Supplement 1B

The Planck Radiation Law

To apply Rayleigh's idea that the radiation field can be represented as a collection of oscillators, we need to calculate the number of oscillators per unit volume for each wavelength λ. The reciprocal of the wavelength, $k = 1/\lambda$, is known as the *wavenumber*, and equals the number of wave oscillations per unit length. The wavenumber actually represents the magnitude of the *wavevector* $\mathbf{k}$, which also determines the direction in which the wave is propagating. Now, all the vectors $\mathbf{k}$ of constant magnitude k in a 3-dimensional space can be considered to sweep out a spherical shell of radius k and infinitesimal thickness dk. The volume (in $\mathbf{k}$-space) of this shell is equal to $4\pi k^2 dk$ and can be identified as the number of modes of oscillation per unit volume (in real space). Expressed in terms of λ, the number of modes per unit volume equals $(4\pi/\lambda^4)d\lambda$. Sir James Jeans recognized that this must be multiplied by 2 to take account of the two possible polarizations of each mode of the electromagnetic field. Assuming equipartition of energy implies that each oscillator has the energy kT, where k here is Boltzmann's constant R/N_{A}. Thus we obtain for the energy per unit volume per unit wavelength range

$$\rho(\lambda) = \frac{8\pi kT}{\lambda^4} \qquad (1)$$

which is known as the Rayleigh-Jeans law. This result gives a fairly accurate account of blackbody radiation for larger values of λ, in the infrared region and beyond. But it does suffer from the dreaded ultraviolet catastrophe, whereby $\rho(\lambda)$ increases without limit as $\lambda \to 0$.

Planck realized that the fatal flaw was equipartition, which is based on the assumption that the possible energies of each oscillator belong to a continuum ($0 \leq E < \infty$). If, instead, the energiy of an oscillator of wavelength λ comes in discrete bundles $h\nu = hc/\lambda$, then the possible energies are given by

$$E_{\lambda,n} = nh\nu = nhc/\lambda, \qquad \text{where } n = 0, 1, 2 \ldots \qquad (2)$$

By the Boltzmann distribution in statistical mechanics, the average energy of an oscillator at temperature T is given by

$$\langle E_\lambda \rangle_{\rm av} = \frac{\sum_n E_{\lambda,n}\, e^{-E_{\lambda,n}/kT}}{\sum_n e^{-E_{\lambda,n}/kT}} \tag{3}$$

Using the formula for the sum of a decreasing geometric progression

$$\sum_{n=0}^{\infty} e^{-nhc/\lambda kT} = \frac{1}{1 - e^{-hc/\lambda kT}} \tag{4}$$

we obtain

$$\langle E_\lambda \rangle_{\rm av} = \frac{hc/\lambda}{e^{hc/\lambda kT} - 1} \tag{5}$$

This implies that the higher-energy modes are less populated than what is implied by the equipartion principle. Substituting this (5), rather than kT, into the Rayleigh-Jeans formula (1), we obtain the Planck distribution law

$$\rho(\lambda) = \frac{8\pi hc}{\lambda^5}\frac{1}{e^{hc/\lambda kT} - 1} \tag{6}$$

Note that, for large values of λ and/or T, the average energy (5) is approximated by $\langle E_\lambda \rangle_{\rm av} \approx kT$ and the Planck formula reduces to the Rayleigh-Jeans approximation. The Planck distribution law accurately accounts for the experimental data on thermal radiation shown in Figure 5. Remarkably, recent measurements of the cosmic microwave background also give a perfect fit with a blackbody distribution at temperature 2.73K, as shown in Figure 1B below. The cosmic microwave background radiation, which was discovered by Penzias and Wilson in 1965, is a relic of the Big Bang about 15 billion years ago.

From the Planck distribution law one can calculate the wavelength at which $\rho(\lambda)$ is a maximum at a given T. The result agrees with the Wien displacement law with

$$\lambda_{\rm max} T = \frac{ch}{4.965\, k} \tag{7}$$

By integration of Eq (6) over all wavelengths λ, we obtain the total radiation energy density per unit volume

$$\mathcal{E} = \int_0^\infty \rho(\lambda)\, d\lambda = \frac{8\pi^5 k^4}{15 c^3 h^3}\, T^4 \tag{8}$$

in accord with the Stefan-Boltzmann law.

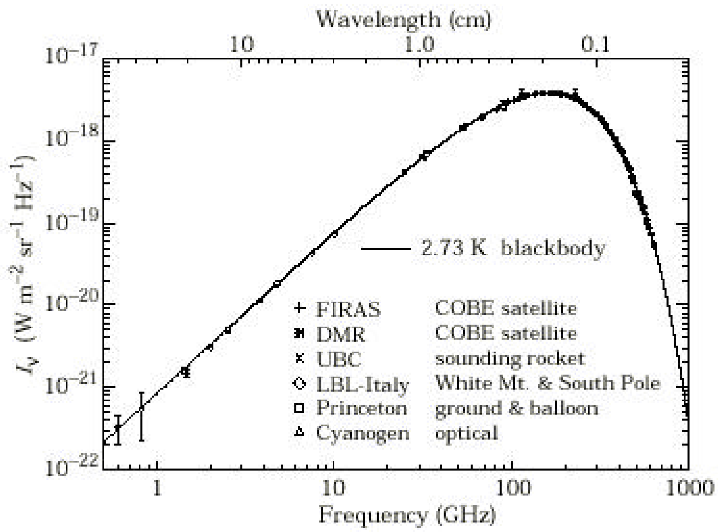

Figure 1B. Cosmic Microwave Background. From G. F. Smoot and D. Scott, `http://pdg.lbl.gov/2001/microwaverpp.pdf`