Gain-Constrained Kalman Filtering for Linear and Nonlinear Systems

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***Abstract*—This paper considers the state-estimation problem with a constraint on the data-injection gain. Special cases of this problem include the enforcing of a linear equality constraint in the state vector, the enforcing of unbiased estimation for systems with unknown inputs, and simplification of the estimator structure for large-scale systems. Both the one-step gain-constrained Kalman predictor and the two-step gain-constrained Kalman filter are presented. The latter is extended to the nonlinear case, yielding the gain-constrained unscented Kalman filter. Two illustrative examples are presented.**

***Index Terms*—Constrained gain, Kalman filter, state estimation, unscented Kalman filter.**

I. Introduction

In classical state estimation, the standard data injection gain is unconstrained in the sense that all measurement residuals are potentially used to directly update all of the state estimates. In fact, it is possible to restrict the form of the data-injection gain. Three distinct motivations for such a restriction can be found in the literature. First, in [6], [8], [9], [16], and [18], the data-injection gain is restricted so that the state estimates are unbiased despite the fact that arbitrary (for example, deterministic or nonzero-mean) unknown exogenous inputs are present. Likewise, in [4] and [5], the data-injection gain is restricted to simplify the estimator structure so as to facilitate multiprocessor implementation for applications involving large-scale systems such as discretized partial differential equations, as well as to handle partial or complete sensor outage. Finally, in [10], the data-injection gain is restricted to obtain state estimates satisfying a linear equality constraint. Alternative state-estimation algorithms for enforcing equality constraints in the state vector are found in [13], [17], [24], and [28]–[31], while the inequality-constrained case is addressed in [20], [22], [25], [28], and [32].

The present paper builds on [5], [6], [10], [16], and [18] by developing a general approach to gain-constrained state estimation, which includes the results of [5], [6], [10], [16], and [18] as special cases. To facilitate implementation to nonlinear systems, we first develop these results for linear systems. We consider both the one-step gain-constrained Kalman predictor (GCKP) as well as the two-step gain-constrained Kalman filter (GCKF). Then we extend GCKF to the nonlinear case based on the unscented Kalman filter [14], a specialized sigma-point filter [33], yielding the gain-constrained unscented Kalman filter (GCUKF). The present paper is based on research in [28].

This paper is organized as follows. Section II develops GCKF. In Section III, the classical Kalman filter [12], the projected Kalman filter by gain projection [10], the unbiased Kalman filter with unknown inputs [6], [16], [18], and the spatially constrained Kalman filter [5] are presented as special cases of GCKF. Section IV presents GCUKF for nonlinear systems. In Section V, the GCKP equations are derived. Finally, two illustrative examples are presented in Section VI.

II. Gain-Constrained Kalman Filter

Consider the stochastic linear discrete-time dynamic system

$$x_k = A_{k-1}x_{k-1} + B_{k-1}u_{k-1} + w_{k-1} \tag{2.1}$$

$$y_k = C_k x_k + v_k \tag{2.2}$$

where $A_{k-1} \in \mathbb{R}^{n\times n}$, $B_{k-1} \in \mathbb{R}^{n\times p}$, and $C_k \in \mathbb{R}^{m\times n}$ are known matrices. Assume that, for all $k \geq 1$, the input $u_{k-1} \in \mathbb{R}^p$ is known and the output $y_k \in \mathbb{R}^m$ is measured. The process noise $w_{k-1} \in \mathbb{R}^n$, which represents unknown input disturbances, and the measurement noise $v_k \in \mathbb{R}^m$, concerning inaccuracies in the measurements, are assumed to be white, Gaussian, zero-mean, and mutually independent with known covariance matrices Q_{k-1} and R_k, respectively. The initial state vector $x_0 \in \mathbb{R}^n$ is assumed to be Gaussian with initial estimate $\hat{x}_{0|0}$ and error-covariance $P_{0|0}^{xx} \triangleq \mathcal{E}[(x_0 - \hat{x}_{0|0})(x_0 - \hat{x}_{0|0})^{\mathrm{T}}]$, where $\mathcal{E}[\cdot]$ denotes expected value, and, for all $k \geq 0$, x_0 is assumed to be uncorrelated with w_k and v_k.

For the system (2.1) and (2.2), we consider a two-step filter with *forecast* step

$$\hat{x}_{k|k-1} = A_{k-1}\hat{x}_{k-1|k-1} + B_{k-1}u_{k-1} \tag{2.3}$$

$$\hat{y}_{k|k-1} = C_k \hat{x}_{k|k-1} \tag{2.4}$$

and *data-assimilation* step

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + L_k(y_k - \hat{y}_{k|k-1}) \tag{2.5}$$

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where the filter gain $L_k \in \mathbb{R}^{n\times m}$ minimizes the cost function

$$J_k(L_k) \triangleq \mathcal{E}\left[(x_k - \hat{x}_{k|k})^{\mathrm{T}} W_k (x_k - \hat{x}_{k|k})\right] \tag{2.6}$$

subject to

$$D_k L_k E_k = F_k \tag{2.7}$$

where, for all $k \geq 1$, the positive-definite weighting matrix $W_k \in \mathbb{R}^{n\times n}$ determines how much the state components should be updated relative to each other. The matrix $D_k \in \mathbb{R}^{q\times n}$ is assumed to be right invertible, $E_k \in \mathbb{R}^{m\times r}$ is assumed to be left invertible, and $F_k \in \mathbb{R}^{q\times r}$. These conditions imply that $q \leq n$ and $r \leq m$, respectively. The constraint (2.7), which is absent in the classical Kalman filter (KF) [12], is used to enforce special properties in the state estimates. The notation $\hat{z}_{k|k-1}$ indicates an estimate of z_k at time k based on information available up to and including time $k-1$. Likewise, $\hat{z}_{k|k}$ indicates an estimate of z_k at time k using information available up to and including time k. Model information is used during the forecast step, while measurement data are injected into the estimates during the data-assimilation step.

Next, define the *forecast error* $e_{k|k-1}$ and the *innovation* $\nu_{k|k-1}$ by

$$e_{k|k-1} \triangleq x_k - \hat{x}_{k|k-1} \tag{2.8}$$
$$\nu_{k|k-1} \triangleq y_k - \hat{y}_{k|k-1} \tag{2.9}$$

as well as the *forecast error covariance* $P^{xx}_{k|k-1}$, *innovation covariance* $P^{yy}_{k|k-1}$, and *cross covariance* $P^{xy}_{k|k-1}$ by

$$P^{xx}_{k|k-1} \triangleq \mathcal{E}\left[e_{k|k-1} e^{\mathrm{T}}_{k|k-1}\right] \tag{2.10}$$
$$P^{yy}_{k|k-1} \triangleq \mathcal{E}\left[\nu_{k|k-1} \nu^{\mathrm{T}}_{k|k-1}\right] \tag{2.11}$$
$$P^{xy}_{k|k-1} \triangleq \mathcal{E}\left[e_{k|k-1} \nu^{\mathrm{T}}_{k|k-1}\right]. \tag{2.12}$$

It follows from (2.1)–(2.4) that

$$e_{k|k-1} = A_{k-1} e_{k-1|k-1} + w_{k-1} \tag{2.13}$$
$$\nu_{k|k-1} = C_k e_{k|k-1} + v_k \tag{2.14}$$

where $e_{k|k}$ is the *data-assimilation error* defined by

$$e_{k|k} \triangleq x_k - \hat{x}_{k|k} \tag{2.15}$$

whose *data-assimilation error covariance* is

$$P^{xx}_{k|k} \triangleq \mathcal{E}\left[e_{k|k} e^{\mathrm{T}}_{k|k}\right]. \tag{2.16}$$

Finally, it follows from (2.4), (2.5), (2.8), and (2.15) that

$$e_{k|k} = (I_{n\times n} - L_k C_k) e_{k|k-1} - L_k v_k. \tag{2.17}$$

The following lemma will be useful.

Lemma 2.1: The forecast error given by (2.8) satisfies

$$\mathcal{E}\left[e_{k|k-1} v^{\mathrm{T}}_k\right] = 0 \tag{2.18}$$

and the data-assimilation error (2.15) satisfies

$$\mathcal{E}\left[e_{k|k} w^{\mathrm{T}}_k\right] = 0. \tag{2.19}$$

Proposition 2.1: For the filter (2.3)–(2.5), the data-assimilation error covariance $P^{xx}_{k|k}$ is updated using

$$P^{xx}_{k|k} = P^{xx}_{k|k-1} - L_k \left(P^{xy}_{k|k-1}\right)^{\mathrm{T}} - P^{xy}_{k|k-1} L^{\mathrm{T}}_k + L_k P^{yy}_{k|k-1} L^{\mathrm{T}}_k \tag{2.20}$$

where

$$P^{xx}_{k|k-1} = A_{k-1} P^{xx}_{k-1|k-1} A^{\mathrm{T}}_{k-1} + Q_{k-1} \tag{2.21}$$
$$P^{yy}_{k|k-1} = C_k P^{xx}_{k|k-1} C^{\mathrm{T}}_k + R_k \tag{2.22}$$
$$P^{xy}_{k|k-1} = P^{xx}_{k|k-1} C^{\mathrm{T}}_k. \tag{2.23}$$

Proof: It follows from (2.13) and (2.14) and Lemma 2.1 that (2.10)–(2.12) are, respectively, given by (2.21)–(2.23). Moreover, using Lemma 2.1, (2.17) implies that (2.16) is given by

$$P^{xx}_{k|k} = (I_{n\times n} - L_k C_k) P^{xx}_{k|k-1} (I_{n\times n} - L_k C_k)^{\mathrm{T}} + L_k R_k L^{\mathrm{T}}_k$$

which, by using (2.22) and (2.23), yields (2.20). ■

Next, using (2.15) and (2.16) in (2.6) yields

$$J_k(L_k) = \operatorname{tr}\left(P^{xx}_{k|k} W_k\right). \tag{2.24}$$

Assume that, for all $k \geq 1$, $P^{yy}_{k|k-1}$ is positive definite. For convenience, we define

$$D^{\mathrm{R}}_k \triangleq D^{\mathrm{T}}_k \left(D_k D^{\mathrm{T}}_k\right)^{-1} \tag{2.25}$$
$$E^{\mathrm{L}}_k \triangleq \left(E^{\mathrm{T}}_k E_k\right)^{-1} E^{\mathrm{T}}_k \tag{2.26}$$

and

$$\Pi_k \triangleq W^{-1}_k D^{\mathrm{T}}_k \left(D_k W^{-1}_k D^{\mathrm{T}}_k\right)^{-1} D_k \tag{2.27}$$
$$\Omega_k \triangleq E_k \left[E^{\mathrm{T}}_k \left(P^{yy}_{k|k-1}\right)^{-1} E_k\right]^{-1} E^{\mathrm{T}}_k \left(P^{yy}_{k|k-1}\right)^{-1} \tag{2.28}$$
$$K_k \triangleq P^{xy}_{k|k-1} \left(P^{yy}_{k|k-1}\right)^{-1}. \tag{2.29}$$

Note that Π_k and Ω_k are oblique projectors, that is, $\Pi^2_k = \Pi_k$ and $\Omega^2_k = \Omega_k$, but are not necessarily symmetric. If $W_k = I_{n\times n}$, then Π_k is an orthogonal projector. Furthermore, if D_k is square, then $\Pi_k = I_{n\times n}$; likewise, if E_k is square, then $\Omega_k = I_{m\times m}$. Also

$$\Pi_k D^{\mathrm{R}}_k = W^{-1}_k D^{\mathrm{T}}_k \left(D_k W^{-1}_k D^{\mathrm{T}}_k\right)^{-1}, \tag{2.30}$$
$$E^{\mathrm{L}}_k \Omega_k = \left[E^{\mathrm{T}}_k \left(P^{yy}_{k|k-1}\right)^{-1} E_k\right]^{-1} E^{\mathrm{T}}_k \left(P^{yy}_{k|k-1}\right)^{-1}. \tag{2.31}$$

Furthermore, note that K_k is equal to the classical Kalman gain [12].

Proposition 2.2: The gain L_k that minimizes (2.24) and satisfies (2.7) is given by

$$L_k = K_k - \Pi_k \left(K_k - D^{\mathrm{R}}_k F_k E^{\mathrm{L}}_k\right) \Omega_k \tag{2.32}$$

where K_k is given by (2.29) with $P^{xx}_{k|k-1}$, $P^{yy}_{k|k-1}$, and $P^{xy}_{k|k-1}$ given by (2.21), (2.22), and (2.23), respectively. Furthermore, $P^{xx}_{k|k}$ in (2.20) is given by the Riccati equation

$$\begin{aligned} P^{xx}_{k|k} =& P^{xx}_{k|k-1} - P^{xy}_{k|k-1}\left(P^{yy}_{k|k-1}\right)^{-1}\left(P^{xy}_{k|k-1}\right)^{\mathrm{T}} \\ &+ \left(\Pi_k D^{\mathrm{R}}_k F_k E^{\mathrm{L}}_k \Omega_k\right) P^{yy}_{k|k-1} \left(\Pi_k D^{\mathrm{R}}_k F_k E^{\mathrm{L}}_k \Omega_k\right)^{\mathrm{T}} \\ &+ \left[\Pi_k P^{xy}_{k|k-1}\left(P^{yy}_{k|k-1}\right)^{-1}\Omega_k\right] P^{yy}_{k|k-1} \\ &\times \left[\Pi_k P^{xy}_{k|k-1}\left(P^{yy}_{k|k-1}\right)^{-1}\Omega_k\right]^{\mathrm{T}} \\ &- \Pi_k\left(\Delta_k + \Delta_k^{\mathrm{T}}\right)\Pi_k^{\mathrm{T}} \end{aligned} \quad (2.33)$$

where

$$\Delta_k \triangleq P^{xy}_{k|k-1}\left(P^{yy}_{k|k-1}\right)^{-1}\Omega_k P^{yy}_{k|k-1}\Omega_k^{\mathrm{T}}\left(D^{\mathrm{R}}_k F_k E^{\mathrm{L}}_k\right)^{\mathrm{T}}. \quad (2.34)$$

Proof: Define the Lagrangian

$$\mathcal{L}_k(L_k) = J_k(L_k) + 2\mathrm{tr}\left[(D_k L_k E_k - F_k)\Lambda^{\mathrm{T}}\right] \quad (2.35)$$

where $\Lambda \in \mathbb{R}^{q\times r}$ is the Lagrange multiplier accounting for the qr constraints in (2.7). The necessary conditions for a minimizer L_k are given by

$$\frac{\partial \mathcal{L}_k}{\partial L_k} = 2\left(-W_k P^{xy}_{k|k-1} + W_k L_k P^{yy}_{k|k-1} + D_k^{\mathrm{T}}\Lambda E_k^{\mathrm{T}}\right) = 0_{n\times m} \quad (2.36)$$

$$\frac{\partial \mathcal{L}_k}{\partial \Lambda} = 2(D_k L_k E_k - F_k) = 0_{q\times r}. \quad (2.37)$$

Note that (2.37) yields (2.7).

In (2.36), using the fact that W_k and $P^{y}_{k|k-1}$ are positive definite yields

$$-P^{xy}_{k|k-1}\left(P^{yy}_{k|k-1}\right)^{-1} + L_k + W_k^{-1}D_k^{\mathrm{T}}\Lambda E_k^{\mathrm{T}}\left(P^{yy}_{k|k-1}\right)^{-1} = 0_{n\times m}. \quad (2.38)$$

Pre-multiplying and post-multiply (2.38) by D_k and E_k, respectively, and substituting (2.7) and (2.29) into the resulting expression yields

$$-D_k K_k E_k + F_k + D_k W_k^{-1} D_k^{\mathrm{T}}\Lambda E_k^{\mathrm{T}}\left(P^{yy}_{k|k-1}\right)^{-1} E_k = 0_{q\times r}$$

which implies

$$\begin{aligned} \Lambda =& \left(D_k W_k^{-1} D_k^{\mathrm{T}}\right)^{-1} D_k K_k E_k \left[E_k^{\mathrm{T}}\left(P^{yy}_{k|k-1}\right)^{-1} E_k\right]^{-1} \\ &- \left(D_k W_k^{-1} D_k^{\mathrm{T}}\right)^{-1} F_k \left[E_k^{\mathrm{T}}\left(P^{yy}_{k|k-1}\right)^{-1} E_k\right]^{-1}. \end{aligned} \quad (2.39)$$

Using (2.25)–(2.31) and substituting (2.39) into (2.38) yields (2.32).

Finally, substituting (2.29) and (2.32) into (2.20) yields (2.33). ■

Proposition 2.3: L_k given by (2.32) is the unique global minimizer of $J_k(L_k)$ (2.24) restricted to the convex constraint set defined by (2.27).

Proof: It follows from [3, p. 286] that, for all $0 < \alpha < 1$, $A_1, A_2 \in \mathbb{R}^{n\times m}$ such that $A_1 \neq A_2$, and positive-definite $B \in \mathbb{R}^{m\times m}$, $\mathrm{tr}[\alpha(1-\alpha)(A_1 - A_2)B(A_1 - A_2)^{\mathrm{T}}] > 0$. Hence, for $A \in \mathbb{R}^{n\times n}$, the mapping $A \rightarrow \mathrm{tr}(ABA^{\mathrm{T}})$ is strictly convex. It thus follows that $J_k(L_k)$ is strictly convex, and hence L_k is the unique global minimizer of $J_k(L_k)$ restricted to (2.7). ■

The *gain-constrained Kalman filter* (GCKF) is expressed in two steps, namely, the forecast step given by (2.3), (2.21), (2.4), (2.22), (2.23), and the data-assimilation step given by (2.25)–(2.29), (2.32), (2.5), and (2.20).

Note that the GCKF equations are identical to the classical KF equations except for the gain expression (2.32). Moreover, note that the first two terms on the right-hand side of (2.33) correspond to the data-assimilation error covariance of KF.

III. Special Cases

A. Kalman Filter

Assume that $D_k = I_{n\times n}$, $E_k = I_{m\times m}$, and $F_k = K_k$, where K_k is given by (2.29). In this case, the optimal gain L_k given by (2.32), which minimizes (2.24) subject to (2.7), is K_k, which is the classical Kalman gain [12]. Furthermore, (2.33) is equal to the Riccati equation of KF, that is

$$P^{xx}_{k|k} = P^{xx}_{k|k-1} - P^{xy}_{k|k-1}\left(P^{yy}_{k|k-1}\right)^{-1}\left(P^{xy}_{k|k-1}\right)^{\mathrm{T}}. \quad (3.1)$$

B. Condition $D_k = I_{n\times n}$

Assume that $D_k = I_{n\times n}$ such that the gain constraint (2.7) is expressed as

$$L_k E_k = F_k. \quad (3.2)$$

In this case, (2.27) yields $\Pi_k = I_{n\times n}$ and (2.25) yields $D^{\mathrm{R}}_k = I_{n\times n}$. Hence, it follows from (2.32) that the optimal gain L_k that minimizes (2.24) and satisfies (3.2) is given by

$$L_k = K_k \Omega_{k\perp} + F_k E^{\mathrm{L}}_k \Omega_k \quad (3.3)$$

where E^{L}_k is given by (2.26), Ω_k is given by (2.28), the complementary oblique projector is given by

$$\Omega_{k\perp} \triangleq I_{m\times m} - \Omega_k, \quad (3.4)$$

and $P^{xx}_{k|k}$ in (2.20) is given by the Riccati equation

$$\begin{aligned} P^{xx}_{k|k} =& P^{xx}_{k|k-1} - P^{xy}_{k|k-1}\left(P^{yy}_{k|k-1}\right)^{-1}\left(P^{xy}_{k|k-1}\right)^{\mathrm{T}} \\ &+ \left(F_k E^{\mathrm{L}}_k \Omega_k\right) P^{yy}_{k|k-1}\left(F_k E^{\mathrm{L}}_k \Omega_k\right)^{\mathrm{T}} \\ &+ \left[P^{xy}_{k|k-1}\left(P^{yy}_{k|k-1}\right)^{-1}\Omega_k\right] P^{yy}_{k|k-1} \\ &\times \left[P^{xy}_{k|k-1}\left(P^{yy}_{k|k-1}\right)^{-1}\Omega_k\right]^{\mathrm{T}} \\ &- \left(\Delta_{1,k} + \Delta_{1,k}^{\mathrm{T}}\right) \end{aligned} \quad (3.5)$$

where

$$\Delta_{1,k} \triangleq P^{xy}_{k|k-1}\left(P^{yy}_{k|k-1}\right)^{-1}\Omega_k P^{yy}_{k|k-1}\Omega_k^{\mathrm{T}}\left(F_k E^{\mathrm{L}}_k\right)^{\mathrm{T}}. \quad (3.6)$$

Recall that, in this particular case, as well as in Section III-A, the minimizer of (2.24) is independent of W_k.

C. Condition $E_k = I_{m\times m}$

Assume that $E_k = I_{m\times m}$ such that the gain constraint (2.7) is expressed as

$$D_k L_k = F_k. \tag{3.7}$$

In this case, (2.28) yields $\Omega_k = I_{m\times m}$ and (2.26) yields $E_k^{\mathrm{L}} = I_{m\times m}$. Hence, it follows from (2.32) that the optimal gain L_k that minimizes (2.24) and satisfies (3.7) is given by

$$L_k = \Pi_{k\perp} K_k + \Pi_k D_k^{\mathrm{R}} F_k \tag{3.8}$$

where D_k^{R} is given by (2.25), Π_k is given by (2.27), the complementary oblique projector is given by

$$\Pi_{k\perp} \triangleq I_{n\times n} - \Pi_k \tag{3.9}$$

and $P_{k|k}^{xx}$ in (2.20) is given by

$$\begin{aligned} P_{k|k}^{xx} = {} & P_{k|k-1}^{xx} - P_{k|k-1}^{xy}\left(P_{k|k-1}^{yy}\right)^{-1}\left(P_{k|k-1}^{xy}\right)^{\mathrm{T}} \\ & + \left(\Pi_k D_k^{\mathrm{R}} F_k\right) P_{k|k-1}^{yy}\left(\Pi_k D_k^{\mathrm{R}} F_k\right)^{\mathrm{T}} \\ & + \Pi_k P_{k|k-1}^{xy}\left(P_{k|k-1}^{yy}\right)^{-1}\left(P_{k|k-1}^{xy}\right)^{\mathrm{T}}\Pi_k^{\mathrm{T}} \\ & - \Pi_k\left\{\left[P_{k|k-1}^{xy}\left(D_k^{\mathrm{R}} F_k\right)^{\mathrm{T}}\right]\right. \\ & \left. + \left[P_{k|k-1}^{xy}\left(D_k^{\mathrm{R}} F_k\right)^{\mathrm{T}}\right]^{\mathrm{T}}\right\}\Pi_k^{\mathrm{T}}. \end{aligned} \tag{3.10}$$

D. Kalman Filter With Equality State Constraint

Consider the system (2.1) and (2.2), whose state vector is known to satisfy the linear equality constraint

$$\tilde{D}_{k-1} x_k = \tilde{d}_{k-1} \tag{3.11}$$

where $\tilde{D}_{k-1} \in \mathbb{R}^{s\times n}$ and $\tilde{d}_{k-1} \in \mathbb{R}^s$ are assumed to be known. Assume that $\mathrm{rank}(\tilde{D}_{k-1}) = s$. We consider the two-step estimator whose forecast step is given by (2.3) and (2.4) and whose data-assimilation step is given by (2.5). In (2.24), assume that

$$W_k = I_{n\times n}. \tag{3.12}$$

We look for a gain L_k in (2.5) that minimizes (2.24) subject to

$$\tilde{D}_{k-1}\hat{x}_{k|k} = \tilde{d}_{k-1} \tag{3.13}$$

where $\hat{x}_{k|k}$ is given by (2.5). By using (2.5), it follows that (3.13) can be written in the form of the gain constraint (2.7), where

$$D_k = \tilde{D}_{k-1} \tag{3.14}$$
$$E_k = y_k - \hat{y}_{k|k-1} \tag{3.15}$$
$$F_k = \tilde{d}_{k-1} - \tilde{D}_{k-1}\hat{x}_{k|k-1} \tag{3.16}$$

where $\hat{x}_{k|k-1}$ is given by (2.3) and $\hat{y}_{k|k-1}$ is given by (2.4). Therefore, L_k is given by (2.32). Substituting (3.12) and (3.14)–(3.16) into (2.32), we obtain

$$\begin{aligned} L_k = {} & K_k + \tilde{D}_{k-1}^{\mathrm{T}}\left(\tilde{D}_{k-1}\tilde{D}_{k-1}^{\mathrm{T}}\right)^{-1}\left(\tilde{d}_{k-1} - \tilde{D}_{k-1}\hat{x}_{k|k}\right) \\ & \times\left[\left(y_k - \hat{y}_{k|k-1}\right)^{\mathrm{T}}\left(P_{k|k-1}^{yy}\right)^{-1}\left(y_k - \hat{y}_{k|k-1}\right)\right]^{-1} \\ & \times\left(y_k - \hat{y}_{k|k-1}\right)^{\mathrm{T}}\left(P_{k|k-1}^{yy}\right)^{-1} \end{aligned} \tag{3.17}$$

which is the gain of the projected Kalman filter by gain projection (PKF-GP) presented in [10].

E. Unbiased Kalman Filter With Unknown Inputs

Consider the stochastic linear discrete-time dynamic system

$$x_k = A_{k-1}x_{k-1} + B_{k-1}u_{k-1} + G_{k-1}d_{k-1} + w_{k-1} \tag{3.18}$$
$$y_k = C_k x_k + H_k\tilde{d}_k + v_k \tag{3.19}$$

where, for all $k \geq 1$, $G_{k-1} \in \mathbb{R}^{n\times s}$ and $H_k \in \mathbb{R}^{m\times t}$ are known matrices. No assumptions are made on the unknown input vectors $d_{k-1} \in \mathbb{R}^s$ and $\tilde{d}_k \in \mathbb{R}^t$. Note that, if we set $s = t$ and $d_k = \tilde{d}_k$, then we have the system with direct feedthrough studied in [6] and [9].

For the system (3.18) and (3.19), consider the two-step estimator whose forecast step is given by (2.3) and (2.4) and whose data-assimilation step is given by (2.5). In this case, $e_{k|k}$ defined by (2.17) is given by

$$\begin{aligned} e_{k|k} = {} & (I_{n\times n} - L_k C_k)A_{k-1}e_{k-1|k-1} + (I_{n\times n} - L_k C_k)w_{k-1} \\ & - L_k v_k + (I_{n\times n} - L_k C_k)G_{k-1}d_{k-1} - L_k H_k\tilde{d}_k. \end{aligned} \tag{3.20}$$

If L_k satisfies

$$(I_{n\times n} - L_k C_k)G_{k-1} = 0_{n\times s} \tag{3.21}$$
$$L_k H_k = 0_{n\times t} \tag{3.22}$$

then $\hat{x}_{k|k}$ in (2.5) is *unbiased*, that is, $\mathcal{E}[e_{k|k}] = 0_{n\times 1}$. Note that (3.21) and (3.22) can be written in the form (3.2), where

$$E_k = [C_k G_{k-1} \quad H_k] \tag{3.23}$$
$$F_k = [G_{k-1} \quad 0_{n\times t}]. \tag{3.24}$$

Substituting (3.23) and (3.24) into (3.3), we obtain the unbiased Kalman filter with unknown inputs (UnbKF-UI) [6], [9], whose error covariance is given by (3.5).

We assume that $E_k \in \mathbb{R}^{m\times(s+t)}$ in (3.23) is left invertible, which is equivalent to the conditions i) $\mathrm{rank}(C_k G_{k-1}) = s$, ii) $\mathrm{rank}(H_k) = t$, and iii) the columns of H_k are linearly independent of the columns of $C_k G_{k-1}$. These conditions are equivalent to $\mathrm{rank}(E_k) = s + t$, which is shown to be a sufficient condition for the unbiasedness of the estimator (2.5) in [6, Lemma 2]. Note that $m \geq s + t$ is necessary for (iii) to hold.

1) Condition $t = 0$: For the system (3.18), (3.19), assume that $t = 0$. In this case, $e_{k|k}$ is given by

$$e_{k|k} = (I_{n\times n} - L_kC_k)A_{k-1}e_{k-1|k-1} + (I_{n\times n} - L_kC_k)w_{k-1} - L_kv_k + (I_{n\times n} - L_kC_k)G_{k-1}d_{k-1} \quad (3.25)$$

while, if (3.21) is satisfied, then $\hat{x}_{k|k}$ in (2.5) is unbiased. Thus, the gain that minimizes (2.24) subject to (3.2) is given by (3.33), where

$$E_k = C_kG_{k-1} \quad (3.26)$$

$$F_k = G_{k-1} \quad (3.27)$$

where, since $\mathrm{rank}(C_kG_{k-1}) = s$, E_k given by (3.26) is left invertible. The error covariance is given by (3.5). The resulting estimator for this case is presented in [16].

2) Condition $s = 0$: For the system (3.18) and (3.19), consider the case $s = 0$. In this case, $e_{k|k}$ is given by

$$e_{k|k} = (I_{n\times n} - L_kC_k)A_{k-1}e_{k-1|k-1} + (I_{n\times n} - L_kC_k)w_{k-1} - L_kv_k - L_kH_k\tilde{d}_k. \quad (3.28)$$

If (3.22) is satisfied, then $\hat{x}_{k|k}$ is unbiased. The gain that minimizes (2.24) and satisfies (3.2) is given by (3.3), where

$$E_k = H_k \quad (3.29)$$

$$F_k = 0_{n\times t} \quad (3.30)$$

where, since $\mathrm{rank}(H_k) = t$, E_k given by (3.29) is left invertible. The error covariance is given by (3.5). The resulting estimator is presented in [18].

F. Kalman Filter With Constrained Output Injection

For the system (2.1) and (2.2), consider the two-step estimator whose forecast step is given by (2.3) and (2.4) and whose data-assimilation step is given by (2.5). We look for a gain L_k that minimizes (2.24) and constrains the estimator so that only estimates in the range of Γ_k are directly updated during data assimilation. Assume that $q < n$ and let $\Gamma_k \in \mathbb{R}^{n\times(n-q)}$ be a full column rank matrix. For example, Γ_k can have the form $\Gamma_k = \left[\begin{smallmatrix} I_{(n-q)\times(n-q)} \\ 0_{q\times(n-q)}\end{smallmatrix}\right]$. If $\Gamma_k = I_{n\times n}$, we have the classical KF. Also, let $\tilde{\Gamma}_k \in \mathbb{R}^{n\times q}$ be such that $[\Gamma_k \; \tilde{\Gamma}_k]$ is nonsingular and

$$\tilde{\Gamma}_k^{\mathrm{T}}\Gamma_k = 0_{q\times(n-q)}. \quad (3.31)$$

That is, we seek a gain L_k that satisfies (3.7), where

$$D_k = \tilde{\Gamma}_k^{\mathrm{T}}, \quad (3.32)$$

$$F_k = 0_{q\times m}. \quad (3.33)$$

Using (3.32) and (3.33), it follows from (3.8) that

$$L_k = \left[I_{n\times n} - W_k^{-1}\tilde{\Gamma}_k\left(\tilde{\Gamma}_k^{\mathrm{T}}W_k^{-1}\tilde{\Gamma}_k\right)^{-1}\tilde{\Gamma}_k^{\mathrm{T}}\right]K_k \quad (3.34)$$

and $P_{k|k}^{xx}$ is given by (3.10). Given (3.31) we have that

$$\Theta_k\left(W_k^{-1}\tilde{\Gamma}_k\left(\tilde{\Gamma}_k^{\mathrm{T}}W_k^{-1}\tilde{\Gamma}_k\right)^{-1}\tilde{\Gamma}_k^{\mathrm{T}} + \Gamma_k\left(\Gamma_k^{\mathrm{T}}W_k\Gamma_k\right)^{-1}\Gamma_k^{\mathrm{T}}W_k\right) = \Theta_k.$$

Since $\Theta_k \triangleq \left[\begin{smallmatrix}\tilde{\Gamma}_k^{\mathrm{T}} \\ \Gamma_k^{\mathrm{T}}W_k\end{smallmatrix}\right]$ is nonsingular, it follows that

$$W_k^{-1}\tilde{\Gamma}_k\left(\tilde{\Gamma}_k^{\mathrm{T}}W_k^{-1}\tilde{\Gamma}_k\right)^{-1}\tilde{\Gamma}_k^{\mathrm{T}} + \Gamma_k\left(\Gamma_k^{\mathrm{T}}W_k\Gamma_k\right)^{-1}\Gamma_k^{\mathrm{T}}W_k = I_{n\times n}$$

which implies that

$$I_{n\times n} - W_k^{-1}\tilde{\Gamma}_k\left(\tilde{\Gamma}_k^{\mathrm{T}}W_k^{-1}\tilde{\Gamma}_k\right)^{-1}\tilde{\Gamma}_k^{\mathrm{T}} = \Gamma_k\left(\Gamma_k^{\mathrm{T}}W_k\Gamma_k\right)^{-1}\Gamma_k^{\mathrm{T}}W_k. \quad (3.35)$$

Postmultiplying (3.35) by K_t and using (3.34) yields

$$L_k = \Gamma_k\tilde{L}_k \quad (3.36)$$

where

$$\tilde{L}_k \triangleq \left(\Gamma_k^{\mathrm{T}}W_k\Gamma_k\right)^{-1}\Gamma_k^{\mathrm{T}}W_kK_k \quad (3.37)$$

is the gain of the spatially constrained Kalman filter (SCKF) presented in [5].

IV. Gain-Constrained Unscented Kalman Filter

Consider the stochastic nonlinear discrete-time dynamic system

$$x_k = f(x_{k-1}, u_{k-1}, k-1) + w_{k-1} \quad (4.1)$$

$$y_k = h(x_k, k) + v_k \quad (4.2)$$

where $f : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{N} \to \mathbb{R}^n$ and $h : \mathbb{R}^n \times \mathbb{N} \to \mathbb{R}^m$ are, respectively, the process and observation models.

For the system (4.1) and (4.2), we consider a suboptimal filter with *forecast* step

$$\hat{x}_{k|k-1} = \sum_{i=0}^{2n}\gamma_i\mathcal{X}_{i,k|k-1} \quad (4.3)$$

$$\hat{y}_{k|k-1} = \sum_{i=0}^{2n}\gamma_i\mathcal{Y}_{i,k|k-1} \quad (4.4)$$

where $\mathcal{X}_{i,k|k-1} \in \mathbb{R}^n$ and $\mathcal{Y}_{i,k|k-1} \in \mathbb{R}^m$, $i = 0, 1, \ldots, 2n$ are sets of sample vectors with weights $\gamma_i \in \mathbb{R}$, and *data-assimilation* step given by (2.5), where L_k is assumed to satisfy (2.7).

We consider the unscented transform (UT) [15] to obtain $\mathcal{X}_{i,k|k-1}$, $\mathcal{Y}_{i,k|k-1}$, and γ_i. UT is a numerical procedure for approximating the posterior mean $\hat{y} \in \mathbb{R}^m$ and covariance $P^{yy} \in \mathbb{R}^{m\times m}$ of a random vector y obtained from the nonlinear transformation $y = h(x)$, where x is a prior random vector whose mean $\hat{x} \in \mathbb{R}^n$ and covariance $P^{xx} \in \mathbb{R}^{n\times n}$ are assumed to be known. UT yields the mean $\hat{y}$ and covariance P^{yy} of y, if $h = h_1 + h_2$, where h_1 is linear and h_2 is quadratic [15]. Otherwise, $\hat{y}$ is a *pseudo mean*, and P^{yy} is a *pseudo covariance*.

UT is based on a set of deterministically chosen sample vectors known as sigma points. To satisfy $\sum_{i=0}^{2n}\gamma_i\mathcal{X}_i = \hat{x}$ and $\sum_{i=0}^{2n}\gamma_i[\mathcal{X}_i - \hat{x}][\mathcal{X}_i - \hat{x}]^{\mathrm{T}} = P^{xx}$, the entries of the sigma-point matrix given by $\mathcal{X} \triangleq [\mathcal{X}_0 \; \mathcal{X}_1 \; \ldots \; \mathcal{X}_{2n}] \in \mathbb{R}^{n\times(2n+1)}$ are chosen as

$$\mathcal{X}_i \triangleq \begin{cases} \hat{x}, & \text{if } i = 0, \\ \hat{x} + \sqrt{\lambda}\,\mathrm{col}_i\left[(P^{xx})^{1/2}\right], & \text{if } i = 1, \ldots, n, \\ \hat{x} - \sqrt{\lambda}\,\mathrm{col}_i\left[(P^{xx})^{1/2}\right], & \text{if } i = n+1, \ldots, 2n \end{cases} \quad (4.5)$$

with weights

$$\gamma_i \triangleq \begin{cases} \frac{\lambda - n}{\lambda}, & \text{if } i = 0, \\ \frac{1}{2\lambda}, & \text{if } i = 1, \ldots, 2n \end{cases} \tag{4.6}$$

where $\mathrm{col}_i[(\cdot)^{1/2}]$ is the ith column of the Cholesky square root and $\lambda > 0$ determines the spread of the sigma points around $\hat{x}$. Propagating each sigma point through h yields $\mathcal{Y}_i = h(\mathcal{X}_i)$ such that $\hat{y} = \sum_{i=0}^{2n} \gamma_i \mathcal{Y}_i$ and $P^{yy} = \sum_{i=0}^{2n} \gamma_i [\mathcal{Y}_i - \hat{y}][\mathcal{Y}_i - \hat{y}]^{\mathrm{T}}$.

Therefore, we apply UT to the GCKF equations to obtain the *gain-constrained unscented Kalman filter* (GCUKF), whose forecast step is given by

$$\mathcal{X}_{k-1|k-1} = \hat{x}_{k-1|k-1} 1_{1\times(2n+1)} + \sqrt{\lambda} \times \left[0_{n\times 1} \ \left(P^{xx}_{k-1|k-1}\right)^{1/2} - \left(P^{xx}_{k-1|k-1}\right)^{1/2} \right] \tag{4.7}$$

$$\mathcal{X}_{i,k|k-1} = f(\mathcal{X}_{i,k-1|k-1}, u_{k-1}, k-1), \ i=0,\ldots,2n \tag{4.8}$$

together with (4.3) and

$$P^{xx}_{k|k-1} = \sum_{i=0}^{2n} \gamma_i [\mathcal{X}_{i,k|k-1} - \hat{x}_{k|k-1}][\mathcal{X}_{i,k|k-1} - \hat{x}_{k|k-1}]^{\mathrm{T}} + Q_{k-1} \tag{4.9}$$

$$\mathcal{X}_{k|k-1} = \hat{x}_{k|k-1} 1_{1\times(2n+1)} + \sqrt{\lambda} \tag{4.10}$$

$$\times \left[0_{n\times 1} \ \left(P^{xx}_{k|k-1}\right)^{1/2} \ - \left(P^{xx}_{k|k-1}\right)^{1/2} \right] \tag{4.11}$$

$$\mathcal{Y}_{i,k|k-1} = h(\mathcal{X}_{i,k|k-1}, \ k), \quad i = 0, \ldots, 2n \tag{4.12}$$

together with (4.4) and

$$P^{yy}_{k|k-1} = \sum_{i=0}^{2n} \gamma_i [\mathcal{Y}_{i,k|k-1} - \hat{y}_{k|k-1}][\mathcal{Y}_{i,k|k-1} - \hat{y}_{k|k-1}]^{\mathrm{T}} + R_k \tag{4.13}$$

$$P^{xy}_{k|k-1} = \sum_{i=0}^{2n} \gamma_i [\mathcal{X}_{i,k|k-1} - \hat{x}_{k|k-1}][\mathcal{Y}_{i,k|k-1} - \hat{y}_{k|k-1}]^{\mathrm{T}} \tag{4.14}$$

and whose data-assimilation step is given by (2.32), (2.5), and (2.20).

Note that the GCUKF equations are identical to the unscented Kalman filter (UKF) equations [14] except for the gain expression (2.32).

The next result shows that the gain (2.32) of GCUKF satisfies the constraint (2.7) with the pseudo error-covariance matrices $P^{yy}_{k|k-1}$ (4.13) and $P^{xy}_{k|k-1}$ (4.14).

Proposition 4.1: Let $P^{yy}_{k|k-1}$ and $P^{xy}_{k|k-1}$ be given, respectively, by (4.13) and (4.14), and let L_k be given by (2.32). Then (2.7) is satisfied.

Proof: Premultiplying (2.27) by D_k and postmultiplying (2.28) by E_k yields

$$D_k \Pi_k = D_k W_k^{-1} D_k^{\mathrm{T}} \left(D_k W_k^{-1} D_k^{\mathrm{T}}\right)^{-1} D_k = D_k,$$

$$\Omega_k E_k = E_k \left[E_k^{\mathrm{T}} \left(P^{yy}_{k|k-1}\right)^{-1} E_k \right]^{-1} E_k^{\mathrm{T}} \left(P^{yy}_{k|k-1}\right)^{-1} E_k = E_k.$$

Also, premultiplying and postmultiplying (2.32) by D_k and E_k and using (2.25) and (2.26) yields

$$\begin{aligned} D_k L_k E_k &= D_k K_k E_k - D_k \Pi_k K_k \Omega_k E_k \\ &\quad + D_k \Pi_k D_k^{\mathrm{R}} F_k E_k^{\mathrm{L}} \Omega_k E_k \\ &= D_k K_k E_k - D_k K_k E_k + D_k D_k^{\mathrm{R}} F_k E_k^{\mathrm{L}} E_k = F_k \end{aligned}$$

which confirms (2.7). ■

V. Gain-Constrained Kalman Predictor

For the system (2.1), (2.2), we now consider a one-step predictor of the form

$$\hat{x}_k = A_{k-1}\hat{x}_{k-1} + B_{k-1}u_{k-1} + L_{k-1}(y_{k-1} - \hat{y}_{k-1}) \tag{5.1}$$

where

$$\hat{y}_{k-1} = C_{k-1}\hat{x}_{k-1} \tag{5.2}$$

and the predictor gain $L_{k-1} \in \mathbb{R}^{n\times m}$ minimizes

$$J_{k-1}(L_{k-1}) \triangleq \mathcal{E}\left[(x_{k-1} - \hat{x}_{k-1})^{\mathrm{T}} W_{k-1} (x_{k-1} - \hat{x}_{k-1})\right] \tag{5.3}$$

subject to the constraint (2.7). The differences between the prediction and filtering algorithms are discussed in [27].

Next, define the *prediction error* e_k and the *innovation* ν_{k-1} by

$$e_k \triangleq x_k - \hat{x}_k \tag{5.4}$$

$$\nu_{k-1} \triangleq y_{k-1} - \hat{y}_{k-1} \tag{5.5}$$

and the *prediction error covariance* P^{xx}_k and the *innovation covariance* P^{yy}_{k-1} by

$$P^{xx}_k \triangleq \mathcal{E}\left[e_k e_k^{\mathrm{T}}\right] \tag{5.6}$$

$$P^{yy}_{k-1} \triangleq \mathcal{E}\left[\nu_{k-1}\nu_{k-1}^{\mathrm{T}}\right]. \tag{5.7}$$

It follows from (2.1) and (2.2) and (5.1) and (5.2) that

$$e_k = (A_{k-1} - L_{k-1}C_{k-1})e_{k-1} + w_{k-1} - L_{k-1}v_{k-1} \tag{5.8}$$

$$\nu_{k-1} = C_{k-1}e_{k-1} + v_{k-1}. \tag{5.9}$$

The following lemma will be useful.

Lemma 5.1: The prediction error given by (5.4) satisfies

$$\mathcal{E}\left[e_k w_k^{\mathrm{T}}\right] = 0 \tag{5.10}$$

$$\mathcal{E}\left[e_k v_k^{\mathrm{T}}\right] = 0. \tag{5.11}$$

Proposition 5.1: For the predictor (5.1), (5.2), the prediction error covariance P^{xx}_k is updated using

$$P^{xx}_k = A_{k-1}P^{xx}_{k-1}A_{k-1}^{\mathrm{T}} + Q_{k-1} + L_{k-1}P^{yy}_{k-1}L_{k-1}^{\mathrm{T}} - \tilde{P}_{k-1}L_{k-1}^{\mathrm{T}} - L_{k-1}\tilde{P}_{k-1}^{\mathrm{T}} \tag{5.12}$$

where

$$P^{yy}_{k-1} = C_{k-1}P^{xx}_{k-1}C_{k-1}^{\mathrm{T}} + R_{k-1} \tag{5.13}$$

and

$$\tilde{P}_{k-1} \triangleq A_{k-1} P_{k-1}^{xx} C_{k-1}^{\mathrm{T}}. \tag{5.14}$$

Proof: It follows from (5.9) and (5.11) that (5.7) is given by (5.13). Moreover, by using (5.10), (5.8) implies that (5.6) is given by

$$P_k^{xx} = (A_{k-1} - L_{k-1}C_{k-1})P_{k-1}^{xx}(A_{k-1} - L_{k-1}C_{k-1})^{\mathrm{T}} + Q_{k-1} + L_{k-1}R_{k-1}L_{k-1}^{\mathrm{T}}$$

which, by using (5.13) and (5.14), is equal to (5.12). ■

Next, using (5.4) and (5.6) in (5.3) yields

$$J_{k-1}(L_{k-1}) = \operatorname{tr}\left(P_{k-1}^{xx} W_{k-1}\right). \tag{5.15}$$

Assume that, for all $k \geq 1$, P_{k-1}^{yy} is positive definite. Then, for convenience, we define

$$\tilde{\Omega}_{k-1} \triangleq E_{k-1}\left[E_{k-1}^{\mathrm{T}}\left(P_{k-1}^{yy}\right)^{-1}E_{k-1}\right]^{-1}E_{k-1}^{\mathrm{T}}\left(P_{k-1}^{yy}\right)^{-1} \tag{5.16}$$

$$\tilde{K}_{k-1} \triangleq \tilde{P}_{k-1}\left(P_{k-1}^{yy}\right)^{-1}. \tag{5.17}$$

Also, let Π_{k-1} be given by (2.27), D_{k-1}^{R} be given by (2.25), and E_{k-1}^{L} be given by (2.26). Note that Π_{k-1} and $\tilde{\Omega}_{k-1}$ are oblique projectors, which satisfy (2.30) and

$$E_{k-1}^{\mathrm{L}}\tilde{\Omega}_{k-1} = \left[E_{k-1}^{\mathrm{T}}\left(P_{k-1}^{yy}\right)^{-1}E_{k-1}\right]^{-1}E_{k-1}^{\mathrm{T}}\left(P_{k-1}^{yy}\right)^{-1} \tag{5.18}$$

respectively. Moreover, note that $\tilde{K}_{k-1}$ is the gain of the classical Kalman predictor [12].

Proposition 5.2: The gain L_{k-1} that minimizes (5.15) and satisfies (2.7) is given by

$$L_{k-1} = \tilde{K}_{k-1} - \Pi_{k-1}\left(\tilde{K}_{k-1} - D_{k-1}^{\mathrm{R}}F_{k-1}E_{k-1}^{\mathrm{L}}\right)\tilde{\Omega}_{k-1} \tag{5.19}$$

where the error covariance P_k^{xx} in (5.12) is updated using the Riccati equation

$$\begin{aligned} P_k^{xx} &= A_{k-1}P_{k-1}^{xx}A_{k-1}^{\mathrm{T}} + Q_{k-1} - \tilde{P}_{k-1}\left(P_{k-1}^{yy}\right)^{-1}(\tilde{P}_{k-1})^{\mathrm{T}} \\ &\quad + \left(\Pi_k D_k^{\mathrm{R}} F_k E_k^{\mathrm{L}} \tilde{\Omega}_k\right) P_{k-1}^{yy} \left(\Pi_k D_k^{\mathrm{R}} F_k E_k^{\mathrm{L}} \tilde{\Omega}_k\right)^{\mathrm{T}} \\ &\quad + \left[\Pi_k \tilde{P}_{k-1}\left(P_{k-1}^{yy}\right)^{-1}\tilde{\Omega}_k\right] P_{k-1}^{yy} \\ &\quad \times \left[\Pi_k \tilde{P}_{k-1}\left(P_{k-1}^{yy}\right)^{-1}\tilde{\Omega}_k\right]^{\mathrm{T}} \\ &\quad - \Pi_k\left(\tilde{\Delta}_k + \tilde{\Delta}_k^{\mathrm{T}}\right)\Pi_k^{\mathrm{T}} \end{aligned} \tag{5.20}$$

where

$$\tilde{\Delta}_k \triangleq \tilde{P}_{k-1}\left(P_{k-1}^{yy}\right)^{-1}\tilde{\Omega}_k P_{k-1}^{yy}\tilde{\Omega}_k^{\mathrm{T}}\left(D_k^{\mathrm{R}}F_kE_k^{\mathrm{L}}\right)^{\mathrm{T}}. \tag{5.21}$$

Proof: The proof is similar to that of Proposition 2.2 and is omitted. For completeness, the proof is presented in [28, Prop. C.1.2]. ■

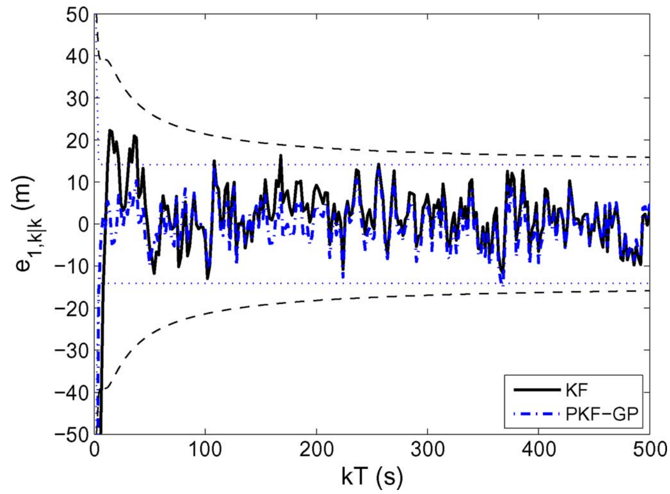

Fig. 1. Comparison of estimation errors for x_1 using KF (——) and PKF-GP ($- \cdot -$) around the associated $\pm 3\sqrt{P_{(1,1),k|k}^{xx}}$ confidence limits given by ($- - -$) and ($\cdots\cdots$), respectively.

Proposition 5.3: L_{k-1} given by (5.19) is the unique global minimizer of $J_{k-1}(L_{k-1})$ given by (5.15) restricted to the convex constraint set defined by (2.7).

Proof: The proof is similar to the proof of Proposition 2.3 and is omitted for brevity. ■

The *gain-constrained Kalman predictor* (GCKP) is given by (5.2), (5.13), (5.14), (2.25)–(2.27), (5.16), (5.17), (5.19), (5.1), and (5.12). Note that the GCKP equations are identical to the classical Kalman predictor equations except for the gain expression (5.19). It is straightforward to derive the special cases shown in Section III in the context of GCKF. However, for the sake of brevity, we omit them.

VI. Examples

A. Tracking a Land-Based Vehicle

We consider a land-based vehicle whose linear dynamics are represented by (2.1) and (2.2), with parameters

$$A_{k-1} = \begin{bmatrix} 1 & 0 & T & 0 \\ 0 & 1 & 0 & T \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad B_{k-1} = \begin{bmatrix} 0 \\ 0 \\ T\sin\theta \\ T\cos\theta \end{bmatrix},$$
$$C_{k-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{6.1}$$

where the state vector $x_k \in \mathbb{R}^4$ is composed of northerly and easterly position and velocity components. For simulation, we set $T = 2$ s, $x_0 = [0\ 0\ 10\tan\theta\ 10]^{\mathrm{T}}$,

$$Q_{k-1} = 10\begin{bmatrix} \sin^2\theta & \sin\theta\cos\theta & 0 & 0 \\ \sin\theta\cos\theta & \cos^2\theta & 0 & 0 \\ 0 & 0 & \sin^2\theta & \sin\theta\cos\theta \\ 0 & 0 & \sin\theta\cos\theta & \cos^2\theta \end{bmatrix}$$

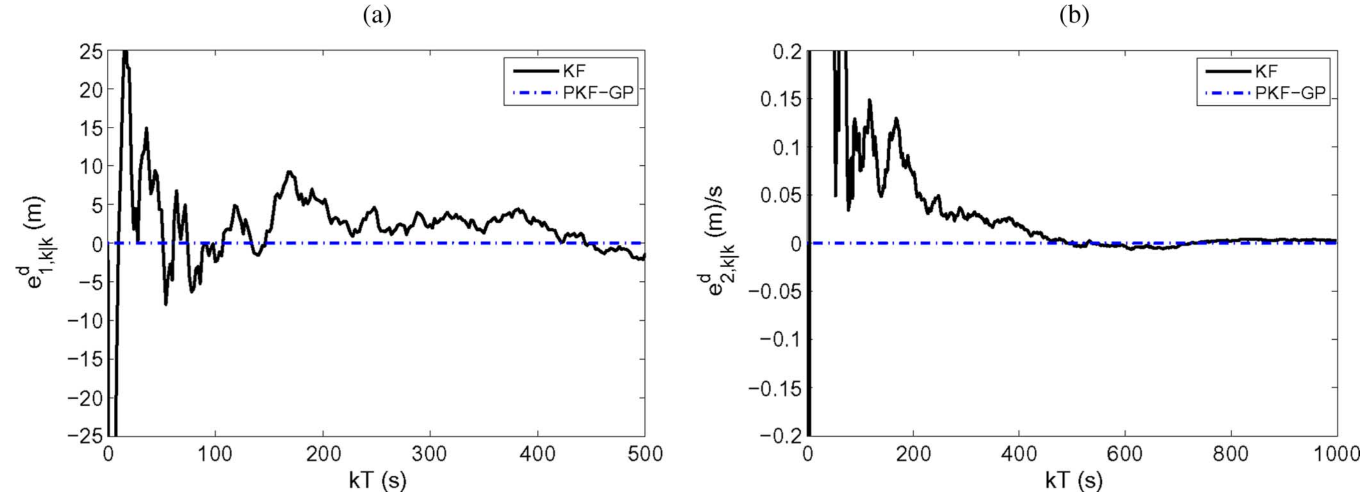

Fig. 2. Comparison of constraint errors (a) $e^d_{1,k|k}$ and (b) $e^d_{2,k|k}$, where $e^d_{k|k} = \tilde{d}_{k-1} - \tilde{D}_{k-1}\hat{x}_{k|k}$, using KF (——) and PKF-GP ($- \cdot -$).

TABLE I
RMS CONSTRAINT ERROR, RMSE (6.3), AND MT (6.4) FOR 100-RUN MONTE CARLO SIMULATION FOR THE LAND-BASED VEHICLE SYSTEM USING THE KF AND PKF-GP ALGORITHMS

KF	PKF-GP
RMSE_i, $i = 1, 2, 3, 4$	
8.79, 2.71 3.60, 2.04	4.74, 2.74, 3.54, 2.04
RMSE of constraint error $e^d_{k\|k} = \tilde{d}_{k-1} - \tilde{D}_{k-1}\hat{x}_{k\|k}$	
7.66	8.65 $\times 10^{-13}$
0.76	2.01 $\times 10^{-15}$
MT	
59.83	46.22

and $R_k = \mathrm{diag}(400, 10)$. It is known that the vehicle is moving in a straight line with a heading of $\theta = 60$ deg. That is, the equality constraint (3.11) with parameters

$$\tilde{D}_{k-1} = \begin{bmatrix} 1 & -\tan\theta & 0 & 0 \\ 0 & 0 & 1 & -\tan\theta \end{bmatrix}, \quad \tilde{d}_{k-1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (6.2)$$

defines the subspace to which the vehicle trajectory is confined. The commanded acceleration u_{k-1}, which is assumed to be known, is alternatively set to $\pm 1\ \mathrm{m/s}^2$, as if the vehicle were accelerating and decelerating in traffic. This example is also investigated in [23] and [24].

Our goal is to obtain state estimates satisfying the equality constraint (3.13). We implement KF (Section III-A) and PKF-GP, which is shown to be a special case of GCKF in Section III-D, to perform state estimation, initialized with $\hat{x}_{0|0} = [500\ (500/\tan\theta)\ 30\ (30/\tan\theta)]^{\mathrm{T}}$ and $P^{xx}_{0|0} = \mathrm{diag}(900, 900, 4, 4)$.

Table I presents a performance comparison between KF and PKF-GP regarding two performance indices, namely, the average of the root-mean-square error of each state component $\hat{x}_{i,k|k,j}$, $j = 1, \ldots, 100$, over a 100-run Monte Carlo simulation (RMSE_i), $i = 1, \ldots, n$,

$$\mathrm{RMSE}_i \triangleq \frac{1}{100}\sum_{j=1}^{100}\left[\sqrt{\frac{1}{N}\sum_{k=1}^{N}(x_{i,k} - \hat{x}_{i,k|k,j})^2}\right] \quad (6.3)$$

where N is the final time, as well as the mean trace (MT) of the error-covariance matrix

$$\mathrm{MT} \triangleq \frac{1}{100}\sum_{j=1}^{100}\left[\frac{1}{N}\sum_{k=1}^{N}\mathrm{tr}\left(P^{xx}_{k|k,j}\right)\right]. \quad (6.4)$$

Fig. 1 shows the estimation error $e_{1,k|k}$ around the confidence interval given by $\pm 3\sqrt{P^{xx}_{(1,1),k|k}}$, while Fig. 2 shows the constraint estimation error $e^d_{k|k} = \tilde{d}_{k-1} - \tilde{D}_{k-1}\hat{x}_{k|k}$.

Results from Table I and Fig. 2 show that, unlike the PKF-GP estimates, the KF estimates do not satisfy the equality constraint (3.13). Moreover, PKF-GP outperforms KF regarding both indexes RMSE and MT; see Table I and Fig. 1.

Similar to the constrained state-estimation problem investigated above, GCKF (or GCUKF) can be used to enforce linear equality constraints on parameters of nonlinear models. This problem is considered in [1], [2] using a constrained least-squares approach.

B. *Van der Pol Oscillator*

We consider the Euler-discretized van der Pol oscillator given by

$$\begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} = \begin{bmatrix} x_{1,k-1} + T x_{2,k-1} \\ -T x_{1,k-1} + \left(T + 1 - T x^2_{1,k-1}\right) x_{2,k-1} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_{k-1} \quad (6.5)$$

where $x_k \triangleq [x_{1,k}\ x_{2,k}]^{\mathrm{T}} \in \mathbb{R}^2$, $u_{k-1} \in \mathbb{R}$ is an exogenous input, and T is the sampling interval, with noisy observation given by (2.2) with $C_k = [1\ 1]$. For simulation, we set $x_0 = [1\ 1]^{\mathrm{T}}$, $T = 0.1$, $R_k = 0.04$, and

$$u_{k-1} = \begin{cases} T\sin(2kT), & \text{if } kT < 10\ \mathrm{s} \text{ or } kT \geq 30\ \mathrm{s} \\ T\left(\sin(2kT) + 0.5\right), & \text{if } 10\ \mathrm{s} \leq kT < 20\ \mathrm{s} \\ T\left(\sin(2kT) - 0.5\right), & \text{if } 20\ \mathrm{s} \leq kT < 30\ \mathrm{s}. \end{cases} \quad (6.6)$$

Now, assuming that u_{k-1} is unknown, our goal is to obtain unbiased state estimates. For comparison, we perform state estimation using UKF in two distinct cases, namely, assuming u_{k-1} is known and assuming u_{k-1} is unknown, as well as using

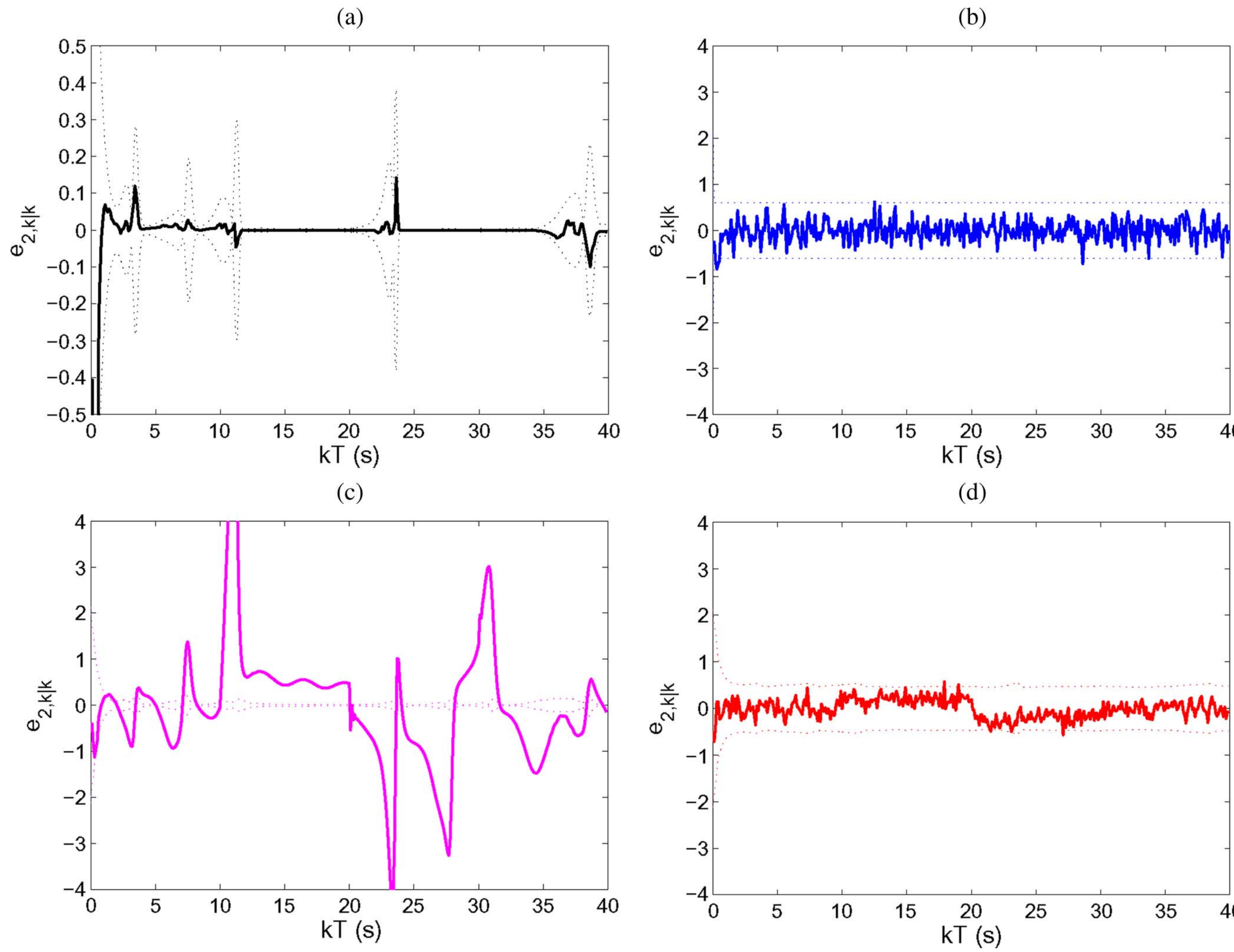

Fig. 3. Comparison of estimation errors for x_2 using (a) UKF with u_{k-1} assumed to be known, (b) GCUKF with u_{k-1} assumed to be unknown, and (c) UKF with u_{k-1} assumed to be unknown. The case of UKF with u_{k-1} assumed to be unknown when a larger value of Q_{k-1} is used is denoted by (d). Dotted lines show the associated $\pm 3\sqrt{P^{xx}_{(2,2),k|k}}$ confidence limits. Observe that these limits for the cases (b) and (d) almost coincide.

GCUKF and assuming that u_{k-1} is unknown. Whenever u_{k-1} is assumed to be unknown, we set $u_{k-1} = 0$ during the forecast step. In this case, u_{k-1} plays the same role as the unknown input d_{k-1} used in the linear model (3.18). We set $\hat{x}_{0|0} = [0.5\ 1.5]^{\rm T}$ and $P^{xx}_{0|0} = 0.5 I_{2\times 2}$. To help convergence of UKF [34] with u_{k-1} assumed to be known, we set $Q_{k-1} = 10^{-6} I_{2\times 2}$. For consistency, this value is used for the remaining two cases. We implement GCUKF using $D_k = I_{2\times 2}$, $E_k = C_k G_{k-1}$, and $F_k = G_{k-1}$, where, according to (6.5), $G_{k-1} = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$; see Section III-E-1.

Fig. 3 shows the estimation error component $e_{2,k|k}$ around the confidence interval $\pm 3\sqrt{P^{xx}_{(2,2),k|k}}$, where $P^{xx}_{k|k}$ is the pseudo error-covariance given by (2.20). Observe that, when we assume u_{k-1} is unknown, $e_{2,k|k}$ given by UKF does not converge. On the other hand, the GCUKF estimates converge, but with a larger confidence interval compared to the case in which UKF is used with u_{k-1} assumed to be known. Likewise, if we set $Q_{k-1} = {\rm diag}(10^{-6}, 0.05)$ using UKF with u_{k-1} assumed to be unknown, then $e_{2,k|k}$ converges with similar accuracy compared to GCUKF; see Fig. 3. This value of Q_{k-1} was heuristically chosen by increasing the original value until the estimation errors remain inside the confidence interval $\pm 3\sqrt{{\rm diag}(P^{xx}_{k|k})}$. Statistical approaches to estimate Q_{k-1} offline are found in [11].

TABLE II

RMSE (6.3) AND MT (6.4) FOR A 100-RUN MONTE CARLO SIMULATION FOR THE VAN DER POL OSCILLATOR USING ALGORITHMS i) UKF WITH KNOWN u_{k-1}, ii) GCUKF WITH ASSUMED UNKNOWN u_{k-1}, iii) UKF WITH UNKNOWN u_{k-1}, AND iv) UKF WITH UNKNOWN u_{k-1} AND A LARGER Q_{k-1}

	case (i)	case (ii)	case (iii)	case (iv)
	RMSE_i, $i = 1, 2$			
$x_{1,k}$	0.0487	0.0728	1.8936	0.0750
$x_{2,k}$	0.0596	0.2134	1.2996	0.2055
	MT			
	0.0080	0.0424	0.0086	0.0382

Table II presents a performance comparison regarding RMSE (6.3) and MT (6.4) over a 100-run Monte Carlo simulation among the following cases: i) UKF with known u_{k-1}; ii) GCUKF with unknown u_{k-1}; iii) UKF with unknown u_{k-1}; and iv) UKF with unknown u_{k-1} and a larger Q_{k-1}. Note that, regarding RMSE, case ii) outperforms case iii). Although case

TABLE III
SUMMARY OF CHARACTERISTICS OF GAIN-CONSTRAINED KALMAN FILTERING ALGORITHMS FOR LINEAR SYSTEMS (2.1) AND (2.2). IT IS SPECIFIED HOW THE GAIN CONSTRAINT (2.7) IS HANDLED BY EACH SPECIAL CASE

Algorithm	Section	D_k	E_k	F_k	References
KF	III-A	$I_{n\times n}$	$I_{m\times m}$	K_k	[12]
PKF-GP	III-D	$\tilde{D}_{k-1}$	$(y_k - \hat{y}_{k\|k-1})$	$(\tilde{d}_{k-1} - \tilde{D}_{k-1}\hat{x}_{k\|k-1})$	[10]
UnbKF-UI	III-E	$I_{n\times n}$	$[\ C_k G_{k-1} \quad H_k\ ]$	$[\ G_{k-1} \quad 0_{n\times t}\ ]$	[6, 9]
UnbKF-UI ($t=0$)	III-E1	$I_{n\times n}$	$C_k G_{k-1}$	G_{k-1}	[16]
UnbKF-UI ($s=0$)	III-E2	$I_{n\times n}$	H_k	$0_{n\times t}$	[18]
SCKF	III-F	$\tilde{\Gamma}_k^{\mathrm{T}}$	$I_{m\times m}$	$0_{q\times m}$	[5]

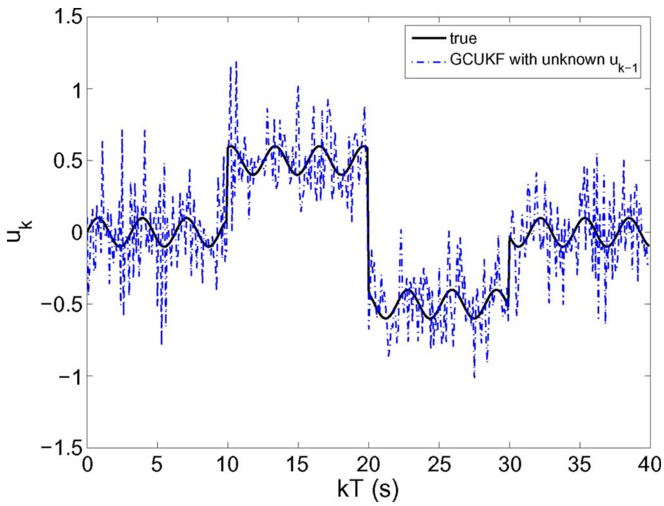

Fig. 4. True (——) and estimated (−·) unknown input u_{k-1} using GCUKF.

iii) exhibits smaller MT than case ii), as illustrated in Fig. 3(c), the estimation error for case iii) do not converge. Also, the indexes RMSE and MT for cases ii) and iv) are close.

Finally, Fig. 4 compares the input u_{k-1} (actual value) to its estimate $\hat{u}_{k-1|k}$ given by [9], [18]

$$\hat{u}_{k-1|k} = \left(G_{k-1}^{\mathrm{T}} G_{k-1}\right)^{-1} G_{k-1}^{\mathrm{T}} L_k (y_k - \hat{y}_{k|k-1}), \qquad (6.7)$$

where $\hat{y}_{k|k-1}$ is given by (4.4) and L_k is given by (2.32) using GCUKF. Alternatively, in [7], [19], a least-squares approach is proposed for estimating unknown inputs of time series, while, in [21] and [26], a dynamical model compensation technique is used.

VII. CONCLUDING REMARKS

This paper derives the optimal solution to the problem of linear state estimation with constrained data-injection gain. Both the one-step gain-constrained Kalman predictor (GCKP) and the two-step gain-constrained Kalman filter (GCKF) are derived. Then, the classical Kalman filter (KF), the projected Kalman filter by gain projection (PKF-GP), the unbiased Kalman filter with unknown inputs (UnbKF-UI), and the spatially constrained Kalman filter (SCKF) are presented as special cases of GCKF. Table III summarizes how the gain constraint is set for all special cases of GCKF.

Using the unscented transform, the gain-constrained unscented Kalman filter (GCUKF) is presented as a nonlinear extension of GCKF. Although the resulting algorithm is an approximate solution to the state-estimation problem for nonlinear systems, its gain exactly satisfies the constraint.

An example of tracking a land-based vehicle is employed to illustrate the use of GCKF to obtain state estimates satisfying a linear equality constraint in the state vector. In addition to satisfying this constraint, the GCKF estimates are improved compared to KF for this example.

Furthermore, an Euler-discretized van der Pol oscillator is used to illustrate a nonlinear application of GCUKF when the input vector is unknown. In this case, improved estimates are obtained using GCUKF compared to the case in which the unscented Kalman filter (UKF) is used with an assumed unknown input vector. Also, GCUKF has performance similar to the case in which UKF is employed with an assumed unknown input vector and a larger process-noise covariance matrix.

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