# WHY TWELVE TONES? THE MATHEMATICS OF MUSICAL TUNING

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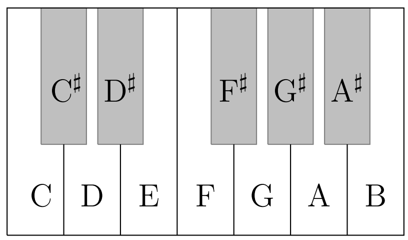
The scale that forms the basis of most Western music consists of twelve tones:




That these twelve pitches form the palette of possible notes is usually taken as axiomatic, or as simply an artifact of cultural convention. In fact, though, it is possible to "discover" the twelve-note scale using just a length of resonant string—an imaginary string will do—and some mathematics. The goal of this article is to give a mathematically-minded exposition of one way to form scales with any number of pitches, the so-called *Pythagorean scales*, wherein the ratio between any two frequencies involves only powers of the prime numbers 2 and 3. Furthermore, we introduce a new property of scales, which we call the *two-step property*, and prove that it is possessed by Pythagorean scales of very special sizes—among them, our familiar scale size of twelve.

## 1. How to invent a scale

Suppose that you hold in your hand a string under constant tension, like those found on a guitar or violin, and that any knowledge you may have of Western music has been wiped away. You notice that plucking the string produces a note, and that other, higher notes are created when the string is cut down to a smaller size.

Perhaps you try cutting the string in half and comparing the new note to the one your original string sounded. In modern musical terminology, these two notes differ by an octave from one another, and we tend to perceive them as "the same"; if one is called a C, the other is called a C, as well. There is a good acoustical reason for this sense of sameness, having to do with the similarity in the Fourier series describing the motion of a vibrating string of length $\ell$ and one of length $\frac{\ell}{2}$.

We will return to this fact later, but for now let us assume that, even in the hypothetical situation wherein no musical knowledge exists, you still hear notes differing by an octave as essentially identical pitches.

The next simple manipulation you might try is to cut your string into thirds. Now you have succeeded in creating a genuinely new note: a string two-thirds the length of your original one produces a pitch that (again borrowing the modern musical terminology) is a fifth higher. This is—more or less—the interval between the notes C and G on a piano keyboard, and the name "fifth" refers to the five white keys spanning this interval.

At this point, a scale is already within reach. If you normalize so that your original string vibrates at a frequency of 1, then by repeatedly cutting a third from the length of your string, you produce notes of frequency $3^b$ for all non-negative integers $b$. Given the assumed equivalence of notes differing by octaves, we may as well divide each of the resulting frequencies by a power of 2 until it lies in the interval $[1, 2)$.

Let us define a *scale* as a collection of frequencies of this form, listed in increasing numerical order. So, for example, the scale formed by the first four powers of 3 is:

$$\frac{3^0}{2^0}, \frac{3^2}{2^3}, \frac{3^4}{2^6}, \frac{3^1}{2^1}, \frac{3^3}{2^4}.$$

How do you know when to stop piling on the powers of three? You might like to stop when you get back to the same pitch you started with (modulo a multiple of 2), but that would mean reaching a nonzero power of 3 that is also a power of 2. This is, of course, impossible, by uniqueness of prime factorizations.

To put the problem another way, the equation

$$2^a = 3^b \tag{1}$$

has no integer solutions besides $(a, b) = (0, 0)$, since it rearranges to

$$\frac{a}{b} = \frac{\log(3)}{\log(2)}, \tag{2}$$

and $\log(3)/\log(2)$ is irrational. This perspective points the way to a good solution: we can find a rational approximation of $\log(3)/\log(2)$, and use that to dictate the length of our scale.

The best rational approximations of irrational numbers are given by continued fractions.[1] These are expressions, for any real number $x$, of

[1]For a precise notion of "best", see [11]. The idea of using continued fractions to determine scale lengths is attributed, in [1], to M.W. Drobisch [6].

the form

$$x = k + \cfrac{1}{k_0 + \cfrac{1}{k_1 + \cfrac{1}{k_2 + \cdots}}}, \tag{3}$$

where $\{k_n\}$ is a possibly infinite sequence of integers for which the truncations of (3) converge to $x$. A common short-hand is

$$x = [k; k_0, k_1, k_2, \ldots].$$

In the case of $\log(3)/\log(2)$, the continued fraction expansion begins:

$$\frac{\log(3)}{\log(2)} = [1; 1, 1, 2, 2, 3, 1, 5, \ldots].$$

Truncating the expansion produces the following successively better approximations of $\log(3)/\log(2)$:

$$\begin{aligned} [1; 1, 1] &= \frac{3}{2}, \\ [1; 1, 1, 2] &= \frac{8}{5}, \\ [1; 1, 1, 2, 2] &= \frac{19}{12}, \\ [1; 1, 1, 2, 2, 3] &= \frac{65}{41}, \\ [1; 1, 1, 2, 2, 3, 1] &= \frac{84}{53}, \end{aligned}$$

and so on. In light of equations (1) and (2), the denominators of these fractions indicate powers of 3 that are close to a power of 2, which are places where it makes sense to stop the process described above and declare the result a scale.

Approximating $\log(3)/\log(2)$ by $19/12$, we find that a scale formed by powers of 3 has approximately returned to its starting pitch after twelve steps. If we list the pitches in increasing numerical order and add the frequency $2 = \frac{3^0}{2^{-1}}$ at the end to give our scale a sense of finality, then we obtain:

$$\frac{3^0}{2^0}, \frac{3^7}{2^{11}}, \frac{3^2}{2^3}, \frac{3^9}{2^{14}}, \frac{3^4}{2^6}, \frac{3^{11}}{2^{14}}, \frac{3^6}{2^9}, \frac{3^1}{2^1}, \frac{3^8}{2^{12}}, \frac{3^3}{2^4}, \frac{3^{10}}{2^{15}}, \frac{3^5}{2^7}, \frac{3^0}{2^{-1}}. \tag{4}$$

Or, we may call these C, C$^\sharp$, D, D$^\sharp$, E, F, F$^\sharp$, G, G$^\sharp$, A, A$^\sharp$, B, C: we have found our way back to the twelve-tone scale.[2]

[2]Or, rather, we have found our way to *some* scale with twelve notes. In fact, as we will see below, this is not quite the present-day Western scale but one of its historical predecessors.

As an enlightening bonus, we may notice that another truncation of the continued fraction approximation of $\log(3)/\log(2)$ is $8/5$, which leads one to a scale of five notes. Such scales are referred to as pentatonic, and are common in Chinese, Scottish, and other musical traditions, as well as in jazz. Of course, a natural question at this point is: where is the musical tradition of the 41-note scale, or the 53-note scale? Though such music has been composed, for example by Dutch guitarist Melle Weijters [22, 9] and others [26], it has not yet achieved the wide popularity of the twelve- or five-note scale.[3]

## 2. The two-step property

Taking a cue from music theorists, let us call the *$B$-note Pythagorean scale* the list, in increasing numerical order, of all numbers $\frac{3^b}{2^a} \in [1,2)$ with $b$ in some range $0, 1, \ldots, B-1$, together with the final pitch $\frac{3^0}{2^{-1}}$.

The discussion of the previous section suggests that Pythagorean scales whose length $B$ is the denominator of a continued fraction approximation of $\log(3)/\log(2)$ are particularly desirable. In fact, these scales all share a property that is not immediately obvious: there are exactly two different possible ratios between successive notes. For example, in the twelve-note Pythagorean scale (4), each frequency is obtained from the frequency of the previous note by multiplication either by $\frac{3^7}{2^{11}}$ or by $\frac{3^{-5}}{2^{-8}}$. Let us call the ratio between successive notes in a scale a *step*, and call this property of the twelve-note Pythagorean scale the *two-step property.*

Musically speaking, the two-step property is a good feature for a scale to have. The reason is that we often like to transpose songs; that is, we take a sequence of notes and shift all of them up or down by some number of scale steps, perhaps to make the song singable by someone with a higher or lower voice. If all of the step sizes in the scale are the same, then the transposed song sounds very much like the original one. If, on the other hand, the scale has steps of many different sizes, then transposition may dramatically change how a melody sounds. Thus, to say that certain Pythagorean scales have the two-step property is to

[3] In a charming *American Mathematical Monthly* article [1] from November 1948, J. M. Barbour discusses, using the notion of "ternary continued fractions", various possible scale lengths beyond those considered here. He treats scales with too many notes, however, as "nothing but mathematical speculation" that is "useless to the musician". In the modern age of computer-generated pitches, though, things have changed: ideas that might previously have been merely academic have become entirely implementable.

say that, in those scales, transposition does not too seriously alter the sound of a song.

With this motivation in mind, let us understand why the two-step property holds for the scales produced by the continued fraction method. The argument relies on just one key fact about continued fractions. Suppose we write the $n$th truncation of the continued fraction approximation of $\log(3)/\log(2)$—or, indeed, of any real number $x \in [1, 2)$—as

$$[1; k_0, k_1, \ldots, k_n] = \frac{a_n}{b_n}$$

for integers $a_n$ and $b_n$. Then

$$a_n = k_n a_{n-1} + a_{n-2} \tag{5}$$

$$b_n = k_n b_{n-1} + b_{n-2}, \tag{6}$$

where the base cases are defined by

$$a_{-1} = 1, \quad a_{-2} = 1,$$
$$b_{-1} = 1, \quad b_{-2} = 0.$$

The proofs of (5) and (6) are elementary, and are left as a challenge to the interested reader.

Consider, now, the $b_n$-note Pythagorean scales for $n \geq 1$. We will not only prove that these scales possess the two-step property, but we will describe the two step sizes that appear. To do so, let

$$I_n = \frac{3^{b_{n-1}}}{2^{a_{n-1}}},$$

and let

$$J_n = \frac{3^{-b_n + b_{n-1}}}{2^{-a_n + a_{n-1}}}.$$

Then the $b_n$-note Pythagorean scale is formed either from

(A) $b_{n-1}$ steps of size $J_n$ and the other steps of size $I_n$; or
(B) $b_{n-1}$ steps of size $J_n^{-1}$ and the other steps of size $I_n^{-1}$,

with the two types alternating as $n$ increases.

The proof of this strengthening of the two-step property is by induction on $n$. As a base case, we have $b_1 = 2$, and the two-note scale

$$\frac{3^0}{2^0}, \frac{3^1}{2^1}, \frac{3^0}{2^{-1}}$$

is indeed of type (B) in the above dichotomy. (Note that $a_1 = 3$, $a_0 = 2$, and $b_0 = 1$.)

To prove the inductive step, we construct the $b_{n+1}$-note scale from the $b_n$-note scale by a process of "filling in". Suppose the $b_n$-note scale is of type (A). Then, to expand it to the $b_{n+1}$-note scale, replace each

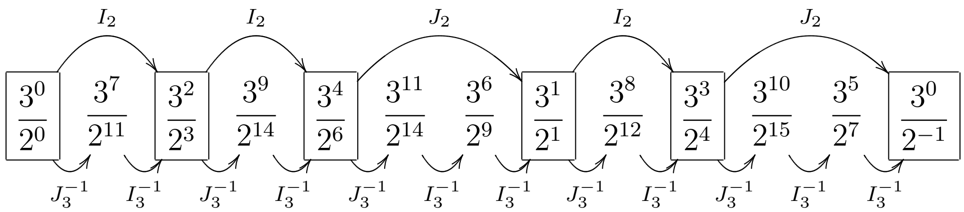



FIGURE 1. Filling in the five-note scale to obtain the twelve-note scale.

step of size $I_n$ by one step of size $J_{n+1}^{-1}$ followed by $k_{n+1}-1$ steps of size $I_{n+1}^{-1}$, and replace each step of size $J_n$ by one step of size $J_{n+1}^{-1}$ followed by $k_{n+1}$ steps of size $I_{n+1}^{-1}$. Figure 1 illustrates the procedure in the case $n=2$, which is the passage from the five-note to the twelve-note scale.

One must now verify a number of facts:

(1) The process described above indeed fills in each step of the $b_n$-note scale.
(2) The new frequencies added to the scale in this way all have numerators of the form $3^b$ with $0 \leq b < b_{n+1}$.
(3) The number of notes added to the scale is $b_{n+1} - b_n$.
(4) All of these new notes are distinct (in other words, no product of the steps $I_{n+1}$ and $J_{n+1}$ produces the step $\frac{3^0}{2^0}$), so the resulting scale has $b_{n+1}$ notes in all.

All four of these facts follow directly from the equations (5) and (6). For example, to prove (1), we may take a step of size $I_n$ in the $b_n$-note scale and attempt to fill it in by one step of size $J_{n+1}^{-1}$ followed by $k_{n+1}-1$ steps of size $I_{n+1}^{-1}$. This works because

$$J_{n+1}^{-1} \cdot (I_{n+1}^{-1})^{k_{n+1}-1} = \frac{3^{b_{n+1}-b_n-(k_{n+1}-1)b_n}}{2^{a_{n+1}-a_n-(k_{n+1}-1)a_n}} = \frac{3^{b_{n-1}}}{2^{a_{n-1}}} = I_n,$$

using equations (5) and (6).

Thus, we have successfully constructed a $b_{n+1}$-note Pythagorean scale of type (B) from a $b_n$-note Pythagorean scale of type (A), and a similar argument constructs a type-(B) scale from a type-(A) scale of the previous size. This completes the induction and confirms the two-step property.

## 3. Other scales

As its name suggests, the method of constructing scales by powers of three has likely been known since the days of Pythagorus. It admits many generalizations, though, which are interesting for both mathematical and music-historical reasons.

For example, one might allow frequencies of the form $3^b 5^c$, adjusted by powers of 2 to lie in the interval $[1, 2)$. The result is what a music theorist would call a *5-limit just scale*. One example of a 5-limit just scale with twelve notes is the following:

$$\frac{3^0 5^0}{2^0}, \frac{3^3 5^1}{2^7}, \frac{3^2 5^0}{2^3}, \frac{3^1 5^2}{2^6}, \frac{3^0 5^1}{2^2}, \frac{3^3 5^2}{2^9}, \frac{3^2 5^1}{2^5}, \frac{3^1 5^0}{2^1}, \frac{3^0 5^2}{2^4}, \frac{3^3 5^0}{2^4}, \frac{3^2 5^2}{2^7}, \frac{3^1 5^1}{2^3}, \frac{3^0 5^0}{2^{-1}}.$$

Playing pitches of these frequencies yields a scale that sounds quite similar to the twelve-note Pythagorean scale.

There is a good reason to prefer the 5-limit scale, though: the numerators and denominators in its frequencies are smaller than those in the Pythagorean scale. When two notes are played simultaneously, they tend to sound more harmonious (at least on traditional string instruments) if the ratio of their frequencies is expressible as a fraction with small numerator and denominator; the explanation is again in terms of the similarity of the Fourier series expressing the motion of strings vibrating at these two frequencies. Consider, then, the fifth note in the above just scale, which is $\frac{5}{4}$, compared to the fifth note in the twelve-note Pythagorean scale, which is $\frac{3^4}{2^6}$. If either one is played simultaneously with a pitch of frequency 1 (forming an interval that, in modern musical terminology, is called a *major third*), then the just option produces a much more pleasing sound.

On the other hand, 5-limit just scales have a downside. If such a scale contains both a "just perfect fifth" (the interval $\frac{3}{2}$) and a "just major third" (the interval $\frac{5}{4}$), it cannot have the two-step property. The twelve-note just scale written above, for example, has steps of three different sizes. Other just scales, even ones with twelve notes, fare even worse. It is an interesting mathematical challenge to describe a property of subsets $A \subset \mathbb{Z}^2$ such that scales of frequencies $3^b 5^c$ with $(b, c) \in A$ have only three step sizes, analogously to the way in which continued fractions yield Pythagorean scales with steps of only two sizes.

Aside from the just scales and their relatives, another possibility is to mimic the Pythagorean process of forming a scale by taking powers of a single number, but to allow that number to be irrational. One historically relevant choice in this vein is to take powers of $5^{1/4}$. A scale consisting of the first twelve non-negative powers of $5^{1/4}$ (adjusted by appropriate powers of 2, and arranged in increasing order) is interesting in that it has the two-step property *and* contains the just major third $\frac{5}{4}$.

This is the so-called *quarter-comma meantone scale* that was popular in Western music of the sixteenth, seventeenth, and eighteenth centuries.

The scale consisting of the first twelve non-negative powers of $2^{1/12}$ is also important, because all of its steps are the same size. This is the 12-tone equal-tempered scale—the Western industry-standard temperament used by piano tuners, guitar manufacturers, and many others today.[4] Harmonically, the equal-tempered scale is in some sense the worst of the options we have considered: not one of the ratios between its frequencies is a fraction with small numerator and denominator. On the other hand, it closely approximates many such ratios while being the only scale with the one-step property, so it perfects the problem of transposition.

Now that we have departed from the Pythagorean framework, moreover, we may as well also depart from the scale size of twelve. Equal-tempered scales, for example, can be formed by dividing an octave into any number of same-sized steps, or the numerators and denominators of all scale notes can be restricted to lie in a certain set as in Harry Partch's tonality diamond [14]. More interestingly, one is not even beholden to octaves as a fundamental unit. Paul Erlich makes the case for relinquishing the primacy of the octave in [8], where he describes how one might arrive at a tuning system by beginning with a lattice of frequencies—all numbers of the form $2^a 3^b$, for instance—and adjusting (literally, "tempering") it so that previously unequal pitches become equal, a process that can and perhaps should involve tempering the octave. See [19, 20] for many further references on the subject of non-octave-based scales.

Do these exotic tuning systems sound unpleasantly dissonant to the untrained ear? They need not, if played on the appropriate instrument. As we have mentioned, the motion of a typical vibrating string can be expressed as a Fourier series whose components ("overtones", in musical parlance) are all integer multiples of a base frequency, and it is due to the similarity in these Fourier decompositions that a pair of frequencies sounds harmonious when related by small integer multiples. With modern technology, however, one can produce sounds with any desired overtone series, and this can be used to engineer timbres in which a tempered octave sounds more pleasant than a true $2:1$ ratio, or in which any predetermined interval becomes a consonance. Research

[4]Stuart Isacoff [10] traces the ascendancy of equal temperament to the influence of French musician Jean-Philippe Rameau and the invention of the modern piano, both in the mid-eighteenth century; by the beginning of the twentieth century, Isacoff writes, "Americans were buying more than 350,000 pianos a year," and "they were all tuned, more or less, in equal temperament."

in this fascinating direction was pioneered by music theorist William A. Sethares [17, 18].

When one considers the relationship between intervals and consonance to be malleable in this way, many new doors open. There are optimization problems related to choosing the best timbre for a given scale [17], and conversely, there are questions about how to measure the dissonance of an interval after a timbre is chosen; Sethares's original work used an experimentally-based measure he calls "sensory dissonance", which is built on sound-perception studies of Plomp and Levelt [15], but another possibility is Erlich's "harmonic entropy" [18, Appendix J], which uses mathematics from information theory to model the extent to which the ear is uncertain about which of a discrete set of ratios a heard interval belongs to. There is work to be done in deciding which among the infinitely-many possible scales is musically desirable, and in developing a music theory—indeed, even a reasonable notation—by which to understand them, work that is being actively carried out by the community of "xenharmonic" musicians [25, 16]. A key role in this process is played by computer programs, including notation software such as Mus2 [13] and the excellent free app Wilsonic [24] dedicated to the tuning theory of Ervin M. Wilson [23].

And all of this is prior to the work of composition itself. Too many composers to name have written works in a wealth of tuning systems, among them Lou Harrison (who constructed, together with his partner Bill Colvig, a number of instruments meant to played in tunings other than 12-tone equal temperament), Ben Johnston (who composed in just scales with many prime factors and proposed a corresponding new notation), Easley Blackwood (who has systematically explored all equal temperaments with between 13 and 24 pitches), Wendy Carlos [5], Brendan Byrnes [4], Tolgahan Çoğulu, Kraig Grady, and many others. Though mathematics led us to the twelve-tone scale, it is becoming consistently clearer that both mathematically and musically speaking, twelve tones are just the beginning.

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