S a graded ring.

\[ S = \bigoplus_{d \geq 0} S_d. \]

\( \text{Proj}(S) \) is a scheme with

- points = homogeneous prime ideals \( \mathfrak{p} \) s.t. \( \mathfrak{p} \not\in \bigoplus_{d \geq 0} S_d. \)

- closed sets = \( \text{V}(g) = \{ p \in \text{Proj}(S) \mid g \in p \} \)

If \( f \in S_d \) \((d \geq 0)\), define:

- \( U_f = \text{Proj}(S) \setminus \text{V}(f) \).

- \( \mathcal{O}_{\text{Proj}(S)}(U_f)_f := (S_f)_0 = \{ \frac{a_i}{f_j^2} \mid a_i, f_j \in S_d \} \) "degree-0 part."
Ex. $A[x_0, \ldots, x_n]$
$\text{deg} \ x_i = 1$

$\text{Proj } (A[x_0,\ldots,x_n]) = \mathbb{P}_A^n$.

Ex: If $S \rightarrow S'$ is a surjection of graded rings then $\text{Proj } (S') \subseteq \text{Proj } (S)$ is a closed embedding.

Ex: If $f, \ldots, f_k \in A[x_0, \ldots, x_n]$ are homogeneous polynomials then $\text{Proj } (A[x_0, \ldots, x_n] \langle f, \ldots, f_k \rangle) \subseteq \mathbb{P}_A^n$.

Remark: There is a map:
$\text{Proj } (S) \rightarrow \text{Spec } (S_0)$
Graded $S$-modules $M = \bigoplus_{d \geq 0} M_d$. Give rise to a coherent sheaf $\tilde{M}$ on $\text{Proj } S$.

(For $f \in S_d : d > 0$, $(M_f)_0 = \tilde{M}(U_f)$.)

**Extended Example**

$S = \mathbb{C}[x_0, x, y]$  \hspace{1cm} $\text{Proj } S = \mathbb{C}P^1$

$M = S_{(m)}$

$0 \cdots 0 \oplus C \oplus C[x, y] \oplus C[x^2, xy, y^2] \cdots$

\hspace{1cm} $\text{degree}$ \hspace{1cm} $-m$ \hspace{1cm} $-m+1$ \hspace{1cm} $-m+2$

$\cdots \oplus C[x^m, x^{m-1}, \cdots, y^n] \oplus \cdots$

$\tilde{M} = \mathcal{O}_{\mathbb{C}P^1}(m)$. 
Proj $S$ has an affine cover by:

$(\nu_x \cdot \text{chart}) S_x = \text{Spec } \mathcal{O}^{\mathbb{Z}_x}$, $y^l_0 = \mathcal{O}^{\mathbb{Z}_x}$.

$\text{Spec } \mathcal{O}^{\mathbb{Z}_x} = A'_c$

$(\nu_y \cdot \text{chart}) S_y = \text{Spec } \mathcal{O}^{\mathbb{Z}_y}$, $y^l_0 = \mathcal{O}^{\mathbb{Z}_y}$.

$\text{Spec } \mathcal{O}^{\mathbb{Z}_y} = A'_c$.

$S(m)_x = \mathcal{O}^{\mathbb{Z}_x}$.

$(S(m)_x)_0 = \bigoplus y^{d+2m}/x^d \quad d \geq m$ (generated by $x^m$: rank 1 tree $\mathcal{O}^{\mathbb{Z}_y}$ mod)

$S(m)_y = \mathcal{O}^{\mathbb{Z}_y}$.

$(S(m)_y)_0 = \bigoplus x^{d+2m}/y^d \quad d \geq m$ (generated by $y^m$)
GLOBAL SECTIONS.

$\Omega_{\mathbb{P}^1/m}(\mathbb{C}P^1)$

can be obtained by gluing sections on $A_{x,y}$ and $A_{y,x}$.

$\Omega_{\mathbb{C}P^1/m}(A_{x,y}) = C_{x^m \otimes C_{x^m y} \otimes \cdots \otimes C_{y^n} \otimes C_{y^n x} \otimes \cdots}$

$\Omega_{\mathbb{C}P^1/m}(A_{y,x}) = \cdots \otimes C_{x^m} \otimes C_{x^m y} \otimes C_{x^m y} \otimes \cdots \otimes C_{y^n}$

$\Omega_{\mathbb{C}P^1/m}(\mathbb{C}P^1) = C_{x^m \otimes C_{x^m y} \otimes \cdots \otimes C_{y^n}}$. 
$\text{Spec}(\mathbb{C}[x,y]) \subseteq \text{Proj}(\mathbb{C}[x,y])$

$A^2 \times \mathbb{P}^1 \cong \{ (a,b) \in A^2 \mid (a,b) \in \mathbb{P}^1 \}$

$\cong V(uy = vx)$

Note: $A^2 \times \mathbb{P}^1 \cong \text{Proj}(\mathbb{C}[x,y] \langle x,y \rangle)$

deg 0 \quad deg + 1$

$uy - vx \text{ is homogeneous}$

Universal line $:= \text{Proj}(\mathbb{C}[x,y] \langle x,y \rangle / uy - vx)$. 

homog.

prime ideal

homogen.

prime ideal.
Red points = "open Möbius strip"

"the blow-up of $A^2$ at a point."
Maps to Projective Space

\[ \mathbb{P}_A^n = \text{Proj}(A[x_0, \ldots, x_n]) \]

has the invertible sheaf

\[ \mathcal{O}(1) = \mathcal{O}(1) \big|_{\mathbb{P}_A^n} = \mathcal{O}(x_0, \ldots, x_n)(1) \]

and global sections:

\[ x_0, \ldots, x_n \in \mathcal{O}(1)(\mathbb{P}_A^n) \]

**Motto**: A map from a scheme \( X/A \) to \( \mathbb{P}_A^n \) is determined by

1. A line bundle (or invertible sheaf) \( \mathcal{L} \) on \( X \)

2. \((n+1)\) global sections

\[ s_0, \ldots, s_n \in \Gamma(X, \mathcal{O}(1)) \]

satisfying \((\ast)\) (global generation)

What is this?
Theorem (Hartshorne 7.1).

(a) If \( \varphi : X \rightarrow \mathbb{P}^n_A \) is an \( A \)-morphism,
then \( \varphi^*(\mathcal{O}(1)) \) is an invertible sheaf
globally generated by the sections:
\[
\varphi^* (\mathcal{O} (x_i)) \in \Gamma (X, \varphi^* \mathcal{O} (1)).
\]

(b) If \( X \) is an invertible sheaf on \( X \)
and \( s_0, \ldots, s_n \in \Gamma (X, \mathcal{O} (1)) \) are sections
that globally generate \( \mathcal{L} \), then \( \mathcal{L} \) is an \( A \)-morphism:

\[
\varphi : X \rightarrow \mathbb{P}^n_A
\]
such that \( \mathcal{L} = \varphi^* \mathcal{O} (1) \) be \( s_i = \varphi^* (x_i) \).

Proof: (a) \( s_i \) globally generates \( \mathcal{L} \)

\[
\begin{array}{c}
\varphi^* \mathcal{O}^\oplus_{n+1} \longrightarrow \mathcal{L} \text{ surjective}. \\
(s_0, \ldots, s_n)
\end{array}
\]

Know: \( \varphi^* \mathcal{O}^\oplus_{n+1} \) surjective
\[
\varphi^* \mathcal{O} (x_i) \text{ locally free}
\]
and pullback is right exact.
(b) **IDEA:** Define the map:

\[ X \rightarrow \mathbb{P}^n_A \]

in pieces. Let:

\[ X_i = X \setminus \{ s_i = 0 \} \]

\[ U_i = \mathbb{P}^n_A \setminus \{ x_i = 0 \} \]

**First:** \( U X_i = X \).

(Proof of the statement)

**Second:** Define:

\[ \varphi_i : X_i \rightarrow U_i = \text{Spec}(\mathbb{A}^n) \]

i.e.: \( \varphi_i : (x_i/y_i) \rightarrow x_i/y_i \).

**Note:**

\( \mathcal{O}_X \rightarrow X \)

restricts to \( \mathcal{O}_X(X_i) \rightarrow \mathcal{L}(X_i) \)

on \( \mathbb{P}^n \) on \( X_i \).

\[ V x_i = \text{m.e.e.} \] so: \( S_j / s_i \in \mathcal{A}_X(X_i) \)
Third: Need to check compatibility.
(exercise/office hours if time available)

If $\mathcal{S}_i \neq 0$ & $\mathcal{S}_e \neq 0$, then:

$$\mathcal{X}_i \rightarrow \mathcal{A}^n$$

$$\mathcal{U}_i \rightarrow \mathcal{A}^n \setminus \{x_i = 0\} = \text{Spec} \mathcal{A} \left[ \frac{x_i}{x_i}, \ldots, \left( \frac{x_e}{x_i} \right)^2, \ldots, \frac{x_e}{x_i} \right]$$

$$\mathcal{X}_e \rightarrow \mathcal{A}^n_2$$

$$\mathcal{Q}_i \left( \frac{x_i}{x_i} \right) = \frac{s_i}{s_i}$$

$$\mathcal{Q}_e \left( \frac{x_i}{x_i} \cdot \frac{x_e}{x_i} \right) = \frac{s_i}{s_e} \cdot \left( \frac{s_i}{s_e} \right)^{-1}$$
Q: Given a map:
\[ \varphi: X \to P_A^N \]

corresponding to
\[ (s_0, \ldots, s_N, \varphi) \]
as above. When is \( \varphi \)
a closed immersion?

**Proposition.** \( \varphi \) is a closed immersion iff

(A) each \( x_i \) is affine, and

(B) the maps:
\[ A[x_0, \ldots, x_{i-1}] \to P(x_i, 0, x_i) \]
above are surjective.

**Proof.** Almost tautological. \( \square \).
Theorem. Let $k = \mathbb{F}$. $X$ a projective $k$-scheme.

$\varphi : X \to \mathbb{P}^n_k$

corresponding to $S_0, \ldots, S_n, X$. Let

$\langle S_0, \ldots, S_n \rangle \in V \subset \mathbb{P}(\mathbb{F}[X_0, X_1])$

$\varphi$ is a closed immersion of $X$.

1. Elements of $V$ separate points

i.e. for any 2 closed pts $P, Q \in X$

$\exists S, T \in V$ s.t. $S + m_P T \neq 0$ but

$S + m_Q T = 0$.

2. Elements of $V$ separate tangent vectors.

At a closed point $P \in X$ the set

$\{ S \in V | S \in m_P T \}$ spans the $k$-vector space

$\mathbb{P}(\mathbb{F}/m_P \mathbb{F})^{\mathbb{P}}$. 


Proof: $(\Rightarrow)$: is less interesting.

$(\Leftarrow)$: Sections in $V$ are pulled back from $\Gamma(P^n, O(1))$. So

1st: $\varphi : X \to P^n$ is injective at the level of $k$-points.

Why?

2nd: Now: $\varphi$ is proper $\Rightarrow$ closed.

$\Rightarrow$ continuous closed injection $\Rightarrow \varphi$ a homeo.

$\Rightarrow \varphi(K) \subseteq P^n$ closed subscheme.

It suffices to show:

$\rho : X \to \varphi(K)$

satisfies $\varphi^{\ast}(\mathcal{O}_K) \cong \rho \ast \mathcal{O}_X$.
Suffice to show:

\[ \mathcal{O}_{\mathfrak{p}} \to \mathfrak{q}, \mathcal{O}_{\mathfrak{x}} \text{ is surjective.} \]

Can be checked locally at closed points.

\[ \mathcal{O}_{\mathfrak{p}_x, \mathfrak{p}} \to \mathcal{O}_{\mathfrak{x}, \mathfrak{p}} \]

1. **SAME RESIDUE FIELD.**

2. \[ \varphi^*: \mathfrak{m}^*_{\mathfrak{p}_x, \mathfrak{p}} \to \mathfrak{m}_{\mathfrak{x}, \mathfrak{p}}^2/\mathfrak{m}_{\mathfrak{x}, \mathfrak{p}} \]

   is surjective.

3. \[ \varphi^* \mathcal{O}_{\mathfrak{x}} \text{ is a coherent } \mathcal{O}_{\mathfrak{p}_x} \text{-module.} \]

---

**Lemma/Exercise/Hintshorne II.7.4/ Application of Nakagawa's Lemma**

f: A \to B local hom. A local Noetherian ring. w/:

- \[ A/\mathfrak{m} \to B/\mathfrak{m}_B \text{ a morph.} \]
- \[ \mathfrak{m}_A \to \mathfrak{m}_B/\mathfrak{m}_B^2 \text{ surj.} \]
- \[ B = f(S) \cdot A \text{-module.} \]

\[ = \text{D + surjective.} \]
Linear Systems

A map $X \to \mathbb{P}^n$ is determined by a line bundle $\mathcal{L}$ and sections $s_0, \ldots, s_n \in \Gamma(\mathcal{L})$. Sections of $\mathcal{L}$ are divisors on $X$.

Assume:

- $X$ a nonsingular projective variety over $k = \mathbb{C}$.
- $\mathcal{L}$ an invertible sheaf on $X$.
- $s_0 \in \Gamma(X, \mathcal{L})$.

There is a divisor (Weil) $D = \text{div}(s_0)$ defined by $s_0 = 0$.

$s : \mathcal{O}_X^* \to \mathcal{L}$ determines $s^* : \mathcal{L}^* \to \mathcal{O}_X^*$

1. $s^* : \mathcal{L}^* \to \mathcal{O}_X^*$ is injective, so defines an ideal sheaf.

2. As $\mathcal{L}^*$ is locally free of rank 1, the ideal is principal.
Propn: $X$ as above. $D; CX$ a divisor.

$X \cong X(D_0)$. 

(a) For $s \in \Gamma(X, X) \setminus 0$, the divisor 

$$D = (s = 0) C X$$

is effective and linearly equivalent to $D_0$.

(b) Every divisor $DCX$ s.t. $D \cong D_0$

is $(s = 0) C X$ for some $s \in \Gamma(X, X) \setminus 0$.

(c) 2 sections $s, s' \in \Gamma(X, X) \setminus 0$.

define the same divisor $c \equiv 0$ if $s = ls'$.

for some $l \in k^*$. 

Aside before proof of (b). 

Q1. How do we define $X(D_0)$? 

$k(X)$ = field of fractions at $X$; is a q-coh. sheaf on $X$. 

- For any open set $U \subset CX$ where $q : \text{Spec}(k(X)) \to X$. 

$$I_{an U} = (f_n) \in \mathcal{O}_{\mathcal{X}, (x)}.$$ 

is principal, consider: 

$$\mathcal{O}_{\mathcal{X}, (x)} \cdot \frac{1}{f_n} \subset k(X).$$

- These glue in $(U, f_n \cdot \frac{1}{f_n}) = \text{S glue to a global section that vanishes on } D_0$. 

The other global sections of \( \mathcal{L}(D_0) \) are the rational functions

\[ h \in k(C) \]

such that for a trivializing cover \( \{U_i\} \) of \( D_0 \),

\[ h \in \mathcal{O}_X(U_i) \cdot \frac{1}{f_{U_i}} \cdot k(X). \]

**Ex.** \( (x^2 = 0) \subset \mathbb{P}^2 = \text{Proj}(k[x,y,z]) \).

\( \frac{1}{x} \)-chart: \( \text{Spec}(k[\frac{y}{x}, \frac{z}{x}]) \)

- generator is \( 1 \).
- Look at \( \frac{1}{y/x}, (\frac{y}{x})^2, (\frac{z}{x})^3 \).

\( \frac{1}{y} \)-chart: \( \text{Spec}(k[\frac{x}{y}, \frac{z}{y}]) \)

- generator is \( (\frac{z}{y})^2 \).

\( \frac{y}{x} = (\frac{x}{y}).(\frac{x}{y})^{-2} \in k[\frac{z}{y}, \frac{z}{y}].(\frac{z}{y})^2 \)

\( \frac{z}{x}^2 = (\frac{x}{y})^2 \cdot (\frac{z}{y})^2 \)

CAN CHECK: GLOBAL SECTIONS OF \( \mathcal{O}(x^2 = 0) \)

restricted to \( \frac{1}{x} \)-chart are

\[ 1, \frac{y}{x}, (\frac{x}{z})^2, (\frac{z}{x})^2, \frac{2y}{x}, \frac{z}{x^2} \]
Lemma/Exercise (Done already?)

Let $L = L(D_0)$. Global sections of $L$ correspond to $k(X)$ such that $(h) + D_0$ is effective.

Proof of (b)

Want to show, if $D$ effective & $D = \text{lm} D_0$

$\exists \phi \in \mathcal{P}(k, L(D_0))$ such that

$(\phi \circ \Delta = 0) = 0 \in \mathcal{O}_X$.

$D = \text{lm} D_0 \implies D - D_0 = (h)$

for some $h \in k(X)$.

But then $(h) + D_0$ is effective

$\implies h$ given a global section of $L(D_0)$

& locally $h = \frac{f_0}{f_0}$, i.e., $f_0$ dividing the

ideal at $D, D_0$ resp.

$\implies$ locally the section $h$ is

$\frac{f_0}{f_0}$ times the generator of $L(D_0)$. 

$\square$
Self divisors \( \mathcal{D} = \mathbb{P} \left( \mathbb{P}^1 \right) \) \( \sim \) a projective space!

**Definition:** A complete linear system on a nonsingular projective variety \( X \) is the projective space of effective divisors linearly equiv. to some \( D_0 \).

**Example:** \( \mathbb{P}^2 = \mathbb{P}^2_k \); \( D_0 = (x_0 = 0) \).

\( \mathcal{O}(D_0) = \mathcal{O}_{\mathbb{P}^2}(1) \).

\( \Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) \cong \mathbb{C} x_0 \otimes \mathbb{C} x_1 \otimes \mathbb{C} x_2 \).

\( |D_0| \cong \mathbb{P}^2 \)
Data: A linear system $\Delta$ on $X$ is a subset of a complete linear system $\mathcal{D}$ on $X$ corresponding to a vector subspace $V \subset \mathcal{P}(X, x)$.

A base point $P$ of $\Delta$ is a point $P \in X$ s.t. $P \not\in D$ for all $D \in \Delta$.

The dimension of the linear system is $\dim_v V - 1$.

**MORAL**

\[
\begin{array}{c}
\text{Non-degenerate maps} \\
X \to P^m
\end{array}
\] \[
\begin{array}{c}
\text{Base point free linear systems on } X \\
\text{of dimension } n
\end{array}
\]

**Ex.** \[V = \langle x^2, y^2, z^2 \rangle \subset \mathcal{P}(\mathbb{P}^2, 0(2))\]

is a base-point free, non-degenerate 2D linear system.

but not complete.
Ex. \( \langle xy, yz, xz \rangle \subset \mathbb{P}(P^2, \mathcal{O}(2)) \)

is 2-dimensional but not base-point-free and not complete.

It defines a map on the complement of \([1:0:0], [0:1:0], [0:0:1]\).

This gives an example of a birational automorphism that is not an automorphism.

**Relative Proj**

\[
\text{Proj}: \text{Input: graded algebra } S = \bigoplus_{d \geq 0} S_d. \\
\text{Output: Proj } S \text{ scheme with a map to } \text{Spec}(S_0).
\]
Proj: Input: sheaf of graded $\mathcal{O}_X$-algebras, $\mathcal{O}_X - S = \oplus_{d \geq 0} S_d$

Output: $\text{Proj}(S)$ a scheme with a map to $X$.

Assumptions:

- $\mathcal{O}_X - S = \oplus_{d \geq 0} S_d$ is a q-coherent $\mathcal{O}_X$-algebra.
- $S_0 = \mathcal{O}_X$, $S_1$ is coherent,
- $S_1$ is generated as an algebra by $S_1$.

Construction:

For each affine open set $U \subseteq X$

let:

- $\text{Proj}(S)_U = \text{Proj}(\mathcal{O}_X(U) \otimes S_0, \ldots)$
  \[ \downarrow \pi_U \]
  \[ U = \text{Spec}(\mathcal{O}_X(U)) \]
0. For any affine open sets $V, U$ with $V \subseteq U$.

\[ \text{Proj}(S)_V \rightarrow \text{Proj}(S)_U \]
\[ \pi_V \downarrow \quad , \quad \downarrow \pi_U \]
\[ V \rightarrow U \]

These glue to a scheme called

\[ \text{Proj}(S)_1 \rightarrow X. \]

0. The invertible sheaf $\mathcal{O}(1)$ defined on each

\[ \text{Proj}(S)_U \text{ affine} \]

extends to an invertible sheaf on all $\text{Proj}(S)_U$.

\[ \text{Remark: Under very mild assumptions:} \]
\[ \pi: \text{Proj}(S) \rightarrow X \text{ is projective,} \]
\[ \text{it is always proper.} \]
Example 1. Let $X$ be Noetherian and let $Z \subset X$ be a closed subscheme with ideal sheaf $I_Z \subset \mathcal{O}_X$. Then

$$\text{Proj}(\bigoplus I_Z^k) \to X \quad (I_Z^0 := \mathcal{O}_X).$$

is the blow-up of $X$ at $Z$.

Example 2. Let $E$ be a locally free sheaf of rank $\geq 0$. Consider the sheaf

$$\text{Sym}^* E = \bigoplus_{k \geq 0} \text{Sym}^k E.$$  

1. $\text{Spec}(\text{Sym}^* E) := \text{Proj}(E^* \to X$ 

\text{the rank $r$ vector bundle associated to $E^*$}.

2. $\text{Proj}(\text{Sym}^* E) := \text{Proj}(E \to X$.

\text{the projective bundle associated to $E$.}
Proposition: Let $X, Z, P(E)$ be as above.

(a) if rank $E \geq 2$ then a canonical

$$Sym^2 E \cong \bigoplus_{i \in \mathbb{Z}} \pi_2(\mathbb{P}(E)),$$

so $\pi_2(\mathbb{P}(E)) = \begin{pmatrix} 0 & 1 \times 0 \\ 0 & 1 - 0 \end{pmatrix}$

(b) $\exists$ a natural surjective morphism

$$\pi_2(E) \to \mathbb{P}(E).$$

(6) Let $\gamma \to X$ be a morphism.

Then there is a natural bijection

$$\begin{cases}
    \{ \gamma \to X \} & \leftrightarrow \\
    \{ \text{line bundle $L$ over $X$} \} & \leftrightarrow \\
    \{ \text{on $Y$ + surjectives} \} & \leftrightarrow \\
    \{ \text{on $Y$ + surjectives} \} & \leftrightarrow \\
\end{cases}$$

Sketch of Proof:

Let $U \subseteq X$ be an affine open set
and $U = \text{Spec } A$ s.t. $E(U)$ is free of rank $r$ on $U$.

Know: $P(E)|_U = \text{Proj } (A \oplus E|_U \oplus Sym^2 E|_U \oplus \ldots) = \mathbb{P}^r_A.$

Hint.: $\text{II.5.13} \implies (E)(\mathcal{O}(E)) \cong Sym^2 E(U)$.
\[ \Rightarrow \] 

subject to 

\[ \text{Hom}(\text{Sym}^2(\xi(U)), \mathcal{O}_U(0)) \]

\[ \text{Hom}(\pi^* \text{Sym}^2(\xi(U)), \mathcal{O}_U(2)). \]

\[ \Rightarrow \exists \text{ a map } \pi^* \xi \to \mathcal{O}_U(3). \]

Locally: over affine open, \( U \subset X \)
we have \( \pi^* \xi \to \mathcal{O}_U(3) \) surjective.

Lastly:

For any map: \( Y \to \text{Pic}(X) \) consider

\[ \varphi^* \xi \to \varphi^*/(\pi^* \xi) \to \mathcal{O}_U(1). \]

Likewise: Given a surjection:

\[ \pi: \varphi^* \xi \to Y. \]

take affine opens \( U \subset X \) that trivialize \( \xi \).

There is a unique map:

\[ \varphi^!(U) \to \mathcal{O}_U = \mathcal{O}(\xi U). \]

These maps glue.
Examples of Projective Bundles

Let \( P' = P_n \).

Let \( E_{ab} = O(a) \otimes O(b) \) for \( a, b \in \mathbb{Z} \).

Q. When is \( P(E_{ab}) \cong P(E_{cd}) \)?

Easier question:

When is \( P(E_{ab}) \cong P(E_{cd}) \) as schemes, \( P' \)?

(\text{HW: Hartshorne II.7.9}).

\[ \Rightarrow P(E_{ab}) \cong P(E_{cd}) \iff \exists x \in P_n(P') \text{ s.t. } E_{ab} \Theta x \]

\[ (\text{Pic } P' = \{0 \times 17 \}) \]

\[ \iff |a - b| = |c - d|. \]

(Aside: If \( k = \mathbb{C} \), the complex

manifolds \( P(E_{ab}) \), \( P(E_{cd}) \) are diffeomorphic

\[ \iff a + b \equiv c + d \pmod{2}. \]
Example \( E = 0 \oplus O(1). \)

\[
\begin{array}{c}
\downarrow \pi \\
\downarrow \\
\text{What are the global sections of } O_{\mathbb{P}^2}(1) \text{?} \\
\end{array}
\]

Have: \( E \cong \mathcal{O}_{\mathbb{P}^2}(1). \)

\[
\Rightarrow \Gamma(\mathbb{P}^2, O(1)) \cong \Gamma(K, \mathcal{O}_{\mathbb{P}^2}(1)) \\
\cong \Gamma(K, E)
\]

Know: \( E \) globally generated.

If \( P' \) has coord. \( x_0 y \):

\[
\begin{array}{c}
\xrightarrow{\text{strict}} \mathcal{O} \\
\end{array}
\]

is surjective.

\[
\Rightarrow \pi^*(\mathcal{O}^{\oplus 3}) \rightarrow E \text{ surj.}
\]

\[
\begin{array}{c}
\xrightarrow{\text{surjective}} \mathcal{O}^{\oplus 3} \\
\xrightarrow{\text{surjective}} E \\
\end{array}
\]

Determine a map: \( P^2 \) → \( P^2 \). \qed
Blowing Up.

\[ \mathbb{C} \times X, \quad \tilde{X} = \text{Proj} \left( \bigoplus_{i=0}^{\infty} \mathcal{O}_X^i \right) \]

\[ \downarrow \pi \]

\[ \downarrow X \]

Defn: let \( f : Y \to X \) be a map and let \( I \subseteq \mathcal{O}_Y \) be an ideal sheaf.

The 
\textit{inversely image ideal} \( f^* I \subseteq \mathcal{O}_X \)

is the image of \( f^* I \to f^* \mathcal{O}_X \cong \mathcal{O}_Y \).
Theorem \[ X \text{ a scheme,} \]

\[ I \text{ a coherent ideal sheaf.} \]

\[ \pi : \tilde{X} \to X \text{ the blow-up.} \]

(a) \( f^{\sim} (I) \) is invertible on \( \tilde{X} \): \( f^{\sim} (I) \) defines the exceptional divisor.

(b) If \( \exists C \subset X \) corresponds to \( I \) then \( U = X \setminus \exists \subset \tilde{X} \) then

\[ \pi \mid_U : \pi^{\sim} (U) \to U. \]

(c) If \( f : Y \to X \) is a map of Noetherian schemes and \( f^{\sim} I : Y \text{-} f^{\sim} I_ Y \).

Let \( \tilde{Y} \) be the blow-up of \( Y \) at \( I_ Y \).

\exists! \text{ morphism:}

\[ \tilde{Y} \xrightarrow{\tilde{f}} \tilde{X} \]

\[ \pi_{\tilde{Y}} \xrightarrow{\downarrow \pi_{\tilde{X}}} \pi_{\tilde{X}}. \]

\[ Y \xrightarrow{f} X. \]

If \( f \) is a closed immersion, then so is \( \tilde{f} \). (In this case, we call \( \tilde{Y} \) the strict transform of \( Y \).)

If \( X \) is a variety, \( k \sqrt{I} \neq 0 \) then:

(d) \( \tilde{X} \) is also a variety.

(e) \( \pi \) is birational/proper, surjective.
Proof:

(a) Want to check \( f: I \) is morphic.

Locally have

\[ U = \text{Spec} A \subset X. \]

\[ I(U) = (f_1, \ldots, f_k) \subset A. \]

\[ \tilde{X}_U = \text{Proj}(A \oplus I^0 \oplus I^0 \oplus \cdots) \]

\[ \overline{U \cup S}. \]

\[ \text{Spec} (S_f)_0. \]

\[ A \to (S_f)_0. \]

\[ I \to (S_f)_0. \]

\[ f_i \to f_i. \]

\[ f_i \to f_i \cdot f_j \cdot f_k \cdot f_j \cdot f_k \cdot f_j. \]

\[ \vdots \]

\[ f_k \to f_k \cdot f_j \cdot f_k \cdot f_i. \]

Observe (as we said)

\[ f: I = \mathcal{O}\left(\frac{I}{f}\right). \]
(c) We'll check this locally for closed dimensions:

\[ \text{Spec} \mathfrak{p} = \mathfrak{p} \to \mathfrak{m} = \text{Spec} A. \]

\[ \uparrow \quad \uparrow \]

\[ \text{Proj}(R \oplus \mathfrak{m} \cdots) \to \text{Proj}(A \oplus \mathfrak{m} \cdots) = \tilde{X} \]

\[ \mathfrak{s}' \quad \mathfrak{s} \]

For \( \mathfrak{f} \in \mathfrak{I} \), we have:

\[ \mathfrak{s} \cdot A \oplus \mathfrak{m} \cdots \to \mathfrak{f} \oplus \mathfrak{m} \cdots = : \mathfrak{s}' \]

Given:

\[ (\mathfrak{s} \cdot)_{\mathfrak{f}} \to (\mathfrak{s}' \cdot)_{\mathfrak{f}} \]

And:

\[ (\mathfrak{s} \cdot)_{\mathfrak{f}} \to (\mathfrak{s}' \cdot)_{\mathfrak{f}} \to 0. \]

(d) \( \tilde{X} \) a variety \( \Rightarrow \tilde{X} \) a variety.

Check locally. \( U = \text{Spec} A.C \times \).

1. \( \mathfrak{s} = A \oplus \mathfrak{m} \oplus \mathfrak{m} \cdots \) is an integral domain.

2. \( \Rightarrow (\mathfrak{s} \cdot)_{\mathfrak{f}} \) is integral.

3. \( \Rightarrow (\mathfrak{s}' \cdot)_{\mathfrak{f}} \) is integral. \( \square \)
Example.

\[ I = (x^2 - y^3) \]
\[ X = A_k^2 \]

\[ \hat{X} = \text{Proj}(k[x,y] \oplus I \oplus I^2 \oplus \ldots) \]

**Note:** There is a quotient map:

\[ k[x,y][u,v] \rightarrow k[x,y] \oplus I \oplus I^2 \oplus \ldots \]

Degree 1:

\[ u \rightarrow x, \quad v \rightarrow y + z \]

\[ \Rightarrow \hat{X} \subseteq \text{Proj}(k[x,y][u,v]) = A^2 \times P^1. \]

Ideal is \( xv = yu \).
\[ K < k[x, y][u, v] \rightarrow k[x, y]/y^2 - k^2 \]
\[ k = (y^2 - x^3, xv - yu, yu - x^2 u, v^2 - xu^2). \]

1st chart
\[ k[x, u]/u^2 = x \]

2nd chart
\[ k[z, y]/1 - x z^2 = 1 - y z^3 \]

\[ y^2. \]