Chapter 2. Exercises

1. In the theory of relativity, space and time variables can be combined to form a 4-dimensional vector thus: $x_1 = x$, $x_2 = y$, $x_3 = z$, $x_4 = ict$. The momentum and energy analogously combine to a 4-vector with $p_1 = p_x$, $p_2 = p_y$, $p_3 = p_z$, $p_4 = iE/c$. By a suitable generalization of the quantization prescription for momentum components, deduce the time-dependent Schrödinger equation:

$$\left\{ -\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r}) \right\} \Psi(\mathbf{r}, t) = i\hbar \frac{\partial \Psi(\mathbf{r}, t)}{\partial t}$$

2. Estimate the number of photons emitted per second by a 100-watt lightbulb. Assume a wavelength of 550 nm (yellow light).

3. Electron diffraction makes use of 40 keV (40,000 eV) electrons. Calculate their de Broglie wavelength.

4. Show that the wavefunction $\Psi(x, t) = e^{i(px-Et)/\hbar}$ is a solution of the one-dimensional time-dependent Schrödinger equation.

5. Show that $\Psi(\mathbf{r}, t) = e^{i(\mathbf{p}\cdot\mathbf{r}-Et)/\hbar}$ is a solution of the three-dimensional time-dependent Schrödinger equation.

6. A certain one-dimensional quantum system in $0 \leq x \leq \infty$ is described by the Hamiltonian:

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - \frac{q^2}{x} \qquad (q = \text{constant})$$

One of the eigenfunctions is known to be

$$\psi(x) = Axe^{-\alpha x}, \qquad \alpha \equiv mq^2/\hbar^2, \qquad A = \text{constant}$$

(i) Write down the Schrödinger equation and carry out the indicated differentiation.

(ii) Find the corresponding energy eigenvalue (in terms of $\hbar$, m and q).

(iii) Find the value of A which normalizes the wavefunction according to

$$\int_0^\infty |\psi(x)|^2 \, dx = 1$$

You may require the definite integrals

$$\int_0^\infty x^n e^{-ax} \, dx = n!/a^{n+1}$$

Answers to Exercises

Don't even think of looking here before you attempt to solve the problems yourself!

1. The components of the momentum operator can be expressed in the form

$$\hat{p}_k = -i\hbar \frac{\partial}{\partial x_k}, \quad k = 1, 2, 3$$

Now extend this relation for $k = 4$ using $p_4 = iE/c$ and $x_4 = ict$. The result is

$$\hat{H} = +i\hbar \frac{\partial}{\partial t}$$

where the energy operator is the Hamiltonian $\hat{H}$. Applying the quantization prescription to the classical energy-momentum relation

$$E = \frac{p^2}{2m} + V(x, y, z) \qquad p^2 = p_1^2 + p_2^2 + p_3^2$$

then leads to the 3-dimensional time-dependent Schrödinger equation (29).

2. 100 watts = 100 J/sec. The energy of a 550 nm photon is given by

$$E = h\nu = \frac{hc}{\lambda} = \frac{(6.626 \times 10^{-34})(2.998 \times 10^8)}{550 \times 10^{-9}} = 3.61 \times 10^{-19}\,\mathrm{J}$$

Thus $100/E = 2.77 \times 10^{20}$ photons/sec.

3. Since 1 eV=1.602×10^{-19} J, each electron has a kinetic energy of $(40 \times 10^3)(1.602 \times 10^{-19})$ J. This is equal to

$$E = \frac{1}{2}mv^2 = \frac{p^2}{2m}$$

The de Broglie relation $\lambda = h/p$, therefore gives

$$\lambda = \frac{h}{\sqrt{2mE}} = \frac{6.626 \times 10^{-34}}{\sqrt{2(9.109 \times 10^{-31})(40 \times 10^3)(1.602 \times 10^{-19})}}$$

$= 6.13 \times 10^{-12}$ m. This gives sufficient resolution to study the geometric structure of molecules. [Since 40 keV electrons travel at a significant fraction of the speed of light, the relativistic energy-momentum relation must be used. The corrected de Broglie wavelength is actually 6.016×10^{-12} m.]

4. Evaluate the partial derivatives

$$\frac{\partial}{\partial x}\Psi(x,t) = \frac{ip}{\hbar}e^{i(px-Et)/\hbar} \qquad \frac{\partial^2}{\partial x^2}\Psi(x,t) = -\frac{p^2}{\hbar^2}e^{i(px-Et)/\hbar}$$

$$\text{and} \qquad \frac{\partial}{\partial t}\Psi(x,t) = -\frac{iE}{\hbar}e^{i(px-Et)/\hbar}$$

Eq (26) then follows from the relation $E = p^2/2m$.

5. Note that $\mathbf{p}\cdot\mathbf{r} = p_x\,x + p_y\,y + p_z\,z$. Then

$$\frac{\partial}{\partial x}\Psi(\mathbf{r},t) = \frac{ip_x}{\hbar}e^{i(\mathbf{p}\cdot\mathbf{r}-Et)/\hbar} \qquad \text{etc.}$$

and Eq (29), with $V(\mathbf{r})$=0, follows from $E = (p_x^2 + p_y^2 + p_z^2)/2m$.

6. Evaluate the derivatives (suppressing A for now):

$$\psi'(x) = e^{-\alpha x} - \alpha x e^{-\alpha x} \qquad \text{and} \qquad \psi''(x) = -2\alpha e^{-\alpha x} + \alpha^2 x e^{-\alpha x}$$

Then the Schrödinger equation $\hat{H}\psi(x) = E\psi(x)$ becomes

$$-\frac{\hbar^2}{2m}(-2\alpha e^{-\alpha x} + \alpha^2 x e^{-\alpha x}) - \frac{q^2}{\cancel{x}}\,\cancel{x}e^{-\alpha x} = Exe^{-\alpha x}$$

Now, cancel out the $e^{-\alpha x}$ and find two independent relations for the terms independent of x and linear in x. The results give $\alpha = mq^2/\hbar^2$, which agrees with the definition and

$$E = -\frac{\hbar^2\alpha^2}{2m} = -\frac{mq^4}{2\hbar^2}$$

To normalize the function

$$\int_0^\infty |\psi(x)|^2\,dx = 1 = A^2\int_0^\infty x^2 e^{-2\alpha x}\,dx = A^2 \times 2!/(2\alpha)^3$$

giving $A = 2\alpha^{3/2}$.