HOMEWORK 2 – MATH 632.

1. Let X be a projective variety over $k = \overline{k}$. Let $\mathcal{E}$ be a locally free sheaf of rank r on X.

a. Show that if $\mathcal{E}$ is globally generated then $\mathcal{O}_{\mathbb{P}(\mathcal{E}}(1)$ is globally generated.
b. Let $\mathcal{L}_1, \cdots, \mathcal{L}_\ell$ be very ample invertible sheaves on X. If there is a surjection:

$$\mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_\ell \to \mathcal{E}$$

show that $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is very ample on $\mathbb{P}(\mathcal{E})$.

2. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be very ample invertible sheaves on X a projective variety. Let $\mathcal{E} = \mathcal{L}_1 \oplus \mathcal{L}_2$. According to the previous problem, $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is very ample on $\mathbb{P}(\mathcal{E})$ so the complete linear system gives a closed immersion:

$$\psi \colon \mathbb{P}(\mathcal{E}) \to \mathbb{P}^N.$$

a. There are two natural sections s_i of the map $\pi \colon \mathbb{P}(\mathcal{E}) \to X$ corresponding to the quotients $\mathcal{E} \to \mathcal{L}_i$. Describe the linear systems on X corresponding to the compositions: $\psi \circ s_i \colon X \to \mathbb{P}^N$.
b. Show that the image of $\mathbb{P}(\mathcal{E})$ can be set theoretically described as follows:

$$\psi(\mathbb{P}(\mathcal{E})) = \left\{ p \in \mathbb{P}^N \;\middle|\; \begin{array}{l} p \text{ is in the linear span} \\ \text{of } \psi \circ s_1(x) \text{ and } \psi \circ s_2(x) \\ \text{for some } x \in X \end{array} \right\}.$$

(Sometimes $\psi(\mathbb{P}(\mathcal{E}))$ is called the **join** of $s_1(\psi(X))$ and $s_2(\psi(X))$.)
c. Use the above description to show that $\mathbb{P}^1 \times \mathbb{P}^1 \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}) \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ is isomorphic to a quadric in $\mathbb{P}^3$.

3. Hartshorne II.5.16.

4. Let $\mathbb{P}^2_k$ be the projective plane and let $T_{\mathbb{P}^2}$ be its tangent bundle.

a. Show that $T_{\mathbb{P}^2}(-1)$ is globally generated.
b. Use part (b) of the previous exercise to prove that $T_{\mathbb{P}^2}(-2)$ has no global sections.
c. Use the previous two parts to prove $T_{\mathbb{P}^2}$ is not isomorphic to $\mathcal{O}_{\mathbb{P}^2}(a) \oplus \mathcal{O}_{\mathbb{P}^2}(b)$ for any $a, b \in \mathbb{Z}$.

(Hint: use the Euler sequence extensively – Hartshorne 8.13.)

5. Let $k = \overline{k}$ be a field. For $n \geq 1$ define

$$C_n := (y^2 - x^n) \subset \mathbb{A}^2_k.$$

a. Show that C_n is smooth $\iff$ $n = 1$.
b. Let $\mu \colon B \to \mathbb{A}^2_k$ be the blow-up of $(0,0) \in \mathbb{A}^2_k$. Show that the strict transform of C_n in B is isomorphic to C_{n-2}. Show that the strict transform of C_2 is smooth.

(So, we can resolve the singularities of C_n by a sequence of $\lfloor n/2 \rfloor$ blow-ups.)

6. Hartshorne II.8.2.

7. Hartshorne II.8.3.

8. Hartshorne II.8.4.

9. Hartshorne II.8.5.