

# A topological obstruction to continuous global stabilization of rotational motion and the unwinding phenomenon

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## Abstract

We show that a continuous dynamical system on a state space that has the structure of a vector bundle on a compact manifold possesses no globally asymptotically stable equilibrium. This result is directly applicable to mechanical systems having rotational degrees of freedom. In particular, the result applies to the attitude motion of a rigid body. In light of this result, we explain how attitude stabilizing controllers obtained using local coordinates lead to unwinding instead of global asymptotic stability. © 2000 Elsevier Science B.V. All rights reserved.

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## 1. Introduction

It is convenient to think of the evolution of a physical system in terms of a dynamical system evolving on the *physical state space*, which is simply the set of all states of the physical system. We assume that the states of the system can be placed into a one-to-one correspondence with the points of an abstract  $m$ -dimensional manifold  $\mathcal{M}$ . Under this correspondence, the dynamics of the physical system give rise to a dynamical system on  $\mathcal{M}$ . The manifold  $\mathcal{M}$  models the physical state space, while the dynamical system on  $\mathcal{M}$  models the dynamics of the physical system. In such a case, we call  $\mathcal{M}$  the *state space*

of the system. Physically relevant quantities such as angular displacement, angular velocity, etc., yield local coordinates on  $\mathcal{M}$ . These local coordinates can be used to locally represent the dynamical system on  $\mathcal{M}$  as a set of differential equations defined on an open subset of  $\mathbb{R}^m$ , even though  $\mathcal{M}$  might be globally quite different from an open subset of  $\mathbb{R}^m$ . While local coordinates are convenient and indeed sufficient for analyzing the local properties of a dynamical system, questions of a global nature require global analysis for satisfactory solutions.

One instance where the above observation is relevant is the problem of stabilizing the attitude of a rigid body. In this case, the state space for the attitude dynamics can be identified with  $\mathcal{M} = \text{SO}(3) \times \mathbb{R}^3$ , where  $\text{SO}(3)$  is the group of rotation matrices, that is,  $3 \times 3$  orthogonal matrices with determinant 1. In Section 6, we consider a continuous feedback controller designed to globally asymptotically stabilize a desired rest attitude of a rigid body. However, a closer examination reveals that this controller, which was designed

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using local coordinates, is not globally well defined on  $\mathcal{M}$  and, in fact, leads to the *unwinding phenomenon* where the body may start at rest arbitrarily close to the desired final attitude and yet rotate through large angles before coming to rest in the desired attitude. Indeed, it has been observed in [17] that due to the global topology of  $\mathcal{M}$ , no continuous vector field on  $\mathcal{M}$  has a globally asymptotically stable equilibrium. In this paper, we generalize and expand upon this observation by identifying a large class of systems for which the global topology of the state space precludes the existence of globally asymptotically stable equilibrium points under continuous dynamics. Besides the rigid body dynamics in terms of both rotation matrices and quaternions, this class of systems also includes any mechanical system that has a rotational degree of freedom. We also outline a general class of situations in which the use of local coordinates to achieve global objectives leads to unwinding.

It is well known that the domain of attraction of an asymptotically stable equilibrium is homeomorphic to  $\mathbb{R}^n$  for some  $n$  [5, Theorem V.3.4]. This fact indicates that unlike local asymptotic stability, global asymptotic stability of an equilibrium depends strongly on the global topology of the state space  $\mathcal{M}$ . Indeed, Theorem 1 in Section 2 states that if  $\mathcal{M}$  has the structure of a vector bundle over a compact manifold  $\mathcal{Q}$ , then no continuous vector field on  $\mathcal{M}$  has a globally asymptotically stable equilibrium. This result is applicable to mechanical systems having rotational degrees of freedom. When applied to the rigid body attitude stabilization problem where the state space is a vector bundle over the compact manifold  $\text{SO}(3)$ , Theorem 1 leads to the observation made in [17] that rigid body attitude cannot be globally stabilized through continuous feedback.

Theorem 1 follows easily from elementary concepts in differential topology. However, given the substantial literature on the global stabilization of rigid body attitude [15,20,21,26–28], we feel that it would be useful to understand the ramifications of this simple but fundamental result.

In Section 4 we illustrate the unwinding phenomenon in the special case of a rigid body rotating about a fixed axis. For this case, continuous globally stabilizing feedback controllers that are designed using local coordinates turn out to be multiple-valued on the state space  $\mathcal{M} = S^1 \times \mathbb{R}$ , where  $S^1$  is the unit circle in the complex plane. Consequently, the closed-loop system exhibits unwinding wherein the body may start at rest in the desired orientation

and yet rotate several times before eventually coming to rest in the initial orientation

Often the state space  $\mathcal{M}$  of the system of interest is related to another manifold  $\mathcal{N}$ , called a *covering manifold*, through a *covering map*  $p: \mathcal{N} \rightarrow \mathcal{M}$ , which is onto and a local diffeomorphism everywhere but globally many to one. Since  $p$  is a local diffeomorphism everywhere, a given control system  $\Sigma$  on  $\mathcal{M}$  may be uniquely lifted to a control system  $\hat{\Sigma}$  on  $\mathcal{N}$  that is locally equivalent to  $\Sigma$ . This property of covering manifolds is particularly useful for modeling since a covering manifold of  $\mathcal{M}$  may have a simpler structure and thus may be more easily coordinatized than  $\mathcal{M}$  itself. For instance, the set of unit quaternions  $S^3$  is a covering manifold for the set of rotation matrices  $\text{SO}(3)$  and the corresponding covering map provides a globally nonsingular parametrization of  $\text{SO}(3)$  in terms of unit quaternions. This parametrization is widely used for modeling attitude dynamics because  $S^3$  is easily parametrized in terms of four parameters subject to one constraint, while a rotation matrix contains nine parameters satisfying six constraints. However, one drawback of using covering manifolds to model control systems is that since a covering map may be many to one, the control system  $\Sigma$  on  $\mathcal{M}$  may not be globally equivalent to the lifted control system  $\hat{\Sigma}$  on the covering manifold  $\mathcal{N}$ . Under certain feedback controllers, two solutions of  $\hat{\Sigma}$  may project onto two distinct curves in  $\mathcal{M}$  passing through the same initial condition. Such feedback controllers will give a family of motions on  $\mathcal{M}$  that exhibit unwinding. These ideas are explained in greater detail in Section 5.

In Section 6, the above ideas are used to explain how the unwinding phenomenon can arise in the case of attitude stabilizing controllers that are designed using quaternions. Although the state space for the attitude stabilization problem is  $\mathcal{M} = \text{SO}(3) \times \mathbb{R}^3$ , it is convenient to represent the problem in terms of unit quaternions on the covering manifold  $\mathcal{N} = S^3 \times \mathbb{R}^3$ . A controller that is defined in terms of quaternions, that is, on  $\mathcal{N}$ , need not determine a well-defined control law in terms of attitude, that is, on  $\mathcal{M}$ . In other words, such a controller may assign more than one control value to a point in  $\mathcal{M}$ . In such a case, the closed loop does not give rise to a well-defined dynamical system on  $\mathcal{M}$ . Thus, not only does such a controller fail to yield global asymptotic stability, but in fact leads to unwinding on  $\mathcal{M}$ . Strictly speaking, global asymptotic stability is not defined (in terms of attitude) for a controller that does not define a dynamical system on the state space  $\mathcal{M}$ .

## 2. A topological obstruction to global stability

We note that by a manifold we mean a smooth, positive dimensional, connected manifold without boundary.

Let  $\mathcal{M}$  be a manifold of dimension  $m$  and consider a continuous vector field  $f$  on  $\mathcal{M}$  with the property that for every  $x \in \mathcal{M}$ , there exists a unique right maximally defined integral curve of  $f$  starting at  $x$ , and, furthermore, every right maximally defined integral curve of  $f$  is defined on  $[0, \infty)$ . In this case, the integral curves of  $f$  are jointly continuous functions of time and initial condition [13, Theorem 3.4, p. 24] and thus define a continuous semiflow  $\psi : [0, \infty) \times \mathcal{M} \rightarrow \mathcal{M}$  on  $\mathcal{M}$  [4] satisfying

$$\psi(0, x) = x, \tag{1}$$

$$\psi(t, (\psi(\tau, x))) = \psi(t + \tau, x) \tag{2}$$

for all  $t, \tau \in [0, \infty)$  and  $x \in \mathcal{M}$ .

A point  $z \in \mathcal{M}$  is an *equilibrium* of  $f$  if  $f(z) = 0$ , or equivalently, if  $\psi(t, z) = z$  for all  $t \geq 0$ . An equilibrium  $z$  of  $f$  is said to be *globally attractive* if, for every  $x \in \mathcal{M}$ ,  $\psi(t, x) \rightarrow z$  as  $t \rightarrow \infty$ . An equilibrium  $z$  of  $f$  is called *Lyapunov stable* if, for every open neighborhood  $\mathcal{U}_\varepsilon \subset \mathcal{M}$  of  $z$ , there exists an open neighborhood  $\mathcal{U}_\delta \subset \mathcal{M}$  of  $z$  such that  $\psi(t, \mathcal{U}_\delta) \subseteq \mathcal{U}_\varepsilon$  for all  $t \geq 0$ , where  $\psi(t, \mathcal{U}_\delta) = \{\psi(t, x) : x \in \mathcal{U}_\delta\}$ . An equilibrium  $z$  of  $f$  is said to be *globally asymptotically stable* if  $z$  is globally attractive and Lyapunov stable.

A manifold  $\mathcal{Q}$  is said to be *contractible* if there exist a point  $q_0 \in \mathcal{Q}$  and a (jointly) continuous mapping  $h : [0, 1] \times \mathcal{Q} \rightarrow \mathcal{Q}$  such that  $h(0, q) = q$  and  $h(1, q) = q_0$  for all  $q \in \mathcal{Q}$ . In other words,  $\mathcal{Q}$  is contractible if the identity map  $q \mapsto q$  on  $\mathcal{Q}$  is homotopic to the constant map  $q \mapsto q_0$ . The following is a simple consequence of mod-2 intersection theory [12, Section 2.4].

**Proposition 1.** *No compact manifold is contractible.*

The following result, which is a straightforward application of Proposition 1, provides a topological obstruction to the global asymptotic stability of an equilibrium of  $f$  in terms of the global structure of  $\mathcal{M}$ . This result generalizes the observation made in [17].

**Theorem 1.** *Suppose  $\pi : \mathcal{M} \rightarrow \mathcal{Q}$  is a vector bundle on  $\mathcal{Q}$ , where  $\mathcal{Q}$  is a compact,  $r$ -dimensional manifold with  $r \leq m$ . Then there exists no equilibrium of  $f$  that is globally asymptotically stable.*

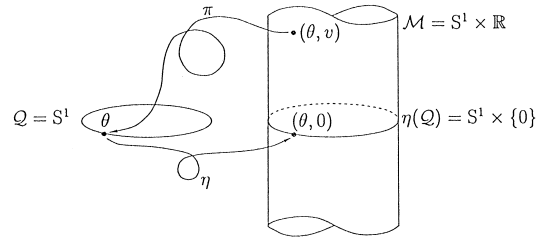


Fig. 1. The zero section of the vector bundle  $S^1 \times \mathbb{R}$ .

**Proof.** Suppose  $z \in \mathcal{M}$  is a globally asymptotically stable equilibrium of  $f$ . Let  $\eta : \mathcal{Q} \rightarrow \mathcal{M}$  denote the zero section and recall that  $\eta$  satisfies  $\pi(\eta(q)) = q$  for all  $q \in \mathcal{Q}$ . Fig. 1 depicts the maps  $\pi$  and  $\eta$  in the case where  $\mathcal{Q} = S^1$ , the unit circle, and  $\mathcal{M}$  is the cylinder  $S^1 \times \mathbb{R}$ .

Define  $h : [0, 1] \times \mathcal{Q} \rightarrow \mathcal{Q}$  by

$$h(\lambda, q) = \begin{cases} \pi(\psi(-\ln(1 - \lambda), \\ \eta(q))), & (\lambda, q) \in [0, 1) \times \mathcal{Q}, \\ \pi(z), & \lambda = 1, q \in \mathcal{Q}. \end{cases} \tag{3}$$

By construction,  $h$  is continuous on  $[0, 1) \times \mathcal{Q}$ , and satisfies  $h(0, q) = q$  and  $h(1, q) = q_0 \triangleq \pi(z)$  for all  $q \in \mathcal{Q}$ . Therefore, to prove that  $h$  is continuous on  $[0, 1] \times \mathcal{Q}$ , it suffices to consider  $q \in \mathcal{Q}$  and a sequence  $\{(\lambda_i, q_i)\}$  in  $[0, 1) \times \mathcal{Q}$  such that  $(\lambda_i, q_i) \rightarrow (1, q)$  as  $i \rightarrow \infty$ . Define the sequence  $\{t_i\}$  in  $[0, \infty)$  and the sequence  $\{x_i\}$  in  $\mathcal{M}$  by  $t_i = -\ln(1 - \lambda_i)$ ,  $x_i = \eta(q_i)$ . Then  $t_i \rightarrow \infty$  and  $x_i \rightarrow z = \eta(q)$  as  $i \rightarrow \infty$ . To prove the continuity of  $h$ , it now suffices to show that  $\psi(t_i, x_i) \rightarrow z$  as  $i \rightarrow \infty$ . Let  $\mathcal{U}_\varepsilon \subset \mathcal{M}$  be an open neighborhood of  $z$ . By Lyapunov stability, there exists an open neighborhood  $\mathcal{U}_\delta$  of  $z$  such that  $\psi(t, \mathcal{U}_\delta) \subseteq \mathcal{U}_\varepsilon$  for all  $t \geq 0$ . By global attractivity, there exists  $\tau > 0$  such that  $\psi(\tau, x) \in \mathcal{U}_\delta$ . Therefore, since  $(\tau, x_i) \rightarrow (\tau, x)$  as  $i \rightarrow \infty$ , it follows by continuity that there exists  $k > 0$  such that  $\psi(\tau, x_i) \in \mathcal{U}_\delta$  for all  $i > k$ . Since  $t_i \rightarrow \infty$  as  $i \rightarrow \infty$ , there exists  $j \geq k$  such that  $t_i > \tau$  for all  $i > j$ . It now follows from (2) and our choice of  $\mathcal{U}_\delta$  that  $\psi(t_i, x_i) = \psi(t_i - \tau, \psi(\tau, x_i)) \in \psi(t_i - \tau, \mathcal{U}_\delta) \subseteq \mathcal{U}_\varepsilon$  for all  $i > j$ . Since  $\mathcal{U}_\varepsilon$  was arbitrary, it follows that  $\psi(t_i, x_i) \rightarrow z$  as  $i \rightarrow \infty$  and thus  $h$  is continuous. It now follows that  $\mathcal{Q}$  is contractible which contradicts Proposition 1. Hence we conclude that there exists no equilibrium of  $f$  that is globally asymptotically stable.  $\square$

**Remark 1.** Theorem 1 is also valid in the more general case of a fibre bundle  $\pi : \mathcal{M} \rightarrow \mathcal{Q}$  that admits a continuous section  $\eta : \mathcal{Q} \rightarrow \mathcal{M}$ ,  $\mathcal{Q}$  being a compact manifold. The proof can be carried out in the general case without any modification. In particular, Theorem 1 applies to state spaces of the form  $\mathcal{M} \times \mathcal{K}$ , where  $\mathcal{M}$  is a vector bundle over a compact manifold and  $\mathcal{K}$  is a manifold. Since the state space for the closed-loop system obtained by applying a dynamic feedback controller to a control system on  $\mathcal{M}$  is of the form  $\mathcal{M} \times \mathcal{K}$ , where  $\mathcal{K}$  is the state space for the feedback controller, Theorem 1 implies that no continuous dynamic feedback controller can globally asymptotically stabilize an equilibrium of a control system whose state space is a vector bundle over a compact manifold.

On letting  $\mathcal{Q} = \mathcal{M}$  in Theorem 1, it follows that no continuous vector field on a compact manifold can have a globally asymptotically stable equilibrium point.

**Remark 2.** Global asymptotic stability is equivalent to local asymptotic stability together with global attractivity. Local asymptotic stability is clearly independent of the global topology of the state space and hence not ruled out by the hypotheses of Theorem 1. More interestingly, while the hypotheses of Theorem 1 rule out the possibility of global asymptotic stability, they do not rule out the possibility of global attractivity. For instance, the differential equation  $\dot{\theta} = \sin^2(\theta/2)$  represents a continuous vector field on the compact manifold  $\mathcal{M} = S^1$  that has  $\theta = 0$  as a globally attractive but Lyapunov unstable equilibrium.

**Remark 3.** Theorem 1 can be restated as saying that under the additional hypothesis of local asymptotic stability, the hypotheses of the theorem rule out global attractivity. Since the hypotheses of the theorem by themselves do not rule out global attractivity, Theorem 1 represents an interesting situation in which the local property of asymptotic stability interacts with the global topology of the state space to rule out global attractivity.

### 3. Mechanical systems with compact configuration manifolds

The *configuration space* of a mechanical system is the set of all configurations of the system, where each configuration refers to a particular arrangement of the

various particles constituting the mechanical system. The configuration space of a mechanical system can often be identified with an  $r$ -dimensional manifold  $\mathcal{Q}$ , the *configuration manifold* of the system, where  $r$  is the number of degrees of freedom of the mechanical system. The state space  $\mathcal{M}$  of a mechanical system with the configuration manifold  $\mathcal{Q}$  has the structure of a vector space over  $\mathcal{Q}$ . In the Lagrangian formulation,  $\mathcal{M}$  is the tangent bundle of  $\mathcal{Q}$ , while, in the Hamiltonian formulation,  $\mathcal{M}$  is the cotangent bundle of  $\mathcal{Q}$ . Hence Theorem 1 implies that a mechanical system having a compact configuration manifold cannot be globally asymptotically stabilized to a rest configuration using continuous state feedback.

A large class of mechanical systems have a configuration manifold that is compact. Some examples are provided in Table 1. Even in the case that a mechanical system with some rotational degrees of freedom has a noncompact configuration manifold, it is still possible to write the state space as a vector bundle over a compact manifold that represents the rotational degrees of freedom. One such mechanical system is the oscillating eccentric rotor ( $\mathcal{Q} = S^1 \times \mathbb{R}$ ,  $\mathcal{M} = S^1 \times \mathbb{R}^3$ ) [3,8,9,14,23].

Theorem 1 is also applicable to the dynamics of a rigid body written in terms of the unit quaternion, since these dynamics evolve on  $\mathcal{M} = S^3 \times \mathbb{R}^3$ , where  $S^3$  is the unit sphere in  $\mathbb{R}^4$ . It is claimed in [20,21] that the feedback controllers presented therein render the quaternion dynamics globally asymptotically stable on  $S^3 \times \mathbb{R}^3$ . This claim is not valid in light of Theorem 1. Indeed, a careful look at the feedback controllers in [20,21] reveals that the closed-loop system in each case possesses two distinct equilibria, thus ruling out global asymptotic stability. The presence of two equilibria is also noted in [15] for the attitude stabilizing controller given therein.

Another control problem that Theorem 1 applies to is the spin-axis stabilization problem involving the design of feedback controllers that cause the motion of a rigid body to asymptotically approach a uniform rotation about a body-fixed spin axis while the spin axis asymptotically points in a desired inertially fixed direction. A special case of this problem is the problem of asymptotically stabilizing the sleeping motion of a spinning top [18,24,25]. The dynamics for the spin-axis stabilization problem are given by the Euler–Poisson equations on  $\mathcal{M} = S^2 \times \mathbb{R}^2$ , where the unit sphere  $S^2$  in  $\mathbb{R}^3$  represents the set of all possible orientations of the spin axis. Since  $S^2$  is compact, it follows from Theorem 1 that continuous global spin-axis

Table 1  
Mechanical systems with compact configuration manifolds

Mechanical system	Configuration manifold $\mathcal{Q}$
Simple pendulum	$S^1$ , the unit circle
Double simple pendulum	$S^1 \times S^1$ , the torus
Planar tree-connected multibody chain	$S^1 \times \cdots \times S^1$ [22]
Spherical pendulum	$S^2$ , the unit sphere in $\mathbb{R}^3$
Double spherical pendulum	$S^2 \times S^2$ [19]
Tree connected chain of spherical pendula	$S^2 \times \cdots \times S^2$
Top or a rigid body with one point inertially fixed	SO(3), the special orthogonal group on $\mathbb{R}^3$ [2]
Single axis Control Moment Gyro (CMG)	$S^1$
Double axis CMG	$S^1 \times S^1$
Double gimbaled gyropendulum	$S^1 \times S^1 \times S^1$ [1,7]
Spacecraft with reaction wheels or CMG's	SO(3) $\times$ $S^1 \times \cdots \times S^1$ [6]

stabilization is not possible. Indeed, the controllers presented in [18,24,25] render the sleeping motion of a top an asymptotically stable equilibrium with a domain of attraction of the form  $\mathcal{U} \times \mathbb{R}^2$ , where  $\mathcal{U} \subset S^2$  is open and hence noncompact. For the controllers given in [18,24] the set  $\mathcal{U}$  is an open hemisphere, while, for the controller given in [25],  $\mathcal{U}$  is all of  $S^2$  with a point removed.

Thus, loosely speaking, Theorem 1 applies to every mechanical system that has at least one rotational degree of freedom. Therefore, Theorem 1 represents a topological obstruction to the global stabilization of any system involving rotational motion.

#### 4. The unwinding phenomenon

To illustrate the unwinding phenomenon, consider a mechanical system consisting of a rigid body rotating about a fixed axis under the action of a control torque. The configuration space of the system can be identified with the unit circle  $S^1 = \{z \in \mathbb{C}: |z| = 1\}$  in the complex plane with  $z = 1$  representing a reference configuration. Consequently, by Theorem 1, the rigid body cannot be globally asymptotically stabilized to a rest position using a continuous state feedback. The angular position  $\theta$  and the angular velocity  $\omega$  of the body represent local coordinates in a neighborhood of  $(1, 0) \in S^1 \times \mathbb{R}$ . Assuming that the moment of inertia about the axis of rotation is unity, the equations of motion of the system can be written in terms of the local coordinates  $\theta$  and  $\omega$  as

$$\dot{\theta}(t) = \omega(t), \quad (4)$$

$$\dot{\omega}(t) = u(t), \quad (5)$$

where  $u$  represents the control torque. A continuous feedback controller that locally asymptotically stabilizes the state  $\theta = 0$ ,  $\omega = 0$  is given by

$$u(t) = \chi(\theta(t), \omega(t)) \triangleq -k\theta(t) - c\omega(t), \quad (6)$$

where  $k > 0$  and  $c > 0$ . The resulting closed-loop system is locally equivalent at  $\theta = 0$ ,  $\omega = 0$  to the second-order system given by

$$\ddot{\theta}(t) + c\dot{\theta}(t) + k\theta(t) = 0. \quad (7)$$

Since  $\theta = 0$  is a globally asymptotically stable equilibrium for the system (7), it might appear that Theorem 1 has been contradicted. However, since values of  $\theta$  that differ by integral multiples of  $2\pi$  represent the same point on the configuration manifold  $S^1$ ,  $\theta$  is multiple-valued on  $S^1$ . Consequently, the control law (6) is not globally well defined on the state space  $S^1 \times \mathbb{R}$ . As a result (7) does not give rise to a well-defined continuous vector field on  $S^1 \times \mathbb{R}$ , thus explaining the apparent contradiction with Theorem 1.

Since the feedback law (6) is not globally well defined on  $S^1 \times \mathbb{R}$ , the closed-loop system exhibits the unwinding phenomenon. Unwinding can be understood by considering the initial condition  $(4\pi, 0)$ . This initial condition coincides with the desired final angular position  $\theta = 0$  of the rigid body and no further control action is needed. However, the feedback controller (6) takes the state  $(\theta, \omega)$  of the system from  $(4\pi, 0)$  to  $(0, 0)$  causing the rigid body to rotate at least twice before coming to rest in the configuration it started in.

Unwinding can be eliminated by replacing  $\theta$  in (6) by the principal value of  $\theta$  in  $[-\pi, \pi)$ . However, the resulting feedback law is discontinuous so that Theorem 1 is not applicable. In general, the feedback law  $u = \chi(\theta, \omega)$  will not lead to unwinding if and only

if  $\chi$  is  $2\pi$ -periodic in  $\theta$ . If, in addition,  $\chi$  is continuous, then the resulting closed-loop system gives rise to a continuous vector field on  $S^1 \times \mathbb{R}$ , in which case Theorem 1 rules out global asymptotic stability.

Similar arguments can be used to show that some of the stabilizing controllers that are proposed in [9,14,23] for the oscillating eccentric rotor give rise to unwinding. Ref. [9] contains an experimental illustration of unwinding in the case of the controlled oscillating eccentric rotor.

## 5. Covering manifolds and unwinding

A smooth surjective (onto) map  $p : \mathcal{N} \rightarrow \mathcal{M}$ , where  $\mathcal{N}$  and  $\mathcal{M}$  are manifolds, is called a *covering map* if every point  $x \in \mathcal{M}$  has an open neighborhood  $\mathcal{U} \subset \mathcal{M}$  such that  $p^{-1}(\mathcal{U})$  is a disjoint union of open sets  $\mathcal{V}_k$  and, for each  $k$ ,  $p$  restricted to  $\mathcal{V}_k$  is a diffeomorphism. If  $p : \mathcal{N} \rightarrow \mathcal{M}$  is a covering map, then  $\mathcal{N}$  is called a *covering manifold* of  $\mathcal{M}$ , and  $\mathcal{N}$  and  $\mathcal{M}$  are locally diffeomorphic everywhere. For instance,  $\mathcal{N} = \mathbb{R}^2$  is a covering manifold of  $\mathcal{M} = S^1 \times \mathbb{R}$  with a covering map  $p : \mathbb{R}^2 \rightarrow S^1 \times \mathbb{R}$  given by

$$p(\theta, \omega) = (e^{i\theta}, \omega), \quad (8)$$

where  $i = \sqrt{-1}$  [11, p. 150]. One can picture  $p$  as the map that wraps the plane  $\mathbb{R}^2$  around the cylinder  $S^1 \times \mathbb{R}$  by mapping each strip of the form  $[2k\pi, 2(k+1)\pi] \times \mathbb{R}$  onto  $S^1 \times \mathbb{R}$ .

Since a covering map  $p : \mathcal{N} \rightarrow \mathcal{M}$  is a local diffeomorphism, a covering map can be used to “lift” curves in  $\mathcal{M}$  to curves in the covering manifold  $\mathcal{N}$ . More precisely, if  $c : [a, b] \rightarrow \mathcal{M}$  represents a continuous curve in  $\mathcal{M}$ , then, for every  $x \in p^{-1}(c(a)) \subset \mathcal{N}$ , there exists a unique curve  $\hat{c} : [a, b] \rightarrow \mathcal{N}$  such that  $\hat{c}(a) = x$  and  $p(\hat{c}(t)) = c(t)$  for all  $t \in [a, b]$  [11, p. 150]. Consequently, a semiflow  $\psi$  on  $\mathcal{M}$  can be lifted to a unique semiflow  $\hat{\psi}$  on  $\mathcal{N}$  such that

$$p(\hat{\psi}(t, x)) = \psi(t, p(x))$$

for all  $(t, x) \in [0, \infty) \times \mathcal{N}$ . In a similar fashion, a vector field (control system)  $f$  on  $\mathcal{M}$  can be uniquely lifted to a vector field (control system)  $\hat{f}$  on  $\mathcal{N}$  such that  $p$  projects the integral curves of  $\hat{f}$  onto the integral curves of  $f$ . However, since a covering map in general is many to one, not every semiflow on the covering manifold  $\mathcal{N}$  projects onto a well-defined semiflow on  $\mathcal{M}$  under the covering map. In particular, a semiflow  $\hat{\psi}$  on  $\mathcal{N}$  projects onto a semiflow on  $\mathcal{M}$  if

$$\text{and only if, for all } x, y \in \mathcal{N} \text{ satisfying } p(x) = p(y), \\ p(\hat{\psi}(t, x)) = p(\hat{\psi}(t, y)) \quad (9)$$

for all  $t \in [0, \infty)$ . If (9) is not satisfied, then the integral curves through  $x$  and  $y$  project onto two distinct curves on  $\mathcal{M}$  passing through the same point  $p(x) = p(y) \in \mathcal{M}$ . In this case, the family of curves  $\{p \circ \hat{\psi}(\cdot, x) : x \in \mathcal{N}\}$  on  $\mathcal{M}$  represents a family of motions that exhibit unwinding. Similar remarks apply to vector fields and control systems on  $\mathcal{N}$ .

To illustrate the above ideas, consider once again a rigid body rotating about a fixed axis with unit moment of inertia under the action of a control torque  $u$ . Such a rigid body can be modeled by the control system

$$\dot{z}(t) = i\omega(t)z(t), \quad (10)$$

$$\dot{\omega}(t) = u(t), \quad (11)$$

which evolves on  $S^1 \times \mathbb{R}$ . Substituting  $z(t) = e^{i\theta(t)}$  in (10) yields (4). Thus (4) and (5) represent the dynamics of the rigid body lifted to  $\mathbb{R}^2$  using the covering map (8). Note that the lifted system (4) and (5) is locally equivalent to (10) and (11) in a neighborhood of  $(\theta, \omega) = (0, 0)$  since  $p$  is a local diffeomorphism. Hence the control law (6) with  $\theta$  restricted to  $(-\pi, \pi)$  locally asymptotically stabilizes the equilibrium  $(z, \omega) = (1, 0)$  of system (10)–(11). However, the closed-loop system (7) obtained from (4) and (5) does not project onto a globally well-defined dynamical system on  $S^1 \times \mathbb{R}$ . For instance, the solutions of (7) passing through  $(4\pi, 0)$  and  $(0, 0)$  project onto two distinct curves in  $S^1 \times \mathbb{R}$  passing through the same initial condition  $(z, \omega) = (1, 0)$ . As a result, controller (6) when applied globally to the system (4)–(5) leads to unwinding as noted in the previous section.

## 6. The unwinding phenomenon in attitude control

The configuration space for the attitude dynamics of a rigid body can be identified with  $\text{SO}(3)$ . Hence the configuration manifold for the attitude dynamics of a rigid body is the three dimensional compact Lie group  $\mathcal{Q} = \text{SO}(3)$ . The Lie group structure of  $\text{SO}(3)$  makes it possible to write the equations of motion on  $\mathcal{M} = \text{SO}(3) \times \mathbb{R}^3$  instead of the tangent bundle  $T\text{SO}(3)$  of  $\text{SO}(3)$ . The equations of motion are given by [16]

$$\dot{R}(t) = -(\omega(t) \times)R(t), \quad (12)$$

$$J\dot{\omega}(t) = -(\omega(t) \times)J\omega(t) + u(t), \quad (13)$$

where  $R \in \text{SO}(3)$  transforms the components of a vector in an inertial frame to its components in a body-fixed frame,  $\omega \in \mathbb{R}^3$  is the angular velocity of the body in the body-fixed frame,  $J$  is the inertia matrix,  $u \in \mathbb{R}^3$  is the external control torque vector and, for each  $\omega \in \mathbb{R}^3$ ,  $(\omega \times)$  is the matrix representation of the linear operation  $v \mapsto \omega \times v$  on  $\mathbb{R}^3$ ,  $\times$  denoting the familiar cross product operation between vectors in  $\mathbb{R}^3$ .

Unit quaternions or Euler parameters provide a popular means of parametrizing matrices in  $\text{SO}(3)$ . We write a quaternion  $q$  as  $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$  with quaternion multiplication defined according to  $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$ ,  $\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$ ,  $\mathbf{jk} = -\mathbf{kj} = \mathbf{i}$  and  $\mathbf{ki} = -\mathbf{ik} = \mathbf{j}$ . The set of unit quaternions  $S^3 = \{q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} : q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1\}$  is a group under quaternion multiplication. Every  $q \in S^3$  gives rise to a linear map  $p(q) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  given by

$$p(q)(v) = q(v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k})\bar{q}, \quad (14)$$

where  $\bar{q} = q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}$ . The matrix of the linear map  $p(q)$  is a rotation matrix for every  $q \in S^3$  [10, Chapter 5]. The map  $p : S^3 \rightarrow \text{SO}(3)$  thus obtained satisfies  $p(q) = p(-q)$  and is a two-to-one covering map of  $\text{SO}(3)$ . Being a local diffeomorphism everywhere,  $p$  yields a globally nonsingular parametrization of rotation matrices in terms of unit quaternions. Consequently, the kinematics (12) can be lifted onto  $S^3$  to yield the differential equations

$$\dot{q}(t) = \frac{1}{2}q(t)(\omega_1(t)\mathbf{i} + \omega_2(t)\mathbf{j} + \omega_3(t)\mathbf{k}), \quad (15)$$

$$J\dot{\omega}(t) = -(\omega(t) \times)J\omega(t) + u(t), \quad (16)$$

on  $\mathcal{N} = S^3 \times \mathbb{R}^3$  for the unit quaternion representation of the attitude of a rigid body [16, p. 58].

The attitude feedback stabilization problem consists of finding a feedback controller  $u(t) = \chi(R(t), \omega(t))$  such that the closed-loop system obtained from (12) and (13) has the state  $(R_e, 0) \in \text{SO}(3) \times \mathbb{R}^3$  corresponding to the desired rest attitude as an asymptotically stable equilibrium. In [15,20,21,26–28], the solution is presented in terms of a feedback  $u(t) = \chi(q(t), \omega(t))$  such that the state  $(q_e, 0) \in S^3 \times \mathbb{R}^3$  is an asymptotically stable equilibrium for the closed-loop system obtained from (15) and (16), where  $q_e$  is such that  $p(q_e) = R_e$ . Since  $p$  is a local diffeomorphism, the asymptotic stability of  $(q_e, 0) \in S^3 \times \mathbb{R}^3$  implies local asymptotic stability of  $(R_e, 0) \in \text{SO}(3) \times \mathbb{R}^3$ . One such controller is given by

$$\chi(q, \omega) = -J\omega - J^{-1}[q_1 \ q_2 \ q_3]^T, \quad (17)$$

where it is assumed that  $q_e = 1$  and  $R_e = I$ , the identity matrix. Local asymptotic stability of the equilibrium  $(q_e, 0) \in S^3 \times \mathbb{R}^3$  and hence of the equilibrium  $(R_e, 0) \in \text{SO}(3) \times \mathbb{R}^3$  follows from the invariant set theorem by considering the Lyapunov function

$$V(q, \omega) = \frac{1}{2}\omega^T J^2 \omega + (q_0 - 1)^2 + q_1^2 + q_2^2 + q_3^2 \quad (18)$$

and its derivative along the closed-loop solutions of (15) and (16) given by

$$\dot{V}(q, \omega) = -\omega^T J^2 \omega. \quad (19)$$

Theorem 1 implies that no point of  $S^3 \times \mathbb{R}^3$  can be a globally asymptotically stable equilibrium of (15) and (16) for any continuous feedback  $u(t) = \chi(q(t), \omega(t))$ . Indeed, under the control law (17), both  $(\pm q_e, 0)$  are equilibria of the closed-loop quaternion equations (15) and (16), so that global asymptotic stability of the equilibrium  $(q_e, 0) \in S^3 \times \mathbb{R}^3$  does not hold. Furthermore,  $(-q_e, 0)$  is a Lyapunov unstable equilibrium, while the solutions from all other initial conditions converge to  $(q_e, 0)$ . Since both  $q_e$  and  $-q_e$  correspond to the same attitude  $R_e$ , the state  $(R_e, 0) \in \text{SO}(3) \times \mathbb{R}^3$  clearly cannot be a Lyapunov stable equilibrium for (12) and (13). Moreover, every solution of (15) and (16) that starts sufficiently close to  $(-q_e, 0)$  diverges from  $(-q_e, 0)$  and converges to the equilibrium  $(q_e, 0)$ . The corresponding attitude response starts close to the desired attitude  $(R_e, 0)$ , diverges from  $(R_e, 0)$  and then converges once again to  $(R_e, 0)$ , thus exhibiting unwinding of up to  $360^\circ$ .

The control law (17) does not satisfy  $\chi(q, \omega) = \chi(-q, \omega)$  for all  $(q, \omega) \in S^3 \times \mathbb{R}^3$  and is thus not well defined on  $\text{SO}(3) \times \mathbb{R}^3$ . As a result the closed-loop equations obtained from (15) and (16) do not give rise to a well-defined vector field on  $\text{SO}(3) \times \mathbb{R}^3$ , while the closed-loop solutions of (15) and (16) lead to attitude responses that exhibit unwinding. To sum up, controller (17) gives rise to a continuous flow on  $S^3 \times \mathbb{R}^3$  and hence does not give global asymptotic stability by Theorem 1. Instead, controller (17) fails to give rise to a well defined flow on  $\text{SO}(3) \times \mathbb{R}^3$  and hence causes unwinding in the attitude responses.

**Remark 4.** Controller (17) belongs to the classes of controllers give in [20,21]. The controllers proposed in [20,21], like the controller (17), also give closed-loop dynamics having two equilibria on  $S^3 \times \mathbb{R}^3$ . This fact is not mentioned in [20,21] and does not appear to have been recognized. In [15,27,28], the existence of a second equilibrium in the closed-loop quaternion dynamics is recognized. However, it is argued in [15]

that since both the quaternion equilibria correspond to the same attitude and all solutions converge to one or the other of the two, the desired attitude is globally asymptotically stable on  $SO(3) \times \mathbb{R}^3$ . As we have argued above, this situation in fact leads to attitude responses that exhibit unwinding. Ref. [26] recognizes the fact that due to the presence of a second unstable equilibrium in the closed-loop quaternion dynamics, the closed-loop dynamics do not give rise to a dynamical system on  $SO(3) \times \mathbb{R}^3$  and, in fact, lead to unwinding in the attitude response.

## 7. Conclusions

Global properties of a dynamical system, such as global asymptotic stability of an equilibrium, depend strongly on the global topology of the underlying state space. For instance, mechanical systems with rotational degrees of freedom cannot be globally asymptotically stabilized to a rest configuration. Locally stabilizing controllers that are designed using local coordinates lead to unwinding when applied globally in a continuous manner. Since unwinding can be highly undesirable in spacecraft applications from the point of view of fuel consumption and vibration suppression, this paper highlights the need for using global considerations in deriving results of a global nature.

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