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Deadbeat Internal Model Control for Command Following and Disturbance Rejection in Discrete-Time Systems

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Abstract— **Internal model control is an established technique in continuous-time linear control but has not been developed for discrete-time systems in the shift and delta domains. In this paper, we present a new linear algebraic solution to the multivariable command following and disturbance rejection problem that unifies continuous and discrete time. Furthermore, we present deadbeat internal model control for discrete-time systems in the shift and delta domains.**

1. Introduction

The internal model principle is a foundation result of control theory [1–9]. Internal model control is essential for both command following and disturbance rejection problems, where the exogenous command and disturbance signals are outputs of an unforced linear system. In short, the internal model principle states that asymptotic command following and disturbance rejection can be achieved by incorporating copies of the exogenous dynamics in the feedback loop.

During the early to mid 1970s, internal model control was an active research area starting with [1–3]. Specifically, [1] treats the disturbance rejection problem for a class of systems for which the range of the disturbance input matrix is contained in the range of the control input matrix. A general treatment of multi-input, multi-output internal model control for both command following and disturbance rejection problems is given in [4–7], including the synthesis of a servocompensator. An alternative geometric approach is given in [8]. In [9], the converse problem is addressed, and necessary conditions for asymptotic regulation are developed.

All of the results in [1–9] on internal model control are confined to continuous-time systems. Analogous results for discrete-time systems are not available in the literature. Furthermore, the results of [4–7] use analytic tools specific to continuous-time models, and thus do not extend to discrete-time systems. In the present paper, we develop an alternative approach to internal model control that is directly applicable to continuous-time systems and discrete-time systems in both the shift and delta domains. In contrast to the analytic approach of [4–7] and the geometric approach of [8, 9], our approach is heavily algebraic. Using this approach, we simultaneously solve the command following and disturbance rejection problem in both continuous and discrete time. We also present sufficient conditions for deadbeat internal model control in the shift and delta domains.

2. Problem Formulation

In this paper, we consider the linear continuous-time system

$$\dot{x}(t) = Ax(t) + Bu(t) + D_1w(t), \tag{2.1}$$
$$z(t) = E_1x(t) + E_2u(t) + E_0w(t), \tag{2.2}$$
$$y(t) = Cx(t) + Du(t) + D_2w(t), \tag{2.3}$$

and the linear discrete-time systems

$$x(t+1) = Ax(t) + Bu(t) + D_1w(t), \tag{2.4}$$
$$z(t) = E_1x(t) + E_2u(t) + E_0w(t), \tag{2.5}$$
$$y(t) = Cx(t) + Du(t) + D_2w(t), \tag{2.6}$$

or

$$x(t+1) = x(t) + h\left[Ax(t) + Bu(t) + D_1w(t)\right], \tag{2.7}$$
$$z(t) = E_1x(t) + E_2u(t) + E_0w(t), \tag{2.8}$$
$$y(t) = Cx(t) + Du(t) + D_2w(t), \tag{2.9}$$

where $x \in \mathbb{R}^n$ is the state, $y \in \mathbb{R}^{l_y}$ is the measured output available to the controller, $z \in \mathbb{R}^{l_z}$ is the performance, $u \in \mathbb{R}^{l_u}$ is the controller output, $w \in \mathbb{R}^{l_w}$ is the exogenous input, and $h > 0$ is the sampling period. Without loss of generality, we assume that rank $B = l_u$ and rank $C = l_y$.

To simultaneously consider continuous-time and discrete-time systems, let $\mathcal{D}$ denote the differential operator $\frac{d}{dt}$, the forward shift operator $\mathbf{q}$, or the delta operator $\delta = \frac{\mathbf{q}-1}{h}$, that is, $\mathcal{D}x(t) = \frac{dx(t)}{dt}$ for continuous time, $\mathcal{D}x(t) = x(t+1)$ for discrete time with the shift operator, and $\mathcal{D}x(t) = \frac{1}{h}\left[x(t+1) - x(t)\right]$ for discrete time with the delta operator. Note that the set containing the argument t depends on the operator $\mathcal{D}$, that is, $t \in \tau$ where

$$\tau \triangleq \begin{cases} [0,\infty), & \mathcal{D} = \frac{d}{dt}, \\ \mathbb{Z}^+, & \mathcal{D} = \mathbf{q}, \\ \{t : \frac{t}{h} \in \mathbb{Z}^+\}, & \mathcal{D} = \delta. \end{cases} \tag{2.10}$$

The objective of this paper is to determine sufficient condition for the existence of a feedback controller that stabilizes the open-loop system (2.1)-(2.3) or (2.4)-(2.6) or (2.7)-(2.9) and regulates the performance z to zero when the exogenous input w is generated by an unforced linear system. This control problem includes both disturbance rejection and command following objectives, that is, the exogenous signal w contains components to be rejected and components to be followed. The problem can be restricted to command following alone by letting $D_1 = 0$ or to disturbance rejection alone by letting $D_2 = 0$ and $E_0 = 0$.

The single-input, single-output (SISO) command following problem in Figure 1 can be written in the standard form (2.1)-(2.3) or (2.4)-(2.6) or (2.7)-(2.9), where the plant G has the realization $G \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, $D_1 = 0$, $E_1 = C$, $E_2 = D$, $E_0 = D_2 = -1$, and $\hat{G}$ is the feedback controller. Then $z = y = Cx + Du + D_2w = y_{\rm out} - w$ is the tracking error where $y_{\rm out} = Cx + Du$.

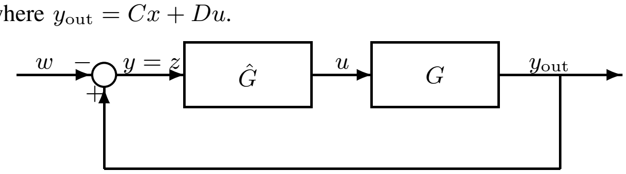

Fig. 1. SISO command following problem

Similarly, the SISO disturbance rejection problem in Figure 2

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can be written in the standard form, where $D_1 = B$, $E_1 = C$, $E_2 = D$, $E_0 = D_2 = 0$, and $G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$.

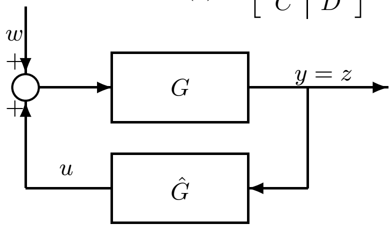

Fig. 2. SISO disturbance rejection problem

Lastly, the combined SISO reference tracking and disturbance rejection problem in Figure 3 can be written in the standard form, where $D_1 = B$, $E_1 = C$, $E_2 = D$, $E_0 = D_2 = \left[\begin{array}{cc} 1 & 0 \end{array}\right]$, and $G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$. Then $z = y = Cx + Du + D_2 w = y_{\rm out} - w_1$ is the tracking error.

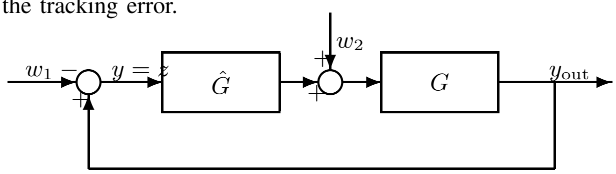

Fig. 3. Combined SISO command following and disturbance rejection problem

Next, define the stable region

$$\mathcal{S} \triangleq \begin{cases} \{\lambda \in \mathbb{C} : \text{Re } \lambda < 0\}, & \mathcal{D} = \frac{d}{dt}, \\ \{\lambda \in \mathbb{C} : |\lambda| < 1\}, & \mathcal{D} = \mathbf{q}, \\ \{\lambda \in \mathbb{C} : \frac{h}{2}|\lambda|^2 + \text{Re } \lambda < 0\}, & \mathcal{D} = \delta, \end{cases} \tag{2.11}$$

and the unstable region

$$\mathcal{U} \triangleq \begin{cases} \{\lambda \in \mathbb{C} : \text{Re } \lambda \geq 0\}, & \mathcal{D} = \frac{d}{dt}, \\ \{\lambda \in \mathbb{C} : |\lambda| \geq 1\}, & \mathcal{D} = \mathbf{q}, \\ \{\lambda \in \mathbb{C} : \frac{h}{2}|\lambda|^2 + \text{Re } \lambda \geq 0\}, & \mathcal{D} = \delta. \end{cases} \tag{2.12}$$

Definition 2.1. *The* spectrum *of A is* $\text{spec}(A) \triangleq \{\lambda : \det(\lambda I - A) = 0\}$.

Definition 2.2. *A is* asymptotically stable *if* $\text{spec}(A) \subset \mathcal{S}$.

Definition 2.3. *A is* completely unstable *if* $\text{spec}(A) \subset \mathcal{U}$.

Let the exogenous signal w be the output of the linear system

$$\mathcal{D}x_w = A_w x_w, \qquad w = C_w x_w, \tag{2.13}$$

where $x_w \in \mathbb{R}^{n_w}$, A_w is completely unstable, and (A_w, C_w) is observable. In the case $l_w = 1$, it follows from [10] that A_w is cyclic. In the case $l_w > 1$, we assume, without loss of generality, that A_w is cyclic.

To achieve closed-loop stability and regulation of the performance variable, consider the feedback controller

$$\mathcal{D}\hat{x} = \hat{A}\hat{x} + \hat{B}y, \qquad u = \hat{C}\hat{x}, \tag{2.14}$$

where $\hat{x} \in \mathbb{R}^{\hat{n}}$. The closed-loop system (2.1)-(2.3) or (2.7)-(2.9) with the feedback controller (2.14) is

$$\mathcal{D}\tilde{x} = \tilde{A}\tilde{x} + \tilde{D}w, \tag{2.15}$$

$$z = \tilde{E}\tilde{x} + E_0 w, \tag{2.16}$$

where

$$\tilde{A} \triangleq \begin{bmatrix} A & B\hat{C} \\ \hat{B}C & \hat{A} + \hat{B}D\hat{C} \end{bmatrix}, \quad \tilde{D} \triangleq \begin{bmatrix} D_1 \\ \hat{B}D_2 \end{bmatrix}, \tag{2.17}$$

$$\tilde{E} \triangleq \begin{bmatrix} E_1 & E_2\hat{C} \end{bmatrix}, \quad \tilde{x} \triangleq \begin{bmatrix} x \\ \hat{x} \end{bmatrix}. \tag{2.18}$$

The following result is presented in [9] for continuous-time systems but is also applicable to both shift and delta operator discrete-time systems. This result is used in the next section to show regulation of the performance variable z. In [9], a geometric proof of this result is given. We provide an alternative proof, which extends the result to discrete-time systems.

Lemma 2.1. *Consider the closed-loop system (2.15)-(2.18) with exogenous input (2.13), and assume that $\tilde{A}$ is asymptotically stable. Then, for all initial conditions $\tilde{x}(0)$ and $x_w(0)$, $\lim_{t\to\infty} z(t) = 0$ if and only if there exists $S \in \mathbb{R}^{(n+\hat{n})\times n_w}$ such that*

$$\tilde{A}S - SA_w = \tilde{D}C_w, \tag{2.19}$$

$$\tilde{E}S = E_0 C_w. \tag{2.20}$$

Proof. The closed-loop system (2.15)-(2.18) with exogenous input (2.13) can be written as

$$\mathcal{D}x_s = A_s x_s, \quad z = E_s x_s, \tag{2.21}$$

where $x_s \triangleq \begin{bmatrix} \tilde{x} & x_w \end{bmatrix}^{\rm T}$ and

$$A_s \triangleq \begin{bmatrix} \tilde{A} & \tilde{D}C_w \\ 0 & A_w \end{bmatrix}, \quad E_s \triangleq \begin{bmatrix} \tilde{E} & E_0 C_w \end{bmatrix}. \tag{2.22}$$

Since $\tilde{A}$ is asymptotically stable and A_w is completely unstable, the Sylvester equation (2.19) has a unique solution. Let $S \in \mathbb{R}^{(n+\hat{n})\times n_w}$ be the unique solution to (2.19). Now define

$$Q \triangleq \begin{bmatrix} I & -S \\ 0 & I \end{bmatrix}, \tag{2.23}$$

and consider the change of basis

$$\bar{A}_s \triangleq Q^{-1}A_s Q = \begin{bmatrix} \tilde{A} & -\tilde{A}S + SA_w + \tilde{D}C_w \\ 0 & A_w \end{bmatrix}$$
$$= \begin{bmatrix} \tilde{A} & 0 \\ 0 & A_w \end{bmatrix}, \tag{2.24}$$

$$\bar{E}_s \triangleq E_s Q = \begin{bmatrix} \tilde{E} & -\tilde{E}S + E_0 C_w \end{bmatrix}. \tag{2.25}$$

To prove necessity for continuous-time systems, suppose that $\lim_{t\to\infty} z(t) = 0$, so that $\lim_{t\to\infty} \left[\tilde{E}e^{\tilde{A}t} + \left(-\tilde{E}S + E_0 C_w\right)e^{A_w t}\right] = 0$. Since $\tilde{A}$ is asymptotically stable, it follows that $\lim_{t\to\infty} \tilde{E}e^{\tilde{A}t} = 0$ and thus $\lim_{t\to\infty} \left(-\tilde{E}S + E_0 C_w\right)e^{A_w t} = 0$. Since A_w is completely unstable, every nonzero entry of $\left(-\tilde{E}S + E_0 C_w\right)e^{A_w t}$ is either a constant or involves exponentials of t, where each coefficient of t has nonnegative real part. Therefore, $\lim_{t\to\infty} \left(-\tilde{E}S + E_0 C_w\right)e^{A_w t} = 0$ implies that $-\tilde{E}S + E_0 C_w = 0$.

To prove necessity for discrete-time systems with the shift operator, suppose that $\lim_{t\to\infty} z(t) = 0$ for all initial conditions, so that $\lim_{t\to\infty} \left[\tilde{E}\tilde{A}^t + \left(-\tilde{E}S + E_0 C_w\right)A_w^t\right] = 0$. Since $\tilde{A}$ is asymptotically stable, it follows that $\lim_{t\to\infty} \tilde{E}\tilde{A}^t = 0$ and thus $\lim_{t\to\infty} \left(-\tilde{E}S + E_0 C_w\right)A_w^t = 0$. Since A_w is completely unstable, $\lim_{t\to\infty} A_w^t$ does not exist and, for all

$t \ge 0$, A_w^t is nonsingular. Assume that $-\tilde{E}S + E_0C_w \neq 0$, and let $\sigma_{\min}(\cdot)$ denote the minimum singular value. Then it follows that $0 = \lim_{t\to\infty} \left\|\left(-\tilde{E}S+E_0C_w\right)A_w^t\right\|_F \ge \left\|-\tilde{E}S+E_0C_w\right\|_F \lim_{t\to\infty}\sigma_{\min}(A_w^t) = \infty$, which is a contradiction. Therefore, $-\tilde{E}S+E_0C_w=0$.

To prove necessity for discrete-time systems with the delta operator, suppose that $\lim_{t\to\infty} z(t) = 0$ for all initial conditions, so that $\lim_{t\to\infty} \bar{E}_s\left(I+h\bar{A}_s\right)^{\frac{t}{h}} = 0$. Thus, $\lim_{t\to\infty}\left[\tilde{E}\left(I+h\tilde{A}\right)^{\frac{t}{h}} + \left(-\tilde{E}S+E_0C_w\right)(I+hA_w)^{\frac{t}{h}}\right] = 0$. Since $\tilde{A}$ is asymptotically stable, it follows that $\lim_{t\to\infty}\tilde{E}\left(I+h\tilde{A}\right)^{\frac{t}{h}} = 0$ and thus $\lim_{t\to\infty}\left(-\tilde{E}S+E_0C_w\right)(I+hA_w)^{\frac{t}{h}} = 0$. Since A_w is completely unstable, $\lim_{t\to\infty}\left(I+h\tilde{A}\right)^{\frac{t}{h}}$ does not exist and, for all $t \ge 0$, $(I+hA_w)^{\frac{t}{h}}$ is nonsingular. Assume that $-\tilde{E}S + E_0C_w \neq 0$, and it follows that $0 = \lim_{t\to\infty}\left\|\left(-\tilde{E}S+E_0C_w\right)(I+hA_w)^{\frac{t}{h}}\right\|_F \ge \left\|-\tilde{E}S+E_0C_w\right\|_F \lim_{t\to\infty} \sigma_{\min}\left((I+hA_w)^{\frac{t}{h}}\right) = \infty$, which is a contradiction. Therefore, $-\tilde{E}S+E_0C_w=0$.

Conversely, since $\tilde{E}S - E_0C_w = 0$, we have $z(t) = \tilde{E}e^{\tilde{A}t}\tilde{x}(0)$ in continuous-time, $z(t) = \tilde{E}\tilde{A}^t\tilde{x}(0)$ in discrete time with the shift operator, and $z(t) = \tilde{E}\left(I+h\tilde{A}\right)^{\frac{t}{h}}\tilde{x}(0)$ in discrete time with the delta operator. Since $\tilde{A}$ is asymptotically stable, $\lim_{t\to\infty} z(t) = 0$. □

3. Internal Model Control

In this section, we consider the command following and disturbance rejection problem for the linear system (2.1)-(2.3) or (2.4)-(2.6) or (2.7)-(2.9). We provide sufficient conditions for the existence of a feedback controller (2.14) that stabilizes the closed-loop system (2.15)-(2.18) and regulates the performance variable z to zero.

Before presenting sufficient conditions for the existence of an internal model controller, we characterize the form of this controller. First, consider the open-loop system (2.1)-(2.3) or (2.4)-(2.6) or (2.7)-(2.9) and cascade its output with an internal model of the exogenous dynamics

$$\mathcal{D}\hat{x}_1 = A_W\hat{x}_1 + B_W y, \tag{3.1}$$

where $A_W \triangleq I_{l_y}\otimes A_w \in \mathbb{R}^{n_w l_y \times n_w l_y}$, $B_W \triangleq I_{l_y}\otimes B_w$, and $B_w \in \mathbb{R}^{n_w\times 1}$ is chosen such that (A_w, B_w) is controllable. There exists B_w such that (A_w, B_w) is controllable since A_w is cyclic. Note that the dynamics of (3.1) contains l_y copies of the exogenous dynamics A_w. The cascade (2.1)-(2.3) or (2.4)-(2.6) or (2.7)-(2.9) and (3.1) is

$$\mathcal{D}\begin{bmatrix} x_1 \\ \hat{x}_1 \end{bmatrix} = \begin{bmatrix} A & 0 \\ B_W C & A_W \end{bmatrix}\begin{bmatrix} x \\ \hat{x}_1 \end{bmatrix} + \begin{bmatrix} B \\ B_W D \end{bmatrix} u + \begin{bmatrix} D_1 \\ B_W D_2 \end{bmatrix} w, \tag{3.2}$$

$$\begin{bmatrix} y \\ \hat{x}_1 \end{bmatrix} = \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} x \\ \hat{x}_1 \end{bmatrix} + \begin{bmatrix} D \\ 0 \end{bmatrix} u + \begin{bmatrix} D_2 \\ 0 \end{bmatrix} w. \tag{3.3}$$

Next, we consider a feedback controller for the augmented system (3.2)-(3.3), of the form

$$\mathcal{D}\hat{x}_2 = A_c\hat{x}_2 + \begin{bmatrix} B_{c1} & B_{c2} \end{bmatrix}\begin{bmatrix} y \\ \hat{x}_1 \end{bmatrix}, \tag{3.4}$$

$$u = C_c\hat{x}_2, \tag{3.5}$$

where $A_c \in \mathbb{R}^{(n+n_w l_y)\times(n+n_w l_y)}$, $B_{c1} \in \mathbb{R}^{(n+n_w l_y)\times l_y}$, $B_{c2} \in \mathbb{R}^{(n+n_w l_y)\times n_w l_y}$, and $C_c \in \mathbb{R}^{l_u\times(n+n_w l_y)}$. Then, the closed-loop system (3.2)-(3.3) and (3.4)-(3.5) is shown in Figure 4 and given by

$$\mathcal{D}\begin{bmatrix} x \\ \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} A & 0 & BC_c \\ B_W C & A_W & B_W D C_c \\ B_{c1}C & B_{c2} & A_c + B_{c1}DC_c \end{bmatrix}\begin{bmatrix} x \\ \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} D_1 \\ B_W D_2 \\ B_{c1}D_2 \end{bmatrix} w, \tag{3.6}$$

$$z = \begin{bmatrix} E_1 & 0 & E_2C_c \end{bmatrix}\begin{bmatrix} x \\ \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + E_0 w. \tag{3.7}$$

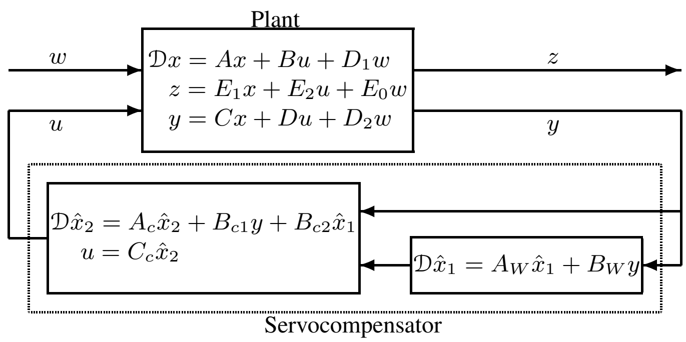

Fig. 4. Internal Model Control

Now, we present the main result of this section.

Theorem 3.1. *Assume that the following conditions hold.*

(i) (A, B, C) *is stabilizable and detectable.*
(ii) $l_u \ge l_y$.
(iii) *For all* $\lambda \in$ *spec* (A_w), *rank* $\left(\begin{bmatrix} A-\lambda I & B \\ C & D \end{bmatrix}\right) = n + \min(l_u, l_y)$.
(iv) *There exists* $R \in \mathbb{R}^{l_z\times l_y}$ *such that* $z = Ry$.

Then there exists a linear time invariant controller of the form (2.14) such that the equilibrium of the closed-loop system (2.15)-(2.18) is asymptotically stable and, for all initial conditions $\tilde{x}(0)$ *and* $x_w(0)$, $\lim_{t\to\infty} z(t) = 0$. *Furthermore, one such linear time invariant controller is given by (2.14) with*

$$\hat{A} \triangleq \begin{bmatrix} A_W & 0 \\ B_{c2} & A_c \end{bmatrix}, \quad \hat{B} \triangleq \begin{bmatrix} B_W \\ B_{c1} \end{bmatrix}, \quad \hat{C} \triangleq \begin{bmatrix} 0 & C_c \end{bmatrix}, \tag{3.8}$$

where A_c, B_{c1}, B_{c2}, *and* C_c *are chosen so that (3.4),(3.5) stabilizes (3.2),(3.3).*

Proof. First, we show that the augmented system (3.2)-(3.3) is detectable and stabilizable. Let $s \in \mathcal{U}$ and $\lambda \in \operatorname{spec}(A_w) \subset \mathcal{U}$.

Since (A,B) is stabilizable, it follows that

$$\begin{aligned}\operatorname{rank}&\begin{bmatrix} A-sI & B & 0\\ B_WC & B_WD & A_W-sI\end{bmatrix}\\ &\geq \operatorname{rank}\begin{bmatrix} A-\lambda I & B & 0\\ B_WC & B_WD & A_W-\lambda I\end{bmatrix}\\ &\geq \operatorname{rank}\left(\begin{bmatrix} I_n & 0 & 0\\ 0 & B_W & A_W-\lambda I\end{bmatrix}\right.\\ &\qquad\times\left.\begin{bmatrix} A-\lambda I & B & 0\\ C & D & 0\\ 0 & 0 & I_{l_yn_w}\end{bmatrix}\right). \qquad (3.9)\end{aligned}$$

Conditions *(ii)* and *(iii)* imply that

$$\operatorname{rank}\begin{bmatrix} A-\lambda I & B & 0\\ C & D & 0\\ 0 & 0 & I_{l_yn_w}\end{bmatrix} = n+l_y+l_yn_w, \qquad (3.10)$$

which is full row rank. Therefore,

$$\begin{aligned} n+l_yn_w &\geq \operatorname{rank}\begin{bmatrix} A-sI & B & 0\\ B_WC & B_WD & A_W-sI\end{bmatrix}\\ &\geq \operatorname{rank}\begin{bmatrix} I_n & 0 & 0\\ 0 & B_W & A_W-\lambda I\end{bmatrix}. \qquad (3.11)\end{aligned}$$

Since (A_W,B_W) is controllable, $\operatorname{rank}\begin{bmatrix} I_n & 0 & 0\\ 0 & B_W & A_W-\lambda I\end{bmatrix} = n+l_yn_w$, and thus

$$\operatorname{rank}\begin{bmatrix} A-sI & B & 0\\ B_WC & B_WD & A_W-sI\end{bmatrix} = n+l_yn_w. \qquad (3.12)$$

Hence $\left(\begin{bmatrix} A & 0\\ B_WC & A_W\end{bmatrix}, \begin{bmatrix} B\\ B_WD\end{bmatrix}\right)$ is stabilizable.

Furthermore, since (A,C) is detectable, it follows that $\left(\begin{bmatrix} A & 0\\ B_WC & A_W\end{bmatrix}, \begin{bmatrix} C & 0\\ 0 & I\end{bmatrix}\right)$ is detectable. Thus, there exist observer-based controllers that stabilize the augmented system (3.2),(3.3). Consider the controller (3.4),(3.5) where the parameters A_c, B_{c1}, B_{c2}, C_c are chosen to stabilize the augmented system (3.2),(3.3). The closed-loop system (3.2),(3.3) and the feedback controller (3.4),(3.5) is asymptotically stable and given by (3.6),(3.7). Furthermore, the closed-loop system (3.6),(3.7) is equivalent to the closed-loop system (2.15)-(2.18) with the controller (2.14), (3.8) where $\hat{x} = \begin{bmatrix}\hat{x}_1\\ \hat{x}_2\end{bmatrix}$. Therefore, we have shown that the closed-loop system (2.15)-(2.18) and (3.8) is asymptotically stable.

Next, we show that $\lim_{t\to\infty} z(t) = 0$. Define

$$T \triangleq \begin{bmatrix} I_n & 0 & 0\\ 0 & 0 & I_{n+l_yn_w}\\ 0 & I_{l_yn_w} & 0\end{bmatrix}, \qquad (3.13)$$

and consider the change of basis $\bar{x} = T\tilde{x}$. In the new basis, the closed-loop system (2.15)-(2.18) and (3.8) has the form

$$\mathcal{D}\bar{x} = \bar{A}\bar{x} + \bar{D}w, \quad z = \bar{E}\bar{x} + E_0w, \qquad (3.14)$$

where

$$\bar{A} \triangleq T\tilde{A}T^{-1} = \begin{bmatrix} A & BC_c & 0\\ B_{c1}C & A_c+B_{c1}DC_c & B_{c2}\\ B_WC & B_WDC_c & A_W\end{bmatrix}, \qquad (3.15)$$

$$\bar{D} \triangleq T\tilde{D} = \begin{bmatrix} D_1\\ B_{c1}D_2\\ B_WD_2\end{bmatrix}, \qquad (3.16)$$

$$\bar{E} \triangleq \tilde{E}T^{-1} = \begin{bmatrix} E_1 & E_2C_c & 0\end{bmatrix}. \qquad (3.17)$$

Next, let $S \triangleq \begin{bmatrix} S_1\\ S_2\end{bmatrix}$ be the unique solution to the Sylvester equation

$$\bar{A}S - SA_w = \bar{D}C_w. \qquad (3.18)$$

Now, it follows from Lemma A.1 in Appendix A with $F_1 = \begin{bmatrix} A & BC_c\\ B_{c1}C & A_c+B_{c1}DC_c\end{bmatrix}$, $F_2 = \begin{bmatrix} 0\\ B_{c2}\end{bmatrix}$, $F_3 = A_w$, $G = B_w$, $H = \begin{bmatrix} C & DC_c\end{bmatrix}$, $J = \begin{bmatrix} D_1C_w\\ B_{c1}D_2C_w\end{bmatrix}$, and $K = D_2C_w$ that $\begin{bmatrix} C & DC_c\end{bmatrix} S_1 = D_2C_w$. Condition *(iv)* implies that there exists $R \in \mathbb{R}^{l_z\times l_y}$ such that $E_1 = RC$, $E_2 = RD$, and $E_0 = RD_2$. Therefore, $\begin{bmatrix} E_1 & E_2C_c\end{bmatrix} S_1 = E_0C_w$, or equivalently

$$\bar{E}S = E_0C_w. \qquad (3.19)$$

Thus, there exists S satisfying (3.18) and (3.19), and Lemma 2.1 implies that $\lim_{t\to\infty} z(t) = 0$. □

Theorem 3.1 provides sufficient conditions for the existence of a linear time invariant controller that stabilizes the continuous-time system (2.1)-(2.3) or the shift operator discrete-time system (2.4)-(2.6) or the delta operator discrete-time system (2.7)-(2.9) and regulates the performance z to zero. The case $z = y$ is considered in [5–7], where it is claimed that the conditions *(i)-(iii)* are necessary and sufficient for continuous-time systems. However it is possible to construct examples for which *(ii)* and *(iii)* are not necessary. For example, consider the single-input, single-output disturbance rejection problem

$$y = G(s)\,(u+w)\,, \quad G(s) \triangleq \frac{s^2+\alpha^2}{p(s)}, \qquad (3.20)$$

where $z = y$, $\alpha \in \mathbb{R}$, deg $p(s) \geq 2$, and $p(s)$ does not have roots at $\pm\jmath\alpha$. Furthermore, assume that w is the output of the linear system (2.13), where A_w has the characteristic polynomial $p_w(s) = s^2+\alpha^2$. Therefore, for every minimal realization of $G(s)$, condition *(iii)* does not hold since $\pm\jmath\alpha$ are eigenvalues of A_w and zeros of $G(s)$. However, consider the feedback controller $u = -\hat{G}(s)y = -\frac{\hat{q}(s)}{\hat{p}(s)}y$ where $\hat{q}(s)$ and $\hat{p}(s)$ are selected so that $\tilde{p}(s) \triangleq p_w(s)\hat{q}(s) + p(s)\hat{p}(s)$ is Hurwitz. Then the final value theorem implies that

$$\begin{aligned}\lim_{t\to\infty} z(t) &= \lim_{s\to 0} s\frac{G(s)}{1+G(s)\hat{G}(s)}\mathcal{L}(w(t))\\ &= \lim_{s\to 0} s\frac{p_w(s)\hat{p}(s)}{\tilde{p}(s)}\frac{q_w(s)}{p_w(s)} = 0, \qquad (3.21)\end{aligned}$$

where $\mathcal{L}(\cdot)$ is the Laplace transform and $\mathcal{L}(w(t)) = \frac{q_w(s)}{p_w(s)}$. In this case, every stabilizing controller drives the performance to zero because the disturbance frequency corresponds to zeros of the open-loop system.

Similar examples can be constructed for the single-input, multi-output disturbance rejection problem with $z = y$, where, for all $\lambda \in \operatorname{spec}(A_w)$,

$$\operatorname{rank}\begin{bmatrix} A-\lambda I & B\\ C & D\end{bmatrix} < n+l_u. \qquad (3.22)$$

In these cases, neither condition *(ii)* nor *(iii)* holds, but since *(i)* holds, it is possible to construct a linear controller that stabilizes the closed-loop system. Furthermore, every stabilizing controller can be shown to satisfy $\lim_{t\to\infty} z(t) = 0$ since the invariant zeros of (A,B,C,D) coincide with the eigenvalues of A_w. More precisely, it suffices to stabilize the closed-loop system because $\lim_{t\to\infty} z(t) = 0$ is trivially true. Therefore, conditions *(ii)* and *(iii)* are not necessary under the assumption made in [5–7] and in the present paper. However, the counterexamples known to us require that the eigenvalues of A_w are also invariant zeros of (A, D_1, E_1, E_0), namely, the invariant zeros of the system from the exogenous signal w to the performance z. To eliminate such counterexamples, assumptions on the invariant zeros of (A, D_1, E_1, E_0) are needed to make conditions *(i)-(iv)* necessary and sufficient. The proof of such a result remains open.

4. Deadbeat Internal Model Control

In this section, a corollary to Theorem 3.1 provides sufficient conditions in discrete-time for the existence of an internal model controller that deadbeats the closed-loop dynamics.

Corollary 4.1. *Consider the discrete-time system (2.4)-(2.6) or (2.7)-(2.9) and assume that the following conditions hold.*

(i) (A, B, C) *is controllable and observable.*
(ii) $l_u \geq l_y$.
(iii) *For all* $\lambda \in \mathrm{spec}\,(A_w)$, *rank* $\left(\begin{bmatrix} A-\lambda I & B \\ C & D \end{bmatrix}\right) = n + \min(l_u, l_y)$.
(iv) *There exists* $R \in \mathbb{R}^{l_z \times l_y}$ *such that* $z = Ry$.

Then there exists a linear time invariant controller (2.14) such that the following conditions hold.

(i) *The equilibrium of the closed-loop system (2.15)-(2.18) is asymptotically stable.*
(ii) $\mathrm{spec}(\tilde{A}) = \begin{cases} 0, & \mathcal{D} = \mathbf{q}, \\ -\frac{1}{h}, & \mathcal{D} = \delta. \end{cases}$
(iii) *For all initial conditions* $\tilde{x}(0)$ *and* $x_w(0)$, *and, for all* $T \geq 2\,(n + n_w l_y)$, $z(T) = 0$ *if* $\mathcal{D} = \mathbf{q}$ *or* $z(hT) = 0$ *if* $\mathcal{D} = \delta$.

Proof. The proof of this result is similar to the proof of Theorem 3.1. Again, we consider the open-loop system (2.7)-(2.9) connected in cascade with l_y copies of the exogenous dynamics A_w given by (3.2)-(3.3). In the proof of Theorem 3.1, we showed that the augmented system (3.2)-(3.3) is detectable and stabilizable. Since, in addition, (A, B) is controllable, it follows that, for all $s \in \mathcal{S}$,

$$\mathrm{rank} \begin{bmatrix} A - sI & B & 0 \\ B_W C & B_W D & A_W - sI \end{bmatrix} = n + l_y n_w, \tag{4.1}$$

and thus $\left(\begin{bmatrix} A & 0 \\ B_W C & A_W \end{bmatrix}, \begin{bmatrix} B \\ B_W D \end{bmatrix}\right)$ is controllable. Since, in addition, (A, C) is observable, it follows that $\left(\begin{bmatrix} A & 0 \\ B_W C & A_W \end{bmatrix}, \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}\right)$ is observable. Thus, there exists an observer-based controller that stabilizes the augmented system (3.2)-(3.3) and yields a closed-loop system with nilpotent dynamics. Using the same argument as in the proof of Theorem 3.1, it follows that there exists a linear time invariant controller (2.14) of order $n+2n_w l_y$, such that the equilibrium of the closed-loop system (2.15)-(2.18) is asymptotically stable, where

$$\mathrm{spec}(\tilde{A}) = \begin{cases} 0, & \mathcal{D} = \mathbf{q}, \\ -\frac{1}{h}, & \mathcal{D} = \delta, \end{cases} \tag{4.2}$$

and, for all initial conditions $\tilde{x}(0)$, $\lim_{t\to\infty} z(t) = 0$.

The closed-loop system (2.15)-(2.18) with exogenous input (2.13) can be written as

$$x_s(t+1) = A_s x_s(t), \quad z(t) = E_s x_s(t), \tag{4.3}$$

where

$$A_s \triangleq \begin{bmatrix} \tilde{A} & \tilde{D} C_w \\ 0 & A_w \end{bmatrix}, \quad E_s \triangleq \begin{bmatrix} \tilde{E} & E_0 C_w \end{bmatrix}, \quad x_s \triangleq \begin{bmatrix} \tilde{x} \\ x_w \end{bmatrix}. \tag{4.4}$$

Since $\lim_{t\to\infty} z(t) = 0$ and $\tilde{A}$ is asymptotically stable, it follows from Lemma 2.1 there exists $S \in \mathbb{R}^{2\left(n+n_w l_y\right)\times n_w}$ such that

$$\tilde{A}S - SA_w = \tilde{D}C_w, \tag{4.5}$$

$$\tilde{E}S = E_0 C_w. \tag{4.6}$$

Now define $Q \triangleq \begin{bmatrix} I & -S \\ 0 & I \end{bmatrix}$, and consider the change of basis

$$\bar{A}_s \triangleq Q^{-1} A_s Q = \begin{bmatrix} \tilde{A} & 0 \\ 0 & A_w \end{bmatrix}, \quad \bar{E}_s \triangleq E_s Q = \begin{bmatrix} \tilde{E} & 0 \end{bmatrix}. \tag{4.7}$$

Then for the shift operator, we have $z(t) = \bar{E}_s \bar{A}_s^t Q^{-1} x_s(0) = \tilde{E}\tilde{A}^t\,[\tilde{x}(0) + Sx_w(0)]$. Since $\tilde{A} \in \mathbb{R}^{2\left(n+n_w l_y\right)\times 2\left(n+n_w l_y\right)}$ is nilpotent, it follows that, for all initial conditions $\tilde{x}(0)$ and $x_w(0)$ and for all $T \geq 2\,(n + n_w l_y)$, $z(T) = 0$.

For the delta operator, we have $z(t) = \bar{E}_s\left(I + h\bar{A}_s\right)^{\frac{t}{h}} Q^{-1}x_s(0) = \tilde{E}\left(I + h\tilde{A}\right)^{\frac{t}{h}}[\tilde{x}(0) + Sx_w(0)]$. Since $\mathrm{spec}(\tilde{A}) = -\frac{1}{h}$, it follows that $\left(I + h\tilde{A}\right) \in \mathbb{R}^{2\left(n+n_w l_y\right)\times 2\left(n+n_w l_y\right)}$ is nilpotent and, for all initial conditions $\tilde{x}(0)$ and $x_w(0)$ and for all $T \geq 2\,(n + n_w l_y)$, $z(hT) = 0$. □

5. Conclusions

This present paper considers internal model control for linear continuous-time and discrete-time systems. Specifically, we presented a linear algebraic solution to the multi-input, multi-output command following and disturbance rejection problem, which simultaneously solves the problem in continuous time and in discrete time with both the shift and delta operators. Lastly, we presented sufficient conditions for the existence of a discrete-time internal model controller that deadbeats the closed-loop dynamics.

Appendix A

Lemma A.1. *Let* $F_1 \in \mathbb{R}^{q\times q}$, $F_2 \in \mathbb{R}^{q\times mp}$, $F_3 \in \mathbb{R}^{m\times m}$, $G \in \mathbb{R}^{m\times 1}$, $H \in \mathbb{R}^{p\times q}$, $J \in \mathbb{R}^{q\times m}$, *and* $K \in \mathbb{R}^{p\times m}$. *Assume that* $\mathrm{spec}\left(\begin{bmatrix} F_1 & F_2 \\ (I_p \otimes G)H & I_p \otimes F_3 \end{bmatrix}\right) \cap \mathrm{spec}(F_3) = \varnothing$, *and the pair* (F_3, G) *is controllable. Let* $S \triangleq \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}$ *be the unique solution to the Sylvester equation*

$$\begin{bmatrix} F_1 & F_2 \\ (I_p \otimes G)H & I_p \otimes F_3 \end{bmatrix}\begin{bmatrix} S_1 \\ S_2 \end{bmatrix} - \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} F_3 = \begin{bmatrix} J \\ (I_p \otimes G)K \end{bmatrix}. \tag{A.1}$$

Then $HS_1 = K$.

Proof. The Sylvester equation (A.1) is equivalent to

$$F_1 S_1 + F_2 S_2 - S_1 F_3 = J, \tag{A.2}$$

$$(I_p \otimes G)HS_1 + (I_p \otimes F_3)S_2 - S_2 F_3 = (I_p \otimes G)K. \tag{A.3}$$

Next let $S_2 = \begin{bmatrix} S_{2,1} \\ \vdots \\ S_{2,p} \end{bmatrix}$, where, for $i = 1, \ldots, p$, $S_{2,i} \in \mathbb{R}^{m\times m}$. It follows from (A.3) that, for $i = 1, \ldots, p$, $F_3 S_{2,i} - S_{2,i} F_3 = G\Lambda_i$, where $\Lambda_i \triangleq e_i\,(K - HS_1)$ and $e_i \triangleq \begin{bmatrix} 0_{1\times(i-1)} & 1 & 0_{1\times(p-i)} \end{bmatrix}$.

Let $M \in \mathbb{R}^{m\times m}$ be such that $\bar{F} \triangleq M^{-1}F_3M$ is in Jordan canonical form. That is, for some $\mu \le m$, $\bar{F} = \operatorname{diag}(\bar{F}_1, \ldots, \bar{F}_\mu)$, where, for $j = 1, \ldots, \mu$, $\bar{F}_j \triangleq \begin{bmatrix} \lambda_j & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & \lambda_j \end{bmatrix} \in \mathbb{R}^{f_j\times f_j}$,and $\lambda_j \in \operatorname{spec}(F_3)$. Furthermore, define

$$\bar{G} \triangleq M^{-1}G = \begin{bmatrix} \bar{G}_1 \\ \vdots \\ \bar{G}_\mu \end{bmatrix}, \tag{A.4}$$

and, for $i = 1, \ldots, p$, define

$$\bar{S}_i \triangleq M^{-1}S_{2,i}M = \begin{bmatrix} \bar{S}_{i,11} & \cdots & \bar{S}_{i,1\mu} \\ \vdots & \ddots & \vdots \\ \bar{S}_{i,\mu 1} & \cdots & \bar{S}_{i,\mu\mu} \end{bmatrix}, \tag{A.5}$$

$$\bar{\Lambda}_i \triangleq \Lambda_i M = \begin{bmatrix} \phi_{i,1} & \cdots & \phi_{i,m} \end{bmatrix}, \tag{A.6}$$

where, for $j = 1, \ldots, \mu$, $\bar{G}_j \in \mathbb{R}^{f_j\times 1}$, and, for $i = 1, \ldots, p$ and for $j, k = 1, \ldots, \mu$, $\bar{S}_{i,jk} \in \mathbb{R}^{f_j\times f_k}$. Therefore, for $i = 1, \ldots, p$, pre-multiplying $F_3S_{2,i} - S_{2,i}F_3 = G\Lambda_i$ by M^{-1} and post-multiplying by M yields

$$\bar{F}\bar{S}_i - \bar{S}_i\bar{F} = \bar{G}\bar{\Lambda}_i. \tag{A.7}$$

Substituting (A.4)-(A.5) into (A.7) and considering only the block diagonal terms implies that, for $i = 1, \ldots, p$ and for $j = 1, \ldots, \mu$,

$$\bar{F}_j\bar{S}_{i,jj} - \bar{S}_{i,jj}\bar{F}_j = \bar{G}_j\bar{\Lambda}_iE_j, \tag{A.8}$$

where $E_j \triangleq \begin{bmatrix} 0_{(f_1+\cdots+f_{j-1})\times f_j} \\ I_{f_j} \\ 0_{(f_{j+1}+\cdots+f_\mu)\times f_j} \end{bmatrix}$, and $f_0 = 0$.

Next, for $i = 1, \ldots, p$ and for $j = 1, \ldots, \mu$, let $\bar{S}_{i,jj} = \begin{bmatrix} s_{ij,1,1} & \cdots & s_{ij,1,f_j} \\ \vdots & \ddots & \vdots \\ s_{ij,f_j,1} & \cdots & s_{ij,f_j,f_j} \end{bmatrix}$, so that

$$\bar{F}_j\bar{S}_{i,jj} - \bar{S}_{i,jj}\bar{F}_j = \begin{bmatrix} s_{ij,2,1} & s_{ij,2,2} - s_{ij,1,1} & s_{ij,2,3} - s_{ij,1,2} & \cdots & s_{ij,2,f_j} - s_{ij,1,f_j-1} \\ s_{ij,3,1} & s_{ij,3,2} - s_{ij,2,1} & s_{ij,3,3} - s_{ij,2,2} & \cdots & s_{ij,3,f_j} - s_{ij,2,f_j-1} \\ s_{ij,4,1} & s_{ij,4,2} - s_{ij,3,1} & s_{ij,4,3} - s_{ij,3,2} & \cdots & s_{ij,4,f_j} - s_{ij,3,f_j-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{ij,f_j,1} & s_{ij,f_j,2} - s_{ij,f_j-1,1} & s_{ij,f_j,3} - s_{ij,f_j-1,2} & \cdots & s_{ij,f_j,f_j} - s_{ij,f_j-1,f_j-1} \\ 0 & -s_{ij,f_j,1} & -s_{ij,f_j,2} & \cdots & -s_{ij,f_j,f_j-1} \end{bmatrix}. \tag{A.9}$$

For $j = 1, \ldots, \mu$, let $g_j \in \mathbb{R}$ denote the last entry of $\bar{G}_j$. For $i = 1, \ldots, p$ and $j = 1, \ldots, \mu$, combining (A.6), (A.8), and (A.9) yields

$$\begin{bmatrix} s_{ij,2,1} & s_{ij,2,2} - s_{ij,1,1} & s_{ij,2,3} - s_{ij,1,2} & \cdots & s_{ij,2,f_j} - s_{ij,1,f_j-1} \\ s_{ij,3,1} & s_{ij,3,2} - s_{ij,2,1} & s_{ij,3,3} - s_{ij,2,2} & \cdots & s_{ij,3,f_j} - s_{ij,2,f_j-1} \\ s_{ij,4,1} & s_{ij,4,2} - s_{ij,3,1} & s_{ij,4,3} - s_{ij,3,2} & \cdots & s_{ij,4,f_j} - s_{ij,3,f_j-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{ij,f_j,1} & s_{ij,f_j,2} - s_{ij,f_j-1,1} & s_{ij,f_j,3} - s_{ij,f_j-1,2} & \cdots & s_{ij,f_j,f_j} - s_{ij,f_j-1,f_j-1} \\ 0 & -s_{ij,f_j,1} & -s_{ij,f_j,2} & \cdots & -s_{ij,f_j,f_j-1} \end{bmatrix} = \begin{bmatrix} \sharp \\ \vdots \\ \sharp \\ g_j \end{bmatrix} \begin{bmatrix} \phi_{i,1+f_1+\cdots+f_{j-1}} & \cdots & \phi_{i,f_1+\cdots+f_j} \end{bmatrix}, \tag{A.10}$$

where $\sharp$ denotes an inconsequential entry.

Now since (F_3, G) is controllable, it follows that $(\bar{F}, \bar{G})$ is controllable, and, for $j = 1, \ldots, \mu$, $(\bar{F}_j, \bar{G}_j)$ is controllable. Therefore, for $j = 1, \ldots, \mu$, $g_j \neq 0$.

First, consider $j = 1$. Since $g_j \neq 0$, inspecting the $(f_j, 1)$ entry of (A.10) yields that for $i = 1, \ldots, p$, $\phi_{i,1+f_1+\cdots+f_{j-1}} = 0$ and thus, for $k = 2, \ldots, f_j$, $s_{ij,k,1} = 0$. Now since $g_j \neq 0$ and $s_{ij,f_j,1} = 0$, inspecting the $(f_j, 2)$ entry of (A.10) yields that for $i = 1, \ldots, p$, $\phi_{i,1+f_1+\cdots+f_{j-1}+1} = 0$ and thus, for $k = 3, \ldots, f_j$, $s_{ij,k,2} = 0$. Now since $g_j \neq 0$ and $s_{ij,f_j,2} = 0$, inspecting the $(f_j, 3)$ entry of (A.10) yields that for $i = 1, \ldots, p$, $\phi_{i,1+f_1+\cdots+f_{j-1}+2} = 0$ and thus for $k = 4, \ldots, f_j$, $s_{ij,k,3} = 0$. Continuing in this manner yields, for $i = 1, \ldots, p$, $\begin{bmatrix} \phi_{i,1+f_1+\cdots+f_{j-1}} & \cdots & \phi_{i,f_1+\cdots+f_j} \end{bmatrix} = 0$. Repeating this for $j = 2, \ldots, \mu$, yields for $i = 1, \ldots, p$, $\bar{\Lambda}_i = 0$, which implies that $HS_1 - K = 0$. □

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