MATH 416, PROBLEM SET 4

Comments about homework.

- Solutions to homework should be written clearly, with justification, in complete sentences. Your solution should resemble something you'd write to teach another student in the class how to solve the problem.
- You are encouraged to work with other 416 students on the homework, but solutions must be written independently. Include a list of your collaborators at the top of your homework.
- You should submit your homework on Gradescope, indicating to Gradescope where the various pieces of your solutions are. The easiest (and recommended) way to do this is to start a new page for each problem.
- Attempting and struggling with problems is **critical** to learning mathematics. Do not search for published solutions to problems. I don't have to tell you that doing so constitutes academic dishonesty; it's also a terrible way to get better at math.

  If you get stuck, ask someone else for a hint. Better yet, go for a walk.

Problem 1. Let a and b be constants. Describe completely and concisely the execution of our polynomial-multiplication algorithm on the inputs

$$A(x) = B(x) = a + bx.$$

Problem 2. Let $\zeta = e^{i\cdot 2\pi/8} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$ and consider the sequence $a = (0, 1, 2, 3, 4, 3, 2, 1)$.

(a) Describe completely and concisely the execution of the Fast Fourier Transform that computes $\mathrm{FFT}(a, \zeta)$. [1] (In particular, you should describe the values of all local variables at all stages of the iteration.)

(b) Conclude by giving the unique polynomial of degree ≤ 7 that interpolates the eight points

$$(1,0), \quad (\tfrac{1}{\sqrt{2}} - i\tfrac{1}{\sqrt{2}}, 1), \quad (-i, 2), \quad (-\tfrac{1}{\sqrt{2}} - i\tfrac{1}{\sqrt{2}}, 3),$$
$$(-1,4), \quad (-\tfrac{1}{\sqrt{2}} + i\tfrac{1}{\sqrt{2}}, 3), \quad (i, 2), \quad (\tfrac{1}{\sqrt{2}} + i\tfrac{1}{\sqrt{2}}, 1).$$

Problem 3. For this problem, fix $n \in \mathbb{N}$ and $\zeta = e^{i\cdot 2\pi/n}$.

(a) Compute $1 + \zeta + \zeta^2 + \cdots + \zeta^{n-1}$.

(b) Compute $1 \cdot \zeta \cdot \zeta^2 \cdot \cdots \cdot \zeta^{n-1}$.

(*Hint:* Your answer might depend on whether n is even or odd.)

(c) Use Euler's formula to prove

$$\cos(x + y) = \mathrm{cos}x\mathrm{cos}y - \mathrm{sin}x\mathrm{sin}y.$$

[1] Recall that we defined the DFT in such a way that $\mathrm{DFT}(a) = \mathrm{FFT}(a, \zeta^{-1})$, so the algorithm should produce $n \cdot \mathrm{IFT}(0, 1, 2, 3, 4, 3, 2, 1)$.

(d) Use Euler's formula to prove that for all $x \in \mathbb{R}$ and all $n \in \mathbb{Z}$

$$(\cos(x) + i\sin(x))^n = \cos(nx) + i\sin(nx).$$

Problem 4. For two sequences $a_\bullet = (a_0, \ldots, a_{n-1})$ and $b_\bullet = (b_0, \ldots, b_{n-1})$, each of length n, we define their *convolution* to be the sequence whose l^{th} term is

$$(a_\bullet * b_\bullet)_l = \sum_{j+k=l} a_j b_k.$$

(a) Explain the connection with polynomials.
(b) Prove that $\mathrm{DFT}(a_\bullet * b_\bullet)$ (a sequence of length $2n-1$) is the componentwise product of $\mathrm{DFT}(a_\bullet, 0, \ldots, 0)$ and $\mathrm{DFT}(b_\bullet, 0, \ldots, 0)$.

Problem 5. Recall that we write DFT for the discrete Fourier transform and IFT for the inverse discrete Fourier transform.

(a) Compute $\mathrm{IFT}(1, 1, \ldots, 1)$.
(b) Compute $\mathrm{DFT}(1, \frac{1}{3}, \frac{1}{3^2}, \ldots, \frac{1}{3^{n-1}})$.
(c) Compute $\mathrm{DFT}\left(\binom{n-1}{0}, \binom{n-1}{1}, \ldots, \binom{n-1}{k}, \ldots, \binom{n-1}{n-1}\right)$.

Define two operations on sequences by

$$\begin{aligned}\mathrm{lshift}(z_0, \ldots, z_{n-1}) &:= (z_1, \ldots, z_{n-1}, z_0)\\ \mathrm{rshift}(z_0, \ldots, z_{n-1}) &:= (z_{n-1}, z_0, \ldots, z_{n-2}).\end{aligned}$$

(d) Find formulas for $\mathrm{DFT}(\mathrm{lshift}(z))$ and $\mathrm{DFT}(\mathrm{rshift}(z))$ in terms of $\mathrm{DFT}(z)$. Make a conclusion that would be intelligible to a linear algebra student.
(e) Use the previous part to find a general formula for the DFT of a cyclic permutation of a sequence in terms of its DFT.
(f) Use the previous parts to find a general formula for the IFT of a cyclic permutation of a sequence in terms of its IFT.

Problem 6. There are n kindergarteners seated in a circle who have each brought in rocks for show and tell. Different kids bring in different numbers of rocks, so let $z = (z_0, \ldots, z_{n-1})$ be the sequence of amounts of rocks brought by the n kids. To prevent jealousy, their teacher has them periodically redistribute the rocks as follows: at regular intervals, each kid hands $1/3$ of their rocks to the kid on their left and the remaining $2/3$ of them to the right. The circular symmetry of the problem suggests that the Discrete Fourier Transform might be a useful tool to analyze what happens to the rocks in the long term.

The redistribution procedure starting with the initial sequence z gives a list of new sequences $z, z^{(1)}, z^{(2)}, \ldots$ satisfying

$$z^{(t+1)} = \tfrac{1}{3}\,\mathrm{lshift}(z^{(t)}) + \tfrac{2}{3}\,\mathrm{rshift}(z^{(t)}).$$

Say $\mathrm{DFT}(z) = (c_0, c_1, \ldots, c_{n-1})$.

(a) Prove that $c_k^{(t)} = (\frac{1}{3}\zeta^k + \frac{2}{3}\zeta^{-k})^t c_k$, where $c_k^{(t)}$ is the k^{th} term of $\text{DFT}(z^{(t)})$.

(*Hint:* DFT is linear. (Why?))

(b) What happens to the rocks as $t \to \infty$?

(*Hint:* Consider separately the cases when n is even and n is odd.)

(c) What would change if for some fixed $p \in (0, 1)$ other than $1/3$, the kindergarteners passed their rocks in proportion p to the left and $1 - p$ to the right?