in G/G_\mathcal{O}} g.X_\mathcal{O}, \qquad \text{where } X_\mathcal{O} = \varprojlim_h X_h \tag{1.1}$$

is a subscheme equal to an inverse limit of affine perfect schemes X_h perfectly finitely presented over $\overline{\mathbb{F}}_q$. Here $X_\mathcal{O}$ carries an action of $G_\mathcal{O} \times T_\mathcal{O}$ and X_h inherits an action of a certain finite (Moy–Prasad) quotient $G_h \times T_h$ of it. Then X_1 is (the perfection of) a classical Deligne–Lusztig variety attached to the reductive quotient of the special fiber of $\mathbf{G}_{\mathcal{O}}$ (isomorphic to $\mathrm{Res}_{\mathbb{F}_{q^{n_0}}/\mathbb{F}_q} \mathrm{GL}_{n'}$), and the deeper-level varieties X_h coincide with the (perfections of) varieties considered in [Lus04] when $\mathbf{G}_{\mathcal{O}} \otimes_{\mathcal{O}_K} \mathbb{F}_q$ is reductive (i.e., $\kappa = 0$), resp. with those in [CI19a] in the general case.

Let Z be the center of G. Then $T = ZT_{\mathcal{O}}$. For a character θ of $T \cong L^\times$ trivial on the h-units U_L^h, (1.1) plus the fact that the fibers of $X_h/\ker(T_h \to T_{h-1}) \to X_{h-1}$ are affine spaces of a fixed dimension, gives $R_T^G(\theta) = \mathrm{cInd}_{ZG_{\mathcal{O}}}^G R_{T_h}^{G_h}(\theta)$, where $R_{T_h}^{G_h}(\theta)$ is the $\theta|_{U_L}$-isotypic component of the ℓ-adic Euler characteristic of X_h (extended to a $ZG_{\mathcal{O}}$-representation by letting $z \in Z \cong K^\times$ act by $\theta(z)$).

The proof of Theorem A consists of five steps:

(1) Show that $\pm R_{T_h}^{G_h}(\theta)$ is an irreducible G_h-representation. See Section 3.
(2) By similar methods as in (1), show for a certain closed $G_h \times T_h$-stable perfect subscheme $X_{h,n'} \subseteq X_h$, that $\pm H_c^*(X_{h,n'})_\theta$ is irreducible and $\pm R_{T_h}^{G_h}(\theta) \cong \pm H_c^*(X_{h,n'})_\theta$. See Section 4.
(3) Show (using (1)) that the induction $\pm R_T^G(\theta) = \mathrm{cInd}_{ZG_{\mathcal{O}}}^G(\pm R_{T_h}^{G_h}(\theta))$ is admissible (equivalently, a finite direct sum of irreducible supercuspidals). See Section 5.
(4) Use [CI19b] to compute the degree $\deg H_c^*(X_{h,n'})_\theta$ of the (finite-dimensional) representation $\pm H_c^*(X_{h,n'})_\theta$, which is then by (2) also equal to $\deg R_{T_h}^{G_h}(\theta)$. See Section 7.1 and [CI19b].
(5) Using (3) together with the traces of $R_T^G(\theta)$ [CI18, Theorem 11.2] and of π_θ on very regular elements (cf. Section 6.2 for a definition) in $T \subseteq G$, conclude by using an argument due to Henniart [Hen92] using linear independence of characters, along with matching $\deg R_{T_h}^{G_h}(\theta)$ from (4) with the explicitly known formal degree of π_θ [CMS90]. See Section 7.2.

Let us briefly comment on steps (1)-(4) here. Step (1) relies on a precise analysis of the quotient $G_h \backslash X_h \times X_h$ (diagonal action) by methods generalizing those from [Lus04] in the special case that $\mathbf{T} \subseteq \mathbf{G}$ is (unramified) elliptic resp., even more specifically, in the case $\mathbf{G} =$ inner form of $\mathbf{GL}_n$ and culminates in showing the following particular Mackey formula for "Deligne–Lusztig induced" G_h-representations.

Theorem B (see Theorem 3.1, Corollary 3.3)**.** *Let θ, θ' be two characters of T_h. Then*

$$\left\langle R_{T_h}^{G_h}(\theta), R_{T_h}^{G_h}(\theta') \right\rangle_{G_h} = \#\left\{ w \in W_{\mathcal{O}}^F : \theta' = \theta \circ \mathrm{ad}(w) \right\},$$

where $W_{\mathcal{O}}$ is the Weyl group of the special fiber of $\mathbf{G}_{\mathcal{O}}$ and F is the Frobenius of $\mathbf{G}$ acting on it. Moreover, if the stabilizer of $\theta|_{U_L^1}$ in $\mathrm{Gal}(L/K)[n']$, the unique subgroup of $\mathrm{Gal}(L/K)$ of order n', is trivial, then $\pm R_{T_h}^{G_h}(\theta)$ is an irreducible G_h-representation and the map

$$\{\textit{characters } \theta \colon T_h \to \overline{\mathbb{Q}}_\ell^\times \textit{ in general position}\}/W_{\mathcal{O}}^F \to \{\textit{irreducible } G_h\textit{-representations}\}$$
$$\theta \mapsto \pm R_{T_h}^{G_h}(\theta)$$

is injective.

The theorem looks like a special case (for given G_h, T_h) of the results of [Lus04], but the assumption "θ regular" (equivalently, primitive) which is crucial to [Lus04] is removed. One can hope that similar methods as used in the proof of Theorem B could lead to a general Mackey formula for all elliptic unramified tori in reductive K-groups.

Step (2) is a technically more elaborated version of the same idea then in step (1). It is (among other things) responsible for the assumptions that $p > n$ and that $\theta|_{U_L^1}$ has trivial stabilizer in $\mathrm{Gal}(L/K)$. See Remark 4.3.

Step (3) relies on the study of the quotient $N_h \backslash X_h$ where $N_h \subseteq G_h$ is a subgroup corresponding to the unipotent radical of a proper parabolic subgroup of $\mathbf{G}$. Once this quotient is described (Lemma 5.8), (3) is easy to show. The main technical role in this description is played by classical minor identities, dating back to 1909 results of Turnbull [Tur09].

Step (4), mainly performed in [CI19b] is based on the determination of the action of Frobenius (over $\mathbb{F}_{q^n}$) in the cohomology of $X_{h,n'}$. This determination is strongly related to the amazing fact that $X_{h,n'}$ is a maximal variety over $\mathbb{F}_{q^n}$, i. e., $\#X_{h,n'}(\mathbb{F}_{q^n})$ attains its Weil–Deligne bound, prescribed by the Lefschetz fix point formula and the dimensions of the single ℓ-adic cohomology groups.

1.1. **Notation.** For a non-archimedean local field M we denote by $\mathcal{O}_M, \mathfrak{p}_M, U_M = \mathcal{O}_M^\times$ resp. $U_M^h = 1 + \mathfrak{p}_M^h$ (with $h \geq 1$) its integers, maximal ideal, units resp. h-units.

Throughout the article we fix a non-archimedean local field K with uniformizer ϖ and residue field $\mathbb{F}_q$ of characteristic p with q elements. We denote by $\breve{K}$ the completion of a fixed maximal unramified extension of K, and by $\mathcal{O}_{\breve{K}}$ the integers of $\breve{K}$. The residue field $\overline{\mathbb{F}}_q$ of $\breve{K}$ is an algebraic closure of $\mathbb{F}_q$, and ϖ is still a uniformizer of $\breve{K}$. We write σ for the Frobenius automorphisms of $\breve{K}/K$ and of $\overline{\mathbb{F}}_q/\mathbb{F}_q$.

Fix an integer $n \geq 2$. We denote by $K \subseteq L \subseteq \breve{K}$ the unique subextension of degree n. Moreover, for any positive divisor r of n we denote by $K \subseteq K_r \subseteq K_n = L$ the unique subextension of degree r over K.

Fix another integer $0 \leq \kappa < n$ and write $n = n'n_0$, $\kappa = n'k_0$, where $n' = \gcd(n, \kappa)$. Then n_0, k_0 are coprime.

Fix a prime $\ell \neq p$ and let $\overline{\mathbb{Q}}_\ell$ be a fixed algebraic closure of $\mathbb{Q}_\ell$. All cohomology groups of (perfections of) quasi-projective schemes of finite type over $\overline{\mathbb{F}}_q$ will be compactly supported étale cohomology groups with coefficients in $\overline{\mathbb{Q}}_\ell$. For such a scheme Y (and more generally, whenever the cohomology groups are defined), we write $H_c^*(Y) := \sum_{i\in\mathbb{Z}} H_c^i(Y, \overline{\mathbb{Q}}_\ell)$ (the coefficients always will be $\overline{\mathbb{Q}}_\ell$, so there is no ambiguity).

Unless otherwise stated, all representations of locally compact groups appearing in this article will be smooth with coefficients in $\overline{\mathbb{Q}}_\ell$.

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2. Coxeter-type loop Deligne–Lusztig scheme in type $\tilde{A}_{n-1}$

Let $n = n'n_0 \geq 2$ and $\kappa = n'k_0$ with $\gcd(k_0, n_0) = 1$ be as in Section 1.1. This notation remains fixed throughout the article.

In this section we review some constructions and results concerning loop Deligne–Lusztig schemes of Coxeter type for inner forms of $\mathbf{GL}_n$ from [CI18], and we simplify the proof of representability (Proposition 2.6).

2.1. **Inner forms of $\mathbf{GL}_n$ and elliptic tori.** Inside the group $\mathbf{GL}_n$ over K we fix a split maximal torus $\mathbf{T}_0$ and the unipotent radicals $\mathbf{U}_0, \mathbf{U}_0^-$ of two opposite K-rational Borel subgroups containing $\mathbf{T}_0$. Let the roots of $\mathbf{T}_0$ in $\mathbf{U}_0$ be the positive roots, determining a set S_0 of simple roots. Conjugating if necessary, we may assume that $\mathbf{T}_0$ is the diagonal torus and $\mathbf{U}_0$ is the group of upper triangular unipotent matrices.

2.1.1. *Forms of* $\mathbf{GL}_n$. The Kottwitz map [Kot85]

$$\kappa_{\mathbf{GL}_n} = \mathrm{val} \circ \det \colon B(\mathbf{GL}_n)_{\mathrm{basic}} \to \mathbb{Z}$$

for $\mathbf{GL}_n$ defines a bijection between the set of basic σ-conjugacy classes in $\mathbf{GL}_n(\breve{K})$ and $\mathbb{Z}$. Fix a basic element $b \in \mathbf{GL}_n(\breve{K})$ with $\kappa_{\mathbf{GL}_n}(b) = \kappa$. Let $\mathbf{G}$ be the K-group defined by

$$\mathbf{G}(R) = \{g \in \mathbf{GL}_n(R \otimes_K \breve{K}) \colon g^{-1} b \sigma(g) = b\}$$

(this is the group J_b from [RZ96, 1.12]). Then $\mathbf{G}$ is an inner form of $\mathbf{GL}_n$ and we may identify $\mathbf{G}(\breve{K}) = \mathbf{GL}_n(\breve{K})$. The Frobenius on $\mathbf{G}(\breve{K})$ is $F_b \colon g \mapsto b\sigma(g)b^{-1}$. The K-points of $\mathbf{G}$ are

$$G := \mathbf{G}(K) \cong \mathrm{GL}_{n'}(D_{k_0/n_0}).$$

We may identify the adjoint Bruhat–Tits building of $\mathbf{G}$ over $\breve{K}$ with that of $\mathbf{GL}_n$. Denote both of them by $\mathscr{B}_{\breve{K}}$. The adjoint Bruhat–Tits building of $\mathbf{G}$ over K is the subcomplex $\mathscr{B}_K = \mathscr{B}_{\breve{K}}^{F_b}$. Let $\mathbf{x}_b \in \mathscr{B}_K$ be a vertex. Bruhat–Tits theory [BT84, 5.2.6] attaches to $\mathbf{x}_b$ a (maximal) parahoric $\mathcal{O}_K$-model $\mathbf{G}_{\mathcal{O}}$ of $\mathbf{G}$, whose $\mathcal{O}_K$-points

$$G_{\mathcal{O}} := \mathbf{G}_{\mathcal{O}}(\mathcal{O}_K) \cong \mathrm{GL}_{n'}(\mathcal{O}_{D_{\kappa_0/n_0}}),$$

form a maximal compact subgroup of G.

Remark 2.1. The groups $\mathbf{G}, \mathbf{G}_{\mathcal{O}}, G, G_{\mathcal{O}}$ depend on the choice of b, but if $b' = h^{-1}b\sigma(h)$ ($h \in \mathbf{GL}_n(\breve{K})$) is another choice inside the same basic σ-conjugacy class, with corresponding groups $\mathbf{G}', G'$, then conjugation with h defines an isomorphism of $\mathbf{G}, G$ and $\mathbf{G}', G'$, and if $\mathbf{x}_b$ is mapped by h to $\mathbf{x}_{b'}$, then conjugation by h maps $\mathbf{G}_{\mathcal{O}}, G'_{\mathcal{O}}$ to $\mathbf{G}_{\mathcal{O}}, G'_{\mathcal{O}}$. As at the end we are interested in isomorphism classes of representations of G (or $G_{\mathcal{O}}$), which are not affected by these isomorphisms, we leave the choice of b unspecified as long as possible. When we need concrete realizations of $\mathbf{G}, \mathbf{G}_{\mathcal{O}}, G, G_{\mathcal{O}}$ (in Sections 3.1, 5.1 and 5.2) we will exploit the freedom of choosing different b's inside the same basic σ-conjugacy class).

2.1.2. *Forms of* $\mathbf{T}_0$. Let $W = W(\mathbf{T}_0, \mathbf{GL}_n)$ be the Weyl group of $\mathbf{T}_0$ in $\mathbf{GL}_n$, then (W, S_0) form a Coxeter system. Let $w_0 = \left(\begin{smallmatrix} 0 & 1 \\ 1_{n-1} & 0 \end{smallmatrix}\right) \in W$. It is a Coxeter element of (W, S_0). Let $\dot{w}_0 \in N_{\mathbf{GL}_n}(\mathbf{T}_0)(\breve{K})$ be a lift of w. Then $\mathrm{Ad}(\dot{w})$ induces an automorphism of the apartment $\mathscr{A}_{\mathbf{T}_0, \breve{K}} \subseteq \mathscr{B}_{\breve{K}}$ of $\mathbf{T}_0$. It has precisely one fixed point $\mathbf{x}_{\dot{w}}$ as w_0 is Coxeter. Let $\mathcal{G}$ be the parahoric $\mathcal{O}_K$-model of $\mathbf{GL}_n$ attached to this fixed point. Let $\mathcal{T}$ be the schematic closure of $\mathbf{T}_0$ in $\mathcal{G}$. Let $\mathbf{T}$ denote the (outer) form of $\mathbf{T}_0$, which splits over $\breve{K}$, and is endowed with the Frobenius $F_{\dot{w}_0} \colon t \mapsto \dot{w}_0 \sigma(t) \dot{w}_0^{-1}$ (independent of the lift $\dot{w}_0$), and similarly let $\mathbf{T}_{\mathcal{O}}$ be the (outer) form of $\mathcal{T}$, which splits over $\mathcal{O}_{\breve{K}}$, and is endowed with the same Frobenius. We get the group

$$T := \mathbf{T}(K) \cong L^\times \quad \text{and its subgroup} \quad T_{\mathcal{O}} := \mathbf{T}_{\mathcal{O}}(\mathcal{O}_{\breve{K}}) \cong \mathcal{O}_L^\times,$$

where L/K is unramified of degree n. In fact, $T = \{\mathrm{diag}(x, \sigma(x), \ldots, \sigma^{n-1}(x)) \colon x \in L^\times\}$ (recall that $\mathbf{T}_0$ is diagonal), and the isomorphism with $L^\times$ is determined up to composition with an element in $\mathrm{Gal}(L/K)$.

2.1.3. *Case* $b = \dot{w}_0$. In the special case $b = \dot{w}_0$ and $\mathbf{x}_b = \mathbf{x}_{\dot{w}_0}$, we have only one Frobenius $F := F_b = F_{\dot{w}_0}$, $\mathbf{G}_{\mathcal{O}}$ is a form of $\mathcal{G}$, and $\mathbf{T}$ is an elliptic maximal torus of $\mathbf{G}$, and $\mathbf{T}_{\mathcal{O}}$ is a maximal torus of $\mathbf{G}_{\mathcal{O}}$. There are unique (closed, reduced) subgroups $\mathbf{U}, \mathbf{U}^-$ of $\mathbf{G}$, such that $\mathbf{U}(\breve{K}) = \mathbf{U}_0(\breve{K})$, $\mathbf{U}^-(\breve{K}) = \mathbf{U}_0^-(\breve{K})$ inside $\mathbf{G}(\breve{K}) = \mathbf{GL}_n(\breve{K})$. Inside $\mathbf{G}_{\mathcal{O}}$ we will need the schematic closures $\mathbf{U}_{\mathcal{O}}$ and $\mathbf{U}_{\mathcal{O}}^-$ of $\mathbf{U}$ and $\mathbf{U}^-$.

The Frobenius F acts on the roots of $\mathbf{T}$ in $\mathbf{G}$, so that there is a unique subgroup $F\,\mathbf{U} \subseteq \mathbf{G}$, satisfying $(F\,\mathbf{U})(\breve{K}) = F(\mathbf{U}(\breve{K}))$, and similarly for $\mathbf{U}^-, \mathbf{U}_{\mathcal{O}}, \mathbf{U}_{\mathcal{O}}^-$. Identifying W with the Weyl group of $\mathbf{T}$ in $\mathbf{G}$, F acts on W. Moreover, $W^F = \langle w_0 \rangle \cong \mathbb{Z}/n\mathbb{Z}$ is the subgroup generated by w_0. It acts on T and the chosen isomorphism $T \cong L^\times$ induces an isomorphism $W^F \cong \mathrm{Gal}(L/K)$, sending w_0 to the image of σ in $\mathrm{Gal}(L/K)$.

The maximal torus in the reductive quotient of the special fiber $\mathbf{T}_{\mathcal{O}} \otimes_{\mathcal{O}_K} \mathbb{F}_q \subseteq (\mathbf{G}_{\mathcal{O}} \otimes_{\mathcal{O}_K} \mathbb{F}_q)^{\mathrm{red}}$ is elliptic. Explicitly, these groups are isomorphic to $\mathrm{Res}_{\mathbb{F}_{q^n}/\mathbb{F}_q} \mathbb{G}_m \subseteq \mathrm{Res}_{\mathbb{F}_{q^{n_0}}/\mathbb{F}_q} \mathrm{GL}_{n',\mathbb{F}_{q^{n_0}}}$. Let $W_{\mathcal{O}}$ be the Weyl group of $\mathbf{T}_{\mathcal{O}} \otimes_{\mathcal{O}_K} \mathbb{F}_q$ in $(\mathbf{G}_{\mathcal{O}} \otimes_{\mathcal{O}_K} \mathbb{F}_q)^{\mathrm{red}}$. It is naturally a subgroup of W, F acts on $W_{\mathcal{O}}$ and $W_{\mathcal{O}}^F$, which is generated by $w_0^{n_0}$, is isomorphically mapped onto $\mathrm{Gal}(L/K_{n_0})$ under the above isomorphism $W^F \cong \mathrm{Gal}(L/K)$.

2.2. **Perfect schemes.** Let k be a perfect field of characteristic p and let X be a k-scheme. Let $\phi = \phi_X \colon X \to X$ be the absolute Frobenius morphism of X, that is ϕ is the identity on the underlying topological space and is given by $x \mapsto x^p$ on $\mathcal{O}_X$. The scheme X is called *perfect* if ϕ is an isomorphism. Let Alg_k denote the category of all k-algebras, and let Perf_k be the full subcategory of perfect k-algebras. Then the restriction functor which sends a perfect k-scheme, regarded as a functor on Alg_k, to a functor on Perf_k is fully faithful [Zhu17, A.12]. Thus we equally may regard a perfect scheme as a functor on Perf_k, which has an open covering by representable functors in the usual sense. Every k-scheme X_0 admits a *perfection*, namely $X_0^{\mathrm{perf}} := \lim_\phi X_0$, which is a perfect scheme. E. g. the perfection of $\mathrm{Spec}\, k[T]$ is $\mathrm{Spec}\, k[T^{1/p^\infty}]$, where $k[T^{1/p^\infty}] = \bigcup_{r \geq 0} k[T^{p^{-r}}]$.

Except stated otherwise, throughout this article we will work with perfect schemes over $k = \overline{\mathbb{F}}_q$ (or $k = \mathbb{F}_q$). So, to simplify notation we write $\mathbb{A}^m = \mathbb{A}^m_k$ resp. $\mathbb{G}_a$ resp. $\mathbb{G}_m$ for the *perfection* of the m-dimensional affine space resp. the additive resp. the multiplicative group over k. A morphism $f\colon \mathrm{Spec}\, A \to \mathrm{Spec}\, B$ of affine perfect schemes is *perfectly finitely presented*, if there is a $A = (A_0)_{\mathrm{perf}}$ for a finitely presented B-algebra A_0 [BS17, 3.10,3.11]. For further results on perfect schemes we refer to [Zhu17, Appendix A.1] and [BS17, §3]. Here we only mention the following lemmas.

Lemma 2.2. *Let $X \subseteq \mathbb{A}^m_k$ be a closed perfect subscheme of the m-dimensional perfect affine space. Then $X \to \mathrm{Spec}\, k$ is perfectly finite presented.*

Proof. Let $T = (T_1, T_2, \ldots, T_m)$ be some coordinates on $\mathbb{A}^m_k$. Let $\mathfrak{a}$ be the ideal of X in the coordinate ring $k[T^{p^{-\infty}}]$ of $\mathbb{A}^m_k$. Then it is easy to check that X is the perfection of $X_0 = \mathrm{Spec}\, k[T]/(\mathfrak{a} \cap k[T])$, which is (reduced and) finitely presented over k. □

Lemma 2.3. *Let $f\colon X \to Y$ be a morphism of perfect k-schemes with X separated. The following are equivalent:*

(i) f is a monomorphism (of fpqc- or étale sheaves on Perf_k*)*
(ii) for every algebraically closed field K/k, $f(K)\colon X(K) \to Y(K)$ is injective.

Proof. Assume (ii). To deduce (i) it is enough to show that for any $R \in \mathrm{Perf}_k$, $f(R)\colon X(R) \to Y(R)$ is injective. Let $x, y\colon \mathrm{Spec}\, R \to X$ be two elements of $X(R)$, such that $fx = fy \in Y(R)$. For each point $p \in \mathrm{Spec}\, R$, choose a morphism $i_p\colon \mathrm{Spec}\, K_p \to \mathrm{Spec}\, R$ with image p, and with K_p an algebraically closed field. Then $fxi_p = fyi_p \in Y(K_p)$ for each p, and from (ii) we deduce $xi_p = yi_p$. As X is separated, the equalizer of x, y is a closed subscheme of $\mathrm{Spec}\, R$, say equal to $\mathrm{Spec}\, R/I$ for some ideal $I \subseteq R$. Now, $xi_p = yi_p$ for *all* field valued points of $\operatorname{Spec} R$ implies that $I \subseteq \bigcap_{\mathfrak{p}\in \operatorname{Spec} R} \mathfrak{p} = \operatorname{rad}(0) = 0$, as R perfect and hence reduced. The other direction is clear. □

2.3. **Witt vectors and loop groups.** If K has positive characteristic, we denote by $\mathbb{W}$ the ring scheme over $\mathbb{F}_q$, where for any $\mathbb{F}_q$-algebra R, $\mathbb{W}(R) = R[\![\varpi]\!]$. If K has mixed characteristic, we denote by $\mathbb{W}$ the K-ramified Witt ring scheme over $\mathbb{F}_q$ so that $\mathbb{W}(\mathbb{F}_q) = \mathcal{O}_K$ and $\mathbb{W}(\overline{\mathbb{F}}_q) = \mathcal{O}_{\breve{K}}$ (see e.g. [FF18, 1.2]). Let $\mathbb{W}_h = \mathbb{W}/V^h\mathbb{W}$ be the truncated ring scheme, where $V\colon \mathbb{W} \to \mathbb{W}$ is the multiplication by ϖ (if $\operatorname{char} K > 0$) resp. the Verschiebung morphism (if $\operatorname{char} K = 0$). We regard $\mathbb{W}_h$ as a functor on $\operatorname{Perf}_{\mathbb{F}_q}$, where it is represented by $\mathbb{A}^h_{\mathbb{F}_q}$. We denote by $\mathbb{W}_h^\times$ the perfect group scheme of invertible elements of $\mathbb{W}$ and for $1 \leq a < h$, we denote by $\mathbb{W}_h^{\times,a} = \ker(\mathbb{W}_h^\times \to \mathbb{W}_a^\times)$ the kernel of the natural projection.

If $\mathbf{X}$ is a $\breve{K}$-scheme, the loop space $L\,\mathbf{X}$ of $\mathbf{X}$ is the functor on $\operatorname{Perf}_{\overline{\mathbb{F}}_q}$,

$$R \mapsto L\,\mathbf{X}(R) = \mathbf{X}(\mathbb{W}(R)[\varpi^{-1}]).$$

If $\mathbf{X}$ is an affine $\breve{K}$-scheme of finite type, $L\,\mathbf{X}$ is represented by an ind-(perfect scheme) [Zhu17, Proposition 1.1]. If $\mathcal{X}$ is a $\mathcal{O}_{\breve{K}}$-scheme, the spaces of (truncated) positive loops of $\mathcal{X}$ are the functors on $\operatorname{Perf}_{\overline{\mathbb{F}}_q}$,

$$R \mapsto L^+\mathcal{X}(R) = \mathcal{X}(\mathbb{W}(R)) \quad \text{resp.} \quad R \mapsto L^+_h\mathcal{X}(R) = \mathcal{X}(\mathbb{W}_h(R)).$$

($h \geq 1$). If $\mathcal{X}$ is an affine $\mathcal{O}_{\breve{K}}$-scheme of finite type, $L^+\mathcal{X}$, $L^+_h\mathcal{X}$ are represented by affine perfect $\overline{\mathbb{F}}_q$-schemes, and $L^+_h\mathcal{X}$ are perfectly finitely presented (by Lemma 2.2). The same holds with $\overline{\mathbb{F}}_q$ replaced by $\mathbb{F}_q$.

Remark 2.4. We could evaluate $\mathbb{W}$, $\mathbb{W}_h$ and $L\,\mathbf{X}$, $L^+\mathcal{X}$, $L^+_h\mathcal{X}$ on all $R \in \operatorname{Alg}_{\overline{\mathbb{F}}_q}$, thus working with schemes $L^+_h\mathcal{X}$ of finite type over $\overline{\mathbb{F}}_q$, instead of perfect schemes. When $\operatorname{char} K > 0$, this causes absolutely no problems, so we equally good could work with schemes of finite type over $\overline{\mathbb{F}}_q$ instead of their perfections. When $\operatorname{char} K = 0$, these objects behave badly on non-perfect algebras (see e.g. [BS17, Remark 9.3], [Zhu17, end of 1.1.1]). Therefore we pass to perfect schemes everywhere. Perfections are universal homeomorphisms, hence do not affect étale cohomology.

2.4. **The perfect $\overline{\mathbb{F}}_q$-space $X^{DL}_{\dot{w}}(b)$.** By a *perfect $\overline{\mathbb{F}}_q$-space* we mean an fpqc-sheaf on $\operatorname{Perf}_{\overline{\mathbb{F}}_q}$. Let b be any basic element with $\kappa_{\mathbf{GL}_n}(b) = \kappa$. Let $\dot{w} \in N_{\mathbf{GL}_n}(\mathbf{T}_0)(\breve{K})$ be any lift of w. Let $\dot{X}^{DL}_{\dot{w}}(b)$ denote the fpqc-sheafification of the presheaf on $\operatorname{Perf}_{\overline{\mathbb{F}}_q}$,

$$R \mapsto \{g \in L\,\mathbf{GL}_n(R)/L\,\mathbf{U}_0(R)\colon g^{-1}b\sigma(g) \in L\,\mathbf{U}_0(R)\dot{w}L\,\mathbf{U}_0(R)\}.$$

If G, T are as in Section 2.1, the group $G \times T$ acts on $\dot{X}^{DL}_{\dot{w}}(b)$ by $g, t\colon x \mapsto gxt$.

Lemma 2.5. *Let b be basic with $\kappa_{\mathbf{GL}_n}(b) = \kappa$ and let $\dot{w}_0$ be any lift of w.*

(i) If $b' = h^{-1}b\sigma(h)$ for some $h \in \mathbf{GL}_n(\breve{K})$, and if $G' = \mathbf{G}'(K)$ is the group attached to b' as in Section 2.1, then $\operatorname{Ad}_h\colon G \to G'$, $g \mapsto h^{-1}gh$ is an isomorphism. Moreover, left multiplication by h induces an isomorphism of $\overline{\mathbb{F}}_q$-spaces $\dot{X}^{DL}_{\dot{w}_0}(b) \cong \dot{X}^{DL}_{\dot{w}_0}(b')$, which is equivariant with respect to the isomorphism $(\operatorname{Ad}_h, \mathrm{id})\colon G \times T \to G' \times T$.

(ii) Let $\dot{w}_0'$ be a second lift of w_0 to $\mathbf{GL}_n(\breve{K})$. Assume that $\kappa_{\mathbf{GL}_n}(\dot{w}_0) = \kappa_{\mathbf{GL}_n}(\dot{w}_0')$. Then there exists a $\tau \in \mathbf{T}_0(\breve{K})$ with $\dot{w}_0' = \tau^{-1}\dot{w}_0\sigma(\tau)$. Let $T' = \mathbf{T}'(K)$ be the group attached to $\dot{w}_0'$ as in Section 2.1. Then $\operatorname{Ad}_\tau\colon T \to T'$, $t \mapsto \tau^{-1}t\tau$ is an isomorphism. Moreover, right multiplication by τ induces an isomorphism of $\overline{\mathbb{F}}_q$-spaces $\dot{X}^{DL}_{\dot{w}_0}(b) \cong \dot{X}^{DL}_{\dot{w}_0'}(b)$, which is equivariant with respect to the isomorphism $(\mathrm{id}, \operatorname{Ad}_\tau)\colon G \times T \to G \times T'$.

(iii) $\dot{X}^{DL}_{\dot{w}_0}(b) = \varnothing$, unless $\kappa_{\mathbf{GL}_n}(\dot{w}_0) = \kappa$.

Proof. (i): is an easy computation. (ii): The fiber over w in $\mathbf{GL}_n(\breve{K})$ is a principal homogeneous space under $\mathbf{T}(\breve{K})$, and it is easy to see that as w is Coxeter, the map $t \mapsto \mathrm{Ad}(w)(t)^{-1}\sigma(t)$ from $\mathbf{T}(\breve{K})$ to $\{\tau \in \mathbf{T}(\breve{K}) \colon \kappa_{\mathbf{GL}_n}(\tau) = 0\}$ is surjective. The rest is an easy computation. (iii): As $\dot{X}_{\dot{w}_0}^{DL}(b)$ is an inverse limit of perfectly finitely presented perfect $\overline{\mathbb{F}}_q$-schemes, it suffices to show that $\dot{X}_{\dot{w}_0}^{DL}(b)(\overline{\mathbb{F}}_q) = \varnothing$. This holds as $\kappa_{\mathbf{GL}_n}(g^{-1}b\sigma(g)) = \kappa_{\mathbf{GL}_n}(b) = \kappa$ and $\kappa_{\mathbf{GL}_n}(LU(\overline{\mathbb{F}}_q)) = 0$. □

2.5. **Representability.** We simplify the proof of representability of $X_{\dot{w}_0}^{DL}(b)$ from [CI18]. Let $b = \dot{w}_0$ be basic with $\kappa_{\mathbf{GL}_n}(b) = \kappa$. Then we are in the setup of Section 2.1.3. Write $F \colon L\mathbf{G} \to L\mathbf{G}$ for the $\overline{\mathbb{F}}_q$-morphism of ind-(perfect schemes) corresponding to $F \colon \mathbf{G}(\breve{K}) \to \mathbf{G}(\breve{K})$, $g \mapsto b\sigma(g)b^{-1}$. Define the fpqc-sheafification X' of the presheaf on $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$,

$$R \mapsto \{x \in L\mathbf{G}(R) \colon x^{-1}F(x) \in F(L\mathbf{U})\}/L(\mathbf{U} \cap F\mathbf{U}).$$

The group $G \times T$ acts on X' by $g, t \colon x \mapsto gxt$. Define $X_{\mathcal{O}}$ as the fpqc-sheafification of the presheaf on $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$,

$$X_{\mathcal{O}} \colon R \mapsto \{x \in L^+\mathbf{G}_{\mathcal{O}}(R) \colon x^{-1}F(x) \in L^+(F\,\mathbf{U}_{\mathcal{O}} \cap \mathbf{U}_{\mathcal{O}}^-)(R)\}.$$

Being the preimage of $L^+(F\,\mathbf{U}_{\mathcal{O}} \cap \mathbf{U}_{\mathcal{O}}^-)$ under the Lang-morphism $\mathrm{Lang}_F \colon L^+\mathbf{G}_{\mathcal{O}} \to L^+\mathbf{G}_{\mathcal{O}}$, $g \mapsto g^{-1}F(g)$, $X_{\mathcal{O}}$ is representable by a perfect $\overline{\mathbb{F}}_q$-scheme. Further, the group $G_{\mathcal{O}} \times T_{\mathcal{O}}$ acts on $X_{\mathcal{O}}$ by $(g,t) \colon x \mapsto gxt$. As T is generated by $T_{\mathcal{O}}$ and the central element $\varpi \in T \subseteq G$, the obvious action of $G \times T_{\mathcal{O}}$ on $\coprod_{G/G_{\mathcal{O}}} g.X_{\mathcal{O}}$ extends to an action of $G \times T$ by letting $(1, \varpi)$ act in the same way as $(\varpi, 1)$.

Proposition 2.6 ([CI18]). *Let $b = \dot{w}_0 \in N_{\mathbf{GL}_n}(\mathbf{T}_0)(\breve{K})$ be basic with $\kappa_{\mathbf{GL}_n}(b) = \kappa$, and mapping to $w_0 \in W$. There are $G \times T$-equivariant isomorphisms of perfect $\overline{\mathbb{F}}_q$-spaces*

$$X_b^{DL}(b) \cong X' \cong \coprod_{g \in G/G_{\mathcal{O}}} g.X_{\mathcal{O}}. \tag{2.1}$$

In particular, $X_b^{DL}(b), X'$ are representable by perfect $\overline{\mathbb{F}}_q$-schemes.

Proof. The same computation as at the end of [CI18, §3] shows that $G \times T$-equivariantly $X_b^{DL}(b) \cong X'$ as $\overline{\mathbb{F}}_q$-spaces. As the right hand side of (2.1) is representable, it suffices to show the second isomorphism in (2.1). Consider the fpqc-sheafification X'' of the presheaf on $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$,

$$R \mapsto \{g \in L\mathbf{G}(R) \colon g^{-1}F(g) \in L(F\mathbf{U} \cap \mathbf{U}^-)(R)\}.$$

As w is Coxeter, the map

$$L(F\mathbf{U} \cap \mathbf{U}) \times L(F\mathbf{U} \cap \mathbf{U}^-) \to L\mathbf{U}, \quad (h,g) \mapsto h^{-1}gF(h)$$

is an isomorphism of fpqc-sheaves (this follows by a concrete calculation – the part of the proof of [CI18, Lemma 2.12] showing equation (7.7) of *loc. cit.* – which can be performed on R-points for any $R \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$. Compare also [HL12]), so that $X' \cong X''$. But X'' is the pull-back of the closed sub-(ind-scheme) $L\mathbf{U}$ under the Lang map $\mathrm{Lang}_F \colon L\mathbf{G} \to L\mathbf{G}$, $g \mapsto g^{-1}F(g)$, which is a morphism of ind-schemes, hence X'' is representable by an ind-(perfect scheme).

For $\tau \in \mathbf{T}(K)$, $x \mapsto \tau^{-1}x\tau$ defines an equivariant isomorphism between X'' and the analogue of X'', where b is replaced by $\tau^{-1}b\tau$. Thus we may take $b = \left(\begin{smallmatrix} 0 & \varpi^\kappa \\ 1_{n-1} & 0 \end{smallmatrix}\right) \cdot \varepsilon$ with $\varepsilon \in \mathbf{T}(\mathcal{O}_{\breve{K}})$. Fix $R \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$. Let $g \in L\mathbf{G}(R) = \mathbf{G}(\mathbb{W}(R)[\varpi^{-1}])$ with $g^{-1}F(g) =: a \in L(F\mathbf{U} \cap \mathbf{U}^-)(R)$. For $1 \leq i \leq n-1$, let $a_i \in L\mathbb{G}_a(R)$ denote the $(i+1,1)$-th entry of the matrix a. Then the matrix g is determined by its first column, denoted v (for $1 \leq i \leq n$ the i-th column is then equal to $(b\sigma)^{i-1}(v)$). Moreover v has to satisfy $(b\sigma)^n(v) = \varpi^\kappa(v + \sum_{i=1}^{n-1} a_i(b\sigma)^i(v))$, an equation which takes place in $L\mathbb{G}_a(R)^n$. Assume R is an algebraically closed field. The valuations of the coefficients of the characteristic polynomial of a σ-linear endomorphism lie over its Newton polygon, which in our case coincide with the Newton polygon of the isocrystal attached to $b\sigma$, and is just the straight line segment connecting the origin and the point (n,κ) in the plane (cf. [CI18, Lemma 6.1] for the precise statement). This shows $\mathrm{val}(a_i) \geq -\frac{i\kappa}{n}$ for $1 \leq i \leq n-1$. But after explicitly determining the affine root subgroups contained in $\mathbf{G}_{\mathcal{O}}(\mathcal{O}_{\breve{K}})$, this translates into the statement that $a \in L^+(F\,\mathbf{U}_{\mathcal{O}} \cap \mathbf{U}_{\mathcal{O}}^-)(R)$. As X'' is a ind-(perfect scheme), this implies that X'' is equal to the fpqc-sheafification of

$$R \mapsto \{g \in L\mathbf{G}(R) \colon g^{-1}F(g) \in L^+(F\,\mathbf{U}_{\mathcal{O}} \cap \mathbf{U}_{\mathcal{O}}^-)(R)\}.$$

Consider the projection $\pi \colon L\mathbf{G} \to L\mathbf{G}/L^+\mathbf{G}_{\mathcal{O}}$. If $g \in X''(R) \subseteq L\mathbf{G}(R)$, then $F(g) \in gL^+(F\,\mathbf{U}_{\mathcal{O}} \cap \mathbf{U}_{\mathcal{O}}^-)(R) \subseteq gL^+\mathbf{G}_{\mathcal{O}}(R)$. Thus X'' maps under π to the discrete subset $(L\mathbf{G}/L^+\mathbf{G}_{\mathcal{O}})^F = G/G_{\mathcal{O}}$. Hence X'' is isomorphic to the right hand side of (2.1), and we are done. □

Corollary 2.7. *Let $b \in \mathbf{GL}_n(\breve{K})$ be basic, $\dot{w}_0$ a lift of w_0 such that $\kappa_{\mathbf{GL}_n}(b) = \kappa_{\mathbf{GL}_n}(\dot{w}_0) = \kappa$. Then $X_{\dot{w}_0}^{DL}(b) \cong \coprod_{G/G_{\mathcal{O}}} gX_{\mathcal{O}}$ is representable by a perfect $\overline{\mathbb{F}}_q$-scheme.*

Proof. This follows from Lemma 2.5 and Proposition 2.6. □

2.6. **Representations $R_T^G(\theta)$ and $R_{T_h}^{G_h}(\theta)$.** Let a basic b and a lift $\dot{w}_0$ as in Section 2.1 with $\kappa_{\mathbf{GL}_n}(b) = \kappa_{\mathbf{GL}_n}(\dot{w}_0) = \kappa$ be fixed. In Section 2.1 we attached to $b, \dot{w}_0$ the locally pro-finite groups G, T and their maximal compact subgroups $G_{\mathcal{O}}, T_{\mathcal{O}}$. In [CI18] we defined families (indexed by $h \geq 1$) of perfectly finitely presented perfect group schemes over $\mathbb{F}_q$, with $\mathbb{F}_q$-points G_h, T_h such that $G_{\mathcal{O}} = \varprojlim_h G_h$ and $T_{\mathcal{O}} = \varprojlim_h T_h$, and showed that $G \times T$-equivariantly,

$$X_{\dot{w}_0}^{DL}(b) \cong \coprod_{G/G_{\mathcal{O}}} g.X_{\mathcal{O}}, \qquad \text{with} \qquad X_{\mathcal{O}} \cong \varprojlim_h X_h$$

such that $X_{\mathcal{O}}$ is acted on by $G_{\mathcal{O}} \times T_{\mathcal{O}}$[1], each X_h is a perfectly finitely presented perfect $\overline{\mathbb{F}}_q$-scheme, acted on by $G_h \times T_h$, and all morphisms are compatible with all actions. Moreover, X_h is the perfection of a smooth affine $\overline{\mathbb{F}}_q$-scheme of finite type. We identify $X_{\dot{w}_0}^{DL}(b)$ with $\coprod_{G/G_{\mathcal{O}}} g.X_{\mathcal{O}}$ via this isomorphism. The groups G_h and T_h are certain Moy–Prasad quotients of $G_{\mathcal{O}}$ and $T_{\mathcal{O}}$, and hence essentially independent of the choice of b, $\mathbf{x}_b$ and $\dot{w}_0$. An explicit presentation of G_h, T_h, X_h is reviewed in Section 3.1 below.

We review the definition of certain étale cohomology groups with compact support of $X_{\dot{w}_0}^{DL}(b)$ and $X_{\mathcal{O}}$ (which are not perfectly finitely presented over $\overline{\mathbb{F}}_q$). First, for $h \geq 1$ and a character $\chi \colon T_h \to \overline{\mathbb{Q}}_\ell^\times$, the χ-isotypic components $H_c^i(X_h)_\chi$ of the ℓ-adic cohomology groups with compact support are defined[2], as X_h is the perfection of smooth scheme of finite type over $\overline{\mathbb{F}}_q$. Second, for $h \geq 1$, the fibers of $X_h/\ker(T_h \to T_{h-1}) \to X_{h-1}$ are isomorphic to $\mathbb{A}^{n-1}$ [CI18, Proposition 7.6]. Let $\chi \colon T_{\mathcal{O}} \to \overline{\mathbb{Q}}_\ell^\times$ be a smooth character. Then there exists an $h \geq 1$, such that χ is trivial on $\ker(T_{\mathcal{O}} \to T_h)$ for some $h \geq 1$. Let $h' \geq h$ and denote the characters induced by χ on T_h and $T_{h'}$ again by χ. Then $H_c^*(X_h)_\chi = H_c^*(X_{h'})_\chi$, where H_c^* is the alternating sum of the cohomology. Thus we can define $H_c^*(X_{\mathcal{O}})_\chi$ as $H_c^*(X_{h'})_\chi$ for any $h' \geq h$ and this is independent of h'[3]. So, if χ is a character of $T_{\mathcal{O}}$ of level h, we have the

[1]Note that T is generated by $T_{\mathcal{O}}$ and a central element of G, when G, T are both regarded as subgroups of $\mathbf{GL}_n(\breve{K})$, so that $\coprod_{G/G_{\mathcal{O}}} g.X_{\mathcal{O}}$ admits also a natural right T-action.

[2]Recall from Section 1.1 that we omit the constant coefficients $\overline{\mathbb{Q}}_\ell$ from the notation.

[3]Note that the single cohomology groups $H_c^i(X_{\mathcal{O}})_\chi$ are not defined, due to a degree shift: $H_c^i(X_{h'})_\chi = H_c^{i-2d}(X_h)_\chi$ for an appropriate $d \geq 0$. One can remedy this by introducing homology groups $H_i(Y) := H_c^{2\dim(Y)-i}(Y)(\dim(Y))$ as in [Lus79], which removes precisely this shift in degree.

G_h-representation

$$R_{T_h}^{G_h}(\chi) := H_c^*(X_{\mathcal{O}})_\chi = H_c^*(X_h)_\chi.$$

and we denote the $G_{\mathcal{O}}$-representation obtained by inflation via $G_{\mathcal{O}} \twoheadrightarrow G_h$ again by $R_{T_h}^{G_h}(\chi)$.

Let $Z \subseteq G$ be the center and let $\widetilde{X}_{\mathcal{O}} := \bigcup_{g \in ZG_{\mathcal{O}}} g.X_{\mathcal{O}}$ be the union in $X_{\dot{w}_0}^{DL}(b)$ of all $ZG_{\mathcal{O}}$-translates of $X_{\mathcal{O}}$. Then $\widetilde{X}_{\mathcal{O}}$ is acted on by $ZG_{\mathcal{O}} \times T$ and is a disjoint union of copies of $X_{\mathcal{O}}$. Exactly as above for $X_{\mathcal{O}}$, for a smooth character $\theta \colon T \to \overline{\mathbb{Q}}_\ell^\times$ we may define the smooth $ZG_{\mathcal{O}}$-representation $H_c^*(\widetilde{X}_{\mathcal{O}})_\theta$.

Lemma 2.8. *Let $\theta \colon T \to \overline{\mathbb{Q}}_\ell^\times$ be a smooth character of level h. As $G_{\mathcal{O}}$-representations, $H_c^*(\widetilde{X}_{\mathcal{O}})_\theta \cong R_{T_h}^{G_h}(\theta)$. As a $ZG_{\mathcal{O}}$-representation, $H_c^*(\widetilde{X}_{\mathcal{O}})_\theta$ is just the $G_{\mathcal{O}}$-representation $R_{T_h}^{G_h}(\theta)$, with action extended to Z by letting $\varpi \in Z \cong K^\times$ act by the scalar $\theta(\varpi)$.*

Proof. This is immediate (see e.g. [Iva16, Lemma 4.5]). □

Justified by this lemma we write $R_{T_h}^{G_h}(\theta)$ for the $ZG_{\mathcal{O}}$-representation $H_c^*(\widetilde{X}_{\mathcal{O}})_\theta$. For schemes Y_i such that $H_c^*(Y_i)$ are defined, put $H_c^*(\coprod_{i \in I} Y_i) := \bigoplus_{i \in I} H_c^*(Y_i)$. We get our main object of study, the smooth G-representation

$$R_T^G(\theta) := H_c^*(X_{\dot{w}_0}^{DL}(b))_\theta = \mathrm{cInd}_{ZG_{\mathcal{O}}}^G R_{T_h}^{G_h}(\theta)$$

(cf. [CI18, Theorem 11.2]).

Remark 2.9. By construction and by Lemma 2.5, the isomorphism class of the G-representation $R_T^G(\theta)$ is independent of the choices of representatives $b, \dot{w}_0$. A similar independence holds for the $ZG_{\mathcal{O}}$-representation $R_{T_h}^{G_h}(\theta)$.

2.7. **Norms and characters.** The following definitions do not depend on the choice of an isomorphism $T \cong L^\times$ (as in Section 2.1.2).

Definition 2.10. We say that a smooth character $\theta \colon T \cong L^\times \to \overline{\mathbb{Q}}_\ell^\times$ is of *level h* if it is trivial on $\ker(T_{\mathcal{O}} \to T_h) \cong U_L^h$, but non-trivial on $\ker(T_{\mathcal{O}} \to T_{h-1}) \cong U_L^{h-1}$.

Recall the subextensions $L \supseteq K_r \supseteq K$ (Section 1.1). Whenever r, s are positive divisors of n such that s divides r, we denote by $N_{r/s} \colon K_r^\times \to K_s^\times$ the norm map for the field extension K_r/K_s. For any $h \geq h' \geq 1$, it induces maps

$$U_{K_r}/U_{K_r}^h \to U_{K_s}/U_{K_s}^h \quad \text{and} \quad U_{K_r}^{h'}/U_{K_r}^h \to U_{K_s}^{h'}/U_{K_s}^h$$

which are surjective (see e.g. [Ser95, Chap. V,§2]), and which we again denote by $\mathrm{N}_{r/s}$.

Definition 2.11.
(i) A character $\theta \colon T \cong L^\times \to \overline{\mathbb{Q}}_\ell^\times$ resp. $\theta \colon T_{\mathcal{O}} \cong U_L \to \overline{\mathbb{Q}}_\ell^\times$ is *in general position*, if the stabilizer of θ in $\mathrm{Gal}(L/K)$ is trivial. We say $\theta|_{U_L^1}$ is in *general position*, if the stabilizer of $\theta|_{U_L^1}$ in $\mathrm{Gal}(L/K)$ is trivial.
(ii) Let $h \geq 1$. A character $\theta \colon T_h \cong U_L/U_L^h \to \overline{\mathbb{Q}}_\ell^\times$ (resp. $\theta|_{T_h^1 = U_h^1/U_L^h}$) is *in general position* if its inflation to $T_{\mathcal{O}}$ (resp. to $\ker(T_{\mathcal{O}} \to T_1)$) is in general position.

Note that $\theta \colon T \cong L^\times \to \overline{\mathbb{Q}}_\ell^\times$ is in general position if and only if $\theta|_{T_{\mathcal{O}}}$ is.

Lemma 2.12. *Let $\theta \colon T \cong L^\times \to \overline{\mathbb{Q}}_\ell^\times$ be a character. Let $s \in \mathbb{Z}$. Then*

$$\theta \circ \sigma^s = \theta \quad \Leftrightarrow \quad \theta \text{ factors through } N_{L/K_{\gcd(n,s)}}.$$

The analogous claim holds for $\theta|_{U_L^1}$. In particular, θ is in general position if and only if θ does not factor through any of the maps $\mathrm{N}_{n/r}$ with $r < n$, and $\theta|_{U_L^1}$ is in general position if and only if $\theta|_{U_L^1}$ does not factor through any of the maps $\mathrm{N}_{n/r}$ with $r < n$.

Proof. $\theta \circ \sigma^s = \theta$ is equivalent to θ being trivial on the image of the map $L^\times \to L^\times$, $x \mapsto x^{-1}\sigma^s(x)$. By Hilbert's Theorem 90, this image is equal to the kernel of the norm map of L over the field stable by σ^s, which is $K_{\gcd(n,s)}$. □

3. Mackey formulas

In this section we prove the following special case of a Mackey-type formula.

Theorem 3.1. *Let $\theta, \theta' \colon T_h \to \overline{\mathbb{Q}}_\ell^\times$ be two characters. Then*

$$\left\langle R_{T_h}^{G_h}(\theta), R_{T_h}^{G_h}(\theta') \right\rangle_{G_h} = \#\left\{w \in W_{\mathcal{O}}^F \colon \theta' = \theta \circ \mathrm{ad}(w)\right\}$$

Remark 3.2. The theorem shows that in the special cases considered in [CI18] and here the assumption in [Lus04, Corollary 2.4] resp. [CI19a, Corollary 4.7] that θ is regular is obsolete.

Corollary 3.3. *Let $\theta \colon T_h \to \overline{\mathbb{Q}}_\ell^\times$ be a character, whose stabilizer in* $\mathrm{Gal}(L/K)[n']$*, the unique subgroup of* $\mathrm{Gal}(L/K)$ *of order n', is trivial. Then $\pm R_{T_h}^{G_h}(\theta)$ is irreducible G_h-representation. In particular,* Fr_{q^n} *acts in $\pm R_{T_h}^{G_h}(\theta)$ by multiplication with a scalar. Moreover, the map*

$$\left\{\textit{Characters } \theta \colon T_h \to \overline{\mathbb{Q}}_\ell^\times \textit{ in general position}\right\}/W_{\mathcal{O}}^F \to \{\textit{irreducible } G_h\textit{-representations}\}$$
$$\theta \mapsto \pm R_{T_h}^{G_h}(\theta)$$

is injective.

Proof. This follows from the description of $W_{\mathcal{O}}^F$ in Section 2.1.3 and Theorem 3.1. □

We prove Theorem 3.1 in four steps (Sections 3.1-3.4). After general preparations in Section 3.1, we show in Section 3.2 that several of the perfect schemes $\widehat{\Sigma}_w$ (as in [Lus04, 1.9]) are empty in our case; then in Sections 3.3,3.4 we generalize Lusztig's argument from [Lus04, 1.9, proof of claim (b)] with the extension of action on $\widehat{\Sigma}''_w$ in two different ways to cover the remaining $\widehat{\Sigma}_w$. The first generalization uses our concrete situation, whereas the second is quite general.

3.1. **General preparations.** In contrast to [CI18] where we worked with Coxeter-type and special representatives for $[b]$ (see [CI18, §5.1]), here it is most convenient to work with a third type of representatives. We put

$$b = \dot{w}_0 = b_0 t_{\kappa,n} \in \mathbf{GL}_n(\breve{K}) \tag{3.1}$$

where

$$b_0 := \begin{pmatrix} 0 & 1 \\ 1_{n-1} & 0 \end{pmatrix}, \qquad \text{and} \qquad t_{\kappa,n} := \begin{cases} \mathrm{diag}(\underbrace{1,\dots,1}_{n-\kappa},\underbrace{\varpi,\dots,\varpi}_{\kappa}) & \text{if } (\kappa,n)=1, \\ \mathrm{diag}(\underbrace{t_{k_0,n_0},\dots,t_{k_0,n_0}}_{n'}) & \text{otherwise.} \end{cases}$$

are as in [CI18, §5.2.1]. In particular, we work in the setup of Section 2.1.3.

Recall the (unique) fixed point $\mathbf{x}_b$ of F in the apartment $\mathscr{A}_{\mathbf{T},\breve{K}}$ of $\mathbf{T}$ in $\mathscr{B}_{\breve{K}}$, and the corresponding maximal parahoric $\mathcal{O}_K$-model $\mathbf{G}_{\mathcal{O}}$ of $\mathbf{G}$. We have the stabilizer $\breve{G}_{\mathbf{x}_b,0} = \mathbf{G}_{\mathcal{O}}(\mathcal{O}_{\breve{K}})$ of $\mathbf{x}_b$ in $\mathbf{G}(\breve{K}) = \mathbf{GL}_n(\breve{K})$ and its Moy–Prasad filtration [MP94] given by subgroups $\breve{G}_{\mathbf{x}_b,r}$ $(r \geq 0)$. Similarly as in [CI18, §5.3], consider the affine perfect group scheme $\mathbb{G}$ over $\mathbb{F}_q$ defined by

$$\mathbb{G}(\overline{\mathbb{F}}_q) = \breve{G}_{\mathbf{x}_b,0}, \qquad \mathbb{G}(\mathbb{F}_q) = \breve{G}_{\mathbf{x}_b,0}^F = G_{\mathcal{O}}.$$

and for $h \in \mathbb{Z}_{\geq 1}$, the affine perfectly finitely presented perfect group scheme $\mathbb{G}_h$ over $\mathbb{F}_q$ such that

$$\mathbb{G}_h(\overline{\mathbb{F}}_q) = \breve{G}_{\mathbf{x}_b,0}/\breve{G}_{\mathbf{x}_b,(h-1)+}, \qquad G_h := \mathbb{G}_h(\mathbb{F}_q) = \breve{G}^F_{\mathbf{x}_b,0}/\breve{G}^F_{\mathbf{x}_b,(h-1)+}.$$

We denote the Frobenii on $\mathbb{G}, \mathbb{G}_h$ again by F. The groups $\mathbb{G}, \mathbb{G}_h$ possess an explicit description in terms of matrices similar to [CI18, §5.3].

Remark 3.4. In [CI18, Section 7], we worked instead with the Coxeter representatives $b' = b_0^{e_{\kappa,n}} t_{\kappa,n}$ as in [CI18, §5.2.1]; but if γ is as in [CI18, §7.6], then $b = \gamma b' \gamma^{-1}$, i.e., b is integrally σ-conjugate to b'. In fact, the groups $\mathbb{G}, \mathbb{G}_h$ used here are equal to $\gamma\mathbb{G}\gamma^{-1}, \gamma\mathbb{G}_h\gamma^{-1}$ with the latter $\mathbb{G}, \mathbb{G}_h$ as in [CI18].

As (perfect) $\mathbb{F}_q$-groups, $\mathbb{G}_1 \cong \mathrm{Res}_{\mathbb{F}_{q^{n_0}}/\mathbb{F}_q} \mathrm{GL}_{n'}$. The above-mentioned description identifies $\mathbb{G}_1$ with a closed $\mathbb{F}_q$-subgroup of $\mathrm{GL}_{n,\mathbb{F}_q}$. In fact, $\mathbb{G}_{1,\overline{\mathbb{F}}_q}$ is the closed subgroup of $\mathrm{GL}_{n,\overline{\mathbb{F}}_q}$ consisting of those $n \times n$-matrices $g = (g_{ij})_{i,j \in \mathbb{Z}/n\mathbb{Z}} \in \mathrm{GL}_{n,\overline{\mathbb{F}}_q}$ for which $X_{ij} = 0$, unless $i \equiv j \mod n_0$; if we now equip $\mathrm{GL}_{n,\overline{\mathbb{F}}_q}$ with the $\mathbb{F}_q$-structure given by the Frobenius $F_0 \colon g \mapsto b_0\sigma(g)b_0^{-1}$ and denote the resulting $\mathbb{F}_q$-group simply by GL_n, then this defines an $\mathbb{F}_q$-embedding $\mathbb{G}_1 \to \mathrm{GL}_n$.

We regard the symmetric group on n letters S_n as the group of set automorphisms of $\mathbb{Z}/n\mathbb{Z}$, and for an element $i \in \mathbb{Z}/n\mathbb{Z}$ let $[i]$ be the unique integer between 1 and n having residue i modulo n. We also identify S_n with the Weyl group of the diagonal torus in GL_n (either over $\overline{\mathbb{F}}_q$ or $\breve{K}$) by sending a permutation $v \in S_n$ to the permutation matrix (again denoted v) whose non-zero entries are $(v(i), i)$ for $1 \leq i \leq n$.

As $\mathbb{G}_1$ is naturally isomorphic to the reductive quotient of the special fiber of $\mathbf{G}_{\mathcal{O}}$, the group $W_{\mathcal{O}}$ is simply the Weyl group of $\mathbb{T}_1$ in $\mathbb{G}_1$. Thus, using the above identifications, $W_{\mathcal{O}}$ is the subgroup of S_n, isomorphic to $S_{n'} \times \cdots \times S_{n'}$ (n_0 times), of those permutations which preserve the residue modulo n_0.

Applying L_h^+ to the inclusions $\mathbf{T}_{\mathcal{O}}, \mathbf{U}_{\mathcal{O}}, \mathbf{U}_{\mathcal{O}}^- \subseteq \mathbf{G}_{\mathcal{O}}$ gives closed subgroups $\mathbb{T}_h, \mathbb{U}_h, \mathbb{U}_h^- \subseteq \mathbb{G}_h$, with $\mathbb{T}_h$ defined over $\mathbb{F}_q$ and $\mathbb{U}_h, \mathbb{U}_h^-$ defined over $\mathbb{F}_{q^n}$ (cf. [CI19a, 2.6]). For a closed subgroup $\mathbb{H}_h \subseteq \mathbb{G}_h$ and $1 \leq a \leq h-1$, we write $\mathbb{H}_h^a := \mathbb{H}_h \cap \ker(\mathbb{G}_h \to \mathbb{G}_a)$. If $\mathbb{H}_h$ is defined over $\mathbb{F}_q$, we write $H := \mathbb{H}(\mathbb{F}_q)$ and $H_h^a := \mathbb{H}_h^a(\mathbb{F}_q)$.

Then we have (by a slight modification – or conjugation with γ from Remark 3.4 – of [CI18, Section 7], in particular, Propositions 7.10,7.11) as perfect $\overline{\mathbb{F}}_q$-spaces

$$X_h \cong \{g \in \mathbb{G}_h \colon g^{-1}F(g) \in \mathbb{U}_h^- \cap F\mathbb{U}_h\} \cong S_h/(\mathbb{U}_h \cap F\mathbb{U}_h), \tag{3.2}$$

where

$$S_h = \{g \in \mathbb{G}_h \colon g^{-1}F(g) \in F\mathbb{U}_h\},$$

and the action of $\mathbb{U}_h \cap F\mathbb{U}_h$ on S_h is by right multiplication (here and in the following: all presheaves have to be sheafified). Moreover, (3.2) is $G_h \times T_h$-equivariant with respect to the $G_h \times T_h$-action on the right hand side given by $(g', t) \colon g \mapsto g'gt$.

The fibers of the projection $S_h \to X_h$ are isomorphic to affine spaces of fixed dimension, so that $R_{T_h}^{G_h}(\theta) = H_c^*(S_h)_\theta$. As in [Lus04, 1.9], if

$$\Sigma = \{(x, x', y) \in F\mathbb{U}_h \times F\mathbb{U}_h \times \mathbb{G}_h \colon xF(y) = yx'\}$$

with the $T_h \times T_h$-action given by $(t, t') \colon (x, x', y) \mapsto (txt^{-1}, t'x't'^{-1}, tyt'^{-1})$, then the map

$$G_h \backslash (X_h \times X_h) \to \Sigma, \tag{3.3}$$

induced by $(g, g') \mapsto (g^{-1}F(g), g'^{-1}F(g'), g^{-1}g')$ is an $T_h \times T_h$-equivariant isomorphism (the quotient of the left side is taken with respect to the diagonal action).

The group $\mathbb{G}_1$ is reductive and $\ker(\mathbb{G}_h \to \mathbb{G}_1)$ is unipotent. Thus the Bruhat decomposition $\mathbb{G}_1 = \coprod_{w\in W_{\mathcal{O}}} \mathbb{U}_1\mathbb{T}_1\dot{w}\mathbb{U}_1$ of $\mathbb{G}_1$ lifts to a decomposition $\mathbb{G}_h = \coprod_{w\in W_{\mathcal{O}}} \mathbb{G}_{h,w}$, with $\mathbb{G}_{h,w} = \mathbb{U}_h\mathbb{T}_h\dot{w}\mathbb{K}_h^1\mathbb{U}_h$, $\mathbb{K}_h^1 = (\mathbb{U}_h^-)^1 \cap w^{-1}(\mathbb{U}_h^-)^1 w$ [CI18, Lemma 8.6]. We then have the locally closed decomposition $\Sigma = \coprod_{w\in W_{\mathcal{O}}} \Sigma_w$, where

$$\Sigma_w = \{(x,x',y) \in F\mathbb{U}_h \times F\mathbb{U}_h \times \mathbb{G}_{h,w} \colon xF(y) = yx'\}.$$

is $T_h \times T_h$-stable. Further, let

$$\widehat{\Sigma}_w = \{(x,x',y_1,\tau,z,y_2) \in F\mathbb{U}_h \times F\mathbb{U}_h \times \mathbb{U}_h \times \mathbb{T}_h \times \mathbb{K}_h^1 \times \mathbb{U}_h \colon xF(y_1\tau\dot{w}zy_2) = y_1\tau\dot{w}zy_2x'\}.$$

where $\dot{w} \in \mathbb{G}_h$ is an (arbitrary but from now on fixed) lift of w. It has a $T_h \times T_h$-action by

$$(t,t') \colon (x,x',y_1,\tau,z,y_2) \mapsto (txt^{-1}, t'x't'^{-1}, ty_1t^{-1}, t\tau\dot{w}t'^{-1}\dot{w}^{-1}, t'zt'^{-1}, t'y_2t'^{-1}). \tag{3.4}$$

Then the map $\widehat{\Sigma}_w \to \Sigma_w$ given by $(x,x',y_1,\tau,z,y_2) \mapsto (x,x',y_1\tau zy_2)$ is a $T_h\times T_h$-equivariant Zariski-locally trivial fibration. All in all, as in [Lus04], using (3.3) it is enough to show that

$$\sum_i (-1)^i \dim_{\overline{\mathbb{Q}}_\ell} H_c^i(\widehat{\Sigma}_w)_{\theta^{-1},\theta'} = \begin{cases} 1 & \text{if } w \in W_{\mathcal{O}}^F \text{ and } \theta' = \theta\circ \mathrm{ad}(w) \\ 0 & \text{otherwise.}\end{cases} \tag{3.5}$$

So far we were essentially following [Lus04, 1.9], but now we have to deviate.

3.2. **Emptyness of certain $\widehat{\Sigma}_w$.** Let $w \in W_{\mathcal{O}}$. As in [Lus04, 1.9], make the change the variables $xF(y_1) \mapsto x$, $x'F(y_2)^{-1} \mapsto x'$. We thus may rewrite

$$\widehat{\Sigma}_w = \{(x,y_1,\tau,z,y_2) \in F\mathbb{U}_h \times \mathbb{U}_h \times \mathbb{T}_h \times \mathbb{K}_h^1 \times \mathbb{U}_h \colon xF(\tau\dot{w}z) \in y_1\tau\dot{w}zy_2F\mathbb{U}_h\} \tag{3.6}$$

with the $T_h \times T_h$-action still given by (3.4).

Lemma 3.5. *Assume that there exists some $2 \leq i \leq n$ such that $[w(i)] > [w(i-1)+1] > 1$. Then $\widehat{\Sigma}_w = \varnothing$.*

Proof. We may assume $h = 1$, and hence we may ignore $z \in \mathbb{K}_h^1$ whose image in $\mathbb{G}_1$ is 1. We use the identification of $\mathbb{G}_1$ with the closed subgroup of GL_n from Section 3.1. Write $y_i = y_{i,1}y_{i,2}$ with $y_{1,1}, y_{2,2} \in \mathbb{U}_1 \cap F\mathbb{U}_1$ and $y_{1,2}, y_{2,1} \in \mathbb{U}_1 \cap F\mathbb{U}_1^-$. Replacing x by $y_{1,1}^{-1}x$ and putting $y_{2,2}$ into the $F\mathbb{U}_1$ on the right hand side, we are reduced to show that there are no $(x, y_{1,2}, y_{2,1}, \tau) \in F\mathbb{U}_1 \times (\mathbb{U}_1 \cap F\mathbb{U}_1^-) \times (\mathbb{U}_1 \cap F\mathbb{U}_1^-) \times \mathbb{T}_1$ with

$$\dot{w}^{-1}\tau y_{1,2}^{-1}xF(\tau\dot{w}) \in y_{2,1}F(\mathbb{U}_1).$$

Replacing everything by appropriate conjugates resp. inverses, it suffices to show that there are no $(x,y,y_{2,1},\tau') \in F\mathbb{U}_1 \times (\mathbb{U}_1 \cap F\mathbb{U}_1^-) \times (\mathbb{U}_1 \cap F\mathbb{U}_1^-) \times \mathbb{T}_1$ satisfying

$$\dot{w}^{-1}yxF(\dot{w}) \in \tau' y_{2,1}F\mathbb{U}_1.$$

For a $n\times n$-matrix X, let $X_{i,j}$ denote its (i,j)th entry. Consider the closed subset

$$M = \{X \in \mathbb{G}_1 \colon X_{i,i} \in \mathbb{G}_m \,\forall\, 2 \leq i \leq n \text{ and } X_{i,j} = 0 \,\forall\, n \geq i > j > 1\}$$

of $\mathbb{G}_1$. We have

$$\mathbb{U}_1 \cap F\mathbb{U}_1^- = \{X \in \mathbb{G}_1 \colon X_{i,i} = 1 \,\forall i \text{ and } X_{i,j} = 0 \,\forall\, (i,j) \text{ with } j \neq 1 \text{ or } i \neq j\}$$

One easily checks that $\mathbb{T}_1 \cdot (\mathbb{U}_1 \cap F\mathbb{U}_1^-) \cdot F\mathbb{U}_1 \subseteq M$. Thus it suffices to check that

$$\dot{w}^{-1}MF(\dot{w}) \cap M = \varnothing.$$

For $X \in \mathbb{G}_1$ (and even more generally for $X \in \mathrm{GL}_n$ and F replaced by F_0 as in Section 3.1), one has the formula

$$(\dot{w}^{-1}XF(\dot{w}))_{i,j} = X_{w(i),[w(j-1)+1]}. \tag{3.7}$$

Let $2 \leq i \leq n$ be such that $[w(i)] > [w(i-1)+1] > 1$. Then for $X \in M$, the (i,i)th diagonal entry of $\dot{w}^{-1}XF(\dot{w})$ is

$$(\dot{w}^{-1}XF(\dot{w}))_{i,i} = X_{w(i),[w(i-1)+1]} = 0,$$

by definition of M. This shows that $X \notin M$ and we are done. □

As mentioned in Section 2.1.3, $W_{\mathcal{O}}^F = \langle w_0^{n_0} \rangle$. Clearly, no element from $W_{\mathcal{O}}^F$ satisfies the condition in Lemma 3.5. Thus Lemma 3.5 implies (3.5) for all w satisfying the condition in the lemma.

3.3. **An extension of action.** It remains to show (3.5) for all $w \in W_{\mathcal{O}} \subseteq S_n$ for which there is no $2 \leq i \leq n$ satisfying $[w(i)] > [w(i-1)+1] > 1$. Consider the closed subgroup

$$H_w = \{(t,t') \in \mathbb{T}_h \times \mathbb{T}_h \colon \dot{w}^{-1}t^{-1}F(t)\dot{w} = t'^{-1}F(t') \text{ centralizes } \mathbb{K}_h = \mathbb{U}_h^- \cap \dot{w}^{-1}\mathbb{U}_h^-\dot{w} \ \}$$

of $\mathbb{T}_h \times \mathbb{T}_h$. It contains $T_h \times T_h$. It is easy to check that the action of $T_h \times T_h$ on Σ_w extends to an action of H_w given by the formula

$$(t,t')\colon (x, y_1, \tau, z, y_2) \mapsto (F(t)xF(t)^{-1}, F(t)y_1F(t)^{-1}, t\tau\dot{w}t'^{-1}\dot{w}^{-1}, t'zt'^{-1}, F(t')y_2F(t')^{-1}).$$

Lemma 3.6. *Let $1 \neq w \in W_{\mathcal{O}}$. Assume that there is no $2 \leq i \leq n$ with $[w(i)] > [w(i-1)+1] > 1$. Then there is a proper Levi subgroup L of $\mathbf{G}_{\breve{K}}$ containing $\mathbf{T}_{\breve{K}}$ such that if $\mathbb{L}_h$ denotes the corresponding subgroup of $\mathbb{G}_h$, then $\mathbb{K}_h \subseteq \mathbb{L}_h$.*

Proof. First we prove the following claim: there is an $s \in \mathbb{Z}_{\geq 1}$ and a sequence $0 =: i_0 < 1 \leq i_1 < \cdots < i_{s-1} < i_s := n$ of integers such that for each $1 \leq j \leq s$, and for each $i_{j-1}+1 \leq i \leq i_j$ (if $j > 1$) resp. for each $1 \leq i \leq i_1$ (if $j = 1$), one has $w(i) = n - i_{j-1} - (i_j - i)$. Indeed, find the $1 \leq i_1 \leq n$ such that $w(i_1) = n$. It follows from the condition on w that $w(i_1 - 1) = n-1$, ..., $w(1) = n - (i_1 - 1)$. The maximal value which w has on $\{i_1+1, \dots, n\}$ is $n - i_1$. Find the $i_1 + 1 \leq i_2 \leq n$ such that $w(i_2) = n - i_1$. It follows from the condition on w that $w(i_2 - 1) = n - i_1 - 1, \dots, w(i_1+1) = n - i_2 + 1$. Then, proceed inductively until $i_s = n$ is reached. The claim is proven.

Note that $i_1 < n$, as $i_1 = n$ would imply $w = 1$, whereas $w \neq 1$ is assumed in the lemma. Let L be the (proper) Levi subgroup of $\mathbf{GL}_{n,\breve{K}} = \mathbf{G}_{\breve{K}}$ containing $\mathbf{T}_{\breve{K}}$ of type $(i_1, i_2 - i_1, \dots, i_s - i_{s-1})$. From the claim it easily follows that $\mathbb{K}_h = \mathbb{U}_h^- \cap w^{-1}\mathbb{U}_h^- w \subseteq \mathbb{L}_h$. □

For $i = 1, 2$ we have the composed maps

$$\pi_i \colon H_w \subseteq \mathbb{T}_h \times \mathbb{T}_h \to \mathbb{T}_h \to \mathbb{T}_1,$$

where the middle map is the projection to the i-th component, and the last map is the natural projection. For $1 \leq i \neq j \leq n$, let $\alpha_{i,j}$ denote the root of $\mathrm{GL}_{n,\mathbb{F}_q}$ corresponding to (i,j)th matrix entry. Recall from Section 3.1 that $\mathbb{T}_1 \subseteq \mathbb{G}_1 \subseteq \mathrm{GL}_{n,\mathbb{F}_q}$ and that $\mathbb{T}_1$ is the diagonal (and in fact elliptic with respect to the Frobenius F_0) torus of $\mathrm{GL}_{n,\mathbb{F}_q}$. Let $\alpha_{i,j}$ be the roots of $\mathbb{T}_1$ in $\mathrm{GL}_{n,\mathbb{F}_q}$ corresponding to (i,j)th entry.

Lemma 3.7. *Let $\delta \colon \mathbb{Z}/n\mathbb{Z} \to \{0,1\}$ be a non-zero function, and let $\chi \colon \mathbb{G}_m \to \mathbb{T}_1$ be the cocharacter $X \mapsto \mathrm{diag}(X^{\delta(1)}, \dots, X^{\delta(n)})$. Then $S_\chi := \{t \in \mathbb{T}_1 \colon t^{-1}F(t) \in \mathrm{im}(\chi)\}$ is a one-dimensional subgroup of $\mathbb{T}_1$. Let $1 \leq j < i \leq n$. If δ does not factor as $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/\gcd(n, i-j)\mathbb{Z} \to \{0,1\}$, then the connected component S_χ° of S_χ is not contained in the subtorus $\ker(\alpha_{i,j})$ of $\mathbb{T}_1$.*

In particular, if for any divisor $d > 1$, δ does not factor as $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/d\mathbb{Z} \to \{0,1\}$, then S_χ° is a not contained in any of the subtori $\ker(\alpha_{i,j})$ $(1 \leq i \neq j \leq n)$ of $\mathbb{T}_1$.

Proof. Assume that δ does not factor through $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/\gcd(n, i-j)\mathbb{Z}$. As $\dim S_\chi = 1$, it suffices to show that $S_\chi \cap \ker(\alpha_{i,j})$ is finite. We write an element in $\mathbb{T}_1$ as an n-tuple $(t_k)_{k=1}^n$

corresponding to the diagonal matrix with entries $t_1, \dots, t_n$. We have $\mathrm{im}(\chi) = \{(a^{-\delta(k)})_{k=1}^n \in \mathbb{T}_1 : a \in \mathbb{G}_m\}$. Thus $(t_k)_{k=1}^n \in \mathbb{T}_1$ lies in S_χ if and only if $t_1^{-1}t_n^q = a^{-\delta(1)}$, $t_2^{-1}t_1^q = a^{-\delta(2)}$, ..., $t_n^{-1}t_{n-1}^q = a^{-\delta(n)}$. Thus S_χ is isomorphic to the one-dimensional subscheme of $\mathbb{G}_m^2$,

$$\{t_1, a \in \mathbb{G}_m^2 : t_1^{1-q^n} = a^{\delta(1)+\sum_{k=2}^n q^{n-k+1}\delta(k)}\}, \tag{3.8}$$

which is embedded into $\mathbb{T}_1$ by sending (t_1, a) to the tuple $(t_k)_{k=1}^n$ with $t_k = t_1^{q^{k-1}} a^{\sum_{\lambda=2}^k q^{k-\lambda}\delta(\lambda)}$. Thus the intersection $S_\chi \cap \ker(\alpha_{i,j})$ is the closed subscheme of (3.8) given by the equation $t_i = t_j$, i.e.,

$$t_1^{q^{i-1}-q^{j-1}} = a^{\sum_{k=2}^j q^{j-k}\delta(k) - \sum_{k=2}^i q^{i-k}\delta(k)}$$

Taking this to $(q^n - 1)$-th power, taking the equation in (3.8) to the power $q^{i-1} - q^{j-1}$, and equalizing the left hand sides, we deduce that on $S_\chi \cap \ker(\alpha_{i,j})$ we must have

$$a^{(q^{i-1}-q^{j-1})(\delta(1)+\sum_{k=2}^n q^{n-k+1}\delta(k))} = a^{(q^n-1)(\sum_{k=2}^i q^{i-k}\delta(k) - \sum_{k=2}^j q^{j-k}\delta(k))}.$$

Thus it suffices to show that

$$(q^{i-1} - q^{j-1})(\delta(1) + \sum_{k=2}^n q^{n-k+1}\delta(k)) \neq (q^n - 1)(\sum_{k=2}^i q^{i-k}\delta(k) - \sum_{k=2}^j q^{j-k}\delta(k)),$$

or equivalently, that

$$\sum_{k=i-1}^{n-1} q^k \delta(n-k+i) - \sum_{k=j-1}^{n-1} q^k \delta(n-k+j) \neq -\sum_{k=0}^{i-2} q^k \delta(i-k) + \sum_{k=0}^{j-2} q^k \delta(j-k),$$

or that

$$\sum_{k=0}^{n-1} q^k(\delta(i-k) - \delta(j-k)) \neq 0.$$

Assume this is wrong, and this sum is 0. All terms $\delta(i-k) - \delta(j-k)$ lie in the set $\{-1, 0, 1\}$ and hence q^{n-1} is bigger than the sum of the absolute values of the remaining summands. It follows that we must have $\delta(i-n+1) - \delta(j-n+1) = 0$. Then we may continue in the same way with q^{n-2} instead of q^{n-1}, etc. All in all we deduce that $\delta(i-k) = \delta(j-k)$ for all $k \in \mathbb{Z}/n\mathbb{Z}$. Or equivalently, that $\delta(k) = \delta(k + (i-j))$ for all $k \in \mathbb{Z}/n\mathbb{Z}$. But this is equivalent to saying that δ factors through $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/\mathrm{gcd}(n, i-j)\mathbb{Z}$, contradicting our assumption. ☐

Now let $1 \neq w \in W_{\mathcal{O}}$, such that there is no $2 \leq i \leq n$ with $[w(i)] > [w(i-1)+1] > 1$. Let $\mathbb{L}_h$ be as in Lemma 3.6 and let $1 \leq i_1 < n$ be the size of its first block (cf. the proof of Lemma 3.6). Let $\delta : \mathbb{Z}/n\mathbb{Z} \to \{0, 1\}$, $i \mapsto 1$ if $i \leq i_1$ and $i \mapsto 0$ otherwise. Let $\chi = (1_{i_1}, 0_{n-i_1})$ be the corresponding cocharacter. We have (again, cf. the proof of Lemma 3.6), $(w\delta)(i) = \delta(i + \lambda)$ for an appropriate $\lambda \in \mathbb{Z}/n\mathbb{Z}$. It follows from Lemma 3.6 and the definition of H_w that $\pi_1(H_w) \supseteq S_{w\chi}$ and $\pi_2(H_w) \supseteq S_\chi$. Hence also

$$\pi_1(H_w^\circ) \supseteq S_{w\chi}^\circ \quad \text{and} \quad \pi_2(H_w^\circ) \supseteq S_\chi^\circ. \tag{3.9}$$

From this together with Lemma 3.7 it follows that for $i = 1, 2$, $\pi_i(H_w^\circ)$ is not contained in any of the $\ker(\alpha_{i,j} : \mathbb{T}_1 \to \mathbb{G}_m)$ $(1 \leq i \neq j \leq n)$. Hence it also holds for $\pi_i(H_{w,\mathrm{red}}^\circ)$, where $H_{w,\mathrm{red}}^\circ$ is the reductive part of H_w° (it is a torus). As in [Lus04] we have now $H_c^*(\widehat{\Sigma}_w)_{\theta^{-1},\theta'} = H_c^*(\widehat{\Sigma}_w^{H_{w,\mathrm{red}}^\circ})_{\theta^{-1},\theta'}$,[4] and because $\pi_i(H_{w,\mathrm{red}}^\circ)$ is not contained in any of the $\ker(\alpha_{i,j})$, we have

$$\widehat{\Sigma}_w^{H_{w,\mathrm{red}}^\circ} \subseteq \{(1, 1, \tau, 1, 1) : \tau \in \mathbb{T}_h, F(\tau\dot{w}) = \tau\dot{w}\},$$

and (3.5) for $\widehat{\Sigma}_w$ easily follows (cf. [Lus04, 1.9, proof of claim (e)]).

[4] The fact that we are working with perfect schemes here does not affect the argument.

3.4. **Another extension of action.** It remains to deal with the case $w = 1$. We prove first a more general result, again generalizing Lusztig's method. The proof does not depend on special properties of GL_n and can be carried out for any group, so we put ourselves – until the end of Section 3.4 only – in the general setup of [CI19a]. Let $\mathbf{G}$ be a reductive group over K, which is split over $\breve{K}$, and let $\mathbf{T}$, $\mathbf{T}'$ be two maximal K-rational, $\breve{K}$-split tori in $\mathbf{G}$. There is a natural inclusion of the reduced Bruhat–Tits building $\mathscr{B}_K$ of $\mathbf{G}$ over K into the reduced Bruhat–Tits building $\mathscr{B}_{\breve{K}}$ of $\mathbf{G}$ over $\breve{K}$. Assume there is a point $\mathbf{y}$ in the intersection of $\mathscr{B}_K$ and the apartments of $\mathbf{T}$ and $\mathbf{T}'$ inside $\mathscr{B}_{\breve{K}}$. We have then the parahoric $\mathcal{O}_K$-model $P_{\mathbf{y}}$ of $\mathbf{G}$ attached to $\mathbf{y}$. Its $\mathcal{O}_{\breve{K}}$-points $P_{\mathbf{y}}(\mathcal{O}_{\breve{K}})$ form the parahoric subgroup of $\mathbf{G}(\breve{K})$ attached to $\mathbf{y}$, which is the stabilizer of $\mathbf{y}$. On $P_{\mathbf{y}}(\mathcal{O}_{\breve{K}})$ we have the descending Moy–Prasad filtration given by certain subgroups $P_{\mathbf{y}}(\mathcal{O}_{\breve{K}})^h$ ($h \geq 0$). Using the truncated loop group construction [CI19a, 2.6], for any $h \geq 1$ one can defined an affine perfectly finitely presented perfect $\mathbb{F}_q$-group $\mathbb{G}_h$ satisfying

$$\mathbb{G}_h(\overline{\mathbb{F}}_q) = P_{\mathbf{y}}(\mathcal{O}_{\breve{K}})/P_{\mathbf{y}}(\mathcal{O}_{\breve{K}})^{(h-1)+}$$

We denote by F the (geometric) Frobenius on $\mathbb{G}_{h,\overline{\mathbb{F}}_q}$ and its closed subgroups. To a closed subgroup $\mathbf{H} \subseteq \mathbf{G}_{\breve{K}}$ one can naturally attach a closed subgroup $\mathbb{H}_h \subseteq \mathbb{G}_h$, by first taking the schematic closure of $\mathbf{H}$ in $P_{\mathbf{y}}$ and then applying L_h^+. We write $\mathbb{H}_h^r := \ker(\mathbb{H}_h \to \mathbb{H}_r)$ for the kernel of the natural projection. We also write $G_h := \mathbb{G}_h(\mathbb{F}_q)$ and $H_h := \mathbb{H}_h(\mathbb{F}_q)$ (the latter only if $\mathbb{H}_h$ is defined over $\mathbb{F}_q$). For more details we refer to [CI19a, 2.6].

Let $\mathbf{U}, \mathbf{U}^-$ resp. $\mathbf{U}', \mathbf{U}'^-$ be the unipotent radicals of a pair of opposite Borel subgroups containing $\mathbf{T}$ resp. $\mathbf{T}'$ and let $\mathbb{U}_h, \mathbb{U}_h^-$ resp. $\mathbb{U}_h', \mathbb{U}_h'^-$ be the corresponding subgroups of $\mathbb{G}_h$. We have the closed perfect subscheme of $\mathbb{G}_h$,

$$S_{T,U,h} = \{g \in \mathbb{G}_h \colon g^{-1}F(g) \in F\mathbb{U}_h\}$$

with a $G_h \times T_h$-action by $(\gamma, t)\colon g \mapsto \gamma g t$. Similarly we have the perfect subscheme $S_{T',U',h} \subseteq \mathbb{G}_h$. As already above, Lusztig's scheme $\Sigma = \{(x, x', y) \in F\mathbb{U}_h \times F\mathbb{U}_h' \times \mathbb{G}_h \colon xF(y) = yx'\}$ is very useful to compute the inner product between the virtual G_h-representations obtained from $S_{T,U,h}$ and $S_{T',U',h}$. More precisely, for $\overline{\mathbb{Q}}_\ell^\times$-valued characters θ resp. θ' of T_h resp. T_h' we have

$$\left\langle H_c^*(S_{T,U,h})_\theta, H_c^*(S_{T',U',h})_{\theta'} \right\rangle_{G_h} = \dim_{\overline{\mathbb{Q}}_\ell} H_c^*(\Sigma)_{\theta^{-1},\theta'}$$

To study $H_c^*(\Sigma)$ Lusztig in [Lus04] (and many authors in follow-up articles) used a locally closed decomposition $\Sigma = \coprod_{w \in W_{\mathbf{y}}(T',T)} \Sigma_w$, where $W_{\mathbf{y}}(T',T) = \{\mathbb{T}_1 v \colon v^{-1}\mathbb{T}_1 v = \mathbb{T}_1'\}$ is the transporter from $\mathbb{T}_1'$ to $\mathbb{T}_1$ in $\mathbb{G}_1$ (= reductive quotient of the special fiber of $P_{\mathbf{y}}$) conjugating $\mathbb{T}_1'$ to $\mathbb{T}_1$. Now, we generalize this construction in a substantial way.

Let $\mathbf{V}$ resp. $\mathbf{V}'$ be the unipotent radical of a second Borel subgroup containing $\mathbf{T}$ resp. $\mathbf{T}'$. We have the corresponding subgroups $\mathbb{V}_h$, $\mathbb{V}_h'$ of $\mathbb{G}_h$. For $v \in W_{\mathbf{y}}(T',T)$ we have the corresponding preimage $\mathbb{V}_h\mathbb{T}_h v \mathbb{K}_{V,V',h}^1 \mathbb{V}_h'$ (with $\mathbb{K}_{V,V',r} := \mathbb{V}_h'^- \cap v^{-1}\mathbb{V}_h^- v$) of the Schubert cell in $\mathbb{G}_1$ attached to v. We consider the following generalization of $\widehat{\Sigma}_w$, Σ_w from [Lus04]:

$$\Sigma_{V,V',v} := \{(x, x', y) \in F\mathbb{U}_h \times F\mathbb{U}_h' \times \mathbb{V}_h\mathbb{T}_h\dot{v}\mathbb{K}_{V,V',h}^1\mathbb{V}_h' \colon xF(y) = yx'\}$$

$$\widehat{\Sigma}_{V,V',v} := \{(x, x', y', \tau, z, y'') \in F\mathbb{U}_h \times F\mathbb{U}_h' \times \mathbb{V}_h \times \mathbb{T}_h \times \mathbb{K}_{V,V',h}^1 \times \mathbb{V}_h' \colon xF(y'\tau\dot{v}zy'') = y'\tau\dot{v}zy''x'\}$$

which have the same alternating sum of cohomology. The action of $T_h \times T_h'$ on $\widehat{\Sigma}_{V,V',v}$ is given by

$$(t, t')\colon (x, x', y', \tau, z, y'') \mapsto (txt^{-1}, t'x't'^{-1}, ty't^{-1}, t\tau\dot{v}t'^{-1}\dot{v}^{-1}, t'zt'^{-1}, t'y''t'^{-1}). \tag{3.10}$$

and by a similar formula for $\Sigma_{V,V',v}$. There is an element $v_0 = v_0(V, V') \in W_{\mathbf{y}}(T', T)$, such that the (generalized) Bruhat cell $\mathbb{V}_1\mathbb{T}_1 v_0 \mathbb{V}_1'$ is generic in $\mathbb{G}_1$, i.e., $v_0^{-1}\mathbb{V}_h v_0 = \mathbb{V}_h'^-$. For this

v_0 we have $\mathbb{K}_{V,V',h} = 1$. We can write $y' \in \mathbb{V}_h$ and $y'' \in \mathbb{V}'_h$ as

$$y' = y'_1 y'_2 \text{ and } y'' = y''_1 y''_2 \quad \text{with} \quad y'_1 \in \mathbb{U}_h \cap \mathbb{V}_h,\ y'_2 \in \mathbb{U}_h^- \cap \mathbb{V}_h,\ y''_1 \in \mathbb{U}'^-_h \cap \mathbb{V}'_h,\ y''_2 \in \mathbb{U}'_h \cap \mathbb{V}'_h$$

(the action of $(t,t') \in \mathbb{T}_h \times \mathbb{T}'_h$ is given by $(t,t')\colon (y'_1, y'_2, y''_1, y''_2) \mapsto (ty'_1t^{-1}, ty'_2t^{-1}, t'y''_1t'^{-1}, t'y''_2t'^{-1})$). Changing the variables $xF(y'_1) \mapsto x$, $x'F(y''_2)^{-1} \mapsto x'$ we can rewrite

$$\widehat{\Sigma}_{V,V',v_0} := \left\{ \begin{array}{r} (x, y'_1, y'_2, \tau, y''_1, y''_2) \in F\mathbb{U}_h \times (\mathbb{U}_h \cap \mathbb{V}_h) \times (\mathbb{U}_h^- \cap \mathbb{V}_h) \times \mathbb{T}_h \times (\mathbb{U}'^-_h \cap \mathbb{V}'_h) \times (\mathbb{U}'_h \cap \mathbb{V}'_h)\colon \\ xF(y'_2\tau\dot{v}y''_1) \in y'_1y'_2\tau\dot{v}y''_1y''_2F\mathbb{U}'_h \end{array} \right\}.$$

Let

$$H'_{v_0} = \{(t,t') \in \mathbb{T}_h \times \mathbb{T}'_h\colon F(t)t^{-1} = \dot{v}_0 t' F(t')^{-1} \dot{v}_0^{-1} \text{ centralizes } \mathbb{U}_h^- \cap \mathbb{V}_h \text{ and } \dot{v}_0(F\mathbb{U}'^-_h \cap \mathbb{V}'_h)\dot{v}_0^{-1}\}.$$

Define an action of H'_{v_0} on $\widehat{\Sigma}_{V,V',v_0}$ by

$$(t,t')\colon (x, y'_1, y'_2, \tau, y''_1, y''_2) \mapsto (F(t)xF(t)^{-1}, F(t)y'_1F(t)^{-1}, ty'_2t^{-1}, t\tau\dot{v}_0t'^{-1}\dot{v}_0^{-1}, t'y''_1t'^{-1}, t'y''_2t'^{-1}).$$

It extends the action of $T_h \times T'_h$. We have to show that it is well-defined, i.e., that if $(x, y'_1, y'_2, \tau, y''_1, y''_2) \in \widehat{\Sigma}_{V,V',v_0}$, then the same holds for $(t,t').(x, y'_1, y'_2, \tau, y''_1, y''_2)$. This reduces to show that

$$xF(y'_2)F(\tau)F(\dot{v}_0)F(y''_1) \in y'_1F(t)^{-1}ty'_2\tau\dot{v}_0y''_1y''_2F\mathbb{U}'_ht'^{-1}F(t')$$

Writing $y'' = y''_1y''_2 \in \mathbb{V}'_h$ as $y'' =: y''_3y''_4$ with $y''_3 \in \mathbb{V}' \cap F\mathbb{U}'^-_h$ and $y''_4 \in \mathbb{V}' \cap F\mathbb{U}'_h$, it suffices to check that $F(t)t^{-1}$ commutes with $y'_2 \in \mathbb{U}_h^- \cap \mathbb{V}_h$ and that $t'^{-1}F(t') = \dot{v}_0^{-1}F(t)t^{-1}\dot{v}_0$ commutes with $y''_3 \in \mathbb{V}' \cap F\mathbb{U}'^-_h$. This holds by definition of H'_{v_0}. We thus have proven the following lemma.

Lemma 3.8. *The action of $T_h \times T'_h$ on $\widehat{\Sigma}_{V,V',v_0}$ extends to an action of the algebraic group H'_{v_0} given by the above formula.*

Returning now to the proof Theorem 3.1, we apply Lemma 3.8 to our $\mathbf{G}$ (= inner form of $\mathbf{GL}_n$), the point $\mathbf{y} = \mathbf{x}_b$, the diagonal (elliptic unramified) torus $\mathbf{T} = \mathbf{T}'$ of $\mathbf{G}$, the subgroup $\mathbf{U} = \mathbf{U}'$ of unipotent upper triangular matrices and to $\mathbf{V} = \mathbf{U}$, $\mathbf{V}' = \mathbf{U}^-$, $v_0 = 1$, in which case $\mathbb{U}_h^- \cap \mathbb{V}_h = 1$ and $\dot{v}_0(F\mathbb{U}_h^- \cap \mathbb{V}'_h)\dot{v}_0^{-1}$ is contained in $\mathbb{L}_h$ for some proper Levi subgroup $\mathbf{L}$ of $\mathbf{G}_{\breve{K}}$, and hence the reductive part $H'^{\circ}_{1,\mathrm{red}}$ of the connected component of H'_1 is big enough in the sense of Lemma 3.7. Note finally that $\Sigma_1 = \Sigma_{U,U,1}$ is a closed subscheme of $\Sigma_{U,U^-,1}$, (in fact, on $\Sigma_{U,U^-,1}$, y varies in $\mathbb{T}_h\mathbb{U}_h\mathbb{U}_h^-$ and Σ_1 is given by the closed condition $y \in \mathbb{T}_h\mathbb{U}_h\mathbb{U}_h^{-,1}$). Let $\widetilde{\Sigma}_1$ denote the pullback of Σ_1 along $\widehat{\Sigma}_{U,U^-,1} \to \Sigma_{U,U^-,1}$. It has the same alternating sum of cohomology as Σ_1, and it is clearly stable under the action of H'_1. Thus the argument from [Lus04, 1.9] applies to $\widetilde{\Sigma}_1$ and we obtain $H_c^*(\widehat{\Sigma}_1) = H_c^*(\Sigma_1) = H_c^*(\widetilde{\Sigma}_1) = H_c^*(\widetilde{\Sigma}_1^{H'^{\circ}_{1,\mathrm{red}}})$ (and the same for $\theta^{-1} \otimes \theta'$-isotypic parts), hence verifying (3.5) in the only remaining case $w = 1$. Theorem 3.1 is proven.

4. A variation of the Mackey formula

We work with exactly the same setup and notation as in Section 3 (and in particular Section 3.1). Recall the presentation (3.2) of X_h. Then

$$X_{h,n'} = \{g \in \mathbb{G}_h\colon g^{-1}F(g) \in \mathbb{U}_h^{-,1} \cap F\mathbb{U}_h^1\},$$

is a closed perfect subscheme of X_h, stable under the action of $G_h \times T_h$. In fact, X_h has a stratification in locally-closed pieces [CI19b] indexed by divisors r of n', and $X_{h,n'}$ is precisely the closed stratum.

Theorem 4.1. *Let $\theta\colon T_h \to \overline{\mathbb{Q}}_\ell^\times$ be a character. Assume that $p > n$, and that $\theta|_{T_h^1}$ has trivial stabilizer in $W_{\mathcal{O}}^F$. Then*

$$\left\langle R_{T_h}^{G_h}(\theta), H_c^*(X_{h,n})_\theta \right\rangle_{G_h} = 1 \tag{a}$$

and

$$\left\langle H_c^*(X_{h,n})_\theta, H_c^*(X_{h,n})_\theta \right\rangle_{G_h} = 1. \tag{b}$$

We prove Theorem 4.1 in Sections 4.1-4.3. From Theorems 3.1 and 4.1 we deduce:

Corollary 4.2. *Under the assumptions of Theorem 4.1, $H_c^*(X_{h,n})_\theta$ is up to sign an irreducible representation of G_h, and $H_c^*(X_{h,n})_\theta \cong R_{T_h}^{G_h}(\theta)$.*

Remark 4.3. There are two general principles used in the proofs of Theorems 3.1 and 4.1: (1) If X is a reasonably nice (perfect) scheme over a field with an action of an algebraic group H, then the induced action of H in $H_c^i(X, \overline{\mathbb{Q}}_\ell)$ is trivial ($\forall i \geq 0$), and (2) if moreover X is (the perfection of a) quasi-projective scheme over a finite field, H is a torus, and $\alpha\colon X \to X$ is a finite order automorphism commuting with the H-action, then $\mathrm{tr}(\alpha, H_c^*(X, \overline{\mathbb{Q}}_\ell)) = \mathrm{tr}(\alpha, H_c^*(X^H, \overline{\mathbb{Q}}_\ell))$. For our purposes (2) is stronger than (1), which for example does not allow quantitative results in Section 4.2. Theorem 4.1 is less general than Theorem 3.1 because in its proof we have to use both (2) and (1), whereas in the proof of Theorem 3.1 we manage to work with (2) only.

4.1. **Proof Theorem 4.1**(a)**: multiplicative extension.** Parts of the proof follows along the same lines as the proof of Theorem 3.1, thus we will be slightly sketchy below. Similar as in [CI18, Lemma 7.12] we have an isomorphism

$$(\mathbb{U}_h^1 \cap F\mathbb{U}_h^1) \times (\mathbb{U}_h^{-,1} \cap F\mathbb{U}_h^1) \to F\mathbb{U}_h^1, \quad (g, x) \mapsto g^{-1}xF(g).$$

Thus we have $G_h \times T_h$-equivariantly $X_{h,n} \cong S_{h,n}/(\mathbb{U}_h^1 \cap F\mathbb{U}_h^1)$, where

$$S_{h,n} = \{g \in \mathbb{G}_h \colon g^{-1}F(g) \in F\mathbb{U}_h^1\}$$

and $G_h \times T_h$ acts on $S_{h,n}$ by $g, t\colon x \mapsto gxt$, and $(\mathbb{U}_h^1 \cap F\mathbb{U}_h^1)$ by right multiplication. Hence $H_c^*(X_{h,n})_\theta \cong H_c^*(S_{h,n})_\theta$. Using Lang's theorem, we have $T_h \times T_h$-equivariantly

$$G_h \backslash (S_h \times S_{h,n}) \xrightarrow{\sim} \Sigma_{(1,n)} := \{(x, x', y) \in F\mathbb{U}_h \times F\mathbb{U}_h^1 \times \mathbb{G}_h \colon xF(y) = yx'\}$$

where $T_h \times T_h$ acts on $\Sigma_{(1,n)}$ by $(t, t')\colon (x, x', y) \mapsto (txt^{-1}, t'x't'^{-1}, tyt'^{-1})$. For $w \in W_{\mathcal{O}}$ let $\Sigma_{(1,n),w} = \{(x, x', y) \in \Sigma_{(1,n)} \colon y \in \mathbb{G}_{h,w}\}$ (it is an $T_h \times T_h$-stable locally closed perfect subscheme) and putting $\mathbb{K}_h = \mathbb{U}_h \cap \dot{w}^{-1}\mathbb{U}_h^- \dot{w}$, we let

$$\widehat{\Sigma}_{(1,n),w} = \{(x, y_1, \tau, z, y_2) \in F\mathbb{U}_h \times \mathbb{U}_h \times \mathbb{T}_h \times \mathbb{K}_h^1 \times \mathbb{U}_h \colon xF(y_1\tau\dot{w}zy_2) \in y_1\tau\dot{w}zy_2F\mathbb{U}_h^1\}$$

be the Zariski-locally trivial covering of $\Sigma_{(1,n),w}$ with $T_h \times T_h$-action given by the same formula as in (3.4). As in Section 3.1 we have $\left\langle R_{T_h}^{G_h}(\theta), H_c^*(X_{h,n})_\theta \right\rangle_{G_h} = \sum_{w \in W_{\mathcal{O}}} \dim H_c^*(\widehat{\Sigma}_{(1,n),w})_{\theta^{-1},\theta}$. We claim that

$$\dim H_c^*(\widehat{\Sigma}_{(1,n),w})_{\theta^{-1},\theta} = \begin{cases} 1 & \text{if } w = 1 \\ 0 & \text{otherwise,} \end{cases} \tag{4.1}$$

which implies the first formula of Theorem 4.1. Assume first w satisfies the condition in Lemma 3.5. Then $\widehat{\Sigma}_{(1,n),w} \subseteq \widehat{\Sigma}_w = \varnothing$ and we are done. Now assume that $w = 1$. Then $\mathbb{G}_{h,1} = \mathbb{U}_h \cdot \mathbb{T}_h \cdot \mathbb{U}_h^{-,1}$, so

$$\widetilde{\Sigma}_{(1,n),1} = \{(x, x', y_1, \tau, z) \in F\mathbb{U}_h \times F\mathbb{U}_h^1 \times \mathbb{U}_h \times \mathbb{T}_h \times \mathbb{U}_h^{-,1} \colon xF(y_1\tau z) \in y_1\tau zx'\}$$

is another a Zariski-locally trivial covering of $\Sigma_{(1,n),1}$ (with obvious $T_h \times T_h$-action), so that $H_c^*(\widehat{\Sigma}_{(1,n),1})_{\theta^{-1},\theta}) = H_c^*(\widetilde{\Sigma}_{(1,n),1})_{\theta^{-1},\theta})$, and we can replace $\widehat{\Sigma}_{(1,n),1}$ by $\widetilde{\Sigma}_{(1,n),1}$. We can write uniquely $z = z_1 z_2$ with $z_1 \in \mathbb{U}_h^{-,1} \cap F\mathbb{U}_h^{-,1}$ and $z_2 \in \mathbb{U}_h^{-,1} \cap F\mathbb{U}_h^1$ and make the change of variables $xF(y_1) \mapsto x$, $z_2 x' \mapsto x'$ (note that the latter indeed works, as $z_2 \in F\mathbb{U}_h^1$!), so that

$$\widetilde{\Sigma}_{(1,n),1} \cong \{(x, y_1, \tau, z_1, z_2) \in F\mathbb{U}_h \times \mathbb{U}_h \times \mathbb{T}_h \times (\mathbb{U}_h^{-,1} \cap F\mathbb{U}_h^{-,1}) \times (\mathbb{U}_h^{-,1} \cap F\mathbb{U}_h^1) \colon xF(\tau z_1 z_2) \in y_1 \tau z_1 F\mathbb{U}_h^1\}$$

The $T_h \times T_h$-action on $\widetilde{\Sigma}_{(1,n),1}$ is given by

$$(t, t') \colon (x, y_1, \tau, z_1, z_2) \mapsto (txt^{-1}, ty_1t^{-1}, t\tau t'^{-1}, t'z_1t'^{-1}, t'z_2t'^{-1}).$$

Let

$$H_1 := \{(t, t') \in \mathbb{T}_h \times \mathbb{T}_h \colon t^{-1}F(t) = t'^{-1}F(t') \text{ centralizes } \mathbb{U}_h \cap F\mathbb{U}_h^-\}.$$

As is checked similar to computations in Sections 3.3 and 3.4, H_1 acts on $\widetilde{\Sigma}_{(1,n),1}$ by

$$(t, t') \colon (x, y_1, \tau, z_1, z_2) \mapsto (F(t)xF(t)^{-1}, F(t)y_1F(t)^{-1}, t\tau t'^{-1}, t'z_1t'^{-1}, t'z_2t'^{-1})$$

(and this action extends the action of $T_h \times T_h$). As $\mathbb{U}_h \cap F\mathbb{U}_h^-$ is contained in the subgroup of $\mathbb{G}_h$ attached to a proper rational Levi subgroup $L \subsetneq \mathbf{G}_{\breve{K}}$, it follows that the connected component $H_{1,\mathrm{red}}^\circ$ of the reductive part of H is big enough (in the sense of Lemma 3.7), so that we deduce $\dim H_c^*(\widetilde{\Sigma}_{(1,n),1})_{\theta^{-1},\theta} = 1$, and hence (4.1) for $w = 1$ (this is the same argument as at the end of Section 3.4).

4.2. **Proof Theorem 4.1**(a)**: additive extension.** It remains to show (4.1) for $1 \neq w \in W_{\mathcal{O}}$ not satisfying the condition from Lemma 3.5. Assume w is such an element. Let

$$H_w^1 := \{(t, t') \in \mathbb{T}_h^1 \times \mathbb{T}_h^1 \colon \dot{w}^{-1}t^{-1}F(t)\dot{w} = t'^{-1}F(t') \text{ centralizes } \mathbb{K}_h^1\}$$

In $\widehat{\Sigma}_{(1,n),w}$ make the change of variables $xF(y_1) \mapsto x$, so that

$$\widehat{\Sigma}_{(1,n),w} = \{(x, y_1, \tau, z, y_2) \in F\mathbb{U}_h \times \mathbb{U}_h \times \mathbb{T}_h \times \mathbb{K}_h^1 \times \mathbb{U}_h \colon xF(\tau\dot{w}z) \in y_1\tau\dot{w}zy_2F(\mathbb{U}_h^1 y_2^{-1})\}$$

with $T_h \times T_h$-action given by the same formula as in (3.4). Now

$$(t, t') \colon (x, y_1, \tau, z, y_2) \mapsto (F(t)xF(t)^{-1}, F(t)y_1F(t)^{-1}, t\tau\dot{w}t'^{-1}\dot{w}^{-1}, t'zt'^{-1}, F(t')y_2F(t')^{-1})$$

defines an action of H_w^1 on $\widehat{\Sigma}_{(1,n),w}$. In order to check this we have to show that if $(t, t') \in H_w^1$ and $(x, y_1, \tau, z, y_2) \in \widehat{\Sigma}_{(1,n),w}$, then also $(t, t').(x, y_1, \tau, z, y_2) \in \widehat{\Sigma}_{(1,n),w}$. After elementary cancellations this reduces to show that

$$xF(\tau\dot{w}zt'^{-1}) \in y_1F(t)^{-1}t\tau\dot{w}zt'^{-1}F(t')y_2F(t')^{-1}F(\mathbb{U}_h^1F(t')y_2^{-1}F(t')^{-1})$$

But as $t' \in \mathbb{T}_h^1$, we have $\mathbb{U}_h^1F(t')y_2^{-1}F(t')^{-1} = \mathbb{U}_h^1y_2^{-1}$, so this reduces to show that

$$xF(\tau\dot{w}z) \in y_1F(t)^{-1}t\tau\dot{w}zt'^{-1}F(t')y_2F(t'^{-1}\mathbb{U}_h^1y_2^{-1}t').$$

Again, using $t' \in \mathbb{T}_h^1$, we deduce that $t'^{-1}\mathbb{U}_h^1y_2^{-1}t' = \mathbb{U}_h^1y_2^{-1}$, and hence that $(t, t').(x, y_1, \tau, z, y_2) \in H_w^1$.

Via the isomorphism $T_h \xrightarrow{\sim} U_L/U_L^h$ mapping a diagonal matrix $t = (t_i)_{i=1}^n$ to its upper left entry t_1, we identify T_h with U_L/U_L^h and T_h^1 with U_L^1/U_L^h. By Lemma 2.12 (and the discussion in Section 2.1.3), the condition that $\theta|_{T_h^1}$ has trivial stabilizer in $W_{\mathcal{O}}^F = \langle w_0^{n_0}\rangle$ translates into the condition that the restriction of θ to U_L^1/U_L^h does not factor through any of the norm maps $N_{n/n_0s} \colon U_L^1/U_L^h \twoheadrightarrow U_{K_{n_0s}}^1/U_{K_{n_0s}}^h$, where $1 \leq s < n'$ goes through all divisors of n'. Let $H_w^{1,\circ}$ be the connected component of H_w^1.

Lemma 4.4. *If (t, t') varies through $(T_h^1 \times T_h^1) \cap H_w^{1,\circ}$, then $t_1^{-1}t_1'$ varies (at least) through all elements of $\ker(\mathrm{N}_{n/n_0s})$ for some divisor $1 \leq s < n'$ of n' (s depends on w).*

Before proving this lemma, we use it to finish the proof of Theorem 4.1(a). Indeed, by assumption on θ for each divisor $s < n'$ of n' there is an element $x = x_s \in \ker \mathrm{N}_{n/n_0 s} \subseteq U_L^1/U_L^h$ such that $\theta(x_s) \neq 1$. By Lemma 4.4 we can find a divisor $s < n'$ of n' and an element $(t,t') \in (T_h^1 \times T_h^1) \cap H_w^{1,\circ}$ such that $t_1^{-1}t_1' = x_s$, and hence $\theta(t_1^{-1}t_1') \neq 1$. Seeing θ as a character of T_h^1 again, this simply means that $\theta(t) \neq \theta(t')$, and it follows that the $T_h \times T_h$-character $\theta^{-1} \otimes \theta$ is non-trivial on $(T_h^1 \times T_h^1) \cap H_w^{1,\circ}$. But the induced action of a connected algebraic group in the cohomology of a separated scheme of finite type over $\overline{\mathbb{F}}_q$ is trivial [DL76, Corollary 6.5] and the same holds after perfection, hence for each $i \geq 0$ we have $H_c^i(\widehat{\Sigma}_{(1,n),w})_{\theta^{-1},\theta} = 0$, which shows claim (4.1) for all remaining elements w, and hence also Theorem 4.1(a).

Remark 4.5. The basic idea in the above arguments is the same as in [DL76, Lemma 6.7]. This gives hope to generalize them to a far more general setup (e.g. all unramified maximal tori in all reductive groups).

Towards the proof of Lemma 4.4, for positive integers s, r such that s divides r, we define morphisms of perfect $\mathbb{F}_q$-schemes

$$\mathrm{Nm}_{r/s} \colon \mathbb{W}_h^{\times,1} \to \mathbb{W}_h^{\times,1} \quad x \mapsto \mathrm{Nm}_{r/s}(x) := \prod_{i=0}^{\frac{r}{s}-1} \sigma^s(x).$$

Proof of Lemma 4.4. By assumption, w does not satisfy the condition of Lemma 3.5. Thus by Lemma 3.6 there is a proper Levi subgroup $L \subseteq \mathbf{G}_{\breve{K}}$ containing $\mathbf{T}_{\breve{K}}$, such that if $\mathbb{L}_h$ is the corresponding subgroup of $\mathbb{G}_h$, we have $\mathbb{K}_h \subseteq \mathbb{L}_h$. We may assume L is maximal, so that there is an $1 \leq m \leq n-1$, such that $L = \mathrm{GL}_{m,\breve{K}} \times \mathrm{GL}_{n-m,\breve{K}}$ (upper left and lower right diagonal blocks). More precisely, we may (and do) choose that m to be the i_1 from the proof of Lemma 3.6. In fact, by our explicit description of $W_{\mathcal{O}} \cong \prod_{i=1}^{n_0} S_{n'}$ in Section 3.1, we see that as $w \in W_{\mathcal{O}}$, our choice $m = w^{-1}(n)$ must be an integer dividing n_0. Let $\chi = (1_m, 0_{n-m})$ be a cocharacter of $\mathbf{T}_{\breve{K}}$. From the explicit form of w determined in Lemma 3.6, we see that $w\chi = (0_{n-m}, 1_m)$. Let $\mathbb{Y}_{h,\chi} \subseteq \mathbb{T}_h$ denote the subgroup of $\mathbb{T}_h$ corresponding to the subgroup $\mathrm{im}(\chi)$ of $\mathbf{T}_{\breve{K}}$ (thus $\mathbb{Y}_{h,\chi} \cong \mathbb{W}_h^{\times}$). As $\mathrm{im}(\chi)$ centralizes L, $\mathbb{Y}_{h,\chi}$ centralizes $\mathbb{L}_h$ and hence also $\mathbb{K}_h$. Thus

$$H_w^1 \supseteq H_{w,\chi}^1 := \{(t,t') \in \mathbb{T}_h^1 \times \mathbb{T}_h^1 \colon \dot{w}^{-1}t^{-1}F(t)\dot{w} = t'^{-1}F(t') \in \mathbb{Y}_{h,\chi}^1\},$$

and the same inclusion holds if we take connected components on both sides. Thus we may replace H_w^1 by $H_{w,\chi}^1$. Let $(t,t') \in \mathbb{T}_h^1 \times \mathbb{T}_h^1$. Write $t = \mathrm{diag}(t_i)_{i=1}^n$ and $t' = \mathrm{diag}(t_i')_{i=1}^n$ with $t_i, t_i' \in \mathbb{W}_h^{\times,1}$. Let x be a $\mathbb{W}_h^{\times,1}$-"coordinate" on $\mathbb{Y}_{h,\chi}^1$ (it is an $(h-1)$-tuple of $\mathbb{A}^1$-coordinates). We can eliminate all "coordinates" t_i ($i \neq n$) and t_i' ($i \neq m$) by expressing them through x and t_m, t_n'. More precisely,

$$H_{w,\chi}^1 \cong \{(x, t_m, t_n') \in \mathbb{W}_h^{\times,1} \times \mathbb{W}_h^{\times,1} \times \mathbb{W}_h^{\times,1} \colon \sigma^n(t_n)t_n^{-1} = \mathrm{Nm}_{m/1}(x) = \sigma^n(t_m')t_m'^{-1}\}.$$

We see that on $H_{w,\chi}^1$, the equation $\sigma(t_n^{-1}t_m') = t_n^{-1}t_m'$ holds, so that $t_n^{-1}t_m'$ can take only finitely many values. On $H_{w,\chi}^{1,\circ}$ we must in particular have $t_n = t_m'$, or equivalently (using the expression of t_1, t_1' through t_n, t_m') we have

$$\sigma^{n-m}(t_1) = t_1' \tag{4.2}$$

on $H_{w,\chi}^{1,\circ}$. Furthermore, $H_{w,\chi}^{1,\circ}$ is contained in the perfect scheme (isomorphic to)

$$\{(x, t_n) \in \mathbb{W}_h^{\times,1} \times \mathbb{W}_h^{\times,1} \colon \sigma^n(t_n)t_n^{-1} = \mathrm{Nm}_{m/1}(x)\}$$

Now let $1 \leq g = \gcd(m,n) < n$. As $\sigma^n(t_n)t_n^{-1} = \mathrm{Nm}_{n/1}(\sigma(t_n)t_n^{-1}) = \mathrm{Nm}_{g/1}(\mathrm{Nm}_{n/g}(\sigma(t_n)t_n^{-1}))$, and $\mathrm{Nm}_{m/1}(x) = \mathrm{Nm}_{g/1}(\mathrm{Nm}_{m/g}(x))$, we have $\mathrm{Nm}_{g/1}(\mathrm{Nm}_{n/g}(\sigma(t_n)t_n^{-1})\,\mathrm{Nm}_{m/g}(x)^{-1}) = 1$ on this scheme, and hence $\mathrm{Nm}_{n/g}(\sigma(t_n)t_n^{-1})\,\mathrm{Nm}_{m/g}(x)^{-1}$ is discrete on it. Hence $H_{w,\chi}^{1,\circ}$ is contained in the perfect scheme (isomorphic to)

$$\{(x,t_n) \in \mathbb{W}_h^{\times,1} \times \mathbb{W}_h^{\times,1}\colon\ \mathrm{Nm}_{n/g}(\sigma(t_n)t_n^{-1}) = \mathrm{Nm}_{m/g}(x)\}.$$

After replacing σ by σ^g, Lemma 4.7 shows that this last perfect $\overline{\mathbb{F}}_q$-scheme is connected, so that it is equal to $H_{w,\chi}^{1,\circ}$. On $H_{w,\chi}^{1}$, $t_1 = \sigma(t_n)$, so that (after replacing $\sigma(x)$ by x which is harmless here), we have

$$H_{w,\chi}^{1,\circ} \cong \{(x,t_1) \in \mathbb{W}_h^{\times,1} \times \mathbb{W}_h^{\times,1}\colon\ \mathrm{Nm}_{n/g}(\sigma(t_1)t_1^{-1}) = \mathrm{Nm}_{m/g}(x)\}$$

Now $H_{w,\chi}^{1,\circ} \cap (T_h^1 \times T_h^1)$ is the locus in $H_{w,\chi}^{1,\circ}$ defined by $x = 1$. Thus we deduce

$$H_{w,\chi}^{1,\circ} \cap (T_h^1 \times T_h^1) = \{(t,t') \in T_h^1 \times T_h^1\colon t_1' = \sigma^{n-m}(t_1) \text{ and } \mathrm{Nm}_{n/g}(\sigma(t_1)t_1^{-1}) = 1\}$$

(recall that in T_h^1, t is determined by its first entry t_1). Note that $\mathrm{Nm}_{n/g}(\sigma(t_1)t_1^{-1}) = 1$ simply means that $\mathrm{Nm}_{n/g}(t_1)$ is σ-stable. As m is divisible by n_0, $T_h^1 = \mathbb{W}_h^{\times,1}(\mathbb{F}_{q^n}) = U_L^1/U_L^h$ and the restriction of $\mathrm{Nm}_{n/g}$ to $T_h^1 \cong U_L^1/U_L^h$ is $\mathrm{N}_{n/g}$, the lemma now follows from Lemma 4.6. □

Lemma 4.6. *Suppose $(n,p)=1$. Let $1 \le m \le n-1$ and put $g = \gcd(n,m)$. Let*

$$\alpha\colon \{y \in U_L^1/U_L^h\colon\ \mathrm{N}_{n/g}(y) \in U_K^1/U_K^h\} \to U_L^1/U_L^h, \quad y \mapsto \sigma^{n-m}(y)y^{-1}.$$

Then $\mathrm{im}(\alpha) = \ker(\mathrm{N}_{n/g}\colon U_L^1/U_L^h \to U_{K_g}^1/U_{K_g}^h)$.

Proof. For arbitrary $a \in \mathbb{Z}$ we have

$$\mathrm{N}_{n/g}(y) \in U_K^1/U_K^h \Rightarrow \mathrm{N}_{n/g}(\sigma^a(y)y^{-1}) = \sigma^a(\mathrm{N}_{n/g}(y))\,\mathrm{N}_{n/g}(y)^{-1} = 1 \Rightarrow \sigma^a(y)y^{-1} \in \ker(\mathrm{N}_{n/g}).$$

Hence $\mathrm{im}(\alpha) \subseteq \ker(\mathrm{N}_{n/g})$. Let $y \in \ker(\alpha)$. Then $\mathrm{N}_{n/g}(y)$ is rational and $\sigma^{n-m}(y) = y$ and $\sigma^n(y) = y$. The last two equalities together are equivalent to $\sigma^g(y) = y$. Hence $\mathrm{N}_{n/g}(y) = \frac{n}{g}y$, and hence y is rational (as $\mathrm{N}_{n/g}(y)$ is, and $(n,p)=1$). Conversely, if y is rational, then surely $y \in \ker(\alpha)$. Thus $\ker(\alpha) = U_K^1/U_K^h$. Now the source of α is the preimage under the (surjective) map $\mathrm{N}_{n/g}\colon U_L^1/U_L^h \to U_{K_g}^1/U_{K_g}^h$ of U_K^1/U_K^h, hence the size of the source of α is $\#\ker(\mathrm{N}_{n/g}) \cdot \#(U_K^1/U_K^h)$. Thus $\#\,\mathrm{im}(\alpha) = \frac{\#(\text{source of }\alpha)}{\#\ker(\alpha)} = \#\ker(\mathrm{N}_{n/g})$. As we already know that $\mathrm{im}(\alpha) \subseteq \ker(\mathrm{N}_{n/g})$ and both sets are finite, we are done. □

For positive integer s define the $\mathbb{F}_q$-morphism

$$\mathrm{tr}_{s/1}\colon \mathbb{G}_a \to \mathbb{G}_a, \quad x \mapsto \mathrm{tr}_{s/1}(x) := \sum_{i=0}^{s-1} x^{q^i}.$$

Lemma 4.7. *Let $r > s \ge 1$ be coprime integers. Suppose $p > s$. The closed perfect subscheme*

$$R_h = \{(y,x) \in \mathbb{W}_h^{\times,1} \times \mathbb{W}_h^{\times,1}\colon\ \mathrm{Nm}_{r/1}(\sigma(y)y^{-1}) = \mathrm{Nm}_{s/1}(x)\}$$

of $\mathbb{W}_h^{\times,1} \times \mathbb{W}_h^{\times,1}$ is connected. More precisely, for $h \ge 2$ the fibers of $R_h \to R_{h-1}$ are isomorphic to $\mathbb{A}^1$ (note that R_1 is a point).

Proof. It suffices to prove that the fibers of $R_h \to R_{h-1}$ are isomorphic to $\mathbb{A}^1$. The fibers of $R_h \to R_{h-1}$ are isomorphic to closed sub-(perfect schemes) of $\mathbb{G}_a^2$ (with coordinates X, Y) given by the equation

$$C\colon\ \mathrm{tr}_{r/1}(Y^q - Y) = \mathrm{tr}_{s/1}(X) + \mathit{const}.$$

where *const* is a constant term depending on the point in R_{h-1}. As $\mathrm{tr}_{r/1}(Y^q - Y) = Y^{q^r} - Y$, one can eliminate this constant term by changing the variable $Y + c \mapsto Y$ (for an appropriate $c \in \overline{\mathbb{F}}_q$). So we assume *const* $= 0$. We may assume $s > 1$, as otherwise we obviously have $C \cong$

$\mathbb{A}^1$. Put $r_0 := r$, $r_1 := s$ and define $r_i \in \mathbb{Z}_{\geq 0}$ $(i \geq 2)$, $\gamma_i \in \mathbb{Z}_{>0}$ $(i \geq 1)$ by $r_i = \gamma_{i+1} r_{i+1} + r_{i+2}$ and $r_{i+2} < r_{i+1}$ for $i \geq 0$. Say this stops at $i = \alpha$, that is $r_{\alpha+1} = \gcd(r,s) = 1$, $r_{\alpha+2} = 0$.

Via the change of variables $X + Y^{q^{r-s+1}} - Y \mapsto X$, C is isomorphic to the curve

$$C_1\colon \operatorname{tr}_{r_1/1}(X) = \operatorname{tr}_{r_2/1}(Y^q - Y).$$

Now $\operatorname{tr}_{r_2/1}(Y^q - Y) = Y^{q^{r_2}} - Y$, so that we can successively make a series of changes of variables of the form $Y + X^{q^\beta} \mapsto Y$ for appropriate $\beta \in \mathbb{Z}_{\geq 0}$, to eliminate all powers of X with exponent greater than q^{r_2}. This shows that C_1 is isomorphic to the curve

$$C_2\colon \operatorname{tr}_{r_3/1}(X) + \gamma_2 \operatorname{tr}_{r_2/1}(X) = \operatorname{tr}_{r_2/1}(Y^q - Y).$$

Now we successively apply the perfection of Lemma 4.8 to C_2 and the initial tuple of integers $(a_1, b_1, c_1, d_1) = (1, \gamma_2, r_3, r_2)$. Consider the operation $(a,b,c,d) \mapsto (b, a + b\gamma, r, c)$ on quadruples of integers (satisfying $0 < c < d$) where $0 \leq r < d$ and $\gamma > 0$ are defined by $d = \gamma c + r$. First of all, if $a, b > 0$, then also $b, a + b\gamma > 0$. Moreover, the operation leaves invariant the sum of products of 1st and 3rd and of 2nd and 4th entries: $ac + bd = br + (a + b\gamma)c$. Thus if (a_i, b_i, c_i, d_i) is the tuple after $(i-1)$th iteration step, we have $a_i c_i + b_i d_i = r_3 + \gamma_2 r_2 = r_1 = s < p$. Also we have $c_i = r_{i+2}$, $d_i = r_{i+1}$, and hence $0 < c_i < d_i$ as long as $i \leq \alpha - 1$. All this implies that $0 < a_i, b_i, c_i, d_i < p$ and $0 < c_i < d_i$ for each $i = 1, 2, \dots, \alpha - 1$, so that Lemma 4.8 indeed applies in each step, as long as $i < \alpha$. The last application (for $i = \alpha - 1$) produces a quadruple $(a_\alpha, b_\alpha, c_\alpha, d_\alpha) = (b_{\alpha-1}, a_{\alpha-1} + b_{\alpha-1} d_{\alpha-1}, 0, 1)$ and C_2 is thus isomorphic to the curve

$$b_\alpha X = Y^q - Y,$$

and by the same preservation property of the sum $ac + bd$ we have that still $0 < b_\alpha < p$ holds. Thus this curve is isomorphic to $\mathbb{A}^1_{\mathbb{F}_q}$, and we are done. □

The following lemma works for schemes of finite type over $\mathbb{F}_p$, so we denote (in this lemma only) by $\mathbb{A}_{\mathbb{F}_p}$ the usual affine space over $\mathbb{F}_p$.

Lemma 4.8. *Let a, b, c, d be positive integers with $a, b < p$ and $c < d$. Write $d = \gamma c + r$ with $0 \leq r < c$. Then the curve in $\mathbb{A}^2_{\mathbb{F}_p}$ given by the equation*

$$C_1 :\ a \operatorname{tr}_{c/1}(x) + b \operatorname{tr}_{d/1}(x) = \operatorname{tr}_{d/1}(y^q - y)$$

is $\mathbb{F}_p$-isomorphic to the curve in $\mathbb{A}^2_{\mathbb{F}_p}$ given by the equation

$$C_2 :\ b \operatorname{tr}_{r/1}(x) + (a + b\gamma) \operatorname{tr}_{c/1}(x) = \operatorname{tr}_{c/1}(y^q - y).$$

Proof. Make the change of variables $x + b^{-1}(y^q - y) \mapsto x$ (by assumption $b < p$, as $0 < c < d$). Thus C_1 is isomorphic to the curve

$$C_1' :\ a \operatorname{tr}_{c/1}(x) + ab^{-1} \operatorname{tr}_{c/1}(y^q - y) + b \operatorname{tr}_{d/1}(x) = 0.$$

Via the change of variables $-a^{-1} b y \mapsto y$, C_1' gets isomorphic to

$$C_1'' :\ a \operatorname{tr}_{c/1}(x) + b \operatorname{tr}_{d/1}(x) = \operatorname{tr}_{c/1}(y^q - y).$$

We have $\operatorname{tr}_{d/1}(y^q - y) = y^{q^d} - y$. Thus we may successively make the changes of variables of the form $y + x^{q^\alpha}$ (for appropriate $\alpha \in \mathbb{Z}_{\geq 0}$), to eliminate all powers of x with exponent greater than q^c. This does not affect the first summand $a \operatorname{tr}_{c/1}(x)$ and after all these changes C_1'' gets isomorphic to the curve

$$C_1''' :\ a \operatorname{tr}_{c/1}(x) + b(\gamma \operatorname{tr}_{c/1}(x) + \operatorname{tr}_{r/1}(x)) = \operatorname{tr}_{c/1}(y^q - y),$$

which is the same as C_2. □

4.3. **Proof Theorem 4.1**(b). Again, we work in the setup of Section 3.1. For $w \in W_{\mathcal{O}}$ put
$\widehat{\Sigma}_{(n,n),w} := \{(x, y_1, \tau, z, y_2) \in F\mathbb{U}_h^1 \times F\mathbb{U}_h^1 \times \mathbb{U}_h \times \mathbb{T}_h \times \mathbb{K}_h^1 \times \mathbb{U}_h \colon xF(y_1\tau\dot{w}z) \in y_1\tau\dot{w}zy_2F(\mathbb{U}_h^1 y_2^{-1})\}$,
and

$$\widetilde{\Sigma}_{(n,n),1} := \{(x, x', y_1, \tau, z) \in F\mathbb{U}_h^1 \times F\mathbb{U}_h^1 \times \mathbb{U}_h \times \mathbb{T}_h \times \mathbb{U}_h^{-,1} \colon xF(y_1\tau z) = y_1\tau z x'\}$$

with natural $T_h \times T_h$-actions (like in Section 4.1). Similar as in the beginning of Section 4.1 it suffices to check that

$$H_c^*(\widehat{\Sigma}_{(n,n),w})_{\theta^{-1},\theta} = 0 \quad \text{for } 1 \neq w \in W_{\mathcal{O}}, \text{ and}$$
$$\dim H_c^*(\widetilde{\Sigma}_{(n,n),1})_{\theta^{-1},\theta} = 1.$$

Consider first the case $w \neq 1$. As $x \in F\mathbb{U}_h^1$ and y_1 varies in $\mathbb{U}_h$, we can not make the change of variables $xF(y_1) \mapsto x$ as in the proof of Theorem 4.1(a). However we can define an action of H_w^1 on $\widehat{\Sigma}_{(n,n),w}$ by

$$(t, t') \colon (x, y_1, \tau, z, y_2) \mapsto$$
$$(F(t)xF(y_1)F(t)^{-1}F(F(t)y_1^{-1}F(t)^{-1}), F(t)y_1F(t)^{-1}, t\tau\dot{w}t'^{-1}\dot{w}^{-1}, t'zt'^{-1}, F(t')y_2F(t')^{-1})$$

Note that $F(t)xF(y_1)F(t)^{-1}F(F(t)y_1^{-1}F(t)^{-1}) \in F\mathbb{U}_h^1$ (on the one side it is contained in $F\mathbb{U}_h$ as $x, F(y_1) \in F\mathbb{U}_h$; on the other side it must lie in $\mathbb{G}_h^1$ as $t, x \in \mathbb{G}_h^1$). The proof that this indeed is an action goes exactly the same way as in Section 4.2. The rest of the argument for $\widehat{\Sigma}_{(1,n),w}$ goes then through exactly as for $\widehat{\Sigma}_{(1,n),w}$ in Section 4.2.

Now let $w = 1$. As $x, x', z \in \mathbb{G}_h^1$, the equation defining $\widetilde{\Sigma}_{(n,n),1}$ modulo $\mathbb{G}_h^1$ reduces to $F(y_1\tau) = y_1\tau$. From this it easily follows that $y_1 \in \mathbb{G}_h^1$. Hence $y_1 \in \mathbb{U}_h^1$. Hence the change of variables $xF(y_1) \mapsto x$ makes sense (such that the new variable x again lives in $F\mathbb{U}_h^1$), and the rest of the argument for $\widetilde{\Sigma}_{(n,n),1}$ goes exactly the same way as for $\widetilde{\Sigma}_{(1,n),1}$ in Section 4.1.

5. Cuspidality

We go back to the setup of Section 2.6. Let θ be a smooth character of $T = L^\times$ of level $h \geq 1$ in general position. Recall that the induced character of T_h is again denoted by θ, and that it is also in general position. By Corollary 3.3, $R_{T_h}^{G_h}(\theta)$ is up to sign an irreducible $G_{\mathcal{O}}$-representation, hence in particular $R_T^G(\theta)$ is up to sign a genuine representation. We write $|R_{T_h}^{G_h}(\theta)|$ resp. $|R_T^G(\theta)|$ for the genuine representation among $\pm R_{T_h}^{G_h}(\theta)$ resp. $\pm R_T^G(\theta)$.

Theorem 5.1. *Let θ be a smooth character of $T = L^\times$ in general position. Then $|R_T^G(\theta)|$ is a finite direct sum of irreducible supercuspidal representations of G.*

Proof. There are many (essentially equivalent) ways to deduce this theorem from Proposition 5.2. By [Bus90, Theorem 1] it suffices to prove that $\Xi_\theta := \mathrm{Ind}_{ZG_{\mathcal{O}}}^G |R_{T_h}^{G_h}(\theta)|$ is admissible. Let $K \subseteq G$ be a compact open subgroup. We have to show that $(\Xi_\theta)^K$ is finite-dimensional. Conjugating K into $G_{\mathcal{O}}$ and making it smaller if necessary, we may assume that $K = \ker(G_{\mathcal{O}} \to G_r)$ for some $r > 0$. Frobenius reciprocity gives

$$(\Xi_\theta)^K = \bigoplus_{g \in G_{\mathcal{O}}Z\backslash G/K} |R_{T_h}^{G_h}(\theta)|^{ZG_{\mathcal{O}} \cap gKg^{-1}}.$$

Thus we have to show that there are only finitely many non-vanishing summands on the right. If $\mathbf{S}$ denotes a maximal split torus of $\mathbf{G}$ whose apartment in $\mathscr{B}_K(\mathbf{G}) = \mathscr{B}_{\breve{K}}^{F_b}$ contains the vertex stabilized by $G_{\mathcal{O}}$, then by the rational Iwahori-Bruhat decomposition, $ZG_{\mathcal{O}}\backslash G/G_{\mathcal{O}} \cong X_*(\mathbf{S}/\mathbf{Z})_{\mathrm{dom}}$. Hence any element of $ZG_{\mathcal{O}}\backslash G/K$ has a representative of the form $g = \varpi^\mu x$ with $x \in G_{\mathcal{O}}$, $\mu \in X_*(\mathbf{S})_{\mathrm{dom}}$. Now K is normal in $G_{\mathcal{O}}$, so $gKg^{-1} = \varpi^\mu K \varpi^{-\mu}$ only depends on μ.

Moreover, any coset $ZG_{\mathcal{O}}\varpi^{\mu}G_{\mathcal{O}}$ contains only finitely many cosets from $ZG_{\mathcal{O}}\backslash G/K$. Thus it suffices to show that for all but finitely many $\mu \in X_*(T_0/\mathbf{Z})_{\mathrm{dom}}$, $|R_{T_h}^{G_h}(\theta)|^{ZG_{\mathcal{O}}\cap\varpi^{\mu}K\varpi^{-\mu}} = 0$. It is easy to see that for all but finitely many such μ, there is a proper K-rational parabolic subgroup $\mathbf{G}$ with unipotent radical $\mathbf{N}$, such that if $N = \mathbf{N}(K)$, then $N \cap G_{\mathcal{O}} \subseteq \varpi^{\mu}K\varpi^{-\mu}$. Thus it is enough to show that for each such N we have $|R_{T_h}^{G_h}(\theta)|^{N\cap G_{\mathcal{O}}} = 0$. As by Corollary 3.3, $|R_{T_h}^{G_h}(\theta)| = \pm R_{T_h}^{G_h}(\theta)$ is a genuine representation, it suffices to show that $R_{T_h}^{G_h}(\theta)^{N\cap G_{\mathcal{O}}} = 0$ (we have the natural map of Grothendieck groups of smooth representations with $\overline{\mathbb{Q}}_\ell$-coefficients $r\colon K_0(G_{\mathcal{O}}) \to K_0(N \cap G_{\mathcal{O}})$ induced by restriction, and $R_{T_h}^{G_h}(\theta)^{N\cap G_{\mathcal{O}}} = 0$ means $\langle \mathbf{1}, r(R_{T_h}^{G_h}(\theta))\rangle = 0$, where $\mathbf{1}$ is the trivial representation). This follows from Proposition 5.2. □

Proposition 5.2. *Let N be the unipotent radical of a proper K-rational parabolic subgroup of $\mathbf{G}$. Then*

$$R_{T_h}^{G_h}(\theta)^{N\cap G_{\mathcal{O}}} = 0.$$

We prove Proposition 5.2 in Section 5.1 in the case $\kappa = 0$, and in Section 5.2 in general. The proof in the general case is more technical, but follows exactly the same idea as in the special case $\kappa = 0$. For reasons of clarity we explain the special case first.

The explicit description in Lemma 5.6 used in the proof of Proposition 5.2 is – to the author's knowledge – already new for classical Deligne–Lusztig varieties, i.e., when $h = 1$ (and $\kappa = 0$). In particular, for the Coxeter-type variety for $\mathrm{GL}_{n,\mathbb{F}_q}$ it gives an alternative and much more direct proof of the cuspidality result for Coxeter-type varieties [DL76, Theorem 8.3], which is the last statement of the following corollary to Proposition 5.2.

Corollary 5.3. *Let $n \geq 1$, and let X be a Deligne–Lusztig variety of Coxeter type attached to $\mathrm{GL}_{n,\mathbb{F}_q}$. Let θ be an* arbitrary *character of $T_1 \cong \mathbb{F}_{q^n}^{\times}$, the corresponding $\mathrm{GL}_n(\mathbb{F}_q)$-representation $R(\theta)$ realized in the cohomology of X, satisfies $R(\theta)^{N(\mathbb{F}_q)} = 0$, for any unipotent radical N of a proper rational parabolic subgroup of GL_n. In particular, if θ is in general position, the genuine $\mathrm{GL}_n(\mathbb{F}_q)$-representation $|R(\theta)|$ is irreducible cuspidal.*

Remark 5.4. The proof of Proposition 5.2 is based on the key lemmas 5.6, 5.8, where the quotient $N_h\backslash X_h$ is determined. If $\overline{X}_h$ denotes the quotient of X_h by the T_h-action, then (the cohomology of) $N_h\backslash\overline{X}_h$ can probably be computed in big generality by same methods as in [Lus76, (2.10)] (where Coxeter-type Deligne–Lusztig varieties in the flag manifold for a reductive group $\mathbb{G}$ over $\mathbb{F}_q$ are studied, in particular $h = 1$). Proofs of Lemmas 5.6, 5.8 suggest that the quotients $N_h\backslash X_h$ are harder to understand than $N_h\backslash\overline{X}_h$.

For $h = 1$ and $\mathbb{G}$ arbitrary reductive group over $\mathbb{F}_q$, a quotient similar to $N_h\backslash X_h$ appears in [BR06, Section 3.2], [Dud13] and a couple of related articles. The methods used in [BR06] are indirect in the sense that the structure of the *tame* fundamental group of the multiplicative group $\mathbb{G}_{m,\overline{\mathbb{F}}_q}$ is used. In our situation these methods would only apply in the case $h = 1$, because for $h > 1$ the natural covering $X_h \to \overline{X}_h$ is wildly ramified.

5.1. **Proof of Proposition 5.2 for $\kappa = 0$.** For N to have a convenient form, we take $b = 1$. We also take $\dot{w}_0$ to be the element b_0 as in (3.1). Then literally $G = \mathrm{GL}_n(K)$, $G_{\mathcal{O}} = \mathrm{GL}_n(\mathcal{O}_K)$ and $G_h = \mathrm{GL}_n(\mathcal{O}_K/(\varpi^h))$. Let N_h denote the image of $N \cap G_{\mathcal{O}}$ in G_h. We can assume that N is the unipotent radical of a *maximal* proper parabolic subgroup. Moreover, conjugating N if necessary, we may assume that there is an $1 \leq i_0 \leq n-1$, such that N consists of matrices $u = (u_{ij})_{1\leq i,j\leq n}$ with $u_{ii} = 1 \forall 1 \leq i \leq n$, and $u_{ij} = 0$ unless $i = j$ or ($1 \leq i \leq i_0$ and $n - i_0 < j \leq n$). As the actions of G_h and T_h on X_h commute, we have

$$R_{T_h}^{G_h}(\theta)^{N_h} = H_c^*(X_h)_\theta^{N_h} = H_c^*(N_h\backslash X_h)_\theta.$$

We introduce some convenient notation. For $r \geq 1$, and an $r \times r$-matrix g, let $|g| := \det g$. For $x = (x_i)_{i=1}^r \in \mathbb{W}_h(R)^r$, write $g_r(x)$ for the $r \times r$-matrix whose ith column is $\sigma^{i-1}(x)$. Also we put

$$Y_{r,h} := \{x \in \mathbb{W}_h^r \colon |g_r(x)| \in \mathbb{W}_h^\times\}.$$

This is a functor on $\mathrm{Perf}_{\mathbb{F}_q}$, which is represented by an affine perfectly finitely presented perfect $\mathbb{F}_q$-scheme. The description of X_h in [CI18, 7.2] says precisely that $X_h \subseteq Y_{n,h}$ is a closed subset defined by the condition $|g_n(x)| \in \mathbb{W}_h^\times(\mathbb{F}_q)$.

Lemma 5.5. *The quotient $N_h \backslash X_h$ exists as a perfect scheme, and $X_h \to N_h \backslash X_h$ is finite étale.*

Proof. X_h is affine and N_h finite, so the quotient exists. As the action has no fixed points the last claim also follows. □

Lemma 5.6. *There is an isomorphism of perfect schemes*

$$\alpha \colon N_h \backslash X_h \to \left\{ (m, x') \in Y_{i_0,h} \times Y_{n-i_0,h} \colon \frac{|g_{i_0}(m)|}{|g_{n-i_0}(x')|^{\sum_{j=1}^{i_0-1} \sigma^j}} \in \mathbb{W}_h^\times(\mathbb{F}_q) \right\},$$

induced by $x = (x_i)_{i=1}^n \mapsto ((m_i(x))_{i=1}^{i_0}, (x_i)_{i=i_0+1}^n)$, where $m_i(x)$ is the $(n-i_0+1)\times(n-i_0+1)$-minor of $g_n(x)$ given by

$$m_i(x) := \begin{vmatrix} x_i & \sigma(x_i) & \dots & \sigma^{n-i_0}(x_i) \\ x_{i_0+1} & \sigma(x_{i_0+1}) & \dots & \sigma^{n-i_0}(x_{i_0+1}) \\ x_{i_0+2} & \sigma(x_{i_0+2}) & \dots & \sigma^{n-i_0}(x_{i_0+2}) \\ \dots & \dots & \dots & \dots \\ x_n & \sigma(x_n) & \dots & \sigma^{n-i_0}(x_n) \end{vmatrix}$$

Proof. It is clear that the assignment in the lemma defines an N_h-equivariant morphism $X_h \to (\mathbb{W}_h)^{i_0} \times (\mathbb{W}_h)^{n-i_0}$ (with trivial N_h-action on the right). Thus it induces a map $N_h \backslash X_h \to (\mathbb{W}_h)^{i_0} \times (\mathbb{W}_h)^{n-i_0}$.

A standard argument shows that for $x = (x_i)_{i=1}^n \in X_h(R)$ with corresponding $x' = (x_i)_{i=i_0+1}^n$ and $m = (m_i(x))_{i=1}^{i_0}$, one has that $g_{n-i_0}(x') \in \mathbb{W}_h^\times(R)$ (see e.g. [CI18, Lemma 6.13]). This combined with Lemma 5.7 below, shows that we also have $|g_{i_0}(m)| \in \mathbb{W}_h^\times(R)$. Thus (using Lemma 5.7 again), we see that α is well-defined.

To prove the lemma, it now suffices to check that α is an isomorphism of étale sheaves on $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$. First we check that as a map of étale sheaves, α is surjective. Let $R \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$. Let Z denote the target of α, and let $m = (m_i)_{i=1}^{i_0}, x' = (x_i')_{i=i_0+1}^n$ be a element of $Z(R)$. We construct a preimage $x = (x_i)_{i=1}^n \in X_h(R')$ for some étale R-algebra R'. Take $x_i = x_i'$ for $i_0 + 1 \leq i \leq n$. Now, we can find an (finite) étale R-algebra R', and for each $1 \leq i \leq i_0$, an $x_i = \sum_{j=0}^{h-1} [x_{i,j}] \varpi^j \in \mathbb{W}_h(R')$ such that

$$m_i = m_i(x) = \sum_{k=0}^{n-i_0} (-1)^{k+1} \sigma^k(x_i) \cdot |g_{n-i_0,k}(x')|, \tag{5.1}$$

holds in $\mathbb{W}_h(R')$, where $g_{n-i_0,k}(x')$ denotes the $(n-i_0) \times (n-i_0)$-matrix whose columns are $x', \sigma(x'), \dots, \widehat{\sigma^k(x')}, \dots, \sigma^{n-i_0}(x')$ (here $\widehat{\cdot}$ means that the vector $\cdot$ is omitted). Indeed, note that for $k = n - i_0$ and for $k = 0$, we have

$$|g_{n-i_0,n-i_0}(x')| = |g_{n-i_0,0}(x')| = |g_{n-i_0}(x')| \in \mathbb{W}_h^\times(\mathbb{F}_q) \tag{5.2}$$

Thus, fixing an i, and proceeding successively for $j = 0, 1, \dots, h-1$, we can take (5.1) modulo ϖ^{j+1} and resolve it for $x_{i,j}$, noting that each time to find a solution we need a (finite) étale extension of R. Thus α is an epimorphism of étale sheaves.

By Lemma 2.3 it remains to show that if R is an algebraically closed field, $\alpha(R)\colon (N_h\backslash X_h)(R) \to Z(R)$ is injective. With notation as above, for a fixed $1 \leq i \leq i_0$ and $x_{i,0}, x_{i,1}, \dots, x_{i,j-1}$, Equation (5.1) gives an equation for $x_{i,j}$ of degree precisely q^{n-i_0} (by (5.2)), which is separable (by (5.2) again). Doing this for each $1 \leq i \leq i_0$ and $0 \leq j < h$, we obtain precisely $q^{i_0(n-i_0)h}$ possible values for $x = (x_i)_{i=1}^n \in \mathbb{W}_h(R)^n$ which map to the given point $(m, x') \in Z(R)$. By Lemma 5.7 all those x automatically lie in $X_h(R)$. This shows that each fiber of the composition of $X_h(R) \to (N_h\backslash X_h)(R)$ with $\alpha(R)$ has precisely $q^{i_0(n-i_0)h} = \#N_h$ points, i.e., that $\alpha(R)$ is injective. The lemma is proven. □

Lemma 5.7. *Let $n \geq 2$, $1 \leq i_0 \leq n-1$. For an $\mathbb{F}_q$-algebra R and $x = (x_i)_{i=1}^n \in Y_{n,h}(R)$, let $m = (m_i(x))_{i=1}^{i_0} \in Y_{i_0,h}(R)$, $x' = (x_i)_{i=i_0+1}^n \in Y_{n-i_0,h}(R)$. Then*

$$|g_{i_0}(m)| = |g_n(x)| \cdot |g_{n-i_0}(x')|^{\sum_{j=1}^{i_0-1}\sigma^j} \tag{5.3}$$

Proof. For $v = (v_j)_{j=1}^r \in Y_{r,h}(R)$, and $1 \leq i \leq r$, let $v^{(i)} = (v_j)_{j=1;j\neq i}^n \in Y_{r-1,h}(R)$ denote the vector v with i-th coordinate omitted. The claim is tautological for $i_0 = 1$ (in particular, we may assume $n > 2$). We use induction on i_0. Expanding along the first column and using the induction hypothesis (for $n-1, i_0-1$), we get

$$|g_{i_0}(m)| = \sum_{i=1}^{i_0}(-1)^{i+1} m_i \sigma\left(|g_{i_0-1}(m^{(i)})|\right) = \sum_{i=1}^{i_0}(-1)^{i+1} m_i \sigma\left(|g_{n-1}(x^{(i)})| \cdot \prod_{j=1}^{i_0-2}\sigma^j\left(|g_{n-i_0}(x')|\right)\right)$$

To show that this equals the right hand side of (5.3) it suffices to show that

$$\sum_{i=1}^{i_0}(-1)^{i+1} m_i \sigma\left(|g_{n-1}(x^{(i)})|\right) = |g_n(x)| \cdot \sigma\left(|g_{n-i_0}(x')|\right). \tag{5.4}$$

This follows from a classical minor identity of Turnbull [Tur09]. We use the more modern source [Lec93]. Let us first recall some notation from [Lec93]. Let S be a ring (commutative, with 1). For $1 \leq i \leq n$, let $a_i, b_i \in S^n$. Then the $2 \times n$-tableau

$$T = \begin{array}{|cccc|} \hline a_1 & a_2 & \dots & a_n \\ \hline b_1 & b_2 & \dots & b_n \\ \hline \end{array} \in S$$

is the product of the determinants of the two $n \times n$-matrices A and B, where the i-th column of A resp. B is a_i resp. b_i. Similarly one defines an $s \times n$-tableau for each positive integer s. The entries of the tableau are the elements a_i, b_i. More generally we need tableaux with boxes containing some of the entries. Let T be a $s \times n$-tableau, let A be a subset of elements of T. For a permutation σ of elements of A, let $\sigma(T)$ denote the tableau obtained from T, where the elements of A were permuted by σ. Then the tableau $\tau = (T$ with boxes around entries in $A)$ is defined as the alternating sum $\sum_\sigma \operatorname{sgn}(\sigma)\sigma(T)$, where the sum is taken over the cosets of the symmetric group on A, modulo the subgroup, which leaves unchanged the rows of T. We give an example for $n = 4$, $s = 2$:

$$\begin{array}{|cccc|} \hline \boxed{a_1} & a_2 & a_3 & a_4 \\ \hline b_1 & b_2 & \boxed{b_3} & \boxed{b_4} \\ \hline \end{array} = \begin{array}{|cccc|} \hline a_1 & a_2 & a_3 & a_4 \\ \hline b_1 & b_2 & b_3 & b_4 \\ \hline \end{array} - \begin{array}{|cccc|} \hline b_3 & a_2 & a_3 & a_4 \\ \hline b_1 & b_2 & a_1 & b_4 \\ \hline \end{array} - \begin{array}{|cccc|} \hline b_4 & a_2 & a_3 & a_4 \\ \hline b_1 & b_2 & b_3 & a_1 \\ \hline \end{array}.$$

To continue with our proof, we take $S = \mathbb{W}_h(R)$. For $1 \leq i \leq n$, let $\mathbf{i} = (0_{i-1}, 1, 0_{n-i}) \in \mathbb{W}_h(R)^n$ denote the i-th coordinate vector. An easy computation shows that

$$|g_n(x)| \cdot \sigma\left(|g_{n-i_0}(x')|\right) - \sum_{i=1}^{i_0} (-1)^{i+1} m_i \sigma\left(|g_{n-1}(x^{(i)})|\right) = \pm \begin{array}{|llllll|} \hline \boxed{\mathbf{1}} \ \boxed{\mathbf{2}} & \dots & \boxed{\mathbf{i_0}} & \sigma(x) & \dots & \sigma^{n-i_0}(x) \\ \hline \boxed{x} \ \sigma(x) & & \dots & & & \sigma^{n-1}(x) \\ \hline \end{array}$$

With other words, to show (5.4) it suffices to show that the tableau on the right side vanishes. Towards this we have

$$\begin{array}{|llllll|} \hline \boxed{\mathbf{1}} \ \boxed{\mathbf{2}} & \dots & \boxed{\mathbf{i_0}} & \sigma(x) & \dots & \sigma^{n-i_0}(x) \\ \hline \boxed{x} \ \sigma(x) & & \dots & & & \sigma^{n-1}(x) \\ \hline \end{array} = \begin{array}{|llllll|} \hline \boxed{\mathbf{1}} \ \boxed{\mathbf{2}} & \dots & \boxed{\mathbf{i_0}} & \boxed{\sigma(x)} & \dots & \boxed{\sigma^{n-i_0}(x)} \\ \hline \boxed{x} \ \sigma(x) & & \dots & & & \sigma^{n-1}(x) \\ \hline \end{array} = 0$$

Here the first equality is immediate from the definition of a tableau with boxes and the fact that the entries $\sigma(x), \dots, \sigma^{n-i_0}(x)$ appear in the second row, and the second equality is an application of Turnbull's identity [Tur09] (see [Lec93, Proposition 1.2.2(i)]), which claims that if the number k of boxed entries satisfies $k > n$, then the tableau vanishes. Indeed, viewed as a function on the boxed entries the tableau is a linear *alternating* (not only skew-symmetric as stated in the proof of [Lec93, Proposition 1.2.2(i)]) form on S^n in k variables, which must therefore vanish, as $\Lambda^k_S M = 0$ for any finitely generated S-module M which can be generated by n elements (in *loc. cit.* the proof is only formulated when S is a field, but it generalizes to all rings). □

We continue with the proof of Proposition 5.2 for $\kappa = 0$. The group $\mathbb{G}^2_{m,\overline{\mathbb{F}}_q}$ acts on $Y_{i_0,h} \times Y_{n-i_0,h}$ by

$$(\tau_1, \tau_2) \colon (y, z) \mapsto (\tau_1 y, \tau_2 z). \tag{5.5}$$

(here $\tau_1 y := (\tau_1 y_i)_{i=1}^{i_0}$ means entry-wise multiplication, and similarly for z). This action restricts to an action of the closed subgroup

$$H_0 := \left\{ (\tau_1, \tau_2) \in \mathbb{G}_m^2 \colon \tau_1^{\sum_{j=0}^{i_0-1} \sigma^j} \left(\prod_{i=0}^{n-i_0-1} \sigma^i(\tau_2) \right)^{-\sum_{j=1}^{i_0-1} \sigma^j} = 1 \right\}$$

on $\alpha_0(N_h \backslash X_h)$, where α_0 is as in Lemma 5.6. By Lemma 5.6 α_0 induces an isomorphism on étale cohomology. Now H is 1-dimensional, hence its connected component H° is a 1-dimensional torus. Therefore the projection of H° to at least one of the $\mathbb{G}_m$-factors of the ambient group $\mathbb{G}_m^2$ is non-constant, hence surjective. Hence $\alpha_0(N_h \backslash X_h)^{H^\circ} = \varnothing$.

The action of $T_h \cong \mathbb{W}_h^\times(\mathbb{F}_{q^n})$ on X_h induces an action on $N_h \backslash X_h$, which under α_0 is compatible with the T_h-action on $\alpha_0(N_h \backslash X_h)$ given by $t \colon (m, x') \mapsto (m \cdot \prod_{j=0}^{i_0-1} \sigma^j(t), x' \cdot t)$ (both products mean scalar multiplication). This action of T_h commutes with the above action of H_0 on $\alpha_0(N_h \backslash X_h)$. The explicit description in Lemma 5.6 also shows that $\alpha_0(N_h \backslash X_h)$ is affine. Thus the T_h-equivariant version of the well-known result [DM91, 10.15 Proposition] gives

$$\dim_{\overline{\mathbb{Q}}_\ell} H_c^*(N_h \backslash X_h)_\theta = \dim_{\overline{\mathbb{Q}}_\ell} H_c^*(\alpha_0(N_h \backslash X_h)^{H^\circ})_\theta = 0.$$

This finishes the proof of Proposition 5.2 in the case $\kappa = 0$.

5.2. **Proof of Proposition 5.2 for arbitrary κ.** Let κ be arbitrary. Let $c := \left(\begin{smallmatrix} 0 & \varpi \\ 1_{n_0-1} & 0 \end{smallmatrix}\right)^{k_0}$, and for $r \geq 1$ let $b_r := \bigoplus_r c$ be the block-diagonal $n_0 r \times n_0 r$-matrix with blocks equal to c. Let $b = b_{n'}$ (it is the special representative corresponding to κ, n as in [CI18, §5.2.2]). Let $\dot{w} = b_0 t_{\kappa,n}$ be as in (3.1). We have then the corresponding groups $\mathbf{G}, \mathbf{G}_{\mathcal{O}}, \mathbf{T}, G_h, \dots$ as in Section 2.1. A maximal rational parabolic subgroup of $\mathbf{G}$ is determined by an integer

$1 \leq i_0 \leq n'-1$. Its unipotent radical $\mathbf{N}$ consists of matrices $(A_{ij})_{1\leq i,j\leq n'}$ where each A_{ij} is a $n_0\times n_0$-matrix, and $A_{ii} = 1_{n_0}$, $A_{ij} = 0$, unless $i = j$ or ($1 \leq i \leq i_0$ and $n'-i_0+1 \leq j \leq n$). Let l denote an integer which modulo n_0 is the multiplicative inverse of k_0. Moreover, for $a \in \mathbb{Z}$ define $[a]_{n_0} \in \mathbb{Z}$ by the requirement that $1 \leq [a]_{n_0} \leq n_0$ and $[a]_{n_0} \equiv a \mod n_0$. The subgroup N_h of G_h corresponding to $\mathbf{N}$ (see Section 3.1) consists of $n \times n$-matrices of the same shape, where now each of the $n_0 \times n_0$-blocks A_{ij} with $1 \leq i \leq i_0$ and $n'-i_0+1 \leq j \leq n$ is of the form $\sum_{\lambda=0}^{n_0-1} \varpi^{-\lfloor \frac{\lambda k_0}{n_0} \rfloor} c^\lambda \operatorname{diag}(a_\lambda, \sigma^{[l]_{n_0}}(a_\lambda), \sigma^{[2l]_{n_0}}(a_\lambda), \dots, \sigma^{[(n_0-1)l]_{n_0}}(a_\lambda))$ with $a_0 \in \mathbb{W}_h(\mathbb{F}_{q^{n_0}})$ and $a_\lambda \in \mathbb{W}_{h-1}(\mathbb{F}_{q^{n_0}})$ for $\lambda > 0$. In particular, $\#N_h = q^{n_0(h+(n_0-1)(h-1))i_0(n'-i_0)}$.

Let $r \geq 1$ and let $Z_{n_0,r,h} = \{(x_i)_{i=1}^{n_0r} \colon\ x_i \in \mathbb{W}_h \text{ if } i \equiv 1 \mod n_0 \text{ and } x_i \in \mathbb{W}_{h-1} \text{ otherwise}\}$. This is a affine, perfectly finitely presented perfect $\mathbb{F}_q$-scheme. For a perfect $\mathbb{F}_q$-algebra R and $x \in Z_{n_0,r,h}(R)$ let $g_{n_0,r}(x)$ denote the $n_0r\times n_0r$-matrix whose i-th column is $\varpi^{-\lfloor \frac{(i-1)k_0}{n_0} \rfloor}(b_r\sigma)^{i-1}(x)$ (the entries of $g_{n_0,r}(x)$ are either in $\mathbb{W}_h(R)$ or in $\mathbb{W}_{h-1}(R)$ or in $\varpi\mathbb{W}_{h-1}(R) \subseteq \mathbb{W}_h(R)$). The determinant $|g_{n_0,r}(x)|$ of $g_{n_0,r}(x)$ is a well-defined element of $\mathbb{W}_h(R)$. Let

$$Y_{n_0,r,h} = \{x \in Z_{n_0,r,h} \colon |g_{n_0,r}(x)| \in \mathbb{W}_h^\times\}$$

The description of X_h in [CI18, 7.2] says precisely that $X_h \subseteq Y_{n_0,n',h}$ is the subset defined by the closed condition that $|g_{n_0,n'}(x)| \in \mathbb{W}_h^\times(\mathbb{F}_q)$.

To simplify notation we write $s := n_0 i_0$ from now on. For $x \in X_h$ and $1 \leq i \leq s$, let $m_i(x)$ denote the $(n-s+1)\times(n-s+1)$-minor obtained from $g_{n_0,n'}(x)$ by removing all rows except for the i-th and $s+1, s+2, \dots, n$-th and all but the first $n-s+1$ columns. Then $m_i(x)$ makes sense as an element of $\mathbb{W}_h$ resp. of $\mathbb{W}_{h-1}$ if $i \equiv 1 \mod n_0$ resp. if $i \not\equiv 1 \mod n_0$. Thus $(m_i(x))_{i=1}^s \in Z_{n_0,i_0,h}$. The analogue of Lemma 5.5 for $N_h\backslash X_h$ holds with the same proof. We have the following generalization of Lemma 5.6.

Lemma 5.8. *The assignment $x = (x_i)_{i=1}^n \in X_h \mapsto m = (m_i(x))_{i=1}^s, x' = (x_i)_{i=s+1}^n$ induces an isomorphism of perfect schemes,*

$$\alpha_\kappa \colon N_h\backslash X_h \to \left\{ (m,x') \in Y_{n_0,i_0,h} \times Y_{n_0,n'-i_0,h} \colon \frac{|g_{n_0,i_0}(m)|}{|g_{n_0,n'-i_0}(x')|^{\sum_{j=1}^{s-1}\sigma^j}} \in \mathbb{W}_h^\times(\mathbb{F}_q) \right\}.$$

Proof. Using the description of N_h given above, one checks that $m_i(x)$ is stable under the N_h-action on X_h. Now the proof goes completely analogous to the proof of Lemma 5.6 (with Lemma 5.7 replaced by its generalization Lemma 5.9). □

Lemma 5.9. *Let $n \geq 2$, $1 \leq i_0 \leq n-1$. For a perfect $\mathbb{F}_q$-algebra R and $x = (x_i)_{i=1}^n \in Y_{n_0,n',h}(R)$, we have $m = (m_i(x))_{i=1}^s \in Y_{n_0,i_0,h}(R)$, $x' = (x_i)_{i=i_0+1}^n \in Y_{n_0,n'-i_0,h}(R)$ and*

$$|g_{n_0,i_0}(m)| = |g_{n_0,n'}(x)| \cdot |g_{n_0,n'-i_0}(x')|^{\sum_{j=1}^{i_0-1}\sigma^j} \tag{5.6}$$

Proof. It is known that for $x \in Y_{n_0,n',h}(R)$, we have $x' \in Y_{n_0,n'-i_0,h}(R)$ (see [CI18, Lemma 6.13]). Thus the similar claim for m follows, once (5.6) is shown. To show (5.6) we first notice that all entries of $g_{n_0,i_0}(m)$ (and not only those in the first column) are in fact $(n-s+1)\times(n-s+1)$-minors of $g_{n_0,n'}(x)$. More precisely, for $1 \leq i,j \leq s$ the (i,j)-th entry of $g_{n_0,i_0}(m)$ is the minor of $g_{n_0,n'}(x)$ obtained by removing all columns except those with numbers $j, j+1, \dots, j+n-s$, and all rows except those with numbers $i, s, s+1, \dots, n$. Let X_i denote the i-th row of $g_{n_0,n'}(x)$. Let also $\mathbf{a}$ denote the a-th standard basis vector of a free rank n module (over an arbitrary ring). Using the formalism of tableaux with boxes (as in the proof of Lemma 5.7), – but now for the *rows* of $g_{n_0,n'}(x)$, we can express $|g_{n_0,i_0}(m)|$ as the $s \times n$-tableau with boxes:

$\boxed{X_1}$	X_{s+1}	X_{s+2}	$\dots$	X_n	$\mathbf{n-s+2}$	$\mathbf{n-s+3}$	$\dots$	$\mathbf{n}$		
$\mathbf{1}$	$\boxed{X_2}$	X_{s+1}	X_{s+2}	$\dots$	X_{n-1}	X_n	$\mathbf{n-s+2}$	$\mathbf{n-s+3}$	$\dots$	$\mathbf{n}$
$\mathbf{1}$	$\mathbf{2}$	$\boxed{X_3}$	X_{s+1}	$\dots$	X_{n-1}	X_n	$\mathbf{n-s+2}$	$\mathbf{n-s+3}$	$\dots$	$\mathbf{n}$
$\dots$					$\dots$					$\dots$
$\mathbf{1}$	$\mathbf{2}$	$\dots$	$\mathbf{s-2}$	$\boxed{X_{s-1}}$	X_{s+1}	X_{s+2}	$\dots$	X_{n-1}	X_n	$\mathbf{n}$
$\mathbf{1}$	$\mathbf{2}$	$\dots$	$\mathbf{s-1}$	$\boxed{X_s}$	X_{s+1}	$\dots$	X_{n-1}	X_n		

As each of the entries $X_{s+1}, X_{s+2}, \dots, X_n$ appears in each row of this tableau, it is equal to

$\boxed{X_1}$	X_{s+1}	X_{s+2}	$\dots$	X_n	$\mathbf{n-s+2}$	$\mathbf{n-s+3}$	$\dots$	$\mathbf{n}$		
$\mathbf{1}$	$\boxed{X_2}$	X_{s+1}	X_{s+2}	$\dots$	X_{n-1}	X_n	$\mathbf{n-s+2}$	$\mathbf{n-s+3}$	$\dots$	$\mathbf{n}$
$\mathbf{1}$	$\mathbf{2}$	$\boxed{X_3}$	X_{s+1}	$\dots$	X_{n-1}	X_n	$\mathbf{n-s+2}$	$\mathbf{n-s+3}$	$\dots$	$\mathbf{n}$
$\dots$					$\dots$					$\dots$
$\mathbf{1}$	$\mathbf{2}$	$\dots$	$\mathbf{s-2}$	$\boxed{X_{s-1}}$	X_{s+1}	X_{s+2}	$\dots$	X_{n-1}	X_n	$\mathbf{n}$
$\mathbf{1}$	$\mathbf{2}$	$\dots$	$\mathbf{s-1}$	$\boxed{X_s}$	$\boxed{X_{s+1}}$	$\dots$	$\boxed{X_{n-1}}$	$\boxed{X_n}$		

Apply (second) Turnbull's identity [Lec93, Proposition 1.2.2(ii)] to the last row of this tableau, deducing that it is equal to

$\boxed{\mathbf{1}}$	X_{s+1}	X_{s+2}	$\dots$	X_n	$\mathbf{n-s+2}$	$\mathbf{n-s+3}$	$\dots$	$\mathbf{n}$		
$\mathbf{1}$	$\boxed{\mathbf{2}}$	X_{s+1}	X_{s+2}	$\dots$	X_{n-1}	X_n	$\mathbf{n-s+2}$	$\mathbf{n-s+3}$	$\dots$	$\mathbf{n}$
$\mathbf{1}$	$\mathbf{2}$	$\boxed{\mathbf{3}}$	X_{s+1}	$\dots$	X_{n-1}	X_n	$\mathbf{n-s+2}$	$\mathbf{n-s+3}$	$\dots$	$\mathbf{n}$
$\dots$					$\dots$					$\dots$
$\mathbf{1}$	$\mathbf{2}$	$\dots$	$\mathbf{s-2}$	$\boxed{\mathbf{s-1}}$	X_{s+1}	X_{s+2}	$\dots$	X_{n-1}	X_n	$\mathbf{n}$
X_1	X_2	$\dots$	X_{s-1}	X_s	X_{s+1}	$\dots$	X_{n-1}	X_n		

Here all boxes can be removed without changing the value of the tableau, as any non-trivial permutation produces a zero $s \times n$-tableau (as at least one row will contain two equal entries and hence be equal to 0). The resulting tableau (without boxes) is precisely the right hand side of (5.6). $\square$

Remark 5.10. In the proof of Lemma 5.8, the fact that the entries of $g_{n_0,i_0}(m)$ are certain minors of $g_{n_0,n'}(x)$ can be shown by a somewhat tedious but straightforward calculation, which we omit here. To illustrate the principle, we give an example. Let $n = 9$, $\kappa = 6$, so that $n' = 3$, $n_0 = 3$, $k_0 = 2$. Let $i_0 = 2$. We have the two minors of $g_{n_0,n'}(x)$,

$$m_2 = \begin{vmatrix} x_2 & \varpi\sigma(x_3) & \sigma^2(x_1) & \sigma^3(x_2) \\ x_7 & \varpi\sigma(x_8) & \varpi\sigma^2(x_9) & \sigma^3(x_7) \\ x_8 & \varpi\sigma(x_9) & \sigma^2(x_7) & \sigma^3(x_8) \\ x_9 & \sigma(x_7) & \sigma^2(x_8) & \sigma^3(x_9) \end{vmatrix} \qquad \text{and} \qquad M := \begin{vmatrix} \varpi\sigma(x_2) & \varpi\sigma^2(x_3) & \sigma^3(x_1) & \varpi\sigma^4(x_2) \\ \varpi\sigma(x_8) & \varpi\sigma^2(x_9) & \sigma^3(x_7) & \varpi\sigma^4(x_8) \\ \varpi\sigma(x_9) & \sigma^2(x_7) & \sigma^3(x_8) & \varpi\sigma^4(x_9) \\ \sigma(x_7) & \sigma^2(x_8) & \sigma^3(x_9) & \sigma^4(x_7) \end{vmatrix}$$

the first corresponding to rows $2, 7, 8, 9$ and columns $1, 2, 3, 4$, and the second corresponding to rows $1, 7, 8, 9$ and $2, 3, 4, 5$. The first of these minors is by definition the $(2,1)$-entry of $g_{n_0,i_0}(m)$, and the fact mentioned above claims that the second minor is equal to the $(1,2)$-entry of $g_{n_0,i_0}(m)$, that is, to $\varpi\sigma(m_2) \in \varpi\mathbb{W}_{h-1} \subseteq \mathbb{W}_h$. First, M makes sense as an element of $\varpi\mathbb{W}_{h-1}$. To compute it, we may lift its entries to elements in $\mathbb{W}$, where we can multiply rows and columns by powers of ϖ, to see that

$$M = \varpi^{-2} \begin{vmatrix} \varpi\sigma(x_2) & \varpi^2\sigma^2(x_3) & \varpi\sigma^3(x_1) & \varpi\sigma^4(x_2) \\ \varpi\sigma(x_8) & \varpi^2\sigma^2(x_9) & \varpi\sigma^3(x_7) & \varpi\sigma^4(x_8) \\ \varpi\sigma(x_9) & \varpi\sigma^2(x_7) & \varpi\sigma^3(x_8) & \varpi\sigma^4(x_9) \\ \sigma(x_7) & \varpi\sigma^2(x_8) & \varpi\sigma^3(x_9) & \sigma^4(x_7) \end{vmatrix} = \varpi \begin{vmatrix} \sigma(x_2) & \varpi\sigma^2(x_3) & \sigma^3(x_1) & \sigma^4(x_2) \\ \sigma(x_8) & \varpi\sigma^2(x_9) & \sigma^3(x_7) & \sigma^4(x_8) \\ \sigma(x_9) & \sigma^2(x_7) & \sigma^3(x_8) & \sigma^4(x_9) \\ \sigma(x_7) & \varpi\sigma^2(x_8) & \varpi\sigma^3(x_9) & \sigma^4(x_7) \end{vmatrix} = \varpi\sigma(m_2)$$

(after reducing modulo $\varpi^h \mathbb{W}$), as claimed.

We continue with the proof of Proposition 5.2. The group $\mathbb{G}_{m,\overline{\mathbb{F}}_q}^2$ acts on $Y_{n_0,i_0,h} \times Y_{n_0,n-i_0,h}$ by the same formula as in (5.5). This action restricts to an action of the closed subgroup

$$H_\kappa := \left\{ (\tau_1, \tau_2) \in \mathbb{G}_m^2 \colon \tau_1^{\sum_{j=0}^{s-1} \sigma^j} \left(\prod_{i=0}^{n-s-1} \sigma^i(\tau_2) \right)^{-\sum_{j=1}^{s-1} \sigma^j} = 1 \right\}$$

on $N_h \backslash X_h \cong \alpha_\kappa(N_h \backslash X_h)$, where α_κ is as Lemma 5.8. Now H is 1-dimensional, hence its connected component H° is a 1-dimensional torus. The rest of the argument is exactly as at the end of Section 5.1. Proposition 5.2 is now proven.

6. Review of some representation theory

We fix an isomorphism $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$ and use it to identify the isomorphism classes of smooth complex with smooth $\overline{\mathbb{Q}}_\ell$-representations of all involved groups. For a finite dimensional (complex or $\overline{\mathbb{Q}}_\ell$-) representation ρ of a group, we denote by $\deg(\rho)$ the degree of ρ.

6.1. **Square-integrable representations.** We recall some well-known results about square-integrable representations of p-adic reductive groups due to Harish-Chandra. For a detailed treatment we refer to [HC70] (see also [Car79]).

In this section let $\mathbf{G}$ be an arbitrary reductive group over K and $G = \mathbf{G}(K)$. Let Z be the (K-valued points of) the maximal split torus contained in the center of $\mathbf{G}$. Let $\psi \colon Z \to \overline{\mathbb{Q}}_\ell^\times$ be a unitary character of Z. We fix now an invariant Haar measure on G/Z (recall that G is unimodular). We work with complex-valued representations of G. Let $\mathscr{E}_2(G,\psi)$ denote the set of equivalence classes of irreducible unitary representations (π, V) of G, which have central character ψ and satisfy

$$\int_{G/Z} |(u, \pi(g)v)|^2 d\bar{g} < +\infty \tag{6.1}$$

where $(\cdot,\cdot)$ denotes the scalar product in the Hilbert space V (the integral makes sense as ψ is unitary). These are the *square-integrable* representations with central character χ. All irreducible supercuspidal representations with unitary central character are square-integrable [HC70, §3].

For a given $\pi \in \mathscr{E}_2(G,\psi)$, the integral (6.1) is equal to $d(\pi, d\bar{g})|u|^2|v|^2$, where the constant $d(\pi, d\bar{g}) > 0$ is independent of u, v (and thus only depends on π and the chosen measure $d\bar{g}$). The constant $d(\pi, d\bar{g})$ is called the *formal degree* of π (with respect to $d\bar{g}$). Let H be a compact open subgroup of G. If $d\bar{g}, d\bar{g}'$ are two invariant Haar measures on G, then $d(\pi, d\bar{g})\mathrm{vol}(HZ/Z, d\bar{g}) = d(\pi, d\bar{g}')\mathrm{vol}(HZ/Z, d\bar{g}')$. Moreover, if $\pi \in \mathscr{E}_2(G,\psi)$ is of the form $\pi = \mathrm{cInd}_{ZH}^G \tau$ for an (automatically finite-dimensional) representation τ on which Z acts by the character ψ, then $d(\pi, d\bar{g})\mathrm{vol}(HZ/Z, d\bar{g}) = \deg \tau$ (cf. [Car79, 1.6]).

For any $\pi \in \mathscr{E}_2(G,\psi)$ and a smooth irreducible representation ρ of H, let $(\pi : \rho)$ denote the multiplicity of ρ in the restriction of π to H. We need the following estimate due to Harish-Chandra.

Theorem 6.1 (see [HC70, p.6]). *Given H, ρ as above, let $\pi \in \mathscr{E}_2(G,\psi)$. Then*

$$\sum_{\pi \in \mathscr{E}_2(G,\psi)} d(\pi, d\bar{g})\mathrm{vol}(HZ/Z, d\bar{g})(\pi : \rho) \leq \deg \rho. \tag{6.2}$$

6.2. **Traces on elliptic elements.** For the moment keep the assumptions of Section 6.1 (in particular, $\mathbf{G}$ is arbitrary reductive). Let $\mathcal{H}(G)$ denote the convolution algebra of locally constant compactly supported functions on G. Fix a Haar measure dg on G. For any smooth G-representation (π, V), $\mathcal{H}(G)$ acts in V by $\pi(f)v = \int_G f(g)\pi(g)vdg$ for all $v \in V$, $f \in \mathcal{H}(G)$. If π is admissible, then $\pi(f)$ has finite dimensional range, and hence a trace. Let $G^{\mathrm{reg,ss}}$ denote the set of regular semi-simple elements of G. It is open dense in G. The following result due to Harish-Chandra and Lemaire ensures the existence of a trace of a finite length G-representation on regular semisimple elements of G.

Theorem 6.2 (see [Hen06, Theorem 1]). *Let π be a finite length (hence admissible) smooth representation of G. Then there is a unique (hence invariant under conjugation) locally constant function $\mathrm{tr}(\pi, \cdot)$ on $G^{\mathrm{reg,ss}}$ of G, locally integrable on G, such that for all $f \in \mathcal{H}(G)$, one has $\mathrm{tr}\,\pi(f) = \int_G \mathrm{tr}(\pi, g)f(g)dg$.*

Now assume again, that $G = \mathbf{G}(K)$ for an inner form $\mathbf{G}$ of $\mathbf{GL}_n$. For $g \in G$, let $P(g)$ denote the reduced characteristic polynomial of g. Two elements of $g_1, g_2 \in G^{\mathrm{reg,ss}}$ are conjugate in G if and only if $P(g_1) = P(g_2)$. All said above applies to $\mathrm{GL}_n(K)$ as a special case. Moreover, for an elements $g \in G^{\mathrm{reg,ss}}$ there is a unique up to conjugation element $g' \in \mathrm{GL}_n(K)^{\mathrm{reg,ss}}$ such that $P(g_1) = P(g_2)$. This has a partial converse. Let $G^{\mathrm{ell}} \subseteq G^{\mathrm{reg,ss}}$ denote the (open) subset of elliptic elements. For any $g' \in \mathrm{GL}_n(K)^{\mathrm{ell}}$ there is a unique up to conjugation $g \in G^{\mathrm{ell}}$ with the same (reduced) characteristic polynomial. The local Jacquet–Langlands correspondence is then the following result, which in its most general form is due to Deligne–Kazhdan–Vigneras [DKV84] and Badulescu [Bad02].

Theorem 6.3 (see [Hen06, Theorem 2]). *There is a unique bijection $\pi' \leftrightarrow \pi = \mathrm{JL}(\pi')$ between the sets of $\mathscr{A}^2(G)$ and $\mathscr{A}^2(\mathrm{GL}_n(K))$ of smooth irreducible square-integrable representations of $\mathrm{GL}_n(K)$ and G, such that $\mathrm{tr}(\pi, g) = (-1)^{n-n'}\,\mathrm{tr}(\pi', g')$ whenever $g \in G^{\mathrm{ell}}$, $g' \in \mathrm{GL}_n(K)^{\mathrm{ell}}$ with $P(g) = P(g')$.*

Now we recall a result from [CI18]. An (elliptic) element $x \in T \cong L^\times$ is called *very regular*, if $x \in \mathcal{O}_L^\times$ and the image of x in the residue field $\mathcal{O}_L/\mathfrak{p}_L \cong \mathbb{F}_{q^n}$ has trivial stabilizer in $\mathrm{Gal}(L/K)$. This definition does not depend on the choice of the isomorphism $T \cong L^\times$ as in Section 2.1.2. Write $\theta^\gamma := \theta \circ \gamma$ for $\gamma \in \mathrm{Gal}(L/K)$, $\theta \colon L^\times \to \overline{\mathbb{Q}}_\ell^\times$.

Proposition 6.4 (Theorem 11.2 of [CI18]). *Let $\theta \colon T \to \overline{\mathbb{Q}}_\ell^\times$ be smooth and $x \in T$ very regular. Then $\mathrm{tr}(R_T^G(\theta), x) = \pm \sum_{\gamma \in \mathrm{Gal}(L/K)} \theta^\gamma(x)$.*

6.3. **Special cases of local Langlands and Jacquet–Langlands correspondences.** As in the introduction, to a character $\theta \colon L^\times \to \overline{\mathbb{Q}}_\ell^\times$ one can attach the n-dimensional representation $\sigma_\theta = \mathrm{Ind}_{\mathcal{W}_L}^{\mathcal{W}_K}(\theta \cdot \mu)$ of the Weil group of K, where we recall that μ is the rectifying character of $L^\times$, given by $\mu|_{U_L} = 1$ and $\mu(\varpi) = (-1)^{n-1}$. The representation σ_θ is irreducible if and only if θ is in general position. In this case, the local Langlands correspondence attaches to σ_θ the irreducible supercuspidal $\mathrm{GL}_n(K)$-representation $\pi_\theta^{\mathrm{GL}_n} := \mathrm{LL}(\sigma_\theta)$. Moreover, the local Jacquet–Langlands correspondence attaches to $\pi_\theta^{\mathrm{GL}_n}$ the irreducible supercuspidal G-representation $\pi_\theta := \mathrm{JL}(\pi_\theta^{\mathrm{GL}_n})$.

Moreover, θ is in general position if and only if it is admissible in the sense of [How77], and the construction of Howe [How77] attaches to it an irreducible supercuspidal $\mathrm{GL}_n(K)$-representation, which is (equivalent to) $\pi_\theta^{\mathrm{GL}_n}$. With other words, with notation as in the introduction, the diagram

$$\begin{array}{ccccc} \mathscr{X} / \mathrm{Gal}_{L/K} & & & & \\ {\scriptstyle \theta\mapsto\sigma_\theta}\downarrow & \overset{\text{Howe}}{\searrow} & & & \\ \mathscr{G}_K^\varepsilon(n) & \xrightarrow{\text{LL}} & \mathscr{A}_K^\varepsilon(n,0) & \xrightarrow{\text{JL}} & \mathscr{A}_K^\varepsilon(n,\kappa) \end{array}$$

commutes.

7. Realization of LL and JL in the cohomology of $X_{\dot{w}}^{DL}(b)$ in some cases

We now will prove Theorem A from the introduction. Let $\theta\colon T \cong L^\times \to \overline{\mathbb{Q}}_\ell^\times$ be a smooth character in general position. Let $\pi_\theta = \mathrm{JL}(\mathrm{LL}(\sigma_\theta)) \in \mathcal{A}_K(n,\kappa)$ be as in Section 6.3.

7.1. **Degree of $R_T^G(\theta)$ and formal degree of π_θ.** First we check that the degree of $R_{T_h}^{G_h}(\theta)$ matches with the formal degree of π_θ (see Section 6.1). Here we use results from [CI19b]. Fix a Howe decomposition for θ: there is a unique tower of fields $L = L_t \supsetneq L_{t-1} \supsetneq \cdots \supsetneq L_1 \supsetneq L_0 = K$ and characters χ, $\phi_1,\dots,\phi_t$ of $K^\times$, $L_1^\times,\dots,L_t^\times$ respectively, such that $\theta = (\chi\circ \mathrm{N}_{L/K})(\phi_1\circ\mathrm{N}_{L/L_1})\dots(\phi_t)$. Denote by $h_1,\dots,h_t$ the levels of $\phi_1,\dots,\phi_t$ respectively and put $d_k = [L:L_k]$, in particular, $d_0 = n$, $d_t = 1$. Also, $\theta|_{U_L^1}$ is in general position if and only if $h_t > 1$.

Lemma 7.1. *Assume $p > n$. Assume $\theta|_{U_L^1}$ is in general position. Then*

$$\deg|R_{T_h}^{G_h}(\theta)| = q^{\frac{1}{2}n\left[n(h_1-1)-(h_t-1)-\sum_{k=1}^{t-1}d_k(h_k-h_{k+1})\right]}\prod_{i=1}^{n'-1}(q^{n_0(n'-i)}-1). \tag{7.1}$$

Proof. As $R_T^G(\theta\cdot(\psi\circ\mathrm{N}_{L/K})) \cong R_T^G(\theta)\otimes(\psi\circ\det)$ [CI18, Lemma 8.4], we may assume $\chi = 1$, i.e., $h = h_1$. The assumptions along with Theorem 4.1 imply that $R_{T_h}^{G_h}(\theta) \cong H_c^*(X_{h,n'})_\theta$. We may assume that b is a Coxeter-type representative (as in [CI18, 5.2.1]). For $t\in T_h$ put $S_{1,t} = \{x \in X_{h,n'}\colon F^n(x) = xt\}$. As in [CI18, Lemma 9.3] we see that $S_{1,t} = \varnothing$, unless $t = 1$ (in *loc. cit.* we worked with the special representative for b and this explains the sign $(-1)^{n'-1}$ appearing there). Further, one has $S_{1,1} = G_h$ [CI19b], and so

$$\#S_{1,1} = \left(\prod_{i=1}^{h-1}\#G_{i+1}^i\right)\cdot\#G_1 = q^{n^2(h-1)}\cdot\prod_{i=0}^{n'-1}(q^{n_0n'}-q^{n_0i}) = q^{n^2(h-1)+\frac{1}{2}n(n'-1)}\cdot\prod_{i=0}^{n'-1}(q^{n_0(n'-i)}-1),$$

as $G_1 \cong (\mathrm{Res}_{\mathbb{F}_{q^{n_0}}/\mathbb{F}_q}\mathrm{GL}_{n',\mathbb{F}_{q^{n_0}}})(\mathbb{F}_q)$ and as $\#G_{i+1}^i = q^{n^2}$ for each $i\geq 1$. Boyarchenko's trace formula [Boy12, Lemma 2.12] and the determination [CI19b] of the scalar by which F^n acts in the non-vanishing cohomology group $H_c^{r_\theta}(X_h)_\theta$ gives

$$\dim|R_{T_h}^{G_h}(\theta)| = \dim|H_c^*(X_{h,n'})_\theta| = \frac{(-1)^{r_\theta}}{(-1)^{r_\theta}q^{\frac{nr_\theta}{2}}\#T_h}\cdot\#S_{1,1},$$

The lemma follows now by an easy calculation, as $\#T_h = (q^n-1)q^{n(h-1)}$, and as $r_\theta = (n'-n)+h_t+(n-2)h+\sum_{k=1}^{t-1}d_k(t_k-t_{k+1})$ by [CI19b]. □

On the other side we use the computation of the formal degree of $\pi_\theta^{\mathrm{GL}_n}$ from [CMS90].

Lemma 7.2. *Assume that $\theta|_{U_L^1}$ is in general position. For any left invariant Haar measure $d\bar{g}$ on G/Z, $d(\pi_\theta,d\bar{g})\mathrm{vol}(ZH/Z,d\bar{g})$ is equal to the right hand side of* (7.1)*. In particular, we have*

$$d(\pi_\theta,d\bar{g})\mathrm{vol}(G_{\mathcal{O}}Z/Z,d\bar{g}) = \deg|R_{T_h}^{G_h}(\theta)|.$$

Proof. The product on the left hand side in the lemma is independent of $d\bar{g}$, so it is enough to show the lemma for a fixed (left invariant) Haar measure. Let $d\bar{x}$ be the Haar measure on G/Z, normalised such that the Steinberg representation St_G of G satisfies $d(\mathrm{St}_G,d\bar{x}) = 1$. Then by Macdonald's formula [SZ96, §3.7] (see also [Kar13, Proposition 5.4]), we have

$$\mathrm{vol}(G_{\mathcal{O}}Z/Z,d\bar{x}) = \frac{1}{n}\prod_{i=1}^{n'-1}(q^{n_0i}-1). \tag{7.2}$$

The normalized formal degree $d(\pi, d\bar{g})$ is stable under the Jacquet–Langlands correspondence [DKV84, BHL10], so we deduce by using (7.2),

$$d(\pi, d\bar{x})\mathrm{vol}(ZH/Z, d\bar{x}) = d(\pi_\theta^{\mathrm{GL}_n}, d\bar{x}^{\mathrm{GL}_n}) \cdot \frac{1}{n}\prod_{i=1}^{n'-1}(q^{n_0 i} - 1),$$

where $d\bar{x}^{\mathrm{GL}_n}$ is the measure $d\bar{x}$ in the special case $n' = n$. Now the normalized formal degree of $\pi_\theta^{\mathrm{GL}_n}$ is determined in [CMS90, Theorem 2.2.8] and coincides with the right hand side of (7.1). □

7.2. **Comparison.** We now prove Theorem A. Assume $p > n$ and assume that $\theta|_{U_L^1}$ is in general position. Let $Z = K^\times$ be the center of G. For a smooth character ϕ of $K^\times$ we have $R_T^G(\theta \cdot (\phi \circ \mathrm{N}_{L/K})) \cong R_T^G(\theta) \otimes (\phi \circ \det)$ [CI18, Lemma 8.4]. An analogous formula holds for π_θ. Hence we may twist both sides of the equality claimed in the theorem by a smooth character ϕ of $K^\times$. Thus we are reduced to the case that $\theta|_Z$ is unitary. Fix an invariant Haar measure $d\bar{g}$ on G/Z.

By Theorem 5.1, there exists a finite set I and an irreducible supercuspidal representation π_i of G for each $i \in I$ such that $|R_T^G(\theta)| \cong \bigoplus_{i=1}^s \pi_i$. It is easy to see (e.g. using [Boy12, Lemma 2.12]) that the central character of $R_T^G(\theta)$ is $\theta|_Z$. From this and the fact that all supercuspidal representations are square-integrable it follows that $\pi_i \in \mathscr{E}_2(G, \theta|_Z)$ for all i. As by assumption $(p, n) = 1$, each π_i is attached to a pair $(E_i/K, \chi_i)$ with E_i/K is a separable degree n extension and χ_i is an admissible character of $E_i^\times$ in the sense of [How77] (indeed, Howe's construction also works for inner forms of GL_n, so that there is no need to pass to the more general constructions of Yu [Yu01] and Kaletha [Kal19]). Let $I_{nr} \subseteq I$ denote the subset of those $i \in I$, for which E_i/K is unramified, i.e., $E_i \cong L$. For each $i \in I$, π_i has a well-defined trace on regular elliptic elements of G, and in particular on the very regular elements of $T \cong L^\times$. If $i \in I \smallsetminus I_{nr}$, then $\pi_i \cong \mathrm{cInd}_{HE_i}^G \tau_i$, where $H \subseteq G_{\mathcal{O}}$ is certain (explicitly determined) compact open subgroup, which is not maximal compact, and $E_i^\times$ is appropriately embedded as a subgroup of $G(K)$ normalizing H. In particular, for $i \in I \smallsetminus I_{nr}$, no conjugate of a very regular element $x \in T$ lies in $HE_i^\times$ (in fact, x has precisely one fixed point on $\mathscr{B}_K$, which has to be a vertex, so it is contained in no stabilizer of a facet of $\mathscr{B}_K$ of dimension ≥ 1). By [BH96, (A.14) Theorem], $\mathrm{tr}(\pi_i, x) = 0$ for $i \notin I_{nr}$, and hence for any very regular element $x \in T \cong L^\times$, we have

$$\pm \sum_{\gamma \in \mathrm{Gal}(L/K)} \theta^\gamma(x) = \mathrm{tr}(|R_T^G(\theta)|, x) = \sum_{i \in I_{nr}} \mathrm{tr}(\pi_i, x) = \sum_{i \in I_{nr}} c_i \sum_{\gamma \in \mathrm{Gal}(L/K)} \chi_i^\gamma(x),$$

where $c_i \in \{\pm 1\}$, the first equality is Proposition 6.4 and the last follows from [Hen92, 3.1 Théorème] (in fact, it shows the claim only for GL_n, but this along with trace relations defining the Jacquet–Langlands correspondence give also the other cases). We use now the argument from [Hen92, 2.8]: if $x \in U_L$ is very regular and $y \in U_L^1$, then $xy \in U_L$ is again very regular. Thus letting x be a fixed very regular element of U_L and varying $y \in U_L^1$ we obtain an equality of finite linear combinations of smooth characters of the group U_L^1. We may find an integer h' such that θ and all χ_i's are trivial on $U_L^{h'}$, and replace U_L^1 by its finite quotient $U_L^1/U_L^{h'}$. As $\theta|_{U_L^1}$ is in general position, the coefficient of $\theta|_{U_L^1}$ on the left hand side is $\theta(x) \neq 0$. By linear independence of characters of a finite group there is at least one $i_0 \in I_{nr}$ with $\chi_{i_0}|_{U_L^1} = \theta|_{U_L^1}$.

Frobenius reciprocity for the compact induction shows that

$$(\pi_i : |R_{T_h}^{G_h}(\theta)|) \geq 1 \quad \text{for } i \in I. \tag{7.3}$$

with notation as in Section 6.1. Fix a Haar measure $d\bar{g}$ on G/Z. By Lemma 7.2 we have $d(\pi_{i_0}, d\bar{g})\mathrm{vol}(G_{\mathcal{O}}Z/Z, d\bar{g}) = \deg|R_{T_h}^{G_h}(\theta)|$, so that Theorem 6.1 implies $(\pi_{i_0} \colon |R_{T_h}^{G_h}(\theta)|) = 1$ and $(\pi \colon |R_{T_h}^{G_h}(\theta)|) = 0$ for all $\pi \in \mathscr{E}_2(G, \theta|_{K^\times})$, $\pi \neq \pi_{i_0}$. Combining this with (7.3) we see that $I = \{i_0\}$. It remains to show that $\chi_{i_0} = \theta$. Either one can apply [CI18, Theorem 11.3] (as we now know that $R_T^G(\theta) \cong \pi_{i_0}$ is irreducible), or alternatively use that we already know $\chi_{i_0} = \theta$ on $K^\times U_L^1$, and then apply the same argument as in [Hen93, 5.3]. Theorem A is proven.

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