1. (Counterexample to Bertini over finite fields) Show that
\[
X = (xw^q - x^q w + yz^q - y^q z = 0) \subset \mathbb{P}^3_{\mathbb{F}_q}
\]
is a smooth surface. Further show that for any plane
\[
H = (ax + by + cz + dw = 0) \subset \mathbb{P}^3_{\mathbb{F}_q}
\]
with coefficients \(a, b, c, d \in \mathbb{F}_q\), the intersection \(H \cap X\) is singular.

2. (The projective dual curve) Let \(k = \mathbb{R}\) and let \(C \subset \mathbb{P}^2_k\) be a smooth curve. Let \(p \in C\) and let \(T_p(C) \in (\mathbb{P}^2)^\vee_k\) be the projective tangent space to \(C\) at \(p\) (thought of as a point in the dual projective space). Show that the map:
\[
T : C \to (\mathbb{P}^2)^\vee_k \text{ given by } p \mapsto T_pC
\]
is an algebraic map. As a challenge, compute the degree of the image curve.

3. (Adjunction for divisors) Let \(X\) be a smooth variety over an algebraically closed field.
   (A) If \(D \subset X\) is a smooth variety, prove that \(\omega_{D/k} \cong (\omega_{X/k} \otimes O_X(D))|_D\).
   (B) Show that if \(X = \mathbb{P}^1 \times \mathbb{P}^1\) then \(O_X(a, b)\) is very ample if and only if \(a, b > 0\).
   (C) If \(k = \mathbb{C}\), use adjunction to compute the genus of a smooth curve \(C \subset \mathbb{P}^1 \times \mathbb{P}^1\).

4. (The Atiyah flop) Consider the Segre embedding:
\[
\mathbb{P}^1 \times \mathbb{P}^1 \to Q = (xw - yz = 0) \subset \mathbb{C}\mathbb{P}^3
\]
given by \([a : b] \times [c : d] \mapsto [ac : ad : bc : bd]\). \(Q\) has two families of lines (the image of \([a : b] \times \mathbb{P}^1\) or \(\mathbb{P}^1 \times [c : d]\); e.g. \(L_1 = (z = w = 0)\) is the image of \([1 : 0] \times \mathbb{P}^1\) and \(L_2 = (x = z = 0)\) is the image of \(\mathbb{P}^1 \times [0 : 1]\)). Let \(X \subset \mathbb{C}\mathbb{P}^4\) be the cone over \(Q\). This is a singular threefold. Consider:
\[
\pi_i : X_i \to X
\]
the blow-up of \(X\) at the the planes \(P_i\) that are the cones over \(L_i\). Prove:
   (A) \(X_1\) and \(X_2\) are smooth.
   (B) The maps \(\pi_i\) are isomorphisms away from the cone point \(P = [0 : 0 : 0 : 1] \in X\).
   (C) \(\pi_i^{-1}(P) \cong \mathbb{P}^1\). (Conclude the planes \(P_i\) are not Cartier divisors in \(X\).)
   (D) Show that the varieties \(X_i\) are not isomorphic over \(X\).

5. Give an example (with proof!) of a smooth projective complex curve that is not isomorphic to a plane curve.

6. Let \(C\) be a smooth projective curve over \(k = \overline{k}\). Show that a line bundle \(\mathcal{L}\) on \(C\) is very ample \iff for every pair of (possibly nondistinct) point \(P, Q \in C\):
\[
\dim_k \Gamma(C, \mathcal{L}(-P - Q)) = \dim_k \Gamma(C, \mathcal{L}) - 2.
\]

7. A curve \(C\) is called hyperelliptic if there is a degree two map \(\pi : C \to \mathbb{P}^1_k\).
   (A) Show that there are hyperelliptic curves of any genus \(g \geq 0\).
   (B) Use the previous exercise to prove that if \(C\) is hyperelliptic then \(\Omega_C\) is not very ample.