R13.2 Internal-Age Distribution, $I(\alpha)$

The relationship between the *internal-age* and *external-age distribution* can be demonstrated by analyzing a continuous reactor operating at steady state that is filled with material of volume V. Consider again that the volume of reactor is filled with maize-colored molecules, and at time $t = 0$ we start to inject blue molecules to replace the maize molecules. By definition of $I(\alpha)$, the volume of molecules inside the reactor that have been there between a time α and $\alpha + d\alpha$ is

$$dV = V[I(\alpha)]\, d\alpha \tag{R13.2-1}$$

At $t = 0$ we will let $(v_0\, d\alpha)$ be the first volume of blue molecules that enter the reactor. We want to consider what has happened to the molecules in this volume at a time α after being injected. Some of the molecules will already have left the system at a time α, while others remain. The fraction of molecules that still remain in the system is $[1 - F(\alpha)]$. Consequently, the volume of molecules that entered the system between $t = 0$ and $t = d\alpha$ and are still in the system at a later time α is

Finding a relation between $E(\alpha)$ and $I(\alpha)$

$$dV = v_0\, d\alpha[1 - F(\alpha)] \tag{R13.2-2}$$

This is the volume of molecules that have an age between α and $(\alpha + d\alpha)$. Equating Equations (R13.2-1) and (R13.2-2) and dividing by V and by $d\alpha$ gives

$$I(\alpha) = \frac{v}{V}[1 - F(\alpha)]$$

Then

Relating $I(\alpha)$ to $F(\alpha)$ and $E(\alpha)$

$$\boxed{I(\alpha) = \frac{1}{\tau}[1 - F(\alpha)] = \frac{1}{\tau}\left[1 - \int_0^{\alpha} E(\alpha)\, d\alpha\right]} \tag{R13.2-3}$$

Differentiating Equation (R13.2-3) and noting that

$$\frac{d[1 - F(\alpha)]}{d\alpha} = -E(\alpha)$$

gives

$$\boxed{E(\alpha) = -\frac{d}{d\alpha}[\tau I(\alpha)]} \tag{R13.2-4}$$

As a brief exercise, the internal-age distribution of a perfectly mixed CSTR will be calculated. Equation (13-27) gives the RTD of the reactor, which upon substitution into Equation (R13.2-3) gives

$$I(\alpha) = \frac{1}{\tau}\left(1 - \int_0^\alpha \frac{1}{\tau}\, e^{-\alpha/\tau}\, d\alpha\right)$$

$$= \frac{1}{\tau}\left(1 + e^{-\alpha/\tau}\Big|_0^\alpha\right)$$

$$= \frac{1}{\tau}\, e^{-\alpha/\tau} \qquad \text{(R13.2-5)}$$

True only for a perfectly mixed CSTR

Thus the internal-age distribution of a perfectly mixed CSTR is identical to the exit-age distribution, or RTD, because the composition of the effluent is identical to the composition of the material anywhere within the CSTR when it is perfectly mixed.

Example R13.2–1 CSTR with Fresh Catalyst Feed

When a catalyst is decaying, fresh catalyst must be fed to a reactor to keep a constant level of activity. The relation between catalyst weight, conversion, and catalyst activity is

$$W = \frac{F_{A0}X}{-r'_A} = \frac{F_{A0}X}{\bar{a}k_0C_A^n} \qquad \text{(RE13.2-1)}$$

where $\bar{a}$ is the mean activity in the reactor. Determine the mean activity for first-order decay in a CSTR.

Solution

Because there will be a distribution of times the various catalyst particles have spent in the reactor, there will be a distribution of activities. The mean activity is the integral of the product of the fraction of the particles that have been in the reactor (i.e., have ages) between time α and $\alpha + \Delta\alpha$, $I(\alpha)\, d\alpha$, and the activity at time α:

$$\bar{a} = \int_0^\infty a(\alpha)I(\alpha)\, d\alpha \qquad \text{(RE13.2-2)}$$

For first-order decay,

$$a = e^{-k\alpha} \qquad \text{(RE13.2-3)}$$

In a well-mixed CSTR,

$$I(\alpha) = \frac{1}{\tau}\, e^{-\alpha/\tau} \qquad \text{(R13.2-5)}$$

Using $I(\alpha)$ and $a(\alpha)$ to find the mean catalyst activity

$$\bar{a} = \int_0^\infty \frac{e^{-k_d\alpha}e^{-\alpha/\tau_c}}{\tau_c}\, d\alpha \qquad \text{(RE13.2-4)}$$

where k_d is the decay constant and τ_c is the mean contact time, such that

$$\tau_c = \frac{W}{F_c} = \frac{\text{weight of catalyst (kg)}}{\text{feed rate of catalyst (kg/s)}} \tag{RE13.2-5}$$

Integrating yields

$$\boxed{\bar{a} = \frac{1}{\tau_c k_d + 1}} \tag{RE13.2-6}$$

We see that for a distribution of activities, each following first-order decay in an ideal CSTR, the form of the mean activity is identical to the integrated form for second-order catalyst decay. See Problem 13-2(a). What if the catalyst decay law in Example 13-6 were second order? Third order? What if the catalyst decay law followed that of West Texas crude in Example 10–7 with t_m = 10 s? What generalizations can you make?