1. Let $X$ be a projective variety over $k = \overline{k}$. Let $\mathcal{E}$ be a locally free sheaf of rank $r$ on $X$.
   a. Show that if $\mathcal{E}$ is globally generated then $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is globally generated.
   b. Let $\mathcal{L}_1, \cdots, \mathcal{L}_\ell$ be very ample invertible sheaves on $X$. If there is a surjection:
      $$\mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_\ell \to \mathcal{E}$$
   show that $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is very ample on $\mathbb{P}(\mathcal{E})$.

2. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be very ample invertible sheaves on $X$ a projective variety. Let $\mathcal{E} = \mathcal{L}_1 \oplus \mathcal{L}_2$. According to the previous problem, $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is very ample on $\mathbb{P}(\mathcal{E})$ so the complete linear system gives a closed immersion:
   $$\psi: \mathbb{P}(\mathcal{E}) \to \mathbb{P}^N.$$ 
   a. There are two natural sections $s_i$ of the map $\pi: \mathbb{P}(\mathcal{E}) \to X$ corresponding to the quotients $\mathcal{E} \to \mathcal{L}_i$. Describe the linear systems on $X$ corresponding to the compositions: $\psi \circ s_i: X \to \mathbb{P}^N$.
   b. Show that the image of $\mathbb{P}(\mathcal{E})$ can be set theoretically described as follows:
      $$\psi(\mathbb{P}(\mathcal{E})) = \left\{ p \in \mathbb{P}^N \mid \begin{array}{l} p \text{ is in the linear span} \\
      \text{of } \psi \circ s_1(x) \text{ and } \psi \circ s_2(x) \\
      \text{for some } x \in X \end{array} \right\}.$$ 
      (Sometimes $\psi(\mathbb{P}(\mathcal{E}))$ is called the join of $s_1(\psi(X))$ and $s_2(\psi(X))$.)
   c. Use the above description to show that $\mathbb{P}^1 \times \mathbb{P}^1 \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ is isomorphic to a quadric in $\mathbb{P}^3$.

3. Hartshorne II.5.16.

4. Let $\mathbb{P}^2_k$ be the projective plane and let $T_{\mathbb{P}^2}$ be its tangent bundle.
   a. Show that $T_{\mathbb{P}^2}(-1)$ is globally generated.
   b. Use part (b) of the previous exercise to prove that $T_{\mathbb{P}^2}(-2)$ has no global sections.
   c. Use the previous two parts to prove $T_{\mathbb{P}^2}$ is not isomorphic to $\mathcal{O}_{\mathbb{P}^2}(a) \oplus \mathcal{O}_{\mathbb{P}^2}(b)$ for any $a, b \in \mathbb{Z}$.
      (Hint: use the Euler sequence extensively – Hartshorne 8.13.)

5. Let $k = \overline{k}$ be a field. For $n \geq 1$ define
   $$C_n := (y^2 - x^n) \subset \mathbb{A}^2_k.$$ 
   a. Show that $C_n$ is smooth $\iff$ $n = 1$.
   b. Let $\mu: B \to \mathbb{A}^2_k$ be the blow-up of $(0, 0) \in \mathbb{A}^2_k$. Show that the strict transform of $C_n$ in $B$ is isomorphic to $C_{n-2}$. Show that the strict transform of $C_2$ is smooth.
      (So, we can resolve the singularities of $C_n$ by a sequence of $[n/2]$ blow-ups.)

6. Hartshorne II.8.2.

7. Hartshorne II.8.3.

8. Hartshorne II.8.4.

9. Hartshorne II.8.5.