D-modules and the Riemann-Hilbert Correspondence

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Abstract

We state and sketch a proof of the Riemann-Hilbert correspondence.

Contents

1 Introduction

Consider $X = \mathbb{C}$ as a complex manifold with its structure sheaf $\mathcal{O}$ of holomorphic functions. Set $\mathcal{O}_0$ the stalk at 0 and $K = \mathrm{Frac}(\mathcal{O}_0)$. Then we set $\tilde{K}$ to be the ring of (possibly multivalued) holomorphic functions defined on an open, punctured disk around 0. For a matrix $A \in M_n(K)$, consider the system of ODEs

$$\frac{d}{dx}u(x) = A(x)u(x)$$

where we take solutions $u(x) \in \tilde{K}^n$. Then the set of solutions forms an n-dimensional vector space over $\mathbb{C}$. Pick a basis for this space of solutions, and let $S(x)$ be the matrix with the chosen basis as columns. Since the analytic continuation of $S(x)$ along a circle around $0 \in \mathbb{C}$ gives another basis of the solution space, there is an invertible matrix $G \in \mathrm{GL}_n(\mathbb{C})$ such that

$$\lim_{t\to 2\pi} S(e^{it}x) = S(x)G$$

Thus, we have obtained (locally) from a differential equation a representation of the fundamental group of the punctured disk. One might hope for a correspondence between these data. However, if $n = 1$ and $A = P(x)$ is a polynomial, then the space of solutions is one-dimensional, generated by an entire function. (Take e.g. $u(x) = \exp(\int_0^x P(t)dt.$) In particular, the representation of the fundamental group we obtain will be the trivial representation. Thus, we must restrict the class of differential equations we consider in order to obtain a meaningful correspondence. Classically, this leads to the notion of a *regular singular point.*

The Riemann-Hilbert correspondence vastly generalizes the above example. In place of differential equations on $\mathbb{C}$, we consider certain D_X-modules on a smooth variety X, and in place of representations of the fundamental group, we consider constructible sheaves on the complex manifold X^{an} associated to X. Analogously to above, they are related by the *solution functor* Sol_X. (For computational reasons, we will prefer to use the *de Rham functor* DR_X instead, but they are closely related by 6.5.)

Theorem 1.1. *The de Rham functor DR_X gives an equivalence of categories:*

$$DR_X : D^b_{rh}(D_X) \xrightarrow{\sim} D^b_c(X)$$

where $D^b_{rh}(D_X)$ is the bounded derived category of D_X-modules consisting of complexes whose cohomology sheaves are regular, holonomic D_X-modules, and $D^b_c(X)$ is the bounded derived category of $\mathbb{C}_{X^{an}}$-modules whose cohomology sheaves are constructible.

To reach the correct conditions on D_X-modules to obtain an equivalence, we proceed as follows. First, we restrict to coherent D_X-modules, which admit well-behaved commutative approximations. This alows us to define the further condition of *holonomicity.* For holonomic D_X-modules, we are already very close: the existence of a duality functor allows us to obtain certain image functors which we did not have for general D_X-modules, which play a crucial role in the proof of the correspondence. Finally, as we saw above, we must impose some notion of *regularity.*

This exposition was written as a Minor thesis at Harvard University under the supervision of Dennis Gaitsgory. The presentation heavily follows [HTT08], and any errors introduced are my own. Unless otherwise stated, varieties in this paper are assumed to be smooth, quasi-projective varieties defined over $\mathbb{C}$.

2 Algebraic D-modules

We begin by defining the category of D-modules on a smooth variety. We then define the inverse and direct image functors for D-modules with respect to a morphism of smooth varieties, and discuss various properties of these functors.

2.1 D-modules

Let X be a smooth variety of dimension $n = \dim X$, $\mathcal{O}_X$ its structure sheaf, and Θ_X its tangent sheaf. We consider Θ_X and $\mathcal{O}_X$ as subsheaves of $\mathscr{E}\mathrm{nd}_{\mathbb{C}_X}(\mathcal{O}_X)$, where Θ_X acts locally by derivations and $\mathcal{O}_X$ acts locally by multiplication. We then define the subsheaf D_X of $\mathscr{E}\mathrm{nd}_{\mathbb{C}_X}(\mathcal{O}_X)$ to be the sheaf generated by $\mathcal{O}_X$ and Θ_X, and we call this the *sheaf of differential operators on* X. It will frequently be convenient to work in (affine) local coordinates on X.

Notation 2.1. Let X be a smooth variety. We will frequently denote the restrictions $\mathcal{O}_X|_U, \Theta_X|_U, D_X|_U$ by $\mathcal{O}_U, \Theta_U, D_U$ respectively.

Example 2.2. Let $U \subset X$ be an open, affine subset of X. Then we can take a local coordinate system $\{x_i \in \mathcal{O}_X(U), \partial_i \in \Theta_X(U)\}_{1 \leq i \leq n}$ such that we have the following local description of Θ_U:

$$\Theta_U = \bigoplus_{i=1}^{n} \mathcal{O}_U \partial_i, \quad [\partial_i, \partial_j] = 0, \quad [\partial_i x_j] = \delta_{ij}$$

From this we obtain the following local description of D_X:

$$D_U = \bigoplus_{\alpha \in \mathbb{N}^n} \mathcal{O}_U \partial^\alpha, \quad \partial^\alpha = \prod_{i=1}^{n} \partial_i^{\alpha_i}$$

We call a sheaf M on X a *left* D_X*-module* if $M(U)$ is a left $D_X(U)$-module for each open $U \subset X$ and these actions commute with the restriction morphisms. The following lemma gives an equivalent characterization of left D_X-modules:

Lemma 2.3. *Let M be an $\mathcal{O}_X$-module. The data of a left D_X-module structure on M extending the $\mathcal{O}_X$-module structure is equivalent to a $\mathbb{C}$-linear morphism*

$$\nabla : \Theta_X \to \mathscr{E}\mathrm{nd}_{\mathbb{C}}(M), \theta \mapsto \nabla_\theta$$

satisfying the following conditions:

1. $\nabla_{f\theta}(s) = f\nabla_\theta(s)$
2. $\nabla_\theta(fs) = \theta(f)s + f\nabla_\theta(s)$
3. $\nabla_{[\theta_1,\theta_2]}(s) = [\nabla_{\theta_1}, \nabla_{\theta_2}](s)$

where $f \in \mathcal{O}_X, \theta, \theta_1, \theta_2 \in \Theta_X, s \in M$ denote local sections, and the D_X-action is given by $\theta s = \nabla_\theta(s)$.

Proof. Given a morphism ∇, the left action defined above commutes with restriction morphisms by definition. Given a left D_X-module M, the morphism ∇ is given by $\theta \mapsto (s \mapsto \theta s)$, and we immediately verify the three conditions:

1. $\nabla_{f\theta}(s) = (f\theta)s = f(\theta s) = f\nabla_\theta(s)$.
2. $\nabla_\theta(fs) = \theta(fs) = (\theta f)s - (f\theta)s + (f\theta)s = [\theta, f]s + (f\theta)s = \theta(f)s + f\nabla_\theta(s)$
3. $\nabla_{[\theta_1,\theta_2]}(s) = [\theta_1, \theta_2]s = (\theta_1\theta_2)s - (\theta_2\theta_1)s = \nabla_1 \circ \nabla_2(s) - \nabla_2 \circ \nabla_1(s) = [\nabla_{\theta_1}, \nabla_{\theta_2}](s)$.

□

For a locally free left $\mathcal{O}_X$-module M of finite rank, such a morphism ∇ as in 2.3 is called an *integrable connection*, and in this situation we will refer also to M itself as an integrable connection. In what follows, it will often be useful to work with *right D_X-modules* as well, and we note the analogous characterization for right D_X-modules.

Lemma 2.4. *Let M be an $\mathcal{O}_X$-module. The data of a right D_X-module structure on M extending the $\mathcal{O}_X$-module structure is equivalent to a $\mathbb{C}$-linear morphism*

$$\nabla : \Theta_X \to \mathscr{E}\mathrm{nd}_{\mathbb{C}}(M), \theta \mapsto \nabla_\theta$$

satisfying the following conditions:

1. $\nabla_{f\theta}(s) = \nabla_\theta(fs)$
2. $\nabla_\theta(fs) = \theta(f)s + f\nabla_\theta(s)$
3. $\nabla_{[\theta_1,\theta_2]}(s) = [\nabla_{\theta_1}, \nabla_{\theta_2}](s)$

where $f \in \mathcal{O}_X, \theta, \theta_1, \theta_2 \in \Theta_X, s \in M$ denote local sections, and the D_X-action is given by $s\theta = -\nabla_\theta(s)$.

The proof is entirely analogous to above.

Notation 2.5. Let X be a smooth variety. We denote by $\mathrm{Mod}(D_X)$ the category of left D_X-modules, and by $\mathrm{Mod}(D_X^{op})$ the category of right D_X-modules.

Proposition 2.6. *Let $M, N \in \mathrm{Mod}(D_X)$ and $M', N' \in \mathrm{Mod}(D_X^{op})$. Then*

1. $M \otimes_{\mathcal{O}_X} N \in \mathrm{Mod}(D_X)$; $\theta(s \otimes t) = \theta s \otimes t + s \otimes \theta t$.
2. $M' \otimes_{\mathcal{O}_X} N \in \mathrm{Mod}(D_X^{op})$; $(s' \otimes t)\theta = s'\theta \otimes t - s' \otimes \theta t$.
3. $\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(M, N) \in \mathrm{Mod}(D_X)$; $(\theta\psi)(s) = \theta(\psi(s)) - \psi(\theta(s))$.
4. $\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(M', N') \in \mathrm{Mod}(D_X)$; $(\theta\psi)(s) = -\psi(s)\theta + \psi(s\theta)$.
5. $\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(M, N') \in \mathrm{Mod}(D_X^{op})$; $(\psi\theta)(s) = \psi(s)\theta + \psi(\theta(s))$

Proof. Using lemmas 2.3 and 2.4, the above can be verified by direct computation. □

As a corollary, we obtain the following isomorphisms.

Corollary 2.7. *Let $M, N \in \mathrm{Mod}(D_X)$ and $M' \in \mathrm{Mod}(D_X^{op})$. Then we have isomorphisms:*

$$(M' \otimes_{\mathcal{O}_X} N) \otimes_{D_X} M \simeq M' \otimes_{D_X} (M \otimes_{\mathcal{O}_X} N) \simeq (M' \otimes_{\mathcal{O}_X} M) \otimes_{D_X} N$$

$$(s' \otimes t) \otimes s \leftrightarrow s' \otimes (s \otimes t) \leftrightarrow (s' \otimes s) \otimes t$$

To translate between left and right D_X-modules, we will use the *canonical sheaf* $\Omega_X = \wedge^n \Omega_X^1$, where Ω_X^1 is the sheaf of 1-forms on X (cotangent sheaf), and its $\mathcal{O}_X$-dual $\Omega_X^{-1} = \mathscr{H}\mathrm{om}_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X)$. Note that Ω_X naturally has the structure of a right D_X-module (locally, $\theta \in \Theta_X$ acts by the Lie derivative: $\omega\theta = -(\mathrm{Lie}\,\theta)\omega$).

Proposition 2.8. *The following functors (which we call the* side-changing functors*) are quasi-inverses:*

$$\Omega_X \otimes_{\mathscr{O}_X} - : \mathrm{Mod}(D_X) \to \mathrm{Mod}(D_X^{op})$$
$$\Omega_X^{-1} \otimes_{\mathscr{O}_X} - = \mathscr{H}\mathrm{om}_{\mathscr{O}_X}(\Omega_X, -) : \mathrm{Mod}(D_X^{op}) \to \mathrm{Mod}(D_X)$$

This follows from 2.6. For the rest of this paper, D-modules will be implicitly assumed to be left D-modules unless otherwise specified.

2.2 Image functors

Throughout this subsection, let $f : X \to Y$ be a morphism of smooth varieties. The main difficulty in defining the inverse and direct images of a D_X-module comes from from the fact that a morphism $X \to Y$ does not induce an obvious relationship between D_X and D_Y. (Morally speaking, given a morphism of commutative rings $A \to B$, there is no reason for a derivation on A to induce a derivation on B or vice versa.)

Inverse images

Let M be a D_Y-module, and consider first its inverse image

$$f^*M = \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}M$$

as an $\mathscr{O}_Y$-module. We give f^*M the structure of a D_X-module as follows. First, consider the morphism of $\mathscr{O}_X$-modules:

$$\mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\Omega_Y^1 \to \Omega_X^1$$

Applying $\mathscr{H}\mathrm{om}_{\mathscr{O}_X}(-, \mathscr{O}_X)$ gives a morphism

$$\Theta_X \to \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\Theta_Y$$

which we denote by $\theta \mapsto \tilde{\theta}$. Then the action of D_X on f^*M is given (locally) by $\theta(\psi \otimes s) = \theta(\psi) \otimes s + \psi\tilde{\theta}(s)$. In case $M = D_Y$, we obtain a left D_X-module

$$f^*D_Y = \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}D_Y$$

and the right multiplication of D_Y on itself induces a right $f^{-1}D_Y$-module structure on f^*D_Y. This bimodule will be important for defining our image functors.

Definition 2.9. The $(D_X, f^{-1}D_Y)$-bimodule $f^*D_X = \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}D_Y$ obtained above is denoted by $D_{X\to Y}$.

We thus have an isomorphism $f^*M \simeq D_{X\to Y} \otimes_{f^{-1}D_Y} f^{-1}M$, from which we obtain a right-exact functor

$$D_{X\to Y} \otimes_{f^{-1}D_Y} f^{-1}- : \mathrm{Mod}(D_Y) \to \mathrm{Mod}(D_X)$$

Example 2.10. We compute $D_{X\to Y}$ in the case of a closed embedding $i : X \to Y$ of smooth varieties. For $p \in X$, we may choose local coordinates $\{y_k, \partial_{y_k}\}_{1\le k\le n}$ as in 2.2 on an affine open subset $p \in U \subset Y$ such that $y_{r+1} = \cdots = y_n = 0$ gives defining equations of X. Set $x_k = y_k \circ i$ for $1 \le k \le r$, giving local coordinates $\{x_k, \partial_{x_k}\}_{1\le k\le r}$ for an affine open subset of X. In this situation, the morphism $\Theta_X \to \mathcal{O}_X \otimes_{i^{-1}\mathcal{O}_Y} i^{-1}\Theta_Y$ is given by $\partial_{x_k} \mapsto \partial_{y_k}$. Now set $D' = \oplus_{m_1,\ldots,m_r} \mathcal{O}_Y \partial_{y_1}^{m_1} \cdots \partial_{y_r}^{m_r} \subset D_Y$. It is a subring of D_Y, and we have that $D_Y \simeq D' \otimes_{\mathbb{C}} \mathbb{C}[\partial_{y_{r+1}}, \ldots, \partial_{y_n}]$ as a left D'-module. Hence

$$D_{X\to Y} \simeq (\mathcal{O}_X \otimes_{i^{-1}\mathcal{O}_Y} i^{-1}D') \otimes_{\mathbb{C}} \mathbb{C}[\partial_{y_{r+1}}, \ldots, \partial_{y_n}]$$

Note that in this situation $\mathcal{O}_X \otimes_{i^{-1}\mathcal{O}_Y} i^{-1}D'$ is isomorphic to D_X.

Direct images

Let M be a right D_X-module. Applying the sheaf theoretic direct image functor f_* to the right $f^{-1}D_Y$-module $M \otimes_{D_X} D_{X\to Y}$ gives a right D_Y-module $f_*(M \otimes_{D_X} D_{X\to Y})$. Thus we have a functor

$$f_*(- \otimes_{D_X} D_{X\to Y}) : \mathrm{Mod}(D_X^{op}) \to \mathrm{Mod}(D_Y^{op})$$

(We later give a slightly refined definition of this functor in terms of derived categories, in order to deal with the fact that tensoring is only right-exact while the pushforward is only left-exact.)

To obtain a direct image functor for left D_X-modules, we use the side-changing functors. To a left D_X-module M, we associate the following left D_Y-module:

$$\Omega_Y^{-1} \otimes_{\mathcal{O}_Y} f_*((\Omega_X \otimes_{\mathcal{O}_X} M) \otimes_{D_X} D_{X\to Y})$$

We then have an isomorphism by 2.7

$$(\Omega_X \otimes_{\mathcal{O}_X} M) \otimes_{D_X} D_{X\to Y} \cong (\Omega_X \otimes_{\mathcal{O}_X} D_{X\to Y}) \otimes_{D_X} M$$

of right $f^{-1}D_Y$-modules. Therefore, we have

$$\begin{aligned}\Omega_Y^{-1} \otimes_{\mathcal{O}_Y} f_*((\Omega_X \otimes_{\mathcal{O}_X} M) \otimes_{D_X} D_{X\to Y}) &\simeq \Omega_Y^{-1} \otimes_{\mathcal{O}_Y} f_*((\Omega_X \otimes_{\mathcal{O}_X} D_{X\to Y}) \otimes_{D_X} M) \\ &\simeq f_*((\Omega_X \otimes_{\mathcal{O}_X} D_{X\to Y} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\Omega_Y^{-1}) \otimes_{D_X} M)\end{aligned}$$

By side changing, we have that $\Omega_X \otimes_{\mathcal{O}_X} D_{X\to Y} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\Omega_Y^{-1}$ is a $(f^{-1}D_Y, D_X)$-bimodule. Thus we define

Definition 2.11. The $(f^{-1}D_Y, D_X)$-bimodule $\Omega_X \otimes_{\mathcal{O}_X} D_{X\to Y} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\Omega_Y^{-1}$ obtained above is denoted by $D_{Y\leftarrow X}$.

Example 2.12. Keeping the notation of 2.10, we obtain by similar computations a local isomorphism $D_{Y\leftarrow X} \simeq \mathbb{C}[\partial_{y_{r+1}}, \ldots, \partial_{y_n}] \otimes_{\mathbb{C}} D_X$. We do not write expicitly the left $f^{-1}D_Y$-structure here.

2.3 D_X-modules as sheaves

Before moving on to derived categories of D-modules, we first list some basic results on D-modules. Most of these facts follow from the corresponding facts about affine varieties, quasi-projective varieties, and (quasi-)coherent $\mathcal{O}_X$-modules.

Notation 2.13. Let X be a smooth variety. We denote by $\mathrm{Mod}_{qc}(D_X)$ the category of D_X-modules which are quasi-coherent as $\mathcal{O}_X$-modules. We denote by $\mathrm{Mod}_c(D_X)$ the category of coherent D_X-modules.

Proposition 2.14. *Assume that* $A = D_X(U)$ *for some affine open subset* $U \subset X$ *or* $A = D_{X,x}$ *for some* $x \in X$*. Then*

1. *A is a left (and right) notherian ring.*
2. *The left and right global dimensions of* A *are at most* $2 \dim X$*.*

Proposition 2.15.

1. D_X *is a coherent sheaf of rings.*
2. *A* D_X*-module is coherent iff it is quasi-coherent over* $\mathcal{O}_X$ *and locally finitely generated over* D_X*.*

We sketch a proof of the following theorem because it makes use of the D_X-action in a meaningful way:

Theorem 2.16. *A* D_X*-module is coherent over* $\mathcal{O}_X$ *iff it is an integrable connection.*

Sketch. We sketch the forward direction. Suppose that a $M \in \mathrm{Mod}(D_X)$ is coherent over $\mathcal{O}_X$. Then it suffices to show that M is locally free over $\mathcal{O}_X$. By a standard fact for coherent $\mathcal{O}_X$-modules, it is equivalent to prove that the stalk M_x for any $x \in X$ is a free $\mathcal{O}_{X,x}$-modules. For this, let us first take local coordinates $\{x_i, \partial_i\}$ around x as in 2.2 such that the $m = (x_1, \dots, x_n)$ is the maximal ideal of $\mathcal{O}_{X,x}$.

By Nakayama's lemma we have $s_1, \dots, s_m \in M_x$ such that M_x is generated over $\mathcal{O}_{X,x}$ by $\{s_1, \dots, s_m\}$, and the images of the generators $\{\overline{s_1}, \dots, \overline{s_m}\}$ under the quotient $V = M_x \to M/mM_x$ form a basis of V as a $\mathcal{O}_{X,x}/m = \mathbb{C}$-module. We claim that in fact $\{s_1, \dots, s_m\}$ are free generators of M_x over $\mathcal{O}_{X,x}$. Suppose to the contrary there is some nontrivial relation

$$\sum_{i=1}^{m} f_i s_i = 0$$

over $\mathcal{O}_{X,x}$, and let $\mathrm{ord}(f_i) = \max\{l \mid f_i \in m^l\}$. Applying ∂_j to the above gives a new relation

$$0 = \sum_{i=1}^{m} (\partial_j f_i) s_i + f_i(\partial_j s_i) = \sum_{i=1}^{m} g_i s_i$$

If each term $\partial_j f_i = 0$ for all j and i, the the original relation immediately descends to a nontrivial relation $\sum_{i=1}^{m} \overline{f_i} \overline{s_i} = 0$, so that each f_i must be 0. Otherwise, because each $\partial_j s_i$ is again a $\mathcal{O}_{X,x}$-linear combination of $\{s_1, \dots, s_m\}$, we may pick some j such that the minimum order of the f_i is larger than the minimum order of the g_i. Repeating this argument until the minimum order reaches 0, we obtain a nontrivial relation $\sum_{i=1}^{m} h_i s_i = 0$ which descends to a nontrivial relation $\sum_{i=1}^{m} \overline{h_i} \overline{s_i} = 0$. □

Definition 2.17. A smooth variety X is called *D-affine* if

1. $\Gamma(X, -) : \mathrm{Mod}_{qc}(D_X) \to \mathrm{Mod}(\Gamma(X, D_X))$ is exact.
2. $\Gamma(X, M) = 0$ for $M \in \mathrm{Mod}_{qc}(D_X) \implies M = 0$.

Note in particular that smooth, affine varieties are D-affine.

Proposition 2.18. *Assume that* X *is D-affine. Then*

1. *Any* $M \in \mathrm{Mod}_{qc}(D_X)$ *is generated over* D_X *by its global sections.*
2. $\Gamma(X, -) : \mathrm{Mod}_{qc}(D_X) \to \mathrm{Mod}(\Gamma(X, D_X))$ *gives an equivalence of categories.*

Proposition 2.19. *Assume that X is D-affine. The equivalence*

$$\mathrm{Mod}_{qc}(D_X) \simeq \mathrm{Mod}(\Gamma(X, D_X))$$

from 2.18(ii) induces an equivalence $\mathrm{Mod}_c(D_X) \simeq \mathrm{Mod}_f(\Gamma(X, D_X))$

Proposition 2.20. *Any $M \in \mathrm{Mod}_{qc}(D_X)$ can be embedded into an injective object I of $\mathrm{Mod}_{qc}(D_X)$ which is flabby.*

Proposition 2.21.

1. *A coherent D_X-module is globally generated by a coherent $\mathcal{O}_X$-submodule.*
2. *Let $M \in \mathrm{Mod}_{qc}(D_X)$ and $U \subset X$ open. Then any coherent D_U-submodule N of $M|_U$ can be extended to a coherent D_X-submodule $\tilde{N}$ of M (s.t. $\tilde{N}|_U = N$).*
3. *Any $M \in \mathrm{Mod}_{qc}(D_X)$ is a union of coherent D_X-submodules.*

Proposition 2.22. *Let X be a smooth quasi-projective variety. Then*

1. *Any $M \in \mathrm{Mod}_{qc}(D_X)$ is a quotient of a locally free (hence locally projective hence locally flat) D_X-module.*
2. *Any $M \in \mathrm{Mod}_c(D_X)$ is a quotient of a locally free D_X-module of finite rank.*

Corollary 2.23. *Let X be a smooth quasi-projective variety. Then*

1. *There is a resolution $\cdots \to P_1 \to P_0 \to M \to 0$ of M by locally free D_X-modules.*
2. *There is a finite resolution $0 \to P_m \to \cdots \to P_1 \to P_0 \to M \to 0$ of M by locally projective D_X-modules.*

3 Derived categories of D-modules

Although it might seem preferable to remain strictly within the category of D_X-modules, the formalism of derived categories will provide many tools to formulate the Riemann-Hilbert correspondence in complete generality. We show that the derived image functors respect composition and preserve quasi-coherence over $\mathcal{O}_X$. In particular, this will allow us to use the strategy of decomposing a morphism $f : X \to Y$ as $X \to X \times Y \to Y$ and studying each piece separately.

3.1 Derived D-module categories

Notation 3.1. Let $\sharp \in \{\varnothing, +, -, b\}$. For a sheaf R of rings on a topological space, we denote the derived category of R-modules $D^\sharp(\mathrm{Mod}(R))$ by $D^\sharp(R)$.

Facts 3.2. Let R be a sheaf of rings on a topological space X. Then for any $M \in \mathrm{Mod}(R)$,

1. there an injective object I of $\mathrm{Mod}(R)$ and a monomorphism $M \to I$, and
2. there is a flat object F of $\mathrm{Mod}(R)$ and an epimorphism $F \to M$.

In particular, any complex $M_\bullet$ of $D^+(R)$ (resp. $D^-(R)$) is quasi-isomorphic to a complex $I_\bullet$ of $D^+(R)$ (resp. $F_\bullet$ of $D^-(R)$) of injective (resp. flat) R-modules.

Notation 3.3. Let $\sharp \in \{\varnothing, +, -, b\}$. We denote by $D^\sharp_{qc}(D_X)$ (resp. $D^\sharp_c(D_X)$) the subcategory of $D^\sharp(D_X)$ consisting of complexes whose cohomology sheaves belong to $\mathrm{Mod}_{qc}(D_X)$ (resp. $\mathrm{Mod}_c(D_X)$).

Proposition 3.4. *Any object of $D^b(D_X)$ (resp. $D^b_{qc}(D_X)$) is represented by a bounded complex of flat D_X-modules (resp. locally projective D_X-modules in $\mathrm{Mod}_{qc}(D_X)$).*

Proof. This follows from 2.14 and 2.23. □

Inverse images

Let $f : X \to Y$ be a morphism of smooth varieties. We can define the left derived functor of the right exact functor f^*:

$$Lf^* : D^b(D_Y) \to D^b(D_X), M_\bullet \mapsto D_{X\to Y} \otimes^L_{f^{-1}D_Y} f^{-1}M_\bullet$$

by using a flat resolution of $M_\bullet$, where $\otimes^L$ denoes the left derived functor of the tensor product. We call Lf^* the *inverse image functor.*

Proposition 3.5. *Lf^* restricts to a functor $Lf^* : D^b_{qc}(D_Y) \to D^b_{qc}(D_X)$.*

Proof. Let $M_\bullet \in D^b_{qc}(D_Y)$. By the forgetful functor $D^b(D_X) \to D^b(\mathcal{O}_X)$, we may consider $M_\bullet \in D^b_{qc}(\mathcal{O}_Y)$. Then computing:

$$\begin{aligned} D_{X\to Y} \otimes^L_{f^{-1}D_Y} f^{-1}M_\bullet &= (\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}D_Y) \otimes^L_{f^{-1}D_Y} f^{-1}M_\bullet \\ &= (\mathcal{O}_X \otimes^L_{f^{-1}\mathcal{O}_Y} f^{-1}D_Y) \otimes^L_{f^{-1}D_Y} f^{-1}M_\bullet \\ &= \mathcal{O}_X \otimes^L_{f^{-1}\mathcal{O}_Y} f^{-1}M_\bullet \end{aligned}$$

Then the desired result follows from the corresponding result for the functor

$$\mathcal{O}_X \otimes^L_{f^{-1}\mathcal{O}_Y} f^{-1}- : D^b(\mathcal{O}_Y) \to D^b(\mathcal{O}_X)$$

which follows from the fact that any $M_\bullet \in D^b_{qc}(\mathcal{O}_Y)$ can be represented by a complex of locally free $\mathcal{O}_Y$-modules. □

Remark 3.6. Lf^* does not necessarily restrict to a functor $Lf^* : D^b_c(D_Y) \to D^b_c(D_X)$. For example, if $M_\bullet = D_Y$, then $Lf^* = D_{X\to Y}$, and if $f : X \to Y$ is a closed embedding with $\dim X < \dim Y$, then $D_{X\to Y}$ is a locally free D_X-module of infinite rank by 2.10.

We also make use of the *shifted inverse image functor*

$$f^\dagger = Lf^*[\dim X - \dim Y] : D^b(D_Y) \to D^b(D_X)$$

Proposition 3.7. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms of smooth varieties. Then*

$$L(g \circ f)^* \simeq Lf^* \circ Lg^*, \qquad (g \circ f)^\dagger \simeq f^\dagger \circ g^\dagger$$

Proof. First, we compute:

$$\begin{aligned}
&D_{X\to Y} \otimes^L_{f^{-1}D_Y} f^{-1}D_{Y\to Z}\\
&\quad= (\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}D_Y) \otimes^L_{f^{-1}D_Y} f^{-1}(\mathcal{O}_Y \otimes_{g^{-1}\mathcal{O}_Z} g^{-1}D_Z)\\
&\quad= (\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}D_Y) \otimes^L_{f^{-1}D_Y} (f^{-1}\mathcal{O}_Y \otimes_{(g\circ f)^{-1}\mathcal{O}_Z} (g\circ f)^{-1}D_Z)\\
&\quad= (\mathcal{O}_X \otimes^L_{f^{-1}\mathcal{O}_Y} f^{-1}D_Y) \otimes^L_{f^{-1}D_Y} (f^{-1}\mathcal{O}_Y \otimes^L_{(g\circ f)^{-1}\mathcal{O}_Z} (g\circ f)^{-1}D_Z)\\
&\quad\simeq \mathcal{O}_X \otimes^L_{(g\circ f)^{-1}\mathcal{O}_Z} (g\circ f)^{-1}D_Z\\
&\quad= \mathcal{O}_X \otimes_{(g\circ f)^{-1}\mathcal{O}_Z} (g\circ f)^{-1}D_Z\\
&\quad= D_{X\to Z}
\end{aligned}$$

Thus, we have

$$\begin{aligned}
L(g\circ f)^*M_\bullet &= D_{X\to Z} \otimes_{(g\circ f)^{-1}D_Y} (g\circ f^{-1})M_\bullet\\
&\simeq (D_{X\to Y} \otimes^L_{f^{-1}D_Y} f^{-1}D_{Y\to Z}) \otimes^L_{f^{-1}g^{-1}D_Y} f^{-1}g^{-1}M_\bullet\\
&= D_{X\to Y} \otimes^L_{f^{-1}D_Y} f^{-1}(D_{Y\to Z} \otimes^L_{g^{-1}D_Y} g^{-1}M_\bullet)\\
&= Lf^*(Lg^*(M_\bullet))
\end{aligned}$$

□

Proposition 3.8. *Let $f : X \to Y$ be a smooth morphism of smooth varieties. Then*

1. *For $M \in \mathrm{Mod}(D_Y)$, we have $H^i(Lf^*M) = 0$ for $i \neq 0$.*
2. *For $M \in \mathrm{Mod}_c(D_Y)$, we have $Lf^*M \in \mathrm{Mod}_c(D_X)$*

Proof.

1. Because f is a smooth morphism, $\mathcal{O}_X$ is flat over $f^{-1}\mathcal{O}_Y$. By the proof of 3.5, $Lf^*M \simeq \mathcal{O}_X \otimes^L_{f^{-1}\mathcal{O}_Y} f^{-1}M$. Thus Lf^*M has trivial cohomology for $i \neq 0$.
2. By 2.15, it suffices to show that the canonical morphism

$$D_X \to D_{X\to Y} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}D_y, P \mapsto P(1\otimes 1)$$

is surjective. This question is local, so we may assume X and Y to be affine. Next, we may choose coordinates $\{x_i, \partial_{x_i}\}_{1\le i\le n}$ on X and $\{y_1, \partial_{y_i}\}_{1\le i\le m}$ as in 2.2. Because f is smooth, these coordinates can be chosen to satisfy the additional condition that $\partial_{x_i} \mapsto 1 \otimes \partial_{y_i}$ for $1 \le i \le m$ and 0 otherwise under the canonical morphism $\Theta_X \to f^*\Theta_Y = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\Theta_Y$. In this situation, we have that

$$D_{X\to Y} = \bigoplus_{r_1,\dots,r_m\ge 0} \mathcal{O}_X \partial_{y_1}^{r_1}\cdots\partial_{y_m}^{r_m}$$

and the canonical morphism $D_X \to D_{X\to Y}$ from above is given by

$$\partial_{x_1}^{r_1}\cdots\partial_{x_n}^{r_n} \mapsto \delta_{r_{m+1}+\cdots+r_n,0}\,\partial_{y_1}^{r_1}\cdots\partial_{y_m}^{r_m}$$

□

Tensor products

The bifunctor

$$- \otimes_{\mathcal{O}_X} - : \mathrm{Mod}(D_X) \times \mathrm{Mod}(D_X) \to \mathrm{Mod}(D_X)$$

is right exact with respect to both factors, and we can thus define its left derived functor

$$- \otimes^L_{\mathcal{O}_X} - : D^b(D_X) \times D^b(D_X) \to D^b(D_X)$$

by using a flat resolution of either factor.

Next let X, Y be smooth varieties, and $p_1 : X \times Y \to X, p_2 : X \times Y \to Y$ the projections. For $M \in \mathrm{Mod}(\mathcal{O}_X)$ and $N \in \mathrm{Mod}(\mathcal{O}_Y)$, we set

$$M \boxtimes N = \mathcal{O}_{X\times Y} \otimes_{p_1^{-1}\mathcal{O}_X \otimes_{\mathbb{C}} p_2^{-1}\mathcal{O}_Y} (p_1^{-1}M \otimes_{\mathbb{C}} p_2^{-1}M) \in \mathrm{Mod}(\mathcal{O}_{X\times Y})$$

This gives a bifunctor

$$- \boxtimes - : \mathrm{Mod}(\mathcal{O}_X) \times \mathrm{Mod}(\mathcal{O}_Y) \to \mathrm{Mod}(\mathcal{O}_{X\times Y})$$

which is exact with respect to both factors, so extends to a functor

$$- \boxtimes - : D^b(\mathcal{O}_X) \times D^b(\mathcal{O}_Y) \to D^b(\mathcal{O}_{X\times Y})$$

For $M \in \mathrm{Mod}(D_X)$ and $N \in \mathrm{Mod}(D_Y)$, the $D_{X\times Y}$-module

$$D_{X\times Y} \otimes_{p_1^{-1}D_x \otimes_{\mathbb{C}} p_2^{-1}D_y} (p_1^{-1}M \otimes_{\mathbb{C}} p_2^{-1}N)$$

is isomorphic as an $\mathcal{O}_{X\times Y}$-module to $M \boxtimes N$ by

$$D_{X\times Y} \simeq \mathcal{O}_{X\times Y} \otimes_{p_1^{-1}\mathcal{O}_X \otimes_{\mathbb{C}} p_2^{-1}\mathcal{O}_Y} p_1^{-1}D_X \otimes_{\mathbb{C}} p_2^{-1}D_Y$$

and we denote this $D_{X\times Y}$-module again by $M \boxtimes Y$, called the *exterior product*. We now obtain a bifunctor

$$- \boxtimes - : \mathrm{Mod}(D_X) \times \mathrm{Mod}(D_Y) \to \mathrm{Mod}(D_{X\times Y})$$

which is again exact with respect to both factors, so extends to a functor

$$- \boxtimes - : D^b(D_X) \times D^b(D_Y) \to D^b(D_{X\times Y})$$

We list some facts about the exterior tensor product:

Facts 3.9.

1. Let X and Y be smooth varieties. $- \boxtimes -$ restricts to give functors $D^b_{qc}(D_X) \times D^b_{qc}(D_X) \to D^b_{qc}(D_{X\times Y})$ and $D^b_c(D_X) \times D^b_c(D_Y) \to D^b_c(D_{X\times Y})$.
2. Let $M \in \mathrm{Mod}(D_X)$. Then $p_1^* M \simeq M \boxtimes \mathcal{O}_Y$.
3. Let $N \in \mathrm{Mod}(D_Y)$. Then $p_2^* N \simeq \mathcal{O}_X \boxtimes N$.
4. Let $\Delta_X : X \to X \times X$ be the diagonal embedding. For $M, N \in \mathrm{Mod}(D_X)$, we have $M \otimes_{\mathcal{O}_X} N \simeq \Delta_X^*(M \boxtimes N)$. Furthermore, for $M_\bullet, N_\bullet \in D^b(D_X)$, we have a canonical isomorphism $M_\bullet \otimes^L_{\mathcal{O}_X} N_\bullet \simeq L\Delta_X^*(M_\bullet \boxtimes N_\bullet)$.
5. If P_i is a flat D_{X_i}-module for $i = 1, 2$ then $P_1 \boxtimes P_2$ is a flat $D_{X_1\times X_2}$-module.
6. Let $f_i : X_i \to Y_i$ be morphisms of smooth varieties. Then for $M_{i,\bullet} \in D^b(Y_i)$, we have $L(f_1 \times f_2)^* M_{1,\bullet} \boxtimes M_{2,\bullet} \simeq Lf_1^* M_{1,\bullet} \boxtimes Lf_2^* M_{2,\bullet}$.
7. Let $f : X \to Y$ be a morphism of smooth varieties. Then for $M_\bullet, N_\bullet \in D^b(D_Y)$, we have $Lf^*(M_\bullet \otimes^L_{\mathcal{O}_Y} N_\bullet) \simeq Lf^* M_\bullet \otimes^L_{\mathcal{O}_X} Lf^* N_\bullet$.

Direct images

Let $f : X \to Y$ be a morphism of smooth varieties. We define functors

$$D^b(D_X) \to D^b(f^{-1}D_Y), \quad M_\bullet \mapsto D_{Y\leftarrow X} \otimes^L_{D_X} M_\bullet$$

$$D^b(f^{-1}D_Y) \to D^b(D_Y), \quad N_\bullet \mapsto Rf_*N_\bullet$$

using a flat resolution of $M_\bullet$ and an injective resolution of $N_\bullet$. We denote the composition by

$$\int_f : D^b(D_X) \to D^b(D_Y), \quad M_\bullet \mapsto Rf_*(D_{Y\leftarrow X} \otimes^L_{D_X} M_\bullet)$$

and for an integer k, we set $\int_f^k M_\bullet = H^k(\int_f M_\bullet)$.

First, we recall a fact about Rf_*:

Proposition 3.10. *The functor $Rf_* : D^b(\mathcal{O}_X) \to D^b(\mathcal{O}_Y)$ restricts to give a functor $D^b_{qc}(\mathcal{O}_X) \to D^b_{qc}(\mathcal{O}_Y)$. If f is proper, it also restricts to a functor $D^b_c(\mathcal{O}_X) \to D^b_c(\mathcal{O}_Y)$.*

Proposition 3.11. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms of smooth varieties. Then we have that $\int_{g\circ f} = \int_g \int_f$.*

Proof. By an analogous computation to the one in the proof of 3.7, we obtain isomorphisms

$$D_{Z\leftarrow X} \simeq f^{-1}D_{Z\leftarrow Y} \otimes_{f^{-1}D_Y} D_{Y\leftarrow X} \simeq f^{-1}D_{Z\leftarrow Y} \otimes^L_{f^{-1}D_Y} D_{Y\leftarrow X}$$

Hence by definition, for $M_\bullet \in D^b(D_X)$, we obtain

$$\int_g \int_f M_\bullet = Rg_*(D_{Z\leftarrow Y} \otimes^L_{D_Y} Rf_*(D_{Y\leftarrow X} \otimes^L_{D_X} M_\bullet))$$

We claim that the canonical morphism

$$F_\bullet \otimes^L_{D_Y} Rf_*G_\bullet \to Rf_*(f^{-1}F_\bullet \otimes^L_{f^{-1}D_Y} G_\bullet)$$

is an isomorphism for any $F_\bullet \in D^b_{qc}(D_Y^{op}), G_\bullet \in D^b(f^{-1}D_Y)$. (The question is local, so take Y affine. Then represent $F_\bullet$ by a complex of free right D_Y-modules, so we reduce to $F_\bullet = D_Y^{\oplus I}$, and

$$F_\bullet \otimes^L_{D_Y} Rf_*G_\bullet \simeq Rf_*(G_\bullet)^{\oplus I} \simeq Rf_*(G_\bullet^{\oplus I}) \simeq Rf_*(f^{-1}F_\bullet \otimes^L_{f^{-1}D_Y} G_\bullet)$$

giving the desired isomorphism.) Hence we compute

$$\begin{aligned}\int_g \int_f M_\bullet &\simeq Rg_*Rf_*(f^{-1}D_{Z\leftarrow Y} \otimes^L_{f^{-1}D_Y} (D_{Y\leftarrow X} \otimes^L_{D_X} M_\bullet)) \\ &\simeq R(g\circ f)_*((f^{-1}D_{Z\leftarrow Y} \otimes^L_{f^{-1}D_Y} D_{Y\leftarrow X}) \otimes^L_{D_X} M_\bullet) \\ &\simeq R(g\circ f)_*(D_{Z\leftarrow X} \otimes^L_{D_X} M_\bullet) \\ &= \int_{g\circ f} M_\bullet\end{aligned}$$

□

Proposition 3.12. *Let $i : X \to Y$ be a closed embedding of smooth varieties.*

1. *For $M \in \mathrm{Mod}(D_X)$, we have $\int_i^k M = 0$ for $k \neq 0$. In particular, the functor $\int_i^0 : \mathrm{Mod}(D_X) \to \mathrm{Mod}(D_Y)$ is exact.*
2. *$\int_i^0$ restricts to a functor $\int_i^0 : \mathrm{Mod}_{qc}(D_X)$ to $\mathrm{Mod}_{qc}(D_Y)$.*

Sketch. By the local computation in 2.12, we have that

$$\int_i M = Ri_*(D_{Y\leftarrow X} \otimes^L_{D_X} M) \simeq Ri_*(\mathbb{C}[\partial_{y_{r+1}}, \ldots, \partial_{y_n}] \otimes_{\mathbb{C}} M) \simeq \mathbb{C}[\partial_{y_{r+1}}, \ldots, \partial_{y_n}] \otimes_{\mathbb{C}} i_* M$$

Ths proves (i). (ii) then follows from a concrete description of the D_Y-module structure on $\mathbb{C}[\partial_{y_{r+1}}, \ldots, \partial_{y_n}] \otimes_{\mathbb{C}} i_* M$. ☐

For the next steps, we will need the following fact.

Lemma 3.13. *There exist locally free resolutions of $\mathcal{O}_X$ as a left D_X-module and Ω_X as a right D_X-module given by*

$$0 \to D_X \otimes_{\mathcal{O}_X} \wedge^n \Theta_X \to \cdots \to D_X \otimes_{\mathcal{O}_X} \wedge^0 \Theta_X \to \mathcal{O}_X \to 0$$

$$0 \to \wedge^0 \Omega_X \otimes_{\mathcal{O}_X} D_X \to \cdots \to \wedge^n \Omega^1_X \otimes_{\mathcal{O}_X} D_X \to \Omega_X \to 0$$

Let Y, Z be smooth varieties, and set $X = Y \times Z$. Let $f, g : X \to Y, Z$ be the projections. To compute $D_{Y\leftarrow X} \otimes^L_{D_X} M$, we use the resolution of the right D_X-module $D_{Y\leftarrow X} = D_Y \boxtimes \Omega_Z$ induced by the resolution of Ω_Z from 3.13. Set $\Omega^k_{X/Y} = \mathcal{O}_Y \boxtimes \Omega^k_Z$ for $0 \leq k \leq \dim Z$. Then for $M \in \mathrm{Mod}_{qc}(D_X)$, we define the relative de Rham complex $DR_{X/Y}(M)$ by $DR_{X/Y}(M)^k = \Omega^{\dim Z + k}_{X/Y} \otimes_{\mathcal{O}_X} M$ for $-\dim Z \leq k \leq 0$ and 0 otherwise. By construction of the relative de Rham complex, we have an equivalence $DR_{X/Y}(M) \simeq D_{Y\leftarrow X} D_{Y\leftarrow X} \otimes^L_{D_X} M$.

Proposition 3.14. *Let Y, Z be smooth varieties, and $f : X = Y \times Z \to Y$ the projection. Then $\int_f$ restricts to a functor $\int_f : D^b_{qc}(D_X) \to D^b_{qc}(D_Y)$.*

Sketch. It suffices to show for $M \in \mathrm{Mod}_{qc}(D_X)$ that $R^i f_*(DR_{X/Y}(M)^k)$ is quasi-coherent for any i and k. Since M is quasi-coherent over $\mathcal{O}_X$, then so is $DR_{X/Y}(M)^k$, and hence by 3.10 so is $R^i f_*(DR_{X/Y}(M)^k)$. ☐

Now we can prove:

Proposition 3.15. *Let $f : X \to Y$ be a morphism of smooth varieties. Then $\int_f$ restricts to a functor $\int_f : D^b_{qc}(D_X) \to D^b_{qc}(D_Y)$.*

Proof. First, factor $f : X \to Y$ into a closed embedding $X \hookrightarrow X \times Y$ followed by a projection $X \times Y \to Y$. Then by 3.11, we may assume that f is either a closed embedding or a projection. The former follows from 3.12 and the latter follows from 3.14. ☐

In fact, direct images corresponding to proper morphisms even preserve coherence:

Theorem 3.16. *Let $f : X \to Y$ be a proper morphism of quasi-projective varieties. Then $\int_f$ restricts to a functor $D^b_c(D_X) \to D^b_c(D_Y)$.*

We mention a fact about direct images and the exterior tensor product:

Facts 3.17. Let $f_i : X_i \to Y_i$ be morphisms of smooth varieties. Then for $M_{i,\bullet} \in D^b_{qc}(D_{X_i})$, the canonical morphism

$$\int_{f_1} M_{1,\bullet} \boxtimes \int_{f_2} M_{2,\bullet} \to \int_{f_1 \times f_2} M_{1,\bullet} \boxtimes M_{2,\bullet}$$

is an isomorphism.

The proofs of these statements again use the technique of splitting f into the composition of a closed embedding followed by a projection.

4 Coherent D-modules

So far, we have encountered the functors $f^\dagger$ and $\int_f$ and seen that they preserve quasi-coherence over $\mathcal{O}_X$. However, as in remark 3.6, we saw that they do not necessarily interact nicely with coherence. In this section, we will focus on coherent D-modules, and commutative approximations of them. We then use this to find a suitable condition on coherent D-modules that is preserved by these functors.

4.1 Good filtrations

We first define the *order filtration* of D_X. Recall that on any affine open $U \subset X$, we have local coordinates $\{x_i, \partial_i\}$ such that $D_U = \oplus_{\alpha \in \mathbb{N}^n} \mathcal{O}_U \partial^\alpha$. We then define $F_l D_u = \sum_{|\alpha|<l} \mathcal{O}_U \partial^\alpha$. Then for an arbitrary open $V \subset X$, we define

$$F_l D_X(V) = \{P \in D_X(V) \mid \mathrm{res}^V_U(P) \in F_l D_X(U) \text{ for any affine open } U \subset V\}$$

We then have the associated graded $\mathrm{gr}^F D_X = \oplus_{i=0}^\infty F_i D_X / F_{i-1} D_X$. We next define filtrations of D_X-modules. Let $M \in \mathrm{Mod}_{qc}(D_X)$. We consider a filtration of M by quasi-coherent $\mathcal{O}_X$-submodules $F_i M$ satisfying

1. $F_i M \subset F_{i+1} M$
2. $F_i M = 0$ for i sufficiently small
3. $M = \cup_{i \in \mathbb{Z}} F_i M$
4. $(F_j D_X)(F_i M) \subset F_{i+j} M$

and define $\mathrm{gr}^F M = \oplus_{i \in \mathbb{Z}} F_i M / F_{i-1} M$ which we consider as a graded $\mathrm{gr}^F D_X$-module. We give a few facts about filtered D_X-modules which will be useful later. These follow from general facts about filtered modules over rings.

Proposition 4.1. *Let (M, F) be a filtered D_X-module. TFAE:*

1. $\mathrm{gr}^F M$ *is coherent over* $\mathrm{gr}^F D_X$.
2. $F_i M$ *is coherent over $\mathcal{O}_X$ for each i and there is i_0 sufficiently large such that $(F_j D_X)(F_i M) = F_{i+j} M$ for $j \geq 0, i \geq i_0$.*

3. *There is locally a surjective D_X-linear morphism $\Phi : D_X^{\oplus m} \to M$ and integers $n_j, 1 \leq j \leq m$ such that*
$$\Phi(\bigoplus_{j=1}^{m} F_{i-n_j} D_X) = F_i M$$

If any of the above equivalent conditions holds for a filtered D_X-module (M, F), then we say that F is a *good filtration* of M.

Theorem 4.2. *Any coherent D_X-module admits a globally defined good filtration. Conversely, a D_X-module with a good filtration is coherent.*

Using good filtrations, we can work with commutative approximations to coherent D-modules. In particular, we will be able to use results directly from classical commutative algebra and algebraic geometry to study D-modules.

4.2 Characteristic varieties

As defined above, $\mathrm{gr}^F D_X$ is a sheaf of commutative algebras finitely generated over $\mathcal{O}_X$. On an open affine $U \subset X$ with local coordinates $\{x_i, \partial_i\}$ as in 2.2, we set $\xi_i = \partial_i \mod F_0 D_U \in \mathrm{gr}_1^F D_U$. Then $\mathrm{gr}_l^F D_U = \oplus_{|\alpha|=l} \mathcal{O}_U \xi^\alpha$, and $\mathrm{gr}^F D_U = \mathcal{O}_U[\xi_1, \ldots, \xi_n]$. For $\pi : T^*X \to X$ the cotangent bundle of X, we may regard $\xi_1, \ldots, \xi_i$ as coordinates on the fibers of the projection over U, and hence we obtain an identification of $\mathcal{O}_U[\xi_1, \ldots, \xi_n]$ with $\pi_* \mathcal{O}_{T^*X}|_U$. This then gives an identification $\mathrm{gr}\, D_X \simeq \pi_* \mathcal{O}_{T^*X}$

Now let M be a coherent D_X-module with a good filtration F by 4.2. By 4.1 and the isomorphism $\mathrm{gr}^F D_X \simeq \pi_* \mathcal{O}_{T^*X}$ obtained above,, we have that $\mathrm{gr}^F M$ is a coherent $\pi_* \mathcal{O}_{T^*X}$-module, and we set
$$\widetilde{\mathrm{gr}^F M} = \mathcal{O}_{T^*X} \otimes_{\pi^{-1} \pi_* T^*X} \pi^{-1} \mathrm{gr}^F M$$
Then $\widetilde{\mathrm{gr}^F M}$ is a coherent $\mathcal{O}_{T^*X}$-module, and we call its support the *characteristic variety of M*, written $\mathrm{Ch}(M)$. The following theorem is a consequence of a more general fact about filtered modules over filtered rings.

Theorem 4.3. 1. *Let M be a coherent D_X-module. Then $\mathrm{Ch}(M)$ does not depend on the choice of a good filtration F.*

2. *For a short exact sequence $0 \to M \to N \to L \to 0$ of coherent D_X-modules, we have $\mathrm{Ch}(N) = \mathrm{Ch}(M) \cup \mathrm{Ch}(L)$*

We also remark the following difficult theorem due originally to Sato-Kawai-Kashiwara:

Theorem 4.4. *The characteristic variety of any coherent D_X-module is involutive with respect to the symplectic structure of the cotanteng bundle T^*X.*

Although a proof of this theorem is beyond the scope of this paper, we note the following important corollary

Corollary 4.5. *Let $M \in \mathrm{Mod}_c(D_X)$. Then any irreducible component of $\mathrm{Ch}(M)$ has dimension at least $\dim X$. In particular, if $M \neq 0$, then $\dim \mathrm{Ch}(M) \geq \dim X$.*

If a coherent D_X-module M has the minimal possible dimension of its characteristic variety ($\dim \mathrm{Ch}(M) \leq \dim X$), then it is called *holonomic*.

Proposition 4.6. *For $M \neq 0 \in \mathrm{Mod}_c(D_X)$, TFAE:*

1. *M is coherent over $\mathcal{O}_X$.*
2. *$\mathrm{Ch}(M) = T^*_X X \simeq X$ (where $T^*_X X$ denotes the zero section of $\pi : T^*X \to X$).*

Proof. Suppose M is coherent over $\mathcal{O}_X$ (hence is it locally free with finite rank $r > 0$ by 2.16). Then the filtration F defined by $F_i M = 0$ for $i < 0$ and $F_i M = M$ for $i \geq 0$ is a good filtration on M, and we have the local isomorphisms $\mathrm{gr}^F M \simeq M \simeq \mathcal{O}_X^r$. Furthermore, Θ_X acts trivially on $\mathrm{gr}^F M$, becausse $\mathrm{gr}^F_l M = 0$ for $l \geq 1$, hence $\mathrm{Ch}(M) = T^*_X X$ is the zero section of $\pi : T^*X \to X$.

Conversely, suppose $\mathrm{Ch}(M) = T^*_X X$. The problem is local on X, so we may take X affine with local coords $\{x_i, \partial_i\}_{1 \leq i \leq n}$ as in 2.2. In this case, $T^*X = X \times \mathbb{C}^n$, and for any good filtration F of M, we have

$$\sqrt{\mathrm{Ann}_{\mathcal{O}_X[\xi_1,\dots,\xi_n]}(\mathrm{gr}^F M)} = (\xi_1, \dots, \xi_n) = I$$

where $(\xi_1, \dots, \xi_n)$ is an ideal of $\mathcal{O}_X[\xi_1, \dots, \xi_n]$. (Recall that $\xi_i = \partial_i \operatorname{Mod} F_0 D_X \in \mathrm{gr}^F_1(M)$, and the identification $\pi_* \mathcal{O}_{T^*X} \simeq \mathcal{O}_X[\xi_1, \dots, \xi_n]$.) Since I is notherian, there is some $m_0 > 0$ such that $I^{m_0} \subset \mathrm{Ann}_{\mathcal{O}_X[\xi_1,\dots,\xi_n]}(\mathrm{gr}^F M)$. Because I^{m_0} is generated by the monomials ξ^α for $|\alpha| = m_0$, we thus have that

$$\partial^\alpha F_j M \subset F_{j+m_0-1} M$$

On the other hand, because F is a good filtration, we have $F_i D_X F_j M = F_{i+j} M$ for j sufficiently large by 4.1(ii). Hence

$$F_{m_0+j} M = (F_{m_0} D_X)(F_j M) \subset F_{j+m_0-1} M$$

for j sufficiently large. Hence $F_j M = F_{j+1} M = M$ for all j sufficiently large, so by 4.1(ii) we have that M is a coherent $\mathcal{O}_X$-module. □

Because $T^*_X X \simeq X$, we have $\dim(T^*_X X) = \dim X$, so we obtain the following corollary:

Corollary 4.7. *Let M be a coherent D_X-module which is also coherent over $\mathcal{O}_X$. (Equivalently, by 2.16, let M be an integrable connection.) Then M is holonomic.*

4.3 Non-characteristic morphisms and inverse images

Although in general inverse images do not preserve coherency, we give a sufficient condition on the morphism $f : X \to Y$ so that the inverse of a coherent D-module will be again coherent. For a morphism $f : X \to Y$ of smooth varieties, we have the induced morphisms $T^*X \xleftarrow{\rho_f} X \times_Y T^*Y \xrightarrow{\varpi_f} T^*Y$.

Definition 4.8. Keeping the notation above, set $T^*_X Y = \rho_f^{-1} T^*_X X \subset X \times_Y T^*Y$. We call a morphism $f : X \to Y$ of smooth varieties *non-characteristic* with respect to a coherent D_Y-module M if $\varpi_f^{-1}(\mathrm{Ch}(M)) \cap T^*_X Y \subset X \times_Y T^*_Y Y$.

The following lemma can be checked by computation.

Lemma 4.9. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of smooth varieties. Then we have a commutative diagram*

$$\begin{array}{ccccc} T^*X & \xleftarrow[\rho_f]{} & X \times_Y T^*Y & \xleftarrow[\varphi]{} & X \times_Z T^*Z \\ & & \downarrow{\scriptstyle \varpi_f} & & \downarrow{\scriptstyle \psi} \\ & & T^*Y & \xleftarrow[\rho_g]{} & Y \times_Z T^*Z \\ & & & & \downarrow{\scriptstyle \varpi_g} \\ & & & & T^*Z \end{array}$$

where $\rho_f \circ \varphi = \rho_{g\circ f}, \varpi_g \circ \psi = \varpi_{g\circ f}$, and the upper right square is cartesian.

We now move on to the main theorem of this subsection.

Lemma 4.10. *Let $f : X \to Y$ be an embedding of a hypersurface, non-characteristic with respect to $M \in \mathrm{Mod}_c(D_Y)$. Then for $u \in M$, there is locally $P \in D_Y$ such that $Pu = 0$ and f is non-characteristic with respect to D_Y/D_YP.*

Proof. We have that $\mathrm{Ch}(D_Y u) \subset \mathrm{Ch}(M)$, so f is also non-characteristic with respect to $D_Y u$. Next, $\mathrm{Ch}(D_Y u)$ is the zero set of $\mathrm{gr}^F I$, where $I = \{Q \in D_Y \mid Qu = 0\}$. Then because $T^*_X Y$ is a line bundle on X (in the case of a closed embedding, $T^*_X Y$ is the conormal bundle of X in Y, which in the case of a hypersurface embedding is a line bundle), we can find locally $P \in I$ such that f is non-characteristic with respect to D_Y/D_YP. □

Theorem 4.11. *Let $f : X \to Y$ be a morphism of smooth varieties non-characteristic with respect to $M \in \mathrm{Mod}_c(D_Y)$.*

1. *$H^j(Lf^*M) = 0$ for all $j \neq 0$.*
2. *$H^0(Lf^*M)$ is a coherent D_X-module.*
3. *$\mathrm{Ch}(H^0(Lf^*M)) \subset \rho_f \varpi_f^{-1} \mathrm{Ch}(M)$.*

Sketch. Factor $f : X \to Y$ into the composition $X \to Y \times X \to Y$. Thus we may reduce to the case where f is a closed embedding of a projection. In the latter case, the assertions follow from the isomorphism $Lf^*M \simeq M \boxtimes \mathcal{O}_Z$.

In the former case, we first consider the closed embedding of a hypersurface. To show (i), pick local coordinates $\{y_i, \partial_{y_i}\}$ on Y as in 2.10 such that $y_1 = 0$ gives a defining equation for X, and $D_{X\to Y} \simeq D_Y/y_1D_Y$. Thus we may compute Lf^* using the resolution $0 \to y_1 D_Y \to D_Y \to D_{X\to Y} \to 0$. Then, Lf^*M is (locally) represented by the complex $f^{-1}M \xrightarrow{\cdot y_1} f^{-1}M$ where the terms are in degree -1 and 0. From here, (i) can be deduced from 4.10.

For (ii) and (iii), take a good filtration F of M. Set $N = H^0(Lf^*M) = f^*M$, and define a filtration F of N by $F_iN = \mathrm{Im}(f^*F_iM \to f^*M)$. It can then be shown that $\mathrm{gr}^F N$ is a coherent $\mathrm{gr}^F D_X$-module such that $\mathrm{Ch}(N) \subset \rho_f \varpi_f^{-1}(\mathrm{Ch}(M))$.

Next, we consider when $f : X \to Y$ is a closed embedding. We proceed by induction on the codimension of X. For $\mathrm{codim}_Y X = 1$, we refer to the previous case. For a more general embedding, we factor $f : X \to Y$ as a composite $X \to Z \to Y$ of closed embeddings of smooth varieties with $\mathrm{codim}_Z X, \mathrm{codim}_Z Y < \mathrm{codim}_Y X$. Then using 4.9 with the induction hypothesis, we can deduce the desired results about f. □

4.4 Duality for D-modules

We have so far used filtrations to obtain commutative approximations to D-modules and to find a condition for inverse images to preserve coherence. In this section, we define the operation of duality, and study how it interacts with holonomicity and the image functors.

Definition 4.12. We define the duality functor $\mathbb{D} = \mathbb{D}_X : D^-(D_X) \to D^+(D_X)^{op}$ by

$$\mathbb{D}M_\bullet = R\mathscr{H}om_{D_X}(M_\bullet, D_X) \otimes_{\mathcal{O}_X} \Omega_X^{-1}[d_X] = R\mathscr{H}om_{D_X}(M_\bullet, D_X \otimes_{\mathcal{O}_X} \Omega_X^{-1}[d_X])$$

where $d_X = \dim X$.

Before we study duality, we first provide a computational lemma.

Lemma 4.13. *Let $M \in \mathrm{Mod}_c(D_X)$. Then for any affine open $U \subset X$,*

$$\mathscr{E}xt^i_{D_X}(M, D_X)(U) = \mathrm{Ext}^i_{D_X(U)}(M(U), D_X(U))$$

To see why the operation $\mathbb{D}$ deserves the name of duality, we have the following lemma:

Proposition 4.14. *1. $\mathbb{D}$ sends $D^b_c(D_X)$ to $D^b_c(D_X)^{op}$*

2. $\mathbb{D}^2 \simeq \mathrm{Id}$ on $D^b_c(D_X)$.

Sketch. 1. First, we may assume that $M_\bullet = M \in \mathrm{Mod}_c(D_X)$. Then we can deduce from 4.13 that $H^i(\mathbb{D}M) \in \mathrm{Mod}_c(D_X)$ for any i. The boundedness follows from 4.13, and 2.14.

2. We construct a canonical morphism $M_\bullet \to \mathbb{D}^2 M_\bullet$ for $M_\bullet \in D^b(D_X)$. By definition,

$$\mathbb{D}^2 M_\bullet \simeq R\mathscr{H}om_{D_X^{op}}(R\mathscr{H}om_{D_X}(M_\bullet, D_X), D_X)$$

Now set $H_\bullet = R\mathscr{H}om_{D_X}(M_\bullet, D_X)$. Then we have

$$R\mathscr{H}om_{D_X \otimes_{\mathbb{C}} D_X^{op}}(M_\bullet \otimes_{\mathbb{C}} H_\bullet, D_X) \simeq R\mathscr{H}om_{D_X}(M_\bullet, R\mathscr{H}om_{D_X^{op}}(H_\bullet, D_X))$$

Applying $H^0(R\Gamma(X, -))$ to the above, we obtain

$$\mathrm{Hom}_{D_X \otimes_{\mathbb{C}} D_X^{op}}(M_\bullet \otimes_{\mathbb{C}} H_\bullet, D_X) \simeq \mathrm{Hom}_{D_X}(M_\bullet, R\mathscr{H}om_{D_X^{op}}(H_\bullet, D_X))$$

From this, we obtain our desired morphism $M \to \mathbb{D}^2 M$ by the above equivalence from the canonical morphism

$$M_\bullet \otimes_{\mathbb{C}} R\mathscr{H}om_{D_X}(M_\bullet, D_X) \to D_X$$

To see that this is an isomorphism, we may first reduce to the case when X is affine (hence D-affine), and then we may replace $M_\bullet$ with D_X by 2.19 by taking a resolution $F_\bullet \simeq M_\bullet$ where $F_\bullet$ is a bounded complex of D_X-modules such that each term of $F_\bullet$ is a direct summand of a free D_X-module of finite rank. In this case, both sides are D_X and the result follows immediately.

□

The following theorem gives further information about characteristic varieties:

Theorem 4.15. *Let X be a smooth variety and M a coherent D_X-module.*

1. $\operatorname{codim}_{T^*X} \operatorname{Ch}(\mathscr{E}\mathrm{xt}^i_{D_X}(M, D_X) \otimes_{\mathcal{O}_X} \Omega_X^{-1}) \geq i$.
2. $\mathscr{E}\mathrm{xt}^i_{D_X}(M, D_X) = 0$ *for* $i < \operatorname{codim}_{T^*X} \operatorname{Ch}(M)$.

Now we can state how $\mathbb{D}$ interacts with holonomicity:

Corollary 4.16. *Let M be a coherent D_X-module.*

1. $H^i(\mathbb{D}M) = 0$ *unless* $-(d_X - \operatorname{codim}_{T^*X} \operatorname{Ch}(m)) \leq i \leq 0$.
2. $\operatorname{codim}_{T^*X} \operatorname{Ch}(H^i(\mathbb{D}M)) \geq d_X + i$.
3. *M is holonomic iff $H^i(\mathbb{D}M) = 0$ for $i \neq 0$.*
4. *If M is holonomic, then $\mathbb{D}M \simeq H^0(\mathbb{D}M)$ is also holonomic.*

Proof. (i) and (ii) follow immediately from 4.15 and the definition of $\mathbb{D}$. (iv) and the forward direction of (iii) follow from the (i) and (ii) and 4.5. (Note that for (iv) to make sense, we also need (iii).)

For the remaining direction of (iii), assume that $H^i(\mathbb{D}M) = 0$ for $i \neq 0$, and set $M^* = H^0(\mathbb{D}M)$. Then $\mathbb{D}M^* = \mathbb{D}^2 M \simeq M$, hence $H^0(\mathbb{D}M) \simeq M$ by 4.14. Then by (ii), $\dim \operatorname{Ch}(H^0(\mathbb{D}M^*)) \geq d_X$, hence $\mathbb{D}M^* \simeq M$ is holonomic. □

We now turn to relationship between duality and the image functors.

Lemma 4.17. *For $M_\bullet \in D^b_c(D_X)$ and $N_\bullet \in D^b(D_X)$, we have*

$$R\mathscr{H}\mathrm{om}_{D_X}(M_\bullet, N_\bullet) \simeq R\mathscr{H}\mathrm{om}_{D_X}(M_\bullet, D_X) \otimes^L_{D_X} N_\bullet$$

Proof. We have a canonical morphism

$$\simeq R\mathscr{H}\mathrm{om}_{D_X}(M_\bullet, D_X) \otimes^L_{D_X} N_\bullet \to R\mathscr{H}\mathrm{om}_{D_X}(M_\bullet, N_\bullet)$$

Thus we may assume that $M_\bullet = D_X$, in which case both side are equal to $N_\bullet$. □

Proposition 4.18. *For $M_\bullet \in D^b_c(D_X)$ and $N_\bullet \in D^b(D_X)$. We have*

$$\begin{aligned} R\mathscr{H}\mathrm{om}_{D_X}(M_\bullet, N_\bullet) &\simeq (\Omega_X \otimes^L_{\mathcal{O}_X} \mathbb{D}_X M_\bullet) \otimes^L_{D_X} N_\bullet[-d_X] \\ &\simeq \Omega_X \otimes^L_{D_X} (\mathbb{D}_X M_\bullet) \otimes^L_{\mathcal{O}_X} N_\bullet)[-d_X] \\ &\simeq R\mathscr{H}\mathrm{om}_{D_X}(\mathcal{O}_X, \mathbb{D}_X M_\bullet \otimes^L_{\mathcal{O}_X} N_\bullet) \end{aligned}$$

in $D^b(\mathbb{C}_X)$. Furthermore, we have

$$R\mathscr{H}\mathrm{om}_{D_X}(\mathcal{O}_X, N_\bullet) \simeq \Omega_X \otimes^L_{D_X} N_\bullet[-d_X]$$

Proof. We begin with the latter statement. By 4.17,

$$R\mathscr{H}\mathrm{om}_{D_X}(\mathcal{O}_X, N_\bullet) \simeq R\mathscr{H}\mathrm{om}_{D_X}(\mathcal{O}_X, D_X) \otimes^L_{D_X} N_\bullet$$

Thus it suffices to compute $R\mathscr{H}\mathrm{om}_{D_X}(\mathcal{O}_X, D_X)$, for which we use the resolution 3.13 of the left D_X-module $\mathcal{O}_X$ and the right D_X-module Ω_X:

$$\begin{aligned} R\mathscr{H}\mathrm{om}_{D_X}(\mathcal{O}_X, D_X) &\simeq [\mathscr{H}\mathrm{om}_{D_X}(D_X \otimes_{\mathcal{O}_X} \wedge^i \Theta_X, D_X)] \\ &\simeq [\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(\wedge^i \Theta_X, D_X)] \\ &\simeq [\wedge^i \Omega^1_X \otimes_{\mathcal{O}_X} D_X] \\ &\simeq \Omega_X[-d_X] \end{aligned}$$

Now let us prove the former statement. By 4.17 and the definition of $\mathbb{D}_X$:

$$\begin{aligned} R\mathscr{H}\mathrm{om}_{D_X}(M_\bullet, N_\bullet) &\simeq R\mathscr{H}\mathrm{om}_{D_X}(M_\bullet, D_X) \otimes^L_{D_X} N_\bullet \\ &\simeq (\Omega_X \otimes^L_{\mathcal{O}_X} \mathbb{D}_X M_\bullet) \otimes^L_{D_X} N_\bullet[-d_X] \end{aligned}$$

The other isomorphisms follow from the derived version of 2.7 and the isomorphism proved in the first half of this proof. □

Applying $R\Gamma(X, -)$ to the first isomorphism of 4.18

Corollary 4.19. *Let $p : X \to \{pt\}$ be the projection. Then for $M_\bullet \in D^b_c(D_X)$ and $N_\bullet \in D^b(D_X)$, we have isomorphisms*

$$\begin{aligned} R\,\mathrm{Hom}_{D_X}(M_\bullet, N_\bullet) &\simeq \int_p (\mathbb{D}_X M_\bullet \otimes^L_{\mathcal{O}_X} N_\bullet)[-d_X] \\ &\simeq R\,\mathrm{Hom}_{D_X}(\mathcal{O}_X, \mathbb{D}_X M_\bullet \otimes^L_{D_X} N_\bullet) \end{aligned}$$

Theorem 4.20. *Let $f : X \to Y$ be a morphism of smooth varieties, and $M \in \mathrm{Mod}_c(D_Y)$.*

1. *If $Lf^* M \in D^b_c(D_X)$, then there is a canonical morphism $\mathbb{D}_X Lf^* M \to Lf^* \mathbb{D}_Y M$.*
2. *Assume f is non-characteristic with respect to M. Then $D_X Lf^* M \simeq Lf^* \mathbb{D}_Y M$.*

We construct the morphism, but do not prove that it is an isomorphism. The proof uses the (by now) standard strategy of factoring f into $X \to \mathbb{P}^n \times Y \to Y$.

Sketch. First, by 4.18, we have

$$\mathrm{Hom}_{D^b(D_Y)}(M, M) \simeq \mathrm{Hom}_{D^b(D_Y)}(\mathcal{O}_Y, \mathbb{D}_Y M \otimes^L_{\mathcal{O}_Y} M)$$

Applying the functor Lf^* and using 3.9, we then have a morphism

$$\mathrm{Hom}_{D^b(D_Y)}(\mathcal{O}_Y, \mathbb{D}_Y M \otimes^L_{\mathcal{O}_Y} M) \to \mathrm{Hom}_{D^b(D_X)}(Lf^*\mathcal{O}_Y, Lf^*(\mathbb{D}_Y M) \otimes^L_{\mathcal{O}_X} Lf^* M)$$

Putting these together and then further computing:

$$\begin{aligned} \mathrm{Hom}_{D^b(D_Y)}(M, M) &\simeq \mathrm{Hom}_{D^b(D_Y)}(\mathcal{O}_Y, \mathbb{D}_Y M \otimes^L_{\mathcal{O}_Y} M) \\ &\to \mathrm{Hom}_{D^b(D_X)}(Lf^*\mathcal{O}_Y, Lf^*(\mathbb{D}_Y M) \otimes^L_{\mathcal{O}_X} Lf^* M) \\ &\simeq \mathrm{Hom}_{D^b(D_X)}(\mathcal{O}_X, Lf^* M \otimes^L_{\mathcal{O}_X} Lf^* \mathbb{D}_Y M) \\ &\simeq \mathrm{Hom}_{D^b(D_X)}(\mathbb{D}_X Lf^* M, Lf^* \mathbb{D}_Y M) \end{aligned}$$

Thus we obtain a canonical morphism $\mathbb{D}_X Lf^* M \to Lf^* \mathbb{D}_Y M$ as the image under the above compositions of id_M. □

Theorem 4.21. *Let $f : X \to Y$ be a proper morphism. Then we have a canonical isomorphism $\int_f \mathbb{D}_X \simeq \mathbb{D}_Y \int_f : D_c^b(D_X) \to D_c^b(D_Y)$.*

As above, we sketch the construction of the morphism, and omit checking that it is actually an isomorphism. The proof again uses the strategy of factoring f into $X \to \mathbb{P}^n \times Y \to Y$.

Sketch. To construct the desired morphism, we will need the trace map

$$\mathrm{Tr}_f : \int_f \mathcal{O}_X[d_X] \to \mathcal{O}_Y[d_Y]$$

This map is constructed in two steps. First, for a closed embedding $i : X \to Y$, we apply the canonical morphism $\int_i i^\dagger \to \mathrm{Id}$ to $\mathcal{O}_Y$ gives a morphism $\int_i i^\dagger \mathcal{O}_Y \to \mathcal{O}_Y$. By local computations, $i^\dagger \mathcal{O}_Y = i^* \mathcal{O}_Y[d_X - d_Y] = \mathcal{O}_X[d_X - d_Y]$ and then shifting everything by d_Y, we get a morphism $\int_i \mathcal{O}_X[d_X] \to \mathcal{O}_Y[d_Y]$. Next, for a projection $\mathbb{P}^n \times Y \to Y$, we may reduce to the situation where Y is a single point. The desired morphism is then induced by the standard trace morphism in algebraic geometry. Finally, we obtain $\mathrm{Tr}_f : \int_f \mathcal{O}_X[d_X] \to \mathcal{O}_Y[d_Y]$ by composing the trace morphisms for a factorization $X \to \mathbb{P}^n \times Y \to Y$. One can then show that Tr_f does not depend on the choice of factorization and is functorial with respect to composition.

Now we construct a canonical morphism $\int_f \mathbb{D}_X \to \mathbb{D}_Y \int_f$. Let $M_\bullet \in D_c^b(D_X)$. Computing gives

$$\begin{aligned}
\int_f \mathbb{D}_X M_\bullet &= Rf_*(R\mathscr{H}\mathrm{om}_{D_X}(M_\bullet, D_X) \otimes^L_{D_X} D_{X\to Y}) \otimes^L_{\mathcal{O}_Y} \Omega_Y^{-1}[d_Y] \\
&= Rf_*(R\mathscr{H}\mathrm{om}_{D_X}(M_\bullet, D_{X\to Y})) \otimes^L_{\mathcal{O}_Y} \Omega_Y^{-1}[d_X] \\
\mathbb{D}_Y \int_f M_\bullet &= R\mathscr{H}\mathrm{om}_{D_X}(\int_f M_\bullet, D_Y) \otimes^L_{\mathcal{O}_Y} \Omega_Y^{-1}[d_Y]
\end{aligned}$$

so it suffices to construct a canonical morphism

$$\Phi(M_\bullet) : Rf_*(R\mathscr{H}\mathrm{om}_{D_X}(M_\bullet, D_{X\to Y}[d_X])) \to R\mathscr{H}\mathrm{om}_{D_Y}(\int_f M_\bullet, D_Y[d_Y])$$

in $D_c^b(D_Y^{op})$. We have

$$\int_f D_{X\to Y}[d_X] = \int_f Lf^* D_Y[d_X] \simeq \int_f \mathcal{O}_X[d_X] \otimes^L_{\mathcal{O}_Y} D_Y$$

so that Tr_F induces a canonical morphism $\int_f D_{X\to Y}[d_X] \to D_Y[d_Y]$. Putting everything together, we may define $\Phi(M_\bullet)$ by the composition

$$\begin{aligned}
&Rf_*(R\mathscr{H}\mathrm{om}_{D_X}(M_\bullet, D_{X\to Y}[d_X])) \\
&\quad \to Rf_* R\mathscr{H}\mathrm{om}_{f^{-1}D_Y}(D_{Y\leftarrow X} \otimes^L_{D_X} M_\bullet, D_{Y\leftarrow X} \otimes^L_{D_X} D_{X\to Y}[d_X]) \\
&\quad \to R\mathscr{H}\mathrm{om}_{D_Y}(Rf_*(D_{Y\leftarrow X} \otimes^L_{D_X} M_\bullet), Rf_*(D_{Y\leftarrow X} \otimes^L_{D_X} D_{X\to Y})[d_X]) \\
&\quad = R\mathscr{H}\mathrm{om}_{D_Y}(\int_f M_\bullet, \int_f D_{X\to Y}[d_X]) \\
&\quad \to R\mathscr{H}\mathrm{om}_{D_Y}(\int_f M_\bullet, D_Y[d_Y])
\end{aligned}$$

□

5 Holonomic D-modules

We now turn to holonomic D-modules. Although we previously required further criteria of a morphism $f : X \to Y$ for its image functors to preserve coherence, we will see that even the image functors of general morphisms $f : X \to Y$ of smooth varieties preserve holonomicity. Furthermore, the duality functor will allow us to identify the previously missing left adjoints to our image functors. Finally, we also single out a particularly simple class of holonomic D_X-modules.

Notation 5.1. Let $\mathrm{Mod}_h(D_X)$ denote the subcategory of $\mathrm{Mod}_c(D_X)$ of holonomic D_X-modules, and let $D^b_h(D_X)$ denote the subcategory of $D^b_c(D_X)$ consisting of $M \in D^b_c(D_X)$ whose cohomology sheaves are holonomic.

5.1 Properties of holonomic D-modules

Proposition 5.2.

1. *For an exact sequence* $0 \to L \to M \to N \to 0$ *in* $\mathrm{Mod}_c(D_X)$, *we have* $N \in \mathrm{Mod}_h(D_X) \iff M, L \in \mathrm{Mod}_h(D_X)$.
2. *Any holonomic* D_X*-module has finite length.*

Sketch. The first statement is an immediate corollary of 4.3. For the second statement, we introduce an invariant called the *total multiplicity*, defined as follows.

Let F be a good filtration of M, so that in particular $\widetilde{\mathrm{gr}^F M}$ is a coherent $\mathcal{O}_{T^*X}$-module. Then for any irreducible component $C \in \mathrm{Ch}(M)$, take an affine open $U \subset T^*X$ such that $\overline{C \cap U} = C$, and let $p_C \subset \mathcal{O}_U(U)$ be the defining ideal of $U \cap C$. Then the stalk $(\widetilde{\mathrm{gr}^F M})_p$ is an artinian $(\mathcal{O}_{T^*X})_p$-module (that does not depend on U), so has a well defined length which we denote $m_C(M)$. Then we define the total multipicity $m(M) = \sum_{C \in \mathrm{Ch}(M)} m_C(M)$, where the sum is over irreducible components C of $\mathrm{Ch}(M)$.

By general facts about filtered rings, $m(M) = m(L) + m(N)$ for any exact sequence $0 \to L \to M \to N \to 0$ of holonomic D_X-modules. Furthermore, $m(M) = 0 \iff M = 0$, so the second statement follows by induction on $m(M)$. ☐

Proposition 5.3. *Let* $M \in \mathrm{Mod}_h(D_X)$. *Then there is an open, dense* $U \subset X$ *such that* $M|_U$ *is coherent on* $\mathcal{O}_U$.

Proof. Let $T^*_X X \subset T^*X$ be the zero section, and set $S = \mathrm{Ch}(M) \setminus T^*_X X$. If $S = \varnothing$, then by 4.6 M is coherent over $\mathcal{O}_X$. If $S \neq \varnothing$, then the fibers of the projection are at least one-dimensional, because in particular each fiber is stable under scaling by $\mathbb{C}$, because $\widetilde{\mathrm{gr}^F M}$ is a graded module over the graded ring $\mathcal{O}_{T^*X}$. (The grading on the latter comes from $\mathrm{gr}^F D_X$.) Hence $\dim \pi(S) < \dim S \leq \dim X$. Thus there is an open subset $U \subset X$ such that $U \cap \pi(S) = \varnothing$, and thus $\mathrm{Ch}(M|_U) \setminus T^*_U U = \varnothing$, so $M|_U$ is coherent over $\mathcal{O}_U$. ☐

We also note the following result:

Proposition 5.4. *Let* $M \in \mathrm{Mod}_{qc}(D_X)$. *For* $U \subset X$ *open, suppose that* N *is a holonomic submodule of* $M|_U$. *Then there is a holonomic submodule* $\tilde{N}$ *of* M *such that* $\tilde{N}|_U = N$.

Sketch. By 2.21, we may assume M coherent and $M|_U = N$. Set $L = H^0(\mathbb{D}_X M)$. Then by 4.16, we have $\operatorname{codim}_{T^*X} \operatorname{Ch}(L) \geq d_X$, hence L is holonomic with $\tilde{N} = \mathbb{D}_X L$ also holonomic. By 4.14, $\tilde{N}|_U = N$, and one can check that the canonical morphism $\tilde{N} \to M$ obtained from the morphism $\mathbb{D}_X M \to L$ is injective. □

Proposition 5.5. *The duality functor $\mathbb{D}_X$ induces isomorphisms*

$$\operatorname{Mod}_h(D_X) \simeq \operatorname{Mod}_h(D_X)^{op}, \quad D_h^b(D_X) \simeq D_h^b(D_X)^{op}$$

Proof. This is an immediate corollary of 4.16 □

We present without proof the following fundamental result on holonomic D-modules. One strategy for proving this theorem first deduces (ii) as a corollary of (i), and then uses the familiar trick of decomposing f as a closed embedding followed by a projection to reduce (i) to the situation in of a projection $\mathbb{C}^{n+1} \to \mathbb{C}^n$, so we can thus work with D-modules on $\mathbb{C}^n$.

Theorem 5.6. *Let $f : X \to Y$ be a morphism of smooth varieties.*

1. *$\int_f$ restricts to a functor $\int_f : D_h^b(D_X) \to D_h^b(D_Y)$.*
2. *$f^\dagger$ restricts to a functor $f^\dagger : D_h^b(D_Y) \to D_h^b(D_X)$.*

5.2 Adjunction formulas

Using duality, we can also introduce new functors:

Definition 5.7. Let $f : X \to Y$ be a morphism of smooth algebraic varieties. We define $\int_{f!}$ and $f^\star$ by

$$\int_{f!} = \mathbb{D}_Y \int_f \mathbb{D}_X : D_h^b(D_X) \to D_h^b(D_Y)$$

$$f^\star = \mathbb{D}_X f^\dagger \mathbb{D}_Y : D_h^b(D_Y) \to D_h^b(D_X)$$

from which we can obtain adjunction formulas:

Theorem 5.8. *For $M_\bullet \in D_h^b(D_X)$ and $N_\bullet \in D_h^b(D_Y)$, we have natural isomorphisms*

$$R\mathscr{H}om_{D_Y}(\int_{f!} M_\bullet, N_\bullet) \simeq Rf_* R\mathscr{H}om_{D_X}(M_\bullet, f^\dagger N_\bullet)$$

$$Rf_* R\mathscr{H}om_{D_X}(f^\star N_\bullet, M_\bullet) \simeq R\mathscr{H}om_{D_Y}(N_\bullet, \int_f M_\bullet)$$

Proof. Note that either isomorphism can be deduced from the other by application of the duality functors. We prove the first:

$$\begin{aligned}
&Rf_* R\mathscr{H}\mathrm{om}_{D_X}(M_\bullet, f^\dagger N_\bullet) \\
&\quad \simeq Rf_*((\Omega_X \otimes^L_{\mathcal{O}_X} \mathbb{D}_X X M_\bullet) \otimes^L_{D_X} f^\dagger N_\bullet)[-d_X] \\
&\quad \simeq Rf_*((\Omega_X \otimes^L_{\mathcal{O}_X} \mathbb{D}_X M_\bullet) \otimes^L_{D_X} D_{X\to Y} \otimes^L_{f^{-1}D_Y} f^{-1} N_\bullet)[d_Y] \\
&\quad \simeq Rf_*((\Omega_X \otimes^L_{\mathcal{O}_X} \mathbb{D}_X M_\bullet) \otimes^L_{D_X} D_{X\to Y}) \otimes^L_{D_Y} N_\bullet[-d_Y] \\
&\quad \simeq (\Omega_Y \otimes^L_{\mathcal{O}_Y} \int_f \mathbb{D}_X M_\bullet) \otimes^L_{D_Y} N_\bullet[-d_Y] \\
&\quad \simeq (\Omega_Y \otimes^L_{\mathcal{O}_Y} \mathbb{D}_Y \int_{f!} M_\bullet) \otimes^L_{D_Y} N_\bullet[-d_Y] \\
&\quad \simeq R\mathscr{H}\mathrm{om}_{D_Y}(\int_{f!} M_\bullet, N_\bullet)
\end{aligned}$$

The first and last equivalences follow from 4.18, and the rest follow from definitions of the functors. □

From this, we obtain adjunctions:

Corollary 5.9. *For $M_\bullet \in D^b_h(D_X)$ and $N_\bullet \in D^b_h(D_Y)$, we have natural isomorphisms*

$$\mathrm{Hom}_{D^b_h(D_Y)}(\int_{f!} M_\bullet, N_\bullet) \simeq \mathrm{Hom}_{D^b_h(D_X)}(M_\bullet, f^\dagger N_\bullet)$$

$$\mathrm{Hom}_{D^b_h(D_X)}(f^\star N_\bullet, M_\bullet) \simeq \mathrm{Hom}_{D^b_h(D_Y)}(N_\bullet, \int_f M_\bullet)$$

Proof. Apply $H^0(R\Gamma(Y,-))$ to the isomorphisms of 5.8. □

Theorem 5.10. *There is a morphism of functors $\int_{f!} \to \int_f : D^b_h(D_X) \to D^b_h(D_Y)$ which is an isomorphism if f is proper.*

Sketch. By Hironaka's desingularization theorem, we can factor $f : X \to Y$ as

$$X \xrightarrow{g} X \times Y \xrightarrow{j} \tilde{X} \times Y \xrightarrow{p} Y$$

where $\tilde{X}$ is a desingularization of $\overline{X}$, g, j are embeddings, and p is the projection. In this situation, g and p are proper, and j is an open embedding, so we may reduce to these cases.

If f is proper, then by 4.21 we have an isomorphism

$$\int_{f!} = \mathbb{D}_Y \int_f \mathbb{D}_X \xrightarrow{\sim} \int_f$$

If f is an open embedding, then for $M_\bullet \in D^b_h(D_X)$, we have

$$\begin{aligned}
\mathrm{Hom}_{D^b_h(D_Y)}(\int_{f!} M_\bullet, \int_j M_\bullet) &\simeq \mathrm{Hom}\, D^b_h(D_X)(M_\bullet, j^\dagger \int_f M_\bullet) \\
&\simeq \mathrm{Hom}_{D^b_h(D_X)}(M_\bullet, M_\bullet)
\end{aligned}$$

by 5.9 and we obtain the desired morphism as the image of $\mathrm{Id} \in \mathrm{Hom}_{D^b_h(D_X)}(M_\bullet, M_\bullet)$. □

5.3 Minimal extensions

A nonzero coherent D-module M is *simple* if it contains no coherent D-submodules other than itself and 0. For any holonomic D-module M, there is a finite sequence $M = M_0 \supset \cdots \supset M_{r+1} = 0$ of holonomic D-submodules such that M_i/M_{i+1} is simple for each i, by 5.2. We now construct simple holonomic D-modules from locally free D-modules of finite rank over $\mathcal{O}$ on locally closed smooth subvarieties.

Let Y be a locally closed smooth subvariety of a smooth variety X, and assume that the inclusion map $i : Y \hookrightarrow X$ is affine. Then $D_{X\leftarrow Y}$ is locally free over D_Y and $Ri_* = i_*$, so for a holonomic D_Y-module M we have $H^j \int_i M = H^j \int_{i!} M = 0$ for $j \neq 0$, so we may thus regard $\int_i M$ and $\int_{i!} M$ as D_X-modules. These are holonomic by 5.6, and by 5.10, we have a morphism $\int_{i!} M \to \int_i M$ in $\mathrm{Mod}_h(D_X)$.

We call the image $L(Y, M)$ of the canonical morphism $\int_{i!} M \to \int_i M$ above the *minimal extension* of M, and $L(Y, M)$ is holonomic by 5.2. We mention the following classification theorem for simple, holonomic D-modules.

Theorem 5.11. *1. Let Y be a locally closed, smooth, connected subvariety of X such that $i : Y \hookrightarrow X$ is affine and let M be a simple holonomic D_Y-module. Then $L(Y, M)$ is also simple, and is the unique simple submodule of $\int_i M$.*

2. Any simple holonomic D_X-module is isomorphic to $L(Y, M)$ for some pair (Y, M), where Y is as in (i) and M is a simple D_Y-module that is locally free and of finite rank over $\mathcal{O}_Y$.

3. Let (Y, M) as in (ii) and (Y', M') another such pair. Then $L(Y, M) \simeq L(Y', M')$ iff $\overline{Y} = \overline{Y'}$ and $M|_U \simeq M'|_U$ for any open dense $U \subset Y \cap Y'$.

Proposition 5.12. *Let Y be a locally closed smooth subvariety of X such that the embedding $i : Y \to X$ is affine, and let M be an integrableconnection on Y. Then $\mathbb{D}_X L(Y, M) \simeq L(Y, \mathbb{D}_Y M)$.*

Proof. By definition of $L(Y, M)$ and exactness of the duality functor:

$$\mathbb{D}_X L(Y, M) = \mathbb{D}_X \operatorname{Im}\left(\int_{i!} M \to \int_i M\right) \simeq \operatorname{Im}\left(\mathbb{D}_X \int_{i!} M \to \mathbb{D}_X \int_i M\right)$$

and $L(Y, \mathbb{D}_Y M) \simeq \operatorname{Im}(\int_{i!} \mathbb{D}_Y M \to \int_i \mathbb{D}_Y M)$. These are isomorphic by 4.21 and the definition of $\int_{i!}$. □

6 Analytic D-modules

Until this point, X has denoted a smooth variety over $\mathbb{C}$. To state the Riemann-Hilbert correspondence, we will also need to study D-modules on complex manifolds. In this section, we give a rapid overview of the theory of D-modules on complex manifolds (much of which will be completely analogous to the algebraic situation).

6.1 D-modules on complex manifolds

Let X be a complex manifold and $\mathcal{O}_X$ its sheaf of holomorphic functions. We will also need the sheaves Θ_X and Ω_X^p of holomorphic vector fields and holomorphic differential forms of degree p. We define D_X to be the subsheaf of $\mathscr{E}\mathrm{nd}_{\mathbb{C}}(\mathcal{O}_X)$ generated by $\mathcal{O}_X$ and Θ_X, and we have the side-changing equivalence

$$\Omega_X \otimes_{\mathcal{O}_X} - : \mathrm{Mod}(D_X) \to \mathrm{Mod}(D_X^{op})$$

We also define the transfer bimodules

$$D_{X\to Y} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}D_Y, \quad D_{Y\leftarrow X} = \Omega_X \otimes_{\mathcal{O}_X} D_{X\to Y} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\Omega_Y^{-1}$$

In local coordinates $\{x_i\}$ on X, we have $D_U = \oplus_{\alpha\in\mathbb{N}^n}\mathcal{O}_U\partial^\alpha$, where $\partial_i = \frac{\partial}{\partial x_i}$, with the order filtration defined locally by $F_l D_U = \sum_{|\alpha|\le l}\mathcal{O}_U\partial^\alpha$. Then the associated graded $\mathrm{gr}^F D_X$ is a sheaf of commutative algebras over $\mathcal{O}_X$, and we will often identify it with a subsheaf of $\pi_*\mathcal{O}_{T^*X}$. (We regard $\xi_i = \partial_i \in \mathrm{gr}_1^F D_X$ as giving coordinates on the fibers of the cotangent bundle $\pi : T^*X \to x$.)

We also have the notion of a good filtration on a D-module M. Unlike in the algebraic case, we no longer have the existence of a global good filtration, but the local version will suffice to define the characteristic variety $\mathrm{Ch}(M)$ of a coherent D_X-module as follows. For an open $U \subset X$ such that $M|_U$ admits a good filtration F, we obtain a coherent $\mathcal{O}_{T^*U}$-module

$$\widetilde{\mathrm{gr}^F(M|_U)} = \mathcal{O}_{T^*U} \otimes_{\pi_U^{-1}\mathrm{gr}^F D_U} \pi_U^{-1}\mathrm{gr}^F M|_U$$

where $\pi_U : T^*U \to U$ is the projection. Set $\mathrm{Ch}(M|_U)$ to be the support of the above $\mathcal{O}_{T^*U}$-module. Then $\mathrm{Ch}(M)$ is the closed subvariety of T^*X such that $\mathrm{Ch}(M)\cap T^*U = \mathrm{Ch}(M|_U)$ for any U and F as above. By an analogous argument as in the algebraic case, $\mathrm{Ch}(M)$ is well-defined, and we have the following theorem.

Theorem 6.1. *For any coherent D_X-module M, $\mathrm{Ch}(M)$ is involutive with respect to the canonical symplectic structure on T^*X. In particular, every irreducible component of $\mathrm{Ch}(M)$ has dimension at least $\dim X$, and $\dim \mathrm{Ch}(M) \ge \dim X$.*

We call M holonomic if $\mathrm{Ch}(M)$ has the minimal dimension $\dim X$, and we define the condition for f to be non-characteristic with respect to a coherent D_X-module M similarly to the algebraic case.

Notation 6.2. We denote by $\mathrm{Mod}_c(D_X)$ and $\mathrm{Mod}_h(D_X)$ the categories of coherent and holonomic D_X-modules, respectively. Furthermore, we denote by $D_c^b(D_X)$ and $D_h^b(D_X)$ the subcategories of $D^b(D_X)$ consisting of complexes whose cohomology sheaves are coherent and holonomic D_X-modules, respectively.

We re-introduce the various functors from before.

Definition 6.3. Let $f : X \to Y$ be a morphism of complex manifolds.

$$\mathbb{D}_X : D_c^b(D_X) \to D_c^b(D_X)^{op}, \quad M_\bullet \mapsto R\mathscr{H}\mathrm{om}_{D_X}(M_\bullet, D_X \otimes_{\mathcal{O}_X} \Omega_X^{-1}[d_X])$$

Note that $\mathbb{D}_X$ also preserves holonomicity: $\mathbb{D}_X : D_h^b(D_X) \mapsto D_h^b(D_X)^{op}$.

$$Lf^* : D^b(D_Y) \to D^b(D_X), \quad M_\bullet \mapsto D_{X\to Y} \otimes^L_{f^{-1}D_Y} f^{-1}M_\bullet$$

$$f^\dagger : D^b(D_Y) \to D^b(D_X), \quad M_\bullet \mapsto Lf^*M_\bullet[d_X - d_Y]$$

$$\int_f : D^b(D_X) \to D^b(D_Y), \quad M_\bullet \mapsto Rf_*(D_{Y\leftarrow X} \otimes^L_{D_X} M_\bullet)$$

While many of the algebraic results carry over immediately to the analytic situation (e.g. under what conditions the functors above preserve coherence, commutivity with duality, etc.), we do not list them here.

6.2 Solution and de Rham functors

In this section, we introduce the de Rham and solution functors. These will be crucial in the Riemann-Hilbert correspondence.

Definition 6.4. Let X be a complex manifold.

$$DR_X : D^b(D_X) \to D^b(\mathbb{C}_X), \quad DR_X M_\bullet = \Omega_X \otimes^L_{D_X} M_\bullet$$

$$\mathrm{Sol}_X : D^b(D_X) \to D^b(\mathbb{C}_X)^{op}, \quad \mathrm{Sol}_X\, M_\bullet = R\mathscr{H}\mathrm{om}_{D_X}(M_\bullet, \mathcal{O}_X)$$

We have the following analogous result to 4.18

Proposition 6.5. *For $M_\bullet \in D^b_c(D_X)$, we have*

$$DR_X M_\bullet \simeq R\mathscr{H}\mathrm{om}_{D_X}(\mathcal{O}_X, M_\bullet)[d_X] \simeq \mathrm{Sol}_X(\mathbb{D}_X M_\bullet)[d_X]$$

By this result, properties of Sol_X can be deduced from properties of DR_X, and vice versa. DR_X has the advantage that we can compute it using a resolution of the right D_X-module Ω_X. Similarly to 3.13, we have a locally free resolution

$$0 \to \Omega^0_X \otimes_{\mathcal{O}_X} D_X \to \cdots \to \Omega^{d_X}_X \otimes_{\mathcal{O}_X} D_X \to \Omega_X \to 0$$

so for $M \in \mathrm{Mod}(D_X)$, we may represent $DR_X(M)[-d_X]$ in $D^b(\mathbb{C}_X)$ by the complex

$$\Omega_{X,\bullet} \otimes_{\mathcal{O}_X} M = [\Omega^0_X \otimes_{\mathcal{O}_X} M \to \cdots \to \Omega^{d_X}_X \otimes_{\mathcal{O}_X} M]$$

In the case where M is an integrable connection of rank m (a coherent D_X-module which is locally free of rank m over $\mathcal{O}_X$), we have that the cohomology sheaf

$$H^0(\Omega_{X,\bullet} \otimes_{\mathcal{O}_X} M) \simeq \mathscr{H}\mathrm{om}_{D_X}(\mathcal{O}_X, M)$$

coincides with the kernel of the morphism

$$\nabla : M \simeq \Omega^0_X \otimes_{\mathcal{O}_X} M \to \Omega^1_X \otimes_{\mathcal{O}_X} M$$

whic is the sheaf

$$M^\nabla = \{s \in M \mid \nabla s = 0\} = \{0 \in M \mid \Theta_X s = 0\}$$

of *horizontal sections* of the integrable connection M. It is a locally free $\mathbb{C}_X$-module of rank m. We call such $\mathbb{C}_X$-modules (locally free $\mathbb{C}_X$-modules of finite rank) *local systems*, and we denote by $\mathrm{Loc}(X)$ the category of local systems on X. Conversely, given a local system L, we can define an integrable connection $M = \mathcal{O}_X \otimes_{\mathbb{C}_X} L$ with $\nabla : M \to \Omega^1_X \otimes \mathbb{C}_X M$ as above given by $d \otimes \mathrm{id}_L$. In fact, we can extend this correspondence to obtain a simple case of the Riemann-Hilbert correspondence:

Theorem 6.6. *Let M be an integrable connection of rank m on a complex manifold X. Then $H^i(DR_X(M)) = 0$ for $i \neq -d_X$, and $H^{-d_X}(DR_X(M))$ is a local system on X. Thus we have an equivalence*

$$H^{-d_X}(DR_X(-)) : \mathrm{Conn}(X) \simeq \mathrm{Loc}(X)$$

where $\mathrm{Conn}(X)$ denotes the category of integrable connections on X.

Theorem 6.7. *Let $f : X \to Y$ be a morphism of complex manifolds. For $M_\bullet \in D^b(D_X)$, we have an isomorphism in $D^b(\mathbb{C}_Y)$:*

$$Rf_* DR_X M_\bullet \simeq DR_Y \int_f M_\bullet$$

If f is non-characteristic with respect to a coherent D_X-module M, then we have

$$DR_Y(Lf^* M) \simeq f^{-1} DR_X(M)[d_Y - d_X]$$

6.3 Constructible sheaves

For a morphism $f : X \to Y$ of analytic spaces, we have functors

$$f^{-1} : \mathrm{Mod}(\mathbb{C}_Y) \to \mathrm{Mod}(\mathbb{C}_X), \quad f_*, f_! : \mathrm{Mod}(\mathbb{C}_X) \to \mathrm{Mod}(\mathbb{C}_Y)$$

The first is exact, and the latter two are left exact. We have their derived functors

$$f^{-1} : D^b(\mathbb{C}_Y) \to D^b(\mathbb{C}_X), \quad Rf_*, Rf_! : D^b(\mathbb{C}_X) \to D^b(\mathbb{C}_Y)$$

and an additional functor $f^! : D^b(\mathbb{C}_Y) \to D^b(\mathbb{C}_X)$, which is right adjoint to $Rf_!$. Furthermore, the tensor product induces a functor

$$- \otimes_{\mathbb{C}} - : D^b(\mathbb{C}_X) \times D^b(\mathbb{C}_X) \to D^b(\mathbb{C}_X)$$

and we also have an exterior tensor product:

Definition 6.8. Let X and Y be analytic spaces. For $K_\bullet \in D^b(\mathbb{C}_X)$ and $L_\bullet \in D^b(\mathbb{C}_Y)$, we define

$$K_\bullet \boxtimes \mathbb{C} L_\bullet = p_1^{-1} K_\bullet \otimes_{\mathbb{C}_{X \times Y}} p_2^{-1} L_\bullet$$

where $p_1, p_2 : X \times Y \to X, Y$ are the projections.

For an analytic space, we set $\omega_{X,\bullet} = a_X^! \mathbb{C} \in D^b(\mathbb{C}_X)$, where $a_X : X \to \{pt\}$ is the unique morphism to the one point space. If X is a complex manifold, then $\omega_{X,\bullet} \simeq \mathbb{C}_X[2 \dim X]$. We define

Definition 6.9. Let X be a complex manifold. We define a functor

$$\mathbf{D}_X : D^b(\mathbb{C}_X) \to D^b(\mathbb{C}_X)^{op}, \quad \mathbf{D}_X F_\bullet = R\mathscr{H}\mathrm{om}_{\mathbb{C}_X}(F_\bullet, \omega_{X,\bullet})$$

and $\mathbf{D}_X F_\bullet$ is called the *Verdier dual* of $F_\bullet \in D^b(\mathbb{C}_X)$.

Proposition 6.10. *Let X be a complex manifold. Let M be a holonomic D_X-module and $\mathbb{D}_X M$ its dual. Then we have isomorphisms*

$$\mathbf{D}_X(DR_X(M)) \simeq DR_X \mathbb{D}_X M, \quad \mathbf{D}_X \mathrm{Sol}_X(M)[d_X] \simeq \mathrm{Sol}_X(\mathbb{D}_X M)[d_X]$$

Let X be an analytic space. A locally finite partition $X = \sqcup_{\alpha \in A} X_\alpha$ by locally closed analytic subsets X_α is a *stratification* of X if for any $\alpha \in A$, X_α is smooth and $\overline{X_\alpha} = \sqcup_{\beta \in B} X_\beta$ some $B \subset A$. We call each X_α a *stratum*.

Definition 6.11. Let X be an analytic space. A $\mathbb{C}_X$-module F is *constructible* on X if there is a stratification $X = \sqcup_{\alpha \in A} X_\alpha$ such that $F|_{X_\alpha}$ is locally free of finite rank for all α. If X is a variety, then a $\mathbb{C}_{X^{an}}$-module F is *algebraically constructible* if there is a stratification $X = \sqcup_{\alpha \in A} X_\alpha$ such that $F|_{X_\alpha^{an}}$ is a locally constant sheaf for all α.

Notation 6.12. For an analytic space X, we denote by $D_c^b(X)$ the subcategory of $D^b(\mathbb{C}_X)$ consisting of complexes whose cohomology sheaves are constructible. For a variety X, we denote by $D_c^b(X)$ the subcategory of $D^b(\mathbb{C}_{X^{an}})$ consisting of complexes whose cohomology sheaves are algebraically constructible.

For a variety X, we write by abuse of notation the sheaf $\omega_{X^{an},\bullet}$ and the functor $\mathbf{D}_{X^{an}} : D^b(\mathbb{C}_{X^{an}}) \to D^b(\mathbb{C}_{X^{an}})^{op}$ simply as $\omega_{X,\bullet}$ and $\mathbf{D}_X$, respectively.

For a morphism $f : X \to Y$ of varieties, we write $(f^{an})^{-1}, (f^{an})^!, Rf_*^{an}, Rf_!^{an}$ as $f^{-1}, f^!, Rf_*, Rf_!$ respectively.

Theorem 6.13. *1. Let X be a variety or an analytic space. Then $\omega_{X,\bullet} \in D^b_c(X)$, $\mathbf{D}_X$ preserves $D^b_c(X)$, and $\mathbf{D}_X^2 \simeq \mathrm{Id}$ on $D^b_c(X)$.*

2. *Let $f : X \to Y$ be a morphism of varieties or analytic spaces. Then f^{-1} and $f^!$ induce $f^{-1}, f^! : D^b_c(Y) \to D^b_c(X)$, and $f^! = \mathbf{D}_X f^{-1} \mathbf{D}_Y$ on $D^b_c(Y)$.*
3. *Let $f : X \to Y$ be a morphism of varieties or analytic spaces. In the latter case, we further assume that f is proper. Then $Rf_*, Rf_!$ induce $Rf_*, Rf_! : D^b_c(X) \to D^b_c(Y)$ and $Rf_! = \mathbf{D}_Y Rf_* \mathbf{D}_X$.*
4. *Let X be a variety or analytic space. Then the tensor product $- \otimes_{\mathbb{C}} -$ induces $- \otimes_{\mathbb{C}} - : D^b_c(X) \times D^b_c(X) \to D^b_c(X)$.*

In fact, we could have taken the above theorem as the definitions of $f_!$ and $f^!$ (i.e. obtained from f^{-1} and f_* by Verdier duality).

Definition 6.14. Let X be a variety or an analytic space. Then $F_\bullet \in D^b_c(X)$ is a *perverse sheaf* if $\dim \operatorname{supp}(H^j(F_\bullet)) \leq -j$ and $\dim \operatorname{supp}(H^j(\mathbf{D}_X F_\bullet)) \leq -j$ for any $j \in \mathbb{Z}$. We denote by $\mathrm{Perv}(\mathbb{C}_X)$ the subcategory of $D^b_c(X)$ consisting of perverse sheaves.

We now mention two remarkable theorems of Kashiwara:

Theorem 6.15. *Let M be a holonomic D_X-module for X a complex manifold. Then $\mathrm{Sol}_X(M) = R\mathscr{H}\mathrm{om}_{D_X}(M, \mathcal{O}_X)$ and $DR_X(M) = \Omega_X \otimes^L_{D_X} M$ are objects in $D^b_c(X)$.*

Theorem 6.16. *Let X be a complex manifold and M a holonomic D_X-module. Then $\mathrm{Sol}_X(M)[d_X] = R\mathscr{H}\mathrm{om}_{D_X}(M, \mathcal{O}_X)[d_X]$ and $DR_X(M) = \Omega_X \otimes^L_{D_X} M$ are perverse sheaves on X.*

6.4 Analytic from algebraic

We turn now to obtaining analytic D-modules from algebraic D-modules on a smooth variety.

For an algebraic variety X, we denote by X^{an} the corresponding analytic space. We have a morphism $i_X : (X^{an}, \mathcal{O}_{X^{an}}) \to (X, \mathcal{O}_X)$ of ringed spaces. If X is a smooth variety, then X^{an} is a complex manifold, and we have a morphism $i_X^{-1} D_X \to D_{X^{an}}$. This gives a functor

$$-^{an} : \mathrm{Mod}(D_X) \to \mathrm{Mod}(D_{X^{an}}), \quad M \mapsto M^{an} = D_{X^{an}} \otimes_{i_X^{-1} D_X} i_X^{-1} M$$

This functor is exact because $D_{X^{an}}$ is faithfully flat over $i_X^{-1} D_X$, so the above functor is exact and extends to a functor

$$-^{an} : D^b(D_X) \to D^b(D_{X^{an}})$$

Note further that $-^{an}$ preserves coherence. We may now define the de Rham and Solution functors for a smooth variety X.

Definition 6.17. Let X be a smooth variety. Then we define functors

$$DR_X : D^b(D_X) \to D^b(\mathbb{C}_{X^{an}}), \quad M_\bullet \mapsto \Omega_{X^{an}} \otimes^L_{D_{X^{an}}} (M_\bullet)^{an}$$

$$\mathrm{Sol}_X : D^b(D_X) \to D^b(\mathbb{C}_{X^{an}})^{op}, \quad M_\bullet \mapsto R\mathscr{H}\mathrm{om}_{D_{X^{an}}}((M_\bullet)^{an}, \mathcal{O}_{X^{an}})$$

Proposition 6.18. *Let $f : X \to Y$ be a morphism of smooth varieties. For $M_\bullet \in D^b_c(D_X)$, there is a canonical morphism $DR_Y(\int_f M_\bullet) \to Rf_*(DR_X M_\bullet)$ which is an isomorphism if f is proper.*

Proposition 6.19. *Let X and Y be smooth algebraic varieties. For $M_\bullet \in D^b_c(D_X)$ and $N_\bullet \in D^b_c(D_Y)$, we have a canonical morphism*

$$DR_X(M_\bullet) \boxtimes_{\mathbb{C}} DY_Y(N_\bullet) \to DR_{X\times Y}(M_\bullet \boxtimes N_\bullet)$$

which is an isomorphism if $M_\bullet \in D^b_h(D_X)$ or $N_\bullet \in D^b_h(D_Y)$.

Proposition 6.20. *Let X be a smooth variety. For $M_\bullet \in D^b_c(D_X)$, we have canonical morphisms*

$$DR_X(\mathbb{D}_X M_\bullet) \to \mathbf{D}_X(DR_X M_\bullet)$$

$$\mathrm{Sol}_X(\mathbb{D}_X M_\bullet)[d_X] \to \mathbf{D}_X(\mathrm{Sol}_X M_\bullet)[d_X]$$

which are isomorphisms if $M_\bullet \in D^b_h(D_X)$.

7 Regular D-modules

While the classical theory of regular integrable connections on a complex manifold provides motivation for the algebraic setting, we instead take the opposite approach and begin with regular integrable connections on an algebraic variety. We then generalize to high dimensional varieties, and mention in passing an analytic result.

7.1 Regularity on curves

Let C be a smooth (algebraic) curve, $p \in C$, $\mathcal{O}_{C,p}$ the local ring, and $K_{C,p}$ its fraction field.

Definition 7.1. Let M be a finite dimensional $K_{C,p}$-module and $\nabla : M \to \Omega^1_{C,p} \otimes_{\mathcal{O}_{C,p}} M$ be a $\mathbb{C}$-linear map. Then (M, ∇) is called an *algebraic meromorphic connection* $p \in C$ if $\nabla(fu) = df \otimes u + f\nabla u$ for $f \in K_{C,p}, u \in M$.

A morphism $\varphi : (M, \nabla_M) \to (N, \nabla_N)$ of algebraic meromorphic connections at $p \in C$ is a $K_{C,p}$-linear map $\varphi : M \to N$ satisfying $\nabla_N \circ \varphi = (id \otimes \varphi) \circ \nabla_M$.

Definition 7.2. An algebraic meromorphic connection (M, ∇) at $p \in C$ is called *regular* if there is a finitely generated $\mathcal{O}_{C,p}$-submodule L of M such that $M = K_{C,p}L$ and $x\nabla(L) \subset \Omega^1_{C,p} \otimes_{\mathcal{O}_{C,p}} L$ for some local parameter x at p. We call such L an $\mathcal{O}_{C,p}$-lattice of (M, ∇).

We now define what it means for a D_C-module to be reguar. Let M be an integrable connection on C. Let $j : C \hookrightarrow \overline{C}$ be a smooth completion, and consider the $D_{\overline{C}}$-module $j_*M = \int_j M$. Because M was locally free over $\mathcal{O}_C$, it is free on a nonempty open $U \subset C$. We set $V = C \setminus U$, and hence $j_*M|_{\overline{C}\setminus V}$ is also free over $j_*\mathcal{O}_C|_{\overline{C}\setminus V}$. Thus j_*M is locally free of finite rank over $j_*\mathcal{O}_C$. Let $p \in \overline{C} \setminus C$. Then the stalk $(j_*M)_p$ is a free module over $K_{\overline{C},p} = (j_*\mathcal{O}_C)_p$ as well as a $D_{\overline{C},p}$-module, and we have a morphism $\nabla : j_*M \to \Omega^1_{\overline{C},p} \otimes_{\mathcal{O}_{\overline{C},p}} j_*M$ given by $m \mapsto dx \otimes \partial m$, where x is a local parameter at p and $\partial = \frac{d}{dx}$. We call the $D_{\overline{C}}$-module j_*M the *algebraic meromorphic extension of* M.

Definition 7.3. Let M be an integrable connection on a smooth curve C. For $p \in \overline{C} \setminus C$, we say that M has a regular singularity at p if $((j_* M)_p, \nabla)$ as defined above is regular. M is called *regular* if M has a regular singularity at all $p \in \overline{C} \setminus C$.

Before we can define regularity for holonomic D_C-modules, we will need the following lemma.

Lemma 7.4. *A coherent D_C-module M is holonomic iff it is generically an integrable connection.*

Proof. The forward direction follows immediately from 5.3. Conversely, suppose $M \in \mathrm{Mod}_c(D_C)$, and there exists an open, dense $U \subset C$ such that $M|_U$ is an integrable connection. In this case, $V = C \setminus U$ is a finite set, and $\mathrm{Ch}(M) \subset T^*_C C \cup (\cup_{p \in V} (T^*C)_p)$, where $(T^*C)_p$ denotes the fiber over p of the projection $\pi : T^*C \to C$. Then because $\dim^*_C C = 1$ and $\dim(T^*C)_p = 1$ and V is finite, we have $\dim \mathrm{Ch}(M) = 1$, so M is holonomic. □

Definition 7.5. Let C be a smooth curve, and $M \in \mathrm{Mod}_h(D_C)$. Then M is *regular* if there is an open, dense $C_0 \subset C$ such that $M|_{C_0}$ is a regular integrable connection on C_0. $M_\bullet \in D^b_h(D_C)$ is regular if all its cohomology sheaves are regular.

7.2 Regularity on general varieties

Let X now denote a smooth variety, and let $j : X \hookrightarrow V$ be an open embedding of X into a smooth variety V such that $D = V \setminus X$ is a divisor on V. We set $\mathcal{O}_V[D] = j_* \mathcal{O}_X$; this is a coherent sheaf of rings. We call a D_V-module an *algebraic meromorphic connection along D* if it is isomorphic as an $\mathcal{O}_V$-module to a coherent $\mathcal{O}_V[D]$-module.

Definition 7.6. An integrable connection M on X is *regular* if for any morphism $i_C : C \to X$ from a smooth curve C, the induced integrable connection $i^*_C M$ on C is regular (as an integrable connection on a smooth curve).

Notation 7.7. We denote by $\mathrm{Conn}(V; D)$ the category of algebraic meromorphic connections along D, $\mathrm{Conn}(X)$ the category of integrable connections on X, and $\mathrm{Conn}^{reg}(X)$ the subcategory of $\mathrm{Conn}(X)$ consisting of regular integrable connections

We now mention a few results which we will need later. Their proofs rely on results for analytic meromorphic connections, and we will not include them here.

Proposition 7.8. *Let $M \in \mathrm{Conn}(X)$. TFAE:*

1. *M is regular.*
2. *There is a smooth completion $j : X \hookrightarrow \overline{X}$ such that $\overline{X} \setminus X$ is a divisor on $\overline{X}$, $(j_* M)^{an}$ is a regular analytic meromorphic connection.*
3. *For any smooth completion $j : X \hookrightarrow \overline{X}$ such that $\overline{X} \setminus X$ is a divisor on $\overline{X}$, $(j_* M)^{an}$ is a regular analytic meromorphic connection.*

The following are due to Deligne:

Theorem 7.9. *Let D be a divisor on a copmlex manifold X and $j : Y = X \setminus D \to D$ the embedding. Let N be a regular (analytic) meromorphic connection along D. Then the following morphisms are isomorphisms*

$$DR_X(N) \to Rj_*j^{-1}DR_X(N)$$

$$R\Gamma(X, DR_X(N)) \to R\Gamma(Y, DR_Y(N|_Y))$$

Theorem 7.10. *Let X be a smooth variety. Then the functor $-^{an}$ induces an equivalence* $\mathrm{Conn}^{reg}(X) \to \mathrm{Conn}(X^{an})$.

7.3 Regular holonomic D-modules

Finally, we define regular, holonomic D-modules.

Definition 7.11. Let X be a smooth variety. $M \in \mathrm{Mod}_h(D_X)$ is *regular* if any composition factor of M is isomorphic to the minimal extension $L(Y, N)$ of some regular integrable connection N on a locally closed smooth subvariety Y of X such that the inclusion $Y \to X$ is affine.

Notation 7.12. We denote by $\mathrm{Mod}_{rh}(D_X)$ the subcategory of $\mathrm{Mod}_h(D_X)$ consisting of regular holonomic D_X-modules, and we denote by $D^b_{rh}(D_X)$ the subcategory of $D^b_h(D_X)$ consisting of objects whose cohomology sheaves are regular holonomic D_X-modules.

We now state a theorem about regular holonomic D-modules which will play a crucial role in the Riemann-Hilbert correspondence.

Theorem 7.13. *Let X be a smooth variety.*

1. *$\mathbb{D}_X$ preserves $D^b_{rh}(D_X)$.*
2. *Let $f : X \to Y$ be a morphism of smooth varieties. Then $\int_f, \int_{f!}$ restrict to functors $D^b_{rh}(D_X) \to D^b_{rh}(D_Y)$ and $f^\dagger, f^\star$ restrict to functors $D^b_{rh}(D_Y) \to D^b_{rh}(D_X)$.*

8 Riemann-Hilbert correspondence

Before we finally state the Riemann-Hilbert correspondence, we first prove a preliminary result about the interactions of the de Rham functor with the other functors we have so far.

Theorem 8.1. *Let $f : X \to Y$ be a morphism of smooth varieties. Then we have the following isomorphisms of functors:*

$$\mathbf{D}_X DR_X \simeq DR_X \mathbb{D}_X : D^b_h(D_X) \to D^b_h(X)$$

$$DR_Y \circ \int_f \simeq Rf^{an}_* \circ DR_X : D^b_{rh}(D_X) \to D^b_c(Y)$$

$$DR_Y \circ \int_{f!} \simeq Rf^{an}_! \circ DR_X : D^b_{rh}(D_X) \to D^b_c(Y)$$

$$DR_X \circ f^\dagger \simeq (f^{an})^! \circ DR_Y : D^b_{rh}(D_Y) \to D^b_c(X)$$

$$DR_X \circ f^\star \simeq (f^{an})^{-1} \circ DR_Y : D^b_{rh}(D_Y) \to D^b_c(X)$$

In the following sketch, we freely use 7.13, which is necessary even to define the functors above.

Sketch. The first isomorphism is 6.20, and we can immediately deduce the third and fifth isomorphisms from the first, second, and fourth using 6.13.

It remains to show the second and fourth isomorphisms. By 6.18, we have the desired morphism

$$DR_Y \circ \int_f \to Rf_* \circ DR_X$$

and we show that it is an isomorphism when restricted to $D^b_{rh}(D_X)$. First, we can factor f as $X \to \overline{X} \to Y$ where the first map is an open embedding such that $\overline{X} \setminus X$ is a normal crossings divisor on $\overline{X}$ (by a result of Hironaka) and the second map is projective. Thus we may assume that f is an open embedding as above or projective. If f is projective, then in particular it is proper, so we have our isomorphism by 6.18.

Now let f is an open embedding as above and $M \in \mathrm{Mod}_{rh}(D_X)$. We proceed by induction on the length of a composition series for M (such a composition series exists by 5.2). In this case, suffices to prove the statement for M simple, and we may reduce to the case $M = \int_i L$ where $i : Z \to X$ is an affine embedding of a smooth locally closed subvariety Z of X and L is a simple regular integrable connection on Z. The isomorphism for L follows from 7.9 and 7.10.

Thus we may compute:

$$DR_Y \int_f M = DR_Y \int_f \int_i L \simeq DR_Y \int_{f\circ i} \simeq R(f \circ i)_* DR_Z L$$

$$\simeq Rf_* Ri_* DR_Z L \simeq Rf_* DR_X \int_i L = Rf_* DR_X M$$

For the fourth isomorphism, we first construct the desired morphism as follows.

$$\begin{aligned}\mathrm{Hom}_{D^b_h(D_X)}(f^\dagger N_\bullet, f^\dagger N_\bullet) &\simeq \mathrm{Hom}_{D^b_H(D_Y)}(\int_{f!} f^\dagger N_\bullet, N_\bullet)\\ &\to \mathrm{Hom}_{D^b_c(Y)}(DR_Y(\int_{f!} f^\dagger N_\bullet), DR_Y N_\bullet)\\ &\simeq \mathrm{Hom}_{D^b_C(Y)}(Rf_! DR_X(f^\dagger N_\bullet), DR_Y N_\bullet)\\ &\simeq \mathrm{Hom}_{D^b_c(X)}(DR_X f^\dagger N_\bullet, f^! DR_Y N_\bullet)\end{aligned}$$

where the first line uses 5.9, the second line is application of DR_Y, the third line comes from the isomorphism proven above, and the final line is again adjunction.

Factor f into $X \to X \times Y \to Y$ to reduce to the cases of a closed embedding and a projection. The projection is in particular smooth, and smooth morphisms are non-characteristic for any coherent D_X-module, so the isomorphism is obtained by 6.7. In the case of a closed embedding $i : X \to Y$, let $j : Y \setminus X \to Y$ be the corresponding open embedding. Then for $N_\bullet \in D^b_{rh}(D_Y)$, we have the following morphism of distinguished triangles:

$$\begin{array}{ccccccc} DR_y \int_i i^\dagger N_\bullet & \longrightarrow & DR_Y N_\bullet & \longrightarrow & DR_Y \int_j j^\dagger N_\bullet & \xrightarrow{+1} & \\ \downarrow \psi & & \downarrow \mathrm{Id} & & \downarrow \varphi & & \\ Ri_* i^! DR_Y N_\bullet & \longrightarrow & DR_Y N_\bullet & \longrightarrow & Rj_* j^! DR_Y N_\bullet & \xrightarrow{+1} & \end{array}$$

Since j is smooth, we have that $DR_Y \int_j j^\dagger N_\bullet \simeq Rj_* j^! DR_Y N_\bullet$, so that φ is an isomorphism. Thus ψ is also an isomorphism. Again by the first isomorphism, we have

$$DR_Y \int_i i^\dagger N_\bullet \simeq Ri_* DR_X i^\dagger N_\bullet$$

Combining this with ψ gives the desired isomorphism (after precomosing with i^{-1}, nothing that $i^{-1}Ri_* = \mathrm{Id}$ since i is a closed embedding). □

Theorem 8.2. *For a smooth variety X, the de Rham functor*

$$DR_X : D^b_{rh}(D_X) \to D^b_c(X)$$

gives an equivalence of categories.

Sketch. First, we show that for $M_\bullet, N_\bullet \in D^b_{rh}(D_X)$,

$$R\,\mathrm{Hom}_{D_X}(M_\bullet, N_\bullet) \simeq R\,\mathrm{Hom}_{\mathbb{C}_{X^{an}}}(DR_X M_\bullet, DR_X N_\bullet)$$

Let $\Delta : X \hookrightarrow X \times X$ be the diagonal embedding, and $p : X \to \{pt\}$ the projection to a point. Then by 4.19, we have the equivalence

$$R\,\mathrm{Hom}_{D_X}(M_\bullet, N_\bullet) \simeq \int_p \Delta^\dagger(\mathbb{D}_X M_\bullet \boxtimes N_\bullet)$$

Next, we have the equivalences for $F_\bullet, G_\bullet \in D^b_c(X)$:

$$\begin{aligned}
\Delta^!(\mathbf{D}_X F_\bullet \boxtimes B_\bullet) &\simeq \Delta^! \mathbf{D}_{X\times X}(F_\bullet \boxtimes \mathbf{D}_X G_\bullet)\\
&\simeq \mathbf{D}_X \Delta^{-1}(F \boxtimes \mathbf{D}_X G_\bullet)\\
&\simeq \mathbf{D}_X(F_\bullet \otimes_{\mathbb{C}} \mathbf{D}_X G_\bullet)\\
&\simeq R\,\mathscr{H}\mathrm{om}_{\mathbb{C}}(F_\bullet \otimes_{\mathbb{C}} \mathbf{D}_X G_\bullet, \omega_{X,\bullet})\\
&\simeq R\,\mathscr{H}\mathrm{om}_{\mathbb{C}}(F_\bullet, R\,\mathscr{H}\mathrm{om}_{\mathbb{C}}(\mathbf{D}_X G_\bullet, \omega_{X,\bullet}))\\
&\simeq R\,\mathscr{H}\mathrm{om}_{\mathbb{C}}(F_\bullet, \mathbf{D}_X^2 G_\bullet)\\
&\simeq R\,\mathscr{H}\mathrm{om}_{\mathbb{C}}(F_\bullet, G_\bullet)
\end{aligned}$$

and applying $Rp_* = R\Gamma(X, -)$ to the first and last terms above gives:

$$R\,\mathrm{Hom}_{\mathbb{C}_{X^{an}}}(F_\bullet, G_\bullet) \simeq Rp_* \Delta^!(\mathbf{D}_X F_\bullet \boxtimes G_\bullet)$$

Thus we obtain:

$$\begin{aligned}
R\,\mathrm{Hom}_{\mathbb{C}_{X^{an}}}(DR_X M_\bullet, DR_X N_\bullet) &\simeq Rp_* \Delta^!((\mathbf{D}_X DR_X M_\bullet) \boxtimes DR_X N_\bullet)\\
6.20 &\simeq Rp_* \Delta^!(DR_X(\mathbb{D}_X M_\bullet) \boxtimes DR_X N_\bullet)\\
6.19 &\simeq Rp_* \Delta^!(DR_{X\times X}((\mathbb{D}_X M_\bullet) \boxtimes N_\bullet))\\
8.1 &\simeq Rp_* DR_X(\Delta^\dagger(\mathbb{D}_X M_\bullet \boxtimes N_\bullet))\\
8.1 &\simeq DR_{pt} \int_p \Delta^\dagger(\mathbb{D}_X M_\bullet \boxtimes N_\bullet)\\
&\simeq \int_p \Delta^\dagger(\mathbb{D}_X M_\bullet \boxtimes N_\bullet)\\
&\simeq R\,\mathrm{Hom}_{D_X}(M_\bullet, N_\bullet)
\end{aligned}$$

thus establishing that DR_X is fully faithful. For essential surjectivity, it suffices to check on generators of $D^b_c(X)$, so we may take $F_\bullet = Ri_*L \in D^b_c(\mathbb{C}_X)$ for an affine embedding $i : Z \to X$ of a locally closed smooth subvariety Z of X and a local system L on Z^{an}. By 7.10, there is a regular integrable connection N on Z such that $DR_Z N \simeq L[\dim Z]$. Set $M_\bullet = \int_i N[-\dim Z] \in D^b_{rh}(D_X)$. Then

$$DR_X(M_\bullet) = DR_X \int_i N[-\dim Z] \simeq Ri_* DR_Z N[-\dim Z] \simeq Ri_* L = F_\bullet$$

□

By 6.5, we obtain the following corollary.

Corollary 8.3. *The solution functor*

$$\mathrm{Sol}_X : D^b_{rh}(D_X) \to D^b_c(X)^{op}$$

gives an equivalence of categories.

Although we will not go into detail here, we can obtain further information from the above correspondence by further investigating the category of perverse sheaves.

Theorem 8.4. *The de Rham functor induces an equivalence*

$$DR_X : \mathrm{Mod}_{rh}(D_X) \to \mathrm{Perv}(\mathbb{C}_X)$$

References

[HTT08] Ryoshi Hotta, Kiyoshi Takeuchi, and Toshiyuki Tanisaki, *D-modules, perverse sheaves, and representation theory*, Progress in Mathematics, vol. 236, Birkhäuser Boston, Inc., Boston, MA, 2008, Translated from the 1995 Japanese edition by Takeuchi. MR 2357361