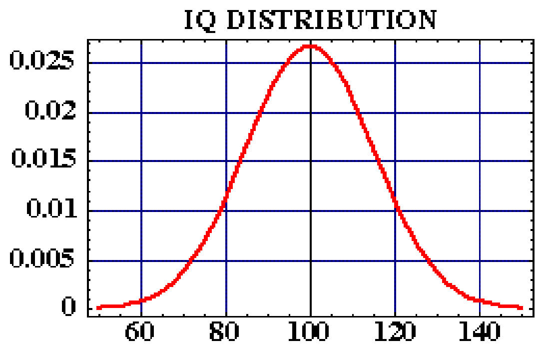
# Supplement 5. Gaussian Integrals

An apocryphal story is told of a math major showing a psychology major the formula for the infamous bell-shaped curve or gaussian, which purports to represent the distribution of intelligence and such:




The formula for a normalized gaussian looks like this:

$$\rho(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-x^2/2\sigma^2}$$

The psychology student, unable to fathom the fact that this formula contained $\pi$, the ratio between the circumference and diameter of a circle, asked "Whatever does $\pi$ have to do with intelligence?" The math student is supposed to have replied, "If your IQ were high enough, you would understand!" The following derivation shows where the $\pi$ comes from.

Laplace (1778) proved that

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi} \tag{1}$$

This remarkable result can be obtained as follows. Denoting the integral by $I$, we can write

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)^2 = \int_{-\infty}^{\infty} e^{-x^2}\, dx \int_{-\infty}^{\infty} e^{-y^2}\, dy \tag{2}$$

where the dummy variable $y$ has been substituted for $x$ in the last integral. The product of two integrals can be expressed as a double integral:

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dx\, dy$$

The differential $dx\,dy$ represents an elementof area in cartesian coordinates, with the domain of integration extending over the entire $xy$-plane. An alternative representation of the last integral can be expressed in plane polar coordinates $r$, $\theta$. The two coordinate systems are related by

$$x = r\cos\theta, \qquad y = r\sin\theta \tag{3}$$

so that

$$r^2 = x^2 + y^2 \tag{4}$$

The element of area in polar coordinates is given by $r\,dr\,d\theta$, so that the double integral becomes

$$I^2 = \int_0^\infty \int_0^{2\pi} e^{-r^2}\, r\,dr\,d\theta \tag{5}$$

Integration over $\theta$ gives a factor $2\pi$. The integral over $r$ can be done after the substitution $u = r^2$, $du = 2r\,dr$:

$$\int_0^\infty e^{-r^2}\, r\,dr = \tfrac{1}{2} \int_0^\infty e^{-u}\,du = \tfrac{1}{2} \tag{6}$$

Therefore $I^2 = 2\pi \times \frac{1}{2}$ and Laplace's result (1) is proven.

A slightly more general result is

$$\int_{-\infty}^\infty e^{-\alpha x^2}\,dx = \left(\frac{\pi}{\alpha}\right)^{1/2} \tag{7}$$

obtained by scaling the variable $x$ to $\sqrt{\alpha}x$.

We require definite integrals of the type

$$\int_{-\infty}^\infty x^n\, e^{-\alpha x^2}\,dx, \qquad n = 1,\, 2,\, 3\ldots \tag{8}$$

for computations involving harmonic oscillator wavefunctions. For odd $n$, the integrals (8) are all zero since the contributions from $\{-\infty, 0\}$ exactly cancel those from $\{0, \infty\}$. The following stratagem produces successive integrals for even $n$. Differentiate each side of (7) wrt the parameter $\alpha$ and cancel minus signs to obtain

$$\int_{-\infty}^{\infty} x^2 \, e^{-\alpha x^2} \, dx = \frac{\pi^{1/2}}{2\,\alpha^{3/2}} \tag{9}$$

Differentiation under an integral sign is valid provided that the integrand is a continuous function. Differentiating again, we obtain

$$\int_{-\infty}^{\infty} x^4 \, e^{-\alpha x^2} \, dx = \frac{3\,\pi^{1/2}}{4\,\alpha^{5/2}} \tag{10}$$

The general result is

$$\int_{-\infty}^{\infty} x^n \, e^{-\alpha x^2} \, dx = \frac{1 \cdot 3 \cdot 5 \cdots |n-1|\,\pi^{1/2}}{2^{n/2}\alpha^{(n+1)/2}}, \quad n = 0, 2, 4 \ldots \tag{11}$$