Math 416

Worksheet 7. Using generating functions to solve recurrence relations

Generating functions In order to prove the D&C Master Theorem, we need a general method to solve recurrence relations.

Definition. The **generating function** of a sequence $a_0, a_1, \ldots$ of complex numbers is the (formal) power series

$$A(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots$$
$$= \sum_{n \geq 0} a_n z^n$$

The Point: A recurrence relation for $(a_n)_{n\in\mathbb{N}}$ often translates into an algebraic or differential equation satisfied by its generating function $A(z)$.

Let's do an example. Consider a sequence $(a_n)_{n\in\mathbb{N}}$ satisfying the following recurrence.

$$a_0 = 0$$
$$a_n = 2a_{n-1} + 2^n$$

Problem 1.

(a) Turn the recurrence into a equation involving $A(z)$.
(*Hint:* Start with $A(z) = \sum_{n\geq 0} a_n z^n$; replace a_n by $2a_{n-1} + 2^n$; reindex, and simplify.)

(b) Solve your equation for $A(z)$! Your answer shouldn't have any $\sum$s in it yet. Don't forget the geometric series formula.

(c) Apropos of nothing, what is $\frac{d}{dz}\frac{1}{1-2z}$?

(d) Use the previous two parts to get an explicit formula for the coefficients of $A(z)$. Conclude by finding an explicit formula for a_n.

We'll mostly view generating functions purely algebraically, i.e., as formal objects, with no worry about convergence. For example, $(1-z)(1+z+z^2+\cdots)$ multiplied as usual gives 1, confirming the geometric series formula; more analysis is needed to determine whether that equation makes sense for any particular value of z.

Example. The function $f(z) = \sum_{n=0}^{\infty} n! z^n = 1 + z + 2z^2 + 6z^3 + 24z^4 + \cdots$ diverges for all $z \neq 0$. But we can still do algebra like

$$f(z)^2 = (1 + z + 2z + \cdots)^2 = 1 + 2z + 5z^2 + 16z^3 + \cdots$$

and make interesting conclusions.

Proof of the Master Theorem. Recall the Master Theorem:

Theorem (The Divide-and-Conquer Master Theorem)**.** Suppose that $a \geq 1$ and $b > 1$ are constants and that $f\colon \mathbb{R} \to \mathbb{R}$ is an increasing, nonnegative function. Suppose further that T is a function satisfying a recurrence

$$T(n) = aT(n/b) + f(n)$$

(where $T(n/b)$ can be taken to mean either $T(\lceil n/b \rceil)$ or $T(\lfloor n/b \rfloor)$).

(a) If $f(n)$ is $O(n^\gamma)$ for some $\gamma < \log_b a$, then $T(n)$ is $\Theta(n^{\log_b a})$.
(Recursion dominates)

(b) If $f(n)$ is $\Theta(n^{\log_b a})$, then $T(n)$ is $\Theta(n^{\log_b a} \log n)$.

(c) If $f(n)$ is $\Omega(n^\gamma)$ for some $\gamma > \log_b a$ AND there is a $c < 1$ for which $af(n/b) \leq cf(n)$ for all sufficiently large n, then $T(n)$ is $\Theta(f(n))$.
(Assembly dominates)

We start by making two simplifying assumptions.

- **Simplification #1:** We worry only about the case in which n is an exact power of b. (To deal with the general case, a lot of careful $\lfloor x \rfloor$ and $\lceil x \rceil$ accounting is necessary.)
- **Simplification #2:** We suppose that $f(n) = Cn^\gamma$ for some γ.
  (Later we will show how to eliminate this assumption.)

Suppose that T is a function satisfying a recurrence $T(n) = aT(n/b) + Cn^\gamma$. Start with a change of variables $S(k) = T(b^k)$.

Problem 2. Translate the recurrence into one satisfied by S:

$$S(k) = \boxed{}$$

Now consider the generating function $F(z) = \sum_{n \geq 0} S(k) z^k$.

Problem 3. Following your strategy from Problem 1, use the recurrence to produce an equation involving $F(z)$.

Problem 4. Show by solving your equation and combining fractions that $F(z)$ takes the form

$$F(z) = \frac{M_1 + M_2 z}{(1 - az)(1 - b^\gamma z)}$$

for some constants M_1 and M_2. (It is not important what these constants are.)

Now the rest of the proof divides neatly into two cases.

Problem 5. First suppose that $a \neq b^\gamma$.

(a) Use the geometric series formula to show that in this case $S(k) = M_3 a^k + M_4 b^{\gamma k}$ for some constants M_3 and M_4 (whose exact values are unimportant).
(b) Assuming $a > b^\gamma$, argue that $T(n)$ is $\Theta(n^{\log_b a})$.
(c) Assuming $a < b^\gamma$, argue that $T(n)$ is $\Theta(n^\gamma)$.
(d) Double-check that this is what the Master Theorem asserts in the first and third cases.

Problem 6. Now suppose that $a = b^\gamma$.

(a) Show that the coefficients of the power series representation of $F(z)$ take the form $S(k) = (M_5 k + M_6) a^k$, for some (unimportant) constants M_5 and M_6.
(b) Conclude that $T(n)$ is $\Theta(n^{\log_b a} \log n)$.
(c) Double-check that this is what the Master Theorem asserts in the second case.

Eliminating a simplifying assumption We would like to eliminate Simplifying Assumption #2 and deal with general functions f.

Problem 7. In the first two cases of the Master Theorem, this is straightforward. Suppose that $f(n) \leq Cn^\gamma$ for all $n \geq N_0$. Let $\tilde{T}$ satisfy the recurrence

$$\tilde{T}(0) = T(0)$$
$$\tilde{T}(n) = a\tilde{T}(n/b) + Cn^\gamma$$

(a) Show that $T(n) \leq \tilde{T}(n)$ for sufficiently large n.
(b) Conclude that $T(n)$ is $O(n^{\log_b a})$.
(c) Use the fact that $T(n) \geq aT(n/b)$ to show that $T(n) \geq T(1) n^{\log_b a}$. Conclude that $T(n)$ is $\Theta(n^{\log_b a})$.

(This shows how to handle the first case; the middle case is similar.)

Problem 8. Go back through the argument from Problem 7(c) under the assumptions of the third case of the Master Theorem. You can conclude that $T(n)$ is $\Omega(\boxed{})$. Is this what we want?

Problem 9. Luckily, on Worksheet 6 you already analyzed an explicit solution to this recurrence. Verify that you can finish the proof for the third case of the Master Theorem using your results from Worksheet 6.

Bonus Exercise. If you haven't seen this before, use this method to establish *Binet's Formula* for the Fibonacci numbers: $F_0 = 0, F_1 = 1,\ F_n = F_{n-1} + F_{n-2}$.

$$F_n = \frac{\varphi^n - \hat{\varphi}^n}{\sqrt{5}},$$

where $\varphi = \frac{1}{2}(1+\sqrt{5})$, the Golden Ratio, and $\hat{\varphi} = 1 - \varphi$.

Cultural aside. One *can* use analytic techniques to understand the growth of coefficients of a power series. Recall:

(i) The **radius of convergence** of a power series $\sum_{n\geq 0} a_n z^n$ is the largest $\rho \geq 0$ for which the power series converges at all $z \in \mathbb{C}$, $|z| < \rho$. It can be computed using the Root Test as $\rho = 1/\limsup_{n\to\infty} |a_n|^{1/n}$.

(ii) If f is a function defined on an open disk $D \subseteq \mathbb{C}$ containing 0 and is (complex-)differentiable on D, then f has a unique power series representation on D. (By this we mean that there are unique $a_0, a_1, a_2, \ldots$ so that $f(z) = \sum_{n\geq 0} a_n z^n$ for all $z \in D$.) The radius of convergence of f is exactly the radius of the largest disk to which f can be extended while remaining differentiable. That is, the radius of convergence is the distance from 0 to the nearest singularity of f.

Now recall the example $a_0 = 0$ and $a_n = 2a_{n-1} + 2^n$. We saw (a_n) had generating function $A(z) = \frac{2z}{(1-2z)^2}$. This function $A(z)$ is complex-differentiable when $|z| < 1/2$ but is undefined at $z = 1/2$ and cannot be extended to $z = 1/2$ in even a continuous way. So the radius of convergence is $1/2 = 1/\limsup_{n\to\infty} |a_n|^{1/n}$, i.e., $2 = \limsup_{n\to\infty} |a_n|^{1/n}$. So $|a_n|$ must 'grow roughly like' 2^n. (Not $\Theta(2^n)$, though!) Indeed, as we saw, $a_n = n2^n$. Our analysis here captured the 2^n but missed the polynomial factor n. More refined analytic techniques yield more refined estimates.