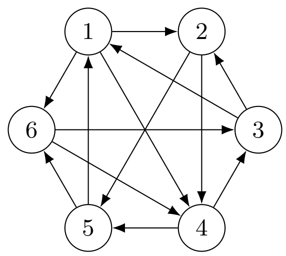
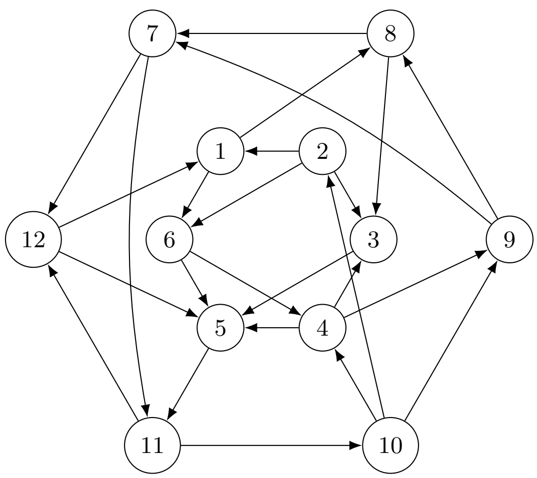
Math 416

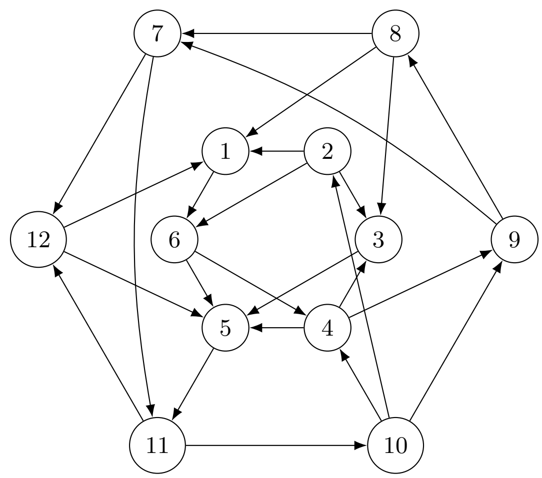
Worksheet 24. Hamiltonian cycles

Definition. A **Hamiltonian cycle** in a (di)graph is a (directed) cycle that visits each vertex exactly once. A natural algorithmic problem is to determine whether a given graph has a Hamiltonian cycle.

Problem 1. Look for a Hamiltonian cycle in the following graphs. Don't spend too much time on the second and third ones, but spend enough time to appreciate the difficulty of the problem. (Yes, the second and third graphs are different.)







Problem 2.

(a) If a digraph has a Hamiltonian cycle, what does that tell you about the metagraph?
(b) Which graphs does part (a) let you solve the Hamiltonian cycle problem for?
(c) Come up with the easiest possible **strongly connected graph** you can think of that has no Hamiltonian cycle.

The Hamiltonian cycle problem probably does not have an efficient solution, as we'll see. Near the end of WS23 we stated:

Theorem. SAT is NP-complete. In fact, 3-SAT is NP-complete.

First we use it to conclude that the Hamiltonian-cycles problem is NP-complete.

Theorem. The Hamiltonian-cycles problem is NP-complete.

(Recall that a problem X is NP-complete, if it is in NP— i.e. it has an efficient certifier — and any other problem in NP can be reduced in polynomial time to X.)

Problem 3.

(a) Explain what an efficient certifier for the Hamiltonian-cycles problem would look like.
(b) An efficient certifier typically gives a brute-force algorithm for a problem. What would that look like here? Can you estimate many steps it would take for the brute force approach to solve the Hamiltonian-cycles problem for the graphs above?

(c) Explain why you need to describe a way of converting 3-SAT problems into graphs so that a Hamiltonian cycle in the graph can be converted back into a solution to the 3-SAT problem.

Problem 4. (The Set-up Graph) Remember that we're given a 3-SAT problem with clauses $C_1, \ldots, C_k$ (using Boolean variables $x_1, \ldots, x_n$) and we're defining a graph. Fix a large number b to be specified later.

We construct n paths $P_1, \ldots, P_n$ where P_i has vertices $v_{i1}, v_{i2}, \ldots, v_{ib}$ There are edges $v_{ij} \to v_{i,j+1}$ and $v_{i,j+1} \to v_{ij}$. Thus P_i can be traversed 'left to right' from v_{i1} to v_{ib}, or 'right to left' from v_{ib} to v_{i1}.

The paths are connected as follows. For each $i = 1, 2, \ldots, n-1$, add edges $v_{i1} \to v_{i+1,1}$ and $v_{i1} \to v_{i+1,b}$. Also add edges $v_{ib} \to v_{i+1,1}$ and $v_{ib} \to v_{i+1,b}$.

Finally (for now), add extra vertices s and t and edges $s \to v_{11}$, $s \to v_{1b}$, and $v_{n1} \to t$, $v_{nb} \to t$. And add an edge from t to s.

(a) Draw a picture.
(b) How many vertices does this graph have?
(c) How many Hamiltonian cycles are there in the graph we've constructed? What do they correspond to?

Problem 5. Now that we have an approximation to our graph, we make it more complicated. We have not yet considered the clauses that actually appear in our 3-SAT instance. Suppose for a concrete example that one of our clauses is $C_1 = x_1 \vee \neg x_2 \vee x_3$.

(a) Translate the satisfiability of this clause into an assertion about a Hamiltonian cycle in our graph.
(b) Suppose that we add a vertex c_1 corresponding to clause C_1. Which edges should we add between c_1 and the paths P_1, P_2, P_3 to ensure that the Hamiltonian path is traversed in the direction dictated by C_1? (Hint: you can guarantee that a Hamiltonian cycle goes leftward on P_1 by adding a vertex c with the only edges coming out of c being $v_{11} \leftarrow c_1 \leftarrow v_{12}$.) Make sure you can account for all Hamiltonian cycles after this addition.
(c) Now do the general case, adding one vertex c_j for each clause C_j.
(d) How large does b need to be for this to work?

Problem 6. Finish the proof that 3-SAT $\leq_P$ Hamiltonian Cycle.

Problem 7. Give a polynomial time reduction from the directed Hamiltonian cycle problem to the undirected Hamiltonian cycle problem.