MATH 732: CUBIC HYPERSURFACES: EXERCISES

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1. Cohomology

Exercise 1. *Assume* $X \subseteq \mathbf{P}^{n+1}$ *is a smooth hypersurface of degree* $d > 1$ *and* $\mathbf{P}^{\ell} \subseteq X$ *is a linear subspace contained in* X*. Show that* $\ell \leq n/2$*.*

Exercise 2. *Conversely, prove that if* $\ell \leq n/2$ *then there exist smooth hypersurfaces of every degree that contain* $\mathbf{P}^{\ell}$*. For which* d *does every degree* d *hypersurface in* $\mathbf{P}^{n+1}$ *contain a* $\mathbf{P}^{\ell}$*?*

Exercise 3. *Assume that* $X \subseteq \mathbf{P}^{n+1}$ *is a smooth hypersurface of degree* $d > 1$*. Let* $h \in \mathrm{H}^2(X, \mathbf{Z})$ *represent the restriction of the hyperplane class. Prove that*

$$\mathrm{H}^{2k}(X, \mathbf{Z}) = \begin{cases} \mathbf{Z}h^k & 0 < 2k < n \\ \mathbf{Z}h^k/d & n < 2k < d. \end{cases}$$

2. Hodge Numbers

Exercise 4. *Prove that if* X *is a variety and*

$$0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{L} \to 0$$

is a short exact sequence of vector bundles (with $\mathcal{L}$ *a line bundle), then for any* $p > 0$

$$0 \to \wedge^p \mathcal{E}_1 \to \wedge^p \mathcal{E}_2 \to (\wedge^{p-1}\mathcal{E}_1) \otimes \mathcal{L} \to 0.$$

Exercise 5. *Assuming the cohomology of line bundles on* $\mathbf{P}$*, prove the Bott Vanishing theorem using the Euler sequence and its exterior powers.*

Exercise 6. *Compute the canonical bundle of* X*:* $\omega_X := \wedge^n \Omega_X$*.*

Exercise 7. *Show that the cotangent bundle* Ω_X *of a smooth hypersurface of degree* $d \geq 3$ *is stable.*

3. Universal Hypersurfaces

Exercise 8. *Filling in some things from class:*

(1) *Prove the map of sheaves*

$$\mathrm{H}^0(\mathbf{P}, \mathcal{O}(d)) \otimes_k \mathcal{O}_{\mathbf{P}} \to \mathcal{O}_{\mathbf{P}}(d-1)^{\oplus n+2} \quad (F \mapsto \oplus \partial_{x_i} F)$$

is surjective.

(2) *Prove that for any $p \in \mathbf{P}$, the generic hypersurface that is singular at p has an ordinary double point at p (i.e. the tangent cone at p is a non-singular quadratic form).*

Exercise 9. *As we mentioned in our example, the discriminant locus $D(2,n)$ of the universal quadratic form corresponds to those whose associated bilinear form is singular (i.e. has nullity ≥ 1).*

(1) *Reprove the theorem on the degree of the discriminant for these forms.*
(2) *Prove that the singular locus of $D(2,n)$ corresponds to the set of bilinear forms whose matrix has nullity ≥ 2.*
(3) *(Optional) Can you figure out what happens in characteristic 2?*

4. Monodromy and Lefschetz Pencils

Exercise 10. (1) *Show that a local system on $[0,1]$ is trivial.*
(2) *Show that any local system L on $B \times [0,1]$ is isomorphic to the inverse image: $p_1^{-1}(L|_{B\times 0})$.*
(3) *Given a local system L on B, conclude that for any 2 homotopic paths between $x, y \in B$:*

$$\gamma_1, \gamma_2 \colon [0,1] \to B$$

there is an induced ismorphism $L_x \simeq L_y$ which is independent of the choice of path.

Exercise 11. *In the case $n = 0$ and $d = 3$, prove that the monodromy group of the family $\mathcal{X}_{U(3,0)} \to U(3,0) \subseteq \mathbf{P}^3$ is $\mathfrak{S}_3$. The discriminant locus $D(3,0) \subseteq \mathbf{P}^3$ is singular along a curve. What is this curve (and prove your answer)?*

5. Classical constructions

Exercise 12. *Let*

$$X = (x_0^3 + \cdots + x_{n+1}^3 = 0) \subseteq \mathbf{P}$$

be an even dimensional Fermat cubic hypersurface and let

$$\Lambda = (x_0 + x_1 = \cdots = x_n + x_{n+1} = 0) \subseteq \mathbf{P}.$$

Show that the corresponding quadric fibration is singular along the union of $n/2 + 1$ hyperplanes and the cubic hypersurface:

$$X \cap (x_0 - x_1 = \cdots x_n - x_{n+1} = 0)$$

thought of as a subset of $\mathbf{P}^{n/2}$.

6. Chern class problems

Exercise 13. *Prove that if $\mathcal{E}$ is a rank 2 vector bundle on X with a section s that is transverse to the 0-section of $\mathcal{E}$. Let*

$$i \colon Y = (s = 0) \hookrightarrow X.$$

Assume $\mathcal{E}$ is an extension of line-bundles:

$$0 \to \mathcal{L}_1 \to \mathcal{E} \to \mathcal{L}_2 \to 0.$$

Define:

$$\xi_i = \mathrm{cl}(\mathcal{L}_i) \in A^2(X).$$

Assume (for convenience of proof) that the induced section of $\mathcal{L}_2$ is transverse to the 0-section. Prove that

$$[Y]_A = \xi_1 \cdot \xi_2 \in A^4(X).$$

(Similarly, one can prove a result for a filtered vector bundle of higher rank.) What does this say if the section is nowhere vanishing?

Exercise 14. *Given rank r vector bundle $\mathcal{E}$ on X, prove that there is a smooth projective variety $\mu \colon Y \to X$ such that*

(1) $\mu^* \mathcal{E}$ is a successive extension of line bundles, and
(2) $\mu^* \colon A^*(X) \to A^*(Y)$ is injective.

Exercise 15. *Let X be a variety and $\mathcal{E}$ a vector bundle of rank r. Use the splitting principle to compute the following Chern classes.*

(1) $c_\bullet(\mathcal{E}^\vee) = 1 - c_1(\mathcal{E}) + c_2(\mathcal{E}) - \cdots$,
(2) $c_\bullet(\mathrm{Sym}^2(\mathcal{E}))$,

(3) $c_\bullet(\mathcal{E}^{\otimes 2})$,
(4) $c_\bullet(\wedge^2 \mathcal{E})$,
(5) $c_\bullet(\wedge^r \mathcal{E})$,
(6) $c_\bullet(\mathcal{E} \otimes \mathcal{L})$ *for a line bundle* $\mathcal{L}$.
(7) Compute the total Chern classes of $T_{\mathbf{P}^n}$ *and* $T_{\mathbf{P}^1 \times \mathbf{P}^1}$.

7. Fano schemes

Exercise 16. *Prove that a subscheme* $T \subseteq \mathbf{G}$ *is contained in* $\mathrm{F}(X, r)$ *if and only if* $s_F|_T \equiv 0$. *In other words:*

$$\mathrm{F}(X, r) = (s_F = 0) \subseteq \mathbf{G}.$$

Moreover, use the fact that $\mathrm{Sym}^d(\mathcal{S}^\vee)$ *is globally generated to show that for general* F *the section* s_F *is transverse to the zero section (so it is smooth and its class computes the top Chern class of* $\mathrm{Sym}^d \mathcal{S}^\vee$*).*

8. 27 lines

Exercise 17. *Prove that the if* $\mathcal{E}$ *is a globally generated vector bundle on a projective variety* X *then a general section* $s \in \mathrm{H}^0(X, \mathcal{E})$ *meets the zero section transversely.*

Exercise 18. *Count the number of lines in a general quintic threefold* $X \subseteq \mathbf{P}^4$.

Exercise 19. *Count the number of lines in a general septic (degree 7) fourfold* $X \subseteq \mathbf{P}^5$.