1. Monodromy and Lefschetz pencils

These notes are based on [Voi03, Ch. 2&3] and [Huy23, §1.2]. See the disclaimer section.

Recall $U = U(d, n) = \mathbb{P}^{N(n,d)} \setminus D(d, n)$ is the set of smooth hypersurfaces. Today we want to study the topology of the family:

$$\pi_U : X_U \to U.$$

**Definition 1.1.** Let $\Lambda$ be an abelian group and let $X$ be a locally connected space. A *local system with stalk* $\Lambda$ is a sheaf $L$ which is locally isomorphic to the constant sheaf with stalk $\Lambda$.

**Example 1.2.** Here’s an example with $B = S^1$ and $\Lambda = \mathbb{Z}_3$.

We consider $\Lambda$ as having the discrete topology. On the left, the trivial local system $\mathbb{Z}_3$ can be considered to be locally constant sections of $S^1 \times \Lambda$. On the right, we quotient by the diagonal action of $\mu_2$ on $S^1 \times \mathbb{Z}_3$, and the sheaf on $S^1$ is *locally constant sections* of $(S^1 \times \mathbb{Z}_3)/\mu_2 \to S^1/\mu_2 \simeq S^1$.

**Lemma 1.3** (Ehresmann’s Lemma). Any smooth projective family of complex varieties

$$\pi : X \to B$$

is locally constant. In other words, for small enough open sets $p \in \Delta \subseteq B$ we have $X_\Delta \simeq X_p \times \Delta$.

**Corollary 1.4.** In the analytic topology, if $B$ is connected then $R^m \pi_* \mathbb{Z}_X$ is a local system on $B$ with stalk $H^m(X_b, \mathbb{Z}_X)$ (for any $b \in B$).
Remark 1.5. We can take inverse images of local systems. Moreover, the geometric local systems described in the Corollary respect the cup product.

Exercise 1. (1) Show that a local system on $[0,1]$ is trivial.
(2) Show that any local system $L$ on $B \times [0,1]$ is isomorphic to the inverse image: $p_1^{-1}(L|_{B\times 0})$.
(3) Given a local system $L$ on $B$, conclude that for any 2 homotopic paths between $x,y \in B$:

$$\gamma_1, \gamma_2 : [0,1] \to B$$

there is an induced isomorphism $L_x \simeq L_y$ which is independent of the choice of path.

Proposition 1.6. If $B$ is simply connected (and locally arcwise connected), then every local system $L$ (with stalk $\Lambda$) is trivial on $B$.

Proof. Fix a basepoint $x \in B$, let $y \in B$ be any other point and let

$$\gamma : [0,1] \to B$$

be a path from $x$ to $y$. By Exercise 1, $\gamma^{-1}L$ is trivial on $[0,1]$ and this gives an isomorphism:

$$L_x \simeq L_y$$

Also by the exercise, this isomorphism is independent of the path.

So for any two points $x,y \in B$ there is a natural isomorphism:

$$L_x \simeq L_y.$$  \hfill (*)

It makes sense then to ask: are these isomorphisms locally constant? (E.g. if the group $\Lambda$ is not discrete, as can happen, we might worry these isomorphisms vary continuously.)

We’ll be a little sketchy here. Let $P$ be the space of paths on $B$. There is a canonical map:

$$\Gamma : P \times [0,1] \to B$$

sending $\gamma \times t \mapsto \gamma(t)$. By the exercise (and some unwinding) we get an isomorphism of local systems:

$$\Gamma_0^{-1}L \simeq \Gamma_1^{-1}L$$

(where $\Gamma_t$ represents the composition $P \to P \times \{t\} \to P \times [0,1] \overset{\Gamma}{\to} B$).

Pointwise this isomorphism of local systems is given by the isomorphism:

$$L_{\gamma(0)} \simeq L_{\gamma(1)}$$
described previously. The fact that this is now an isomorphism of local systems, implies that the isomorphisms \((\ast)\) vary continuously. (The condition \textit{locally arcwise connected} implies that the maps \(\Gamma_t\) are open, which is useful in proving the sketchy part.)

\[\square\]

**Theorem 1.7.** Let \(B\) be a locally simply connected (and arcwise connected) space with basepoint \(x \in B\). Fix a group \(\Lambda\). There is a bijection:

\[
\begin{cases}
\text{local systems on } B \text{ with group } \\
\Lambda \text{ plus a choice of } \Lambda \simeq L_x
\end{cases}
\leftrightarrow
\begin{cases}
\text{representations } \\
\pi_1(B, x) \to \text{Aut}(\Lambda)
\end{cases}
\]

**Remark 1.8.** So, our short-term goal then will be to understand the representation

\[\pi_1(U, [X]) \to \text{Aut}(H^n(X, \mathbb{Z})).\]

**Proof.** Let \(L\) be a local system on \(B\) with stalk \(\Lambda\) and choose an isomorphism:

\[\alpha : L_x \simeq \Lambda.\]

Consider the universal cover

\[\mu : \tilde{B} \to B.\]

Then, by Proposition 1, \(\mu^{-1}L\) is locally constant. Moreover, for any chosen point \(x' \in \tilde{B}\) over \(x \in B\), there is a unique isomorphism \(\beta : \mu^{-1}L \simeq \Lambda\) so that the induced isomorphism:

\[(\mu^{-1}L)_{x'} \xrightarrow{\beta_{x'}} \Lambda\]

equals the isomorphism:

\[(\mu^{-1}L)_{x'} \simeq L_x \xrightarrow{\alpha_{x'}} \Lambda.\]

For any \(\gamma \in \pi_1(B, x), \gamma \cdot x' \in \tilde{B}\) also maps to \(x \in B\). The same isomorphism \(\beta\) gives an isomorphism:

\[\mu^{-1}L_y \xrightarrow{\beta_{y'x'}} \Lambda,\]

but we no longer necessarily have that:

\[\Lambda \xrightarrow{\beta^{-1}_{y'x'}} \mu^{-1}L_y \simeq L_x \xrightarrow{\alpha} \Lambda\]

is the identity. Let \(\rho(\gamma)\) denote this composition. Then:

\[\rho \pi_1(X, x) \to \text{Aut}(\Lambda)\]

is the associated group homomorphism (we omit the proof that the map respects composition). This shows that local systems give rise to \(\pi_1\)-representations.
In the reverse direction, we start with a representation
\[ \rho \circlearrowright \pi_1(B, x) \to \Lambda. \]
Note that \( \pi_1(B, x) \) acts freely on \( B' \) with quotient \( B \). The local system \( L_\rho \) on \( B \) assigns to each open set \( U \subseteq B \) the set of equivariant sections of \( \Lambda \) on \( \pi_1(B, x) \):
\[
L_\rho(U) = \left\{ s \in \Lambda_{B'}(\mu^{-1}U) \mid \rho(\gamma) \circ s = s \circ \gamma \quad \forall \gamma \in \pi_1(B, x) \right\}.
\]

\[ \square \]

**Remark 1.9.** The representation associated to a local system is called the *monodromy representation*. It is very reasonable to think of a local system as a sheaf that has parallel transport. Following a loop in the base, the parallel transport map induces the representation.

**Definition 1.10.** Recall, a *Lefschetz pencil* of degree \( d \) hypersurface in \( \mathbb{P}^{n+1} \) is a pencil \( C^2 \cong \lambda \subseteq H^0(\mathbb{P}, \mathcal{O}_\mathbb{P}(d)) \) such that
1. the base locus of \( \lambda \) has codimension 2 in \( \mathbb{P} \), and
2. any singular hypersurface \( \lambda \) has a single singular point which is an ordinary double point.

**Remark 1.11.** Here’s a cartoon of a Lefschetz pencil of quadrics.

The singular points form a finite subset of \( \mathbb{P}^1 \).
Previously we showed there are \((d - 1)^{n+1}(n + 2)\) singular points \(\Sigma \subseteq \mathbb{P}^1\). Computing the monodromy of a Lefschetz pencil means computing the monodromy action for the family:

\[
\mathcal{X}_{\mathbb{P}^1 \setminus \Sigma} \rightarrow (\mathbb{P}^1 \setminus \Sigma).
\]

As \(\pi_1(\mathbb{P}^1 \setminus \Sigma)\) is generated by the loops in the picture, it amounts to understanding how these loops act on cohomology.

**Definition 1.12.** A *Lefschetz degeneration* is a map

\[
f: Y \rightarrow \Delta \subseteq \mathbb{C}
\]

where \(Y\) is a smooth, \(n + 1\) dimensional (analytic) variety, \(f\) is a projective morphism, smooth away from \(0 \in \Delta\) such that the fiber \(Y_0\) has a single singularity which is an ordinary double point.

**Remark 1.13.** So it’s like a tiny neighborhood of a singular point in a Lefschetz pencil.

**Theorem 1.14** (Picard-Lefschetz formula). Let \(f: Y \rightarrow \Delta\) be a Lefschetz degeneration. Let \(T \in \text{Aut}(H^n(Y_1, \mathbb{Z}))\) be the image of a generator of \(\pi_1(\Delta^*, 1)\). There exists a class \(\delta \in H^n(Y_1, \mathbb{Z})\) (called a vanishing sphere) such that for every \(\alpha \in H^n(Y_1, \mathbb{Z})\),

\[
T = \alpha + \epsilon_n((\alpha, \delta))\delta.
\]

(Here \(\epsilon_n = -(-1)^{\frac{n(n-1)}{2}}\) and \(\langle -, - \rangle\) is the intersection product)

**Example 1.15.** Consider the elliptic curve:

\[
y^2 = (x^2 - t)(x - 1).
\]

(for \(t\) small). This has a double root when \(t = 0\), and we want to consider the monodromy around the loop \(t = \epsilon e^{i\theta}\).
In this case the green loop is the *vanishing sphere* $\delta$, because as $t = 0$, $\delta$ becomes homologous to 0. We see that the magenta loop maps to the green loop under the monodromy representation. Note, that Ehresmann’s lemma also gives rise to a diffeomorphism of the torus (that depends on some trivialization choices). The diffeomorphism here is called a Dehn twist.

**Remark 1.16.** The *vanishing sphere* in the Picard-Lefschetz formula is defined in several steps.

1. Analytic locally, the map $f$ looks like:
   \[ C^{n+1} \to C \quad (z_1, \ldots, z_n) \mapsto z_1^2 + \cdots + z_n^2. \]
   at the singular point in the fiber.

2. If $B \subseteq C^{n+1}$ is a ball of radius $r$, then for $t = se^{i\theta}$ small, the fiber $B_t$ contains the sphere
   \[ S^n = \{ (z_1, \ldots, z_n, z_{n+1}) \in B | z_i = \sqrt{s}e^{i\theta}x_i, x_i \in \mathbb{R}, \sum x_i^2 = 1. \}. \]
   Note that as $t \to 0$, this sphere shrinks to 0. The claim here is that the fiber $B_t$ deformation retracts onto the sphere $S^{n-1}$. (See the picture in the example above.)

3. For the Lefschetz degeneration: $f: Y \to \Delta$, the fundamental class of $S^{n-1}$ (choosing an orientation) generates the kernel of the composition:
   \[ H^n(Y, \mathbb{Z}) \cong H_n(Y_\epsilon, \mathbb{Z}) \to H_n(Y, \mathbb{Z}). \]
   The class $\delta$ is this generator in $H^n(Y_\epsilon, \mathbb{Z}) \cong H(Y_1, \mathbb{Z})$. 
**Definition 1.17.** For a smooth projective family \( \mathcal{X} \rightarrow B \) with marked fiber \( X \), the \( m \)th monodromy group is defined to be the image of the monodromy representation:

\[
\pi_1(B) \rightarrow \text{Aut}(H^m(X, \mathbb{Z})).
\]

When \( \mathcal{X}_{U(d,n)} \rightarrow U(d,n) \) is the universal family, we set

\[
\Gamma(d,n) = \text{Im}(\pi_1(U(d,n)) \rightarrow \text{Aut}(H^n(X, \mathbb{Z}))).
\]

**Theorem 1.18.** Restricting to the case of cubic hypersurfaces, the monodromy group \( \Gamma(3,n) \) of the universal smooth cubic is

\[
\Gamma(3,n) = \begin{cases} 
\tilde{O}^+(H^n(X, \mathbb{Z})) & \text{if } n \text{ is even} \\
\text{SpO}(H^n(X, \mathbb{Z}), q) & \text{if } n \text{ is odd}
\end{cases}
\]

**Remark 1.19.** I won’t define these groups precisely. Note there is a natural intersection bilinear form on \( H^n(X, \mathbb{Z}) \), which is preserved by the monodromy action. The bilinear form is symmetric when \( n \) is even and alternating when \( n \) is odd. This explains the \( O \) and the \( \text{Sp} \).

Moreover, in the case \( n \) is even, the hyperplane class \( h^{n/2} \) is a monodromy invariant of \( H^n(X, \mathbb{Z}) \). It follows that there is a representation:

\[
\pi_1(U(d,n)) \rightarrow \text{Aut}(H^n(X, \mathbb{Z})_{\text{prim}}),
\]

and \( \tilde{O}^+(H^n(X, \mathbb{Z})) \) is a finite index subgroup of \( O(H^n(X, \mathbb{Z})_{\text{prim}}) \). (In fact, \( H^n(X, \mathbb{Z}) \neq H^n(X, \mathbb{Z})_{\text{prim}} \oplus \mathbb{Z} h^{n/2} \) as lattices, and this accounts – to some extent – for why it is only a finite index subgroup.)

In the case \( n \) is odd, there is a \( \mathbb{Z}_2 \)-valued quadratic form \((\text{Kervaire invariant?})\) in the picture, and that is the reason for the \( O \).

**Big points in the Proof of Theorem.** We proceed in a few steps:

1. First show that for a Lefschetz pencil \( \mathbb{P}^1 \subseteq \mathbb{P}^{N(n,d)} \) with singularities \( \Sigma \subseteq \mathbb{P}^1 \), the mapping:

\[
\pi(\mathbb{P}^1 \setminus \Sigma) \rightarrow \pi_1(U(n,d)).
\]

So the monodromy group of \( U(n,d) \) is the same as the monodromy group of the Lefschetz pencil.

2. The punchline here is that (for hypersufaces) the primitive cohomology is generated by the vanishing spheres. In a sentence, this is an application of Morse Theory / the Lefschetz theorems.

3. Presumably, then some computation is necessary. I do not know the details of this computation. I assume it is proved that the simple loops from the Lefschetz pencil generate these groups directly (by explicitly describing these groups).
Theorem 1.20. The monodromy representation
\[ \Gamma(d, n) \to \text{Aut}(H^n(X, Q)_{\text{prim}}) \]
is irreducible.

Proof. Again we consider the case of a Lefschetz pencil. We need a couple of facts. First the pairing on \( H^n(X, Q)_{\text{prim}} \) is non-degenerate and second the vanishing spheres \( \delta_i \) generate the primitive cohomology.

Suppose that \( F \subseteq H^n(X, Q)_{\text{prim}} \) is a non-zero subrepresentation. Let \( \alpha \in F \) be any vector. Then for the loop \( \gamma_i \in \pi_1(\mathbb{P}^1 \setminus \Sigma) \) we have:
\[
\rho(\gamma_i)(\alpha) = \alpha \pm (\alpha, \delta_i)\delta_i.
\]
There exists some \( \delta_i \) such that \( (\alpha, \delta_i) \neq 0 \). So:
\[
\pm (\alpha, \delta_i)\delta_i = \alpha - \rho(\gamma_i)(\alpha) \in F \implies \delta_i \in F.
\]

Now we want to show that the monodromy action acts transitively on the vanishing spheres, at least up to sign. More globally, a vanishing sphere can be constructed as follows. Let \( 0 \in U(d, n) \) be a marked point in the space of smooth hypersurfaces.

1. Choose a point \( y \in D(d, n)^0 \) (the smooth locus of the discriminant divisor), and make a small normal disk \( \Delta_y \subseteq \mathbb{P}^{N(d, n)} \) to \( D(d, n) \) at \( y \). Choose a point \( y' \in \Delta_y^* \).
2. Choose a path \( \gamma \) from 0 to \( y' \).

Then we get a vanishing sphere by choosing a generator of the kernel of the composition:
\[
H^n(X_0, Z) \xrightarrow{\rho(\gamma)} H^n(X_{y'}, Z) \simeq H_n(X_{y'}, Z) \to H_n(X_{\Delta_y}, Z).
\]
We can call such a vanishing sphere \( \delta_{\gamma, y} \) (and let’s denote the composition \( \phi_{\gamma, y} \)). Note that all vanishing spheres arise this way.

First, different choices of paths (up to homotopy) differ by pre-composing with an element in \( \gamma' \in \pi_1(U(d, n)) \). The vanishing sphere obtained by this different vector is given by a generator of the kernel of the map \( \phi_{\gamma, y} \circ \rho(\gamma') \). Thus:
\[
\delta_{\gamma \circ \gamma', y} = \rho(\gamma'^{-1}) \circ \delta_{\gamma, y}.
\]
So we see that monodromy can be used to transport one vanishing sphere at \( y \) to another.
Finally, we must consider what happens when we choose a different point $z \in D(d,n)^0$ and ANY path $0 \to z'$. Now $D(d,n)^0$ is irreducible, so we choose a path $\gamma_{y \to z} \in D(d,n)^0$ from $y \to z$ and we may make a tubular neighborhood and use it to construct a path $\gamma : y' \to z'$.

The claim is that
\[
\ker(\phi_{\gamma,y}) = \ker(\phi_{\gamma'_{y' \to z'} \circ \gamma,z}).
\]
which shows $\delta_{\gamma,y} = \delta_{\gamma'_{y' \to z'} \circ \gamma,z}$.

\[\square\]

**Remark 1.21.** In the case of cubic surfaces, we have $H^2(X, \mathbb{C}) = H^{1,1}(X)$. It follows by the Hodge index theorem that the primitive cohomology is a negative definite lattice. As a consequence, there are only finitely many automorphisms of the lattice: $H^2(X, \mathbb{Z})_{\text{prim}}$ (choose any basis $\{\beta_i\}$, there are only finitely many elements $\alpha \in H^2(X, \mathbb{Z})_{\text{prim}}$ with
\[
|\langle \alpha, \alpha \rangle| < \max\{|\langle \beta_i, \beta_i \rangle|\}.
\]
This shows that $\Gamma(3,2)$ is finite, and in fact $\Gamma(3,2) = W(E_6)$!

**Exercise 2.** In the case $n = 0$ and $d = 3$, prove that the monodromy group of the family $\mathcal{X}_{U(3,0)} \to U(3,0) \subseteq \mathbb{P}^3$ is $\mathfrak{S}_3$. The discriminant locus $D(3,0) \subseteq \mathbb{P}^3$ is singular along a curve. What is this curve (and prove your answer)?

**References**