Ramsey Theory: Order From Chaos

Justen Holl, Elizabeth Tso, and Julia Balla

Fall 2020

Abstract

Ramsey Theory is the study of how specific patterns inevitably emerge in sufficiently large systems. This paper provides an overview of three key theorems in Ramsey Theory: Ramsey's Theorem, Van der Waerden's Theorem, and Rado's Theorem, which deal with finding patterns in mathematical objects such as graphs, the number line, and systems of linear equations respectively. While each theorem is distinctly different, they share a common objective of discerning order in chaos.

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1 Introduction

A simple yet powerful example of finding order in chaos is the pigeonhole principle, which is at the core of much of Ramsey Theory.

Theorem 1.0.1 (**Pigeonhole Principle**)**.** *If there exist m pigeonholes containing n pigeons, where $n > m$, then at least one of the pigeonholes must contain at least 2 pigeons.*

The following diagram demonstrates the pigeonhole principle for placing $n = m + 1$ pigeons (represented by circles) into m pigeonholes (squares), where one pigeonhole is guaranteed to contain 2 pigeons.
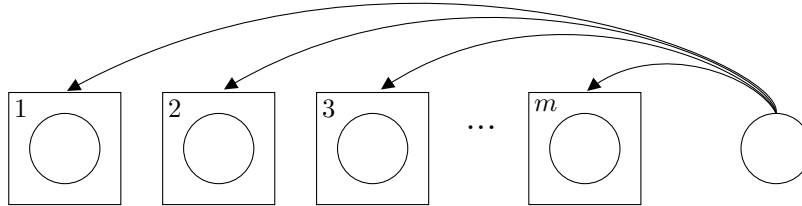
Theorem 1.0.2 (**Infinite Pigeonhole Principle**). *If there exist a finite number of pigeonholes containing an infinite number of pigeons, then at least one of the pigeonholes must contain an infinite number of pigeons.*

This type of counting argument allows us to make conclusions about sets of objects partitioned into a finite number of classes, namely that one of these classes must have a certain size. One illustrative application of the pigeonhole principle is in solving the following classic puzzle in Ramsey Theory.

Friends and Enemies Puzzle: *Suppose there is a crowd of* 6 *people, where any* 2 *people are either friends or enemies. Show that there are always either at least* 3 *people who are pairwise mutual friends, or at least* 3 *who are pairwise mutual enemies.*

We can easily express this problem using graph-theoretic terms:

Definition 1.0.1. A **graph** is a pair $G = (V, E)$, where V is a set of vertices and E is a set of size-2 subsets of V. Each edge in E connects 2 vertices in V.

Example 1.0.1. The following graph G contains vertices $V = \{v_0, v_1, v_2, v_3\}$ and edges $E = \{\{v_0, v_1\}, \{v_1, v_2\}, \{v_2, v_3\}\}$.
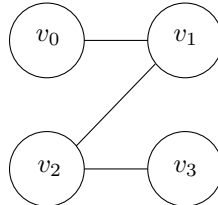
Definition 1.0.2. A **complete graph** is a graph where for any 2 distinct vertices $v_1, v_2 \in V$, there exists an edge $\{v_1, v_2\} \in E$ connecting them. We denote the complete graph on n vertices by K_n.
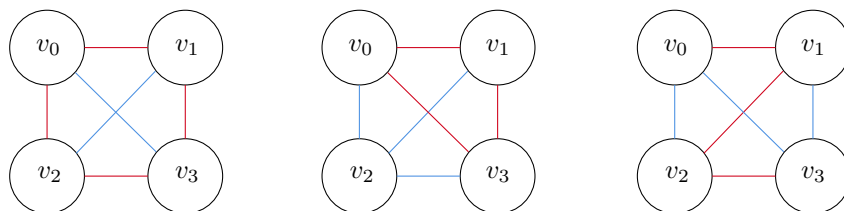
Example 1.0.2. K_4 is the complete graph on 4 vertices.
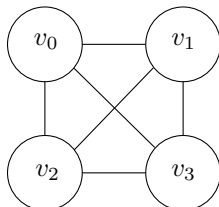
Definition 1.0.3. A graph is **r-colored** if each edge $e \in E$ is assigned a color from one of r colors.

Note that this definition is different from the usual notion of graph coloring, in which vertices are colored instead of edges.

Example 1.0.3. The following are three examples of 2-colorings of K_4, colored red and blue.

Returning to the Friends and Enemies Puzzle, we can express the crowd of 6 as the complete graph K_6. If 2 people are friends, color the edge connecting the corresponding vertices red. Otherwise, they are enemies, so color the edge blue. Solving the puzzle is then equivalent to proving the following statement: *A 2-colored K_6, colored red and blue, must either contain a red K_3 or blue K_3.* In section 3.1, we show that the proof is simply a direct application of the pigeonhole principle to colored graphs, where edges in the graph are sorted into red or blue pigeonholes.

Suppose we had successfully proved the above result. Then any 2-colored complete graph on more than 6 vertices must also contain either a red K_3 or blue K_3, since it contains K_6 by definition. The following question then arises: is 6 the smallest size for a complete graph to guarantee red K_3 or blue K_3 subgraph, or can we do better?

The answer to this question is precisely the definition of a Ramsey Number, as described in **Ramsey's Theorem**. The theorem states that sufficiently large, finitely colored, complete graphs must contain a specific monochromatic subgraph. We first prove Ramsey's Theorem for graphs of finite size, and then extend it to infinite graphs and hypergraphs.

Next, we provide a proof of **Van der Waerden's Theorem**, which is concerned with finding monochromatic substructures called arithmetic progressions in finite colorings of the set of natural numbers, $\mathbb{N}$. An arithmetic progression is defined to be a sequence of numbers with the same common difference.

Finally, we discuss **Rado's Theorem**, which deals with finding monochromatic solutions to systems of linear equations using finite colorings of $\mathbb{N}$. In fact, by representing an arithmetic progression using a system of equations, we show that Van der Waerden's Theorem is a specific case of Rado's which states that there must exist a monochromatic solution to the aforementioned system of equations.

2 History and Philosophy of Ramsey Theory

We as the human race have always been devoted to finding structure within chaos. For thousands of years, religions, philosophies, and disciplines have been devoted to finding signs within naturally occurring events. Ancient Greek philosophy centered around *logos*, the *order* which they believed was embedded in the universe; the philosopher Heraclitus defined the term to encapsulate the belief that there was a universal law to impose order on the cosmos.

In fact, we have explicit records from as far back as 3,500 years ago of people finding specific patterns in the seemingly random. According to surviving cuneiform text, an ancient Sumerian scholar saw in the heavens that the stars seemed to form a lion, a bull, and a scorpion. Today, we also group the stars into shapes, constellations like the Big Dipper, Little Dipper, or Orion. These shapes we see in the sky at night beg the question: what are the chances that the stars would fall in the shape of a person, or even just in a straight line? How can it be possible that of all the configurations these astronomical bodies could form, the stars form shapes familiar to us? Are we, as humans, projecting a desire for order and imagining patterns where there are none, or does the universe actually contain structure?

In fact, mathematics holds the answer. Frank Plumpton Ramsey began the study of the eponymous Ramsey Theory, a branch of mathematics devoted to the study of order within chaos. While Ramsey Theory contains many theorems, the essential overarching result is that in large enough groups, structure does emerge. This is a fascinating result; because there are so many stars in the sky, we will be able in certainty to find lines or shapes among them.

3 Ramsey's Theorem

3.1 Ramsey's Theorem for colored graphs

Definition 3.1.1. The **Ramsey Number**, $R(s,t)$, is the number of vertices in the smallest complete graph which, when 2-colored red and blue, must contain a red K_s or a blue K_t, where we denote the complete graph on n vertices by K_n.

Example 3.1.1. R(3,3) = 6.
As discussed in the introduction, showing that 6 is the smallest size of a crowd that guarantees at least 3 mutual friends or 3 mutual enemies is equivalent to showing $R(3,3) = 6$, for which we will now provide a proof.

Proof. First, we show that $R(3,3) > 5$ (or $R(3,3) \geq 6$) by exhibiting a complete graph on 5 vertices that does not contain a red K_3 or blue K_3:
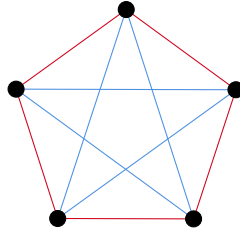
We now show that K_6 must always contain a red K_3 or blue K_3. Recall that this is equivalent to the statement of the Friends and Enemies Puzzle.

First, pick any vertex v and consider the edges incident to it:
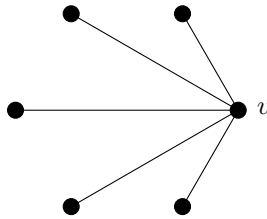
Since there are 5 edges and only 2 possible colors for each edge, by the pigeonhole principle, at least 3 of these edges must have the same color. Without loss of generality, assume there are 3 blue edges connecting v to 3 other vertices.
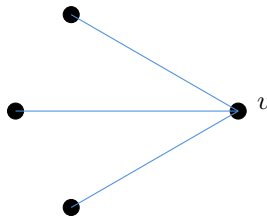
Consider the K_3 subgraph generated by the 3 adjacent vertices. If all edges in the subgraph are red, then we have found a red K_3.
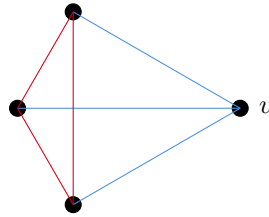
Otherwise, at least one of the edges must be blue. This edge completes a blue K_3 with the original set of 3 blue edges incident to v.
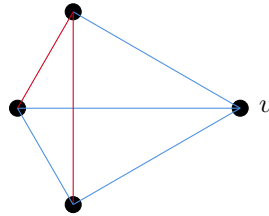
Therefore, $R(3,3) = 6$. □

This example is a specific case of the more general Ramsey's Theorem.

Theorem 3.1.1 (**Ramsey**). *For any two natural numbers, s and t, there exists a natural number, $R(s,t) = n$, such that any 2-colored complete graph of order at least n, colored red and blue, must contain a monochromatic red K_s or blue K_t.*

Proof. This proof will follow a similar outline to how Taylor formulated it in [1]. It suffices to show that $R(s,t)$ exists by proving it is upper-bounded. The proof will be by induction.

Base Cases:

- $R(s,2) = R(2,s) = s$. Either every edge is colored the same, or at least one is colored differently. If they are all colored the same then a K_s exists, otherwise at least a K_2 exists.

- $R(s,1) = R(1,s) = 1$. This holds as there is no innate difference between coloring a vertex red or blue.

Inductive Hypothesis: Assume $R(s-1,t)$ and $R(s,t-1)$ exist.

Claim. $R(s,t) \leq R(s-1,t) + R(s,t-1)$
First look at a 2-colouring of a complete graph with $n = R(s-1,t) + R(s,t-1)$ vertices. Then pick a vertex in K_n. Let's call that vertex x.

Define R_x such that every edge connecting a vertex in R_x to x is red. Similarly, let B_x be the set of vertices adjacent to x such that every edge connecting a vertex in B_x to x is blue. Since K_n is a complete graph, $B_x = [n] \setminus (R_x \cup \{x\})$ and so $|R_x| + |B_x| = n - 1$. If $|R_x| \leq R(s-1, t)$ and $|B_x| \leq R(s-1, t)$ then since $n = R(s-1, t) + R(s, t-1)$ we must have $|R_x| + |B_x| \leq n - 2$, a contradiction. So $|B_x| \geq R(s, t-1)$ or $|R_x| \geq R(s-1, t)$.

If $|B_x| \geq R(s, t-1)$ and B_x induces a red K_s we are done. If B_x induces a blue K_{t-1} then K_n must contain a blue K_t since $B_x \cup \{x\}$ must induce a blue K_t. Indeed, each edge $\{x, y\}$ is blue for all $y \in B_x$, from the definition of B_x. So $B_x \cup \{x\}$ must induce a blue K_t if B_x contains a blue K_{t-1}. The case for R_x is completely symmetric.

We have shown that a 2-coloured complete graph of order $R(s-1, t) + R(s, t-1)$ must contain a red K_s or a blue K_t, proving that $R(s, t) \leq R(s-1, t) + R(s, t-1)$. This completes our induction.

$\square$

There exist multiple generalizations of Ramsey's Theorem, for example to graphs colored with 3 or more colors. We examine two other generalizations: Ramsey's for graphs of infinite size and for hypergraphs.

3.2 Infinite Ramsey's Theorem for colored graphs

Definition 3.2.1. $K_{\mathbb{N}}$ is the complete graph whose vertex set is countably infinite.

In other words, the vertices of $K_{\mathbb{N}}$ can be enumerated using the set of natural numbers, as shown below:
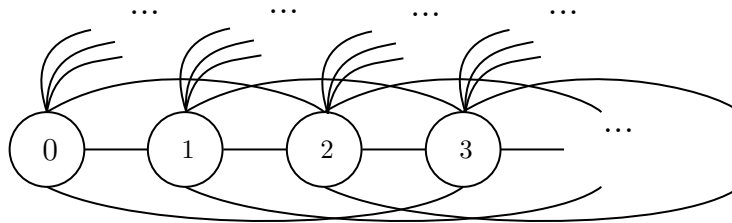
We first show Infinite Ramsey's Theorem for 2 colors, which will then help us generalize to any finite number of colors.

Theorem 3.2.1 (**Infinite Ramsey's for 2 colors**)**.** *Every 2-colored $K_{\mathbb{N}}$ must contain a countably infinite monochromatic complete graph.*

Proof. Fix some 2-coloring of $K_{\mathbb{N}} = (V, E)$, where each edge is colored either red or blue. We will use this coloring to construct an infinite monochromatic complete subgraph by iteratively restricting the vertices we include in the vertex

set. The key will lie in the infinite pigeonhole principle, which will allow us to continually choose infinite subsets of an infinite set of vertices.

Consider some vertex $v_0 \in V$ and all edges incident to it.
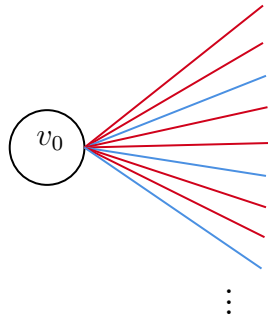
There are infinitely many such edges, and only 2 possible colors for each edge. Therefore, by the infinite pigeonhole principle, either the set of red edges incident to v_0 is infinite, or the set of blue edges is infinite. Since these 2 cases are symmetric, assume without loss of generality that there are infinitely many red edges incident to v_0.
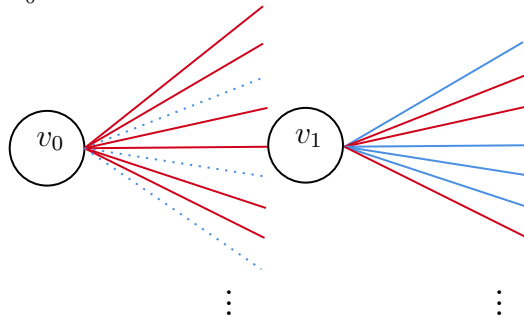
Let S_0 be the infinite set of vertices connected to v_0 by a red edge. Pick some vertex $v_1 \in S_0$ and consider the edges incident to v_1 and some other vertex in S_0.

There are infinitely many such edges since S_0 is infinite, and only 2 possible colors for each edge. Again, by the infinite pigeonhole principle, there must either be an infinite set of red edges or of blue edges. For example, assume that there are infinitely many blue edges incident to v_1 and some other vertex in S_0.

Let S_1 be the infinite set of vertices in S_0 connected to v_1 by a blue edge. Note that all vertices in S_1 are also connected to v_0 by a red edge, since $S_0 \supset S_1$. Pick some vertex $v_2 \in S_1$ and consider the edges incident with v_2 and some other vertex in S_1.
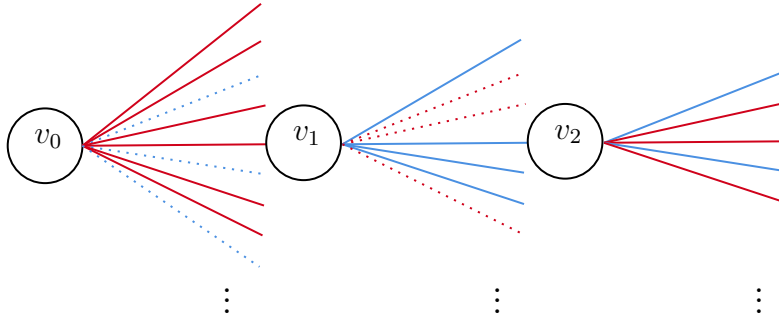
There are again infinitely many such edges and only 2 possible colors for each edge, so by the infinite pigeonhole principle, there must either be an infinite set of red edges or of blue edges. For example, assume red.

Let S_2 be the infinite set of vertices in S_1 connected to v_2 by a red edge. Note that all vertices in S_2 are also connected to v_1 by a blue edge and to v_0 by a red edge, since $S_0 \supset S_1 \supset S_2$.

We can continue this procedure of picking a successive vertex and applying the infinite pigeonhole principle to generate an another infinite subset of vertices indefinitely because $K_\mathbb{N}$ is infinite. This will result in an infinite set of vertices $V^* = \{v_0, v_1, v_2, \ldots\}$.

Let $E^* = \{\{v_0, v_1\}, \{v_0, v_2\}, \ldots, \{v_1, v_2\}, \ldots\}$ be the set of edges connecting the vertices in V^*. Observe that the color of each edge in E^* is determined by the vertex with the smaller index. For $\{v_a, v_b\} \in E^*$ where $a < b$, vertex v_a is chosen first, so only the vertices connected to v_a by a red (or blue, depending on which color corresponded to the infinite set) edge are included in the set S_a. Since v_b is chosen at a successive step, then $v_b \in S_{b-1} \subset \ldots \subset S_a$, so v_b must be connected to v_a by a red (or blue) edge. The color of edge $\{v_a, v_b\}$ is thus determined by v_a.

Therefore, all edges $\{v_a, v_b\} \in E^*$ where v_a is the vertex with the smaller index must have the same color. Color each vertex $v_a \in V^*$ with the color corresponding to these edges: $V^* = \{v_0, v_1, v_2, \ldots\}$. The set V^* contains infinitely many vertices, and each vertex has 2 possible colors. By the infinite pigeonhole principle, V^* must contain either an infinite set of red vertices or of blue vertices. Call this set M.

The graph induced by the vertices in M is a subgraph of $K_\mathbb{N}$. Furthermore, this subgraph must be complete, since each successive vertex $v_b \in V^*$ is always connected to all of the previous vertices $v_a \in V^*$ for $a < b$. Hence, the subgraph induced by M is a countably infinite monochromatic complete graph, as desired. □

We can now generalize Infinite Ramsey's Theorem for 2 colors to r colors.

Theorem 3.2.2 (**Infinite Ramsey's for r colors**). *Every r-colored $K_\mathbb{N}$ must contain a countably infinite monochromatic complete graph, where $1 \leq r < \infty$.*

Proof. Proof by induction on r.

Base cases:

- $r = 1$: The theorem holds trivially, since a 1-colored $K_\mathbb{N}$ is itself a countably infinite monochromatic complete graph.

- $r = 2$: The theorem holds by Infinite Ramsey's Theorem for 2 colors, shown previously.

Inductive Hypothesis: Assume that for some r such that $1 \leq r < \infty$, every r-colored $K_\mathbb{N}$ contains a countably infinite monochromatic complete graph. We show that the theorem holds for $r + 1$.

Suppose $K_\mathbb{N}$ is colored with $r + 1$ colors $c_1, c_2, \ldots, c_{r+1}$. Take all edges that are colored one of $c_1, c_2, c_3, \ldots, c_{r-1}$ and color them all c_r. We now have a 2-coloring of $K_\mathbb{N}$, where each edge is colored either c_r or c_{r+1}. By Infinite Ramsey's for 2 colors, $K_\mathbb{N}$ must contain a monochromatic countably infinite complete subgraph. In the new coloring, this subgraph must be colored either:

1. c_{r+1}: In this case, the edges are also colored c_{r+1} in the original coloring of $K_\mathbb{N}$, so they form a countably infinite monochromatic complete subgraph. Therefore, the inductive hypothesis holds for $r + 1$.

2. c_r: The edges colored by c_r in the new coloring are colored by one of $c_1, c_2, \ldots, c_r$ in the original coloring of $K_\mathbb{N}$. Since these edges form a countably infinite complete r-colored graph, by the inductive hypothesis, this graph must contain a countably infinite monochromatic subgraph. Thus, $K_\mathbb{N}$ contains a countably infinite monochromatic subgraph, as desired.

In both cases, the inductive hypothesis holds for $r + 1$ colors. Conclude that for any finite number of colors r, every r-colored $K_\mathbb{N}$ must contain a countably infinite monochromatic complete graph. $\square$

Infinite Ramsey's Theorem for 2 colors can be used to generate a second proof of the finite case, shown here.

Second Proof of Ramsey's Theorem. Recall the theorem statement: *For any two natural numbers, s and t, there exists a natural number, $R(s, t) = n$, such that any 2-colored complete graph of order at least n, colored red and blue, must contain a monochromatic red K_s or blue K_t.*

Proof. For contradiction, assume that for some $s, t \in \mathbb{N}$, there is a 2-colored K_n without a red K_s or blue K_t for all $n \in \mathbb{N}$. In other words, we can always find an n large enough such that for every complete graph on at least n vertices, there exists a 2-coloring that does not contain a red K_s or blue K_t. Hence,

$R(s, t) = \infty$. We then have an infinite sequence of 2-colorings of $K_2, K_3, K_4, \ldots$ which don't contain a red K_s or blue K_t. We denote this sequence of colorings by $C(K_2), C(K_3), C(K_4), \ldots$.

The idea of the proof is to use these colorings to color $K_\mathbb{N}$ such that it has no red or blue countably infinite monochromatic complete subgraph, which would contradict Infinite Ramsey's Theorem for 2 colors. The graph $K_\mathbb{N}$ that we will color is shown below.
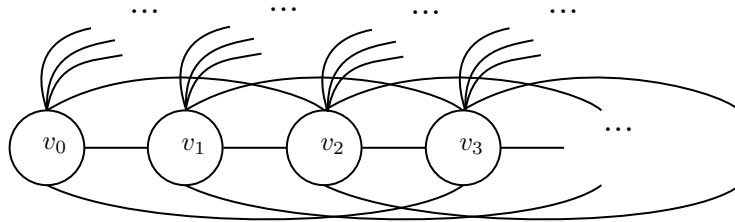
Let $K_\mathbb{N} = (V, E)$, where $V = \{v_0, v_1, v_2, \ldots\}$.

First, consider edge $\{v_0, v_1\} \in E$. There are infinitely many colorings in the sequence $C(K_2), C(K_3), C(K_4), \ldots$, and each colors $\{v_0, v_1\}$ in one of 2 ways. By the infinite pigeonhole principle, there must either be infinitely many colorings that color $\{v_0, v_1\}$ red, or that color it blue. Assume for example that there are infinitely many colorings where $\{v_0, v_1\}$ is red, and remove all colorings where $\{v_0, v_1\}$ is blue. The sequence of colorings is still infinite, since we've only removed a finite number of colorings.
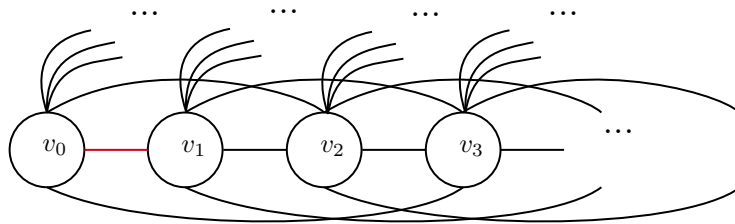
Next, consider edge $\{v_1, v_2\} \in E$. There are infinitely many colorings in the sequence, and 2 possible colors for $\{v_1, v_2\}$ in each coloring. Again, by the infinite pigeonhole principle, there must be infinitely many colorings where $\{v_1, v_2\}$ is red, or where it is blue. For example, assume blue. Remove the finitely many colorings where $\{v_1, v_2\}$ is red, and also remove $C(K_2)$, since K_2 does not contain a third vertex v_2. The sequence of colorings is again still infinite.
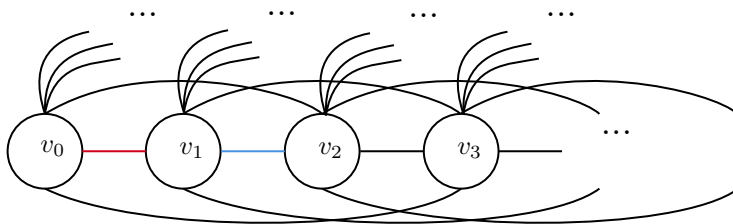
Repeat this procedure for edge $\{v_2, v_3\}$. By the infinite pigeonhole principle, there must either be infinitely many remaining colorings that color $\{v_2, v_3\}$ red, or that color it blue. For example, assume red, and remove $C(K_3)$ and the colorings where $\{v_2, v_3\}$ is blue.
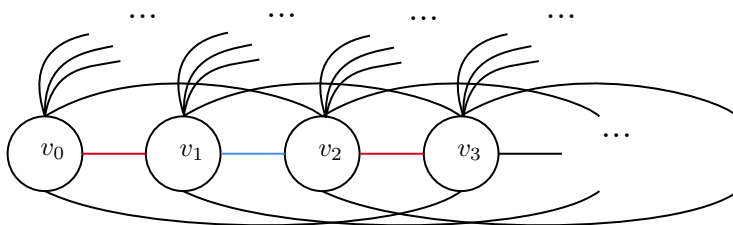
We can continue this process of coloring each edge indefinitely since the sequence of colorings will always stay infinite after removing a finite number of them. This will result in some 2-coloring of $K_{\mathbb{N}}$. Moreover, this coloring does not contain a red K_s or blue K_t, since it is constructed from colorings that do not contain a red K_s or blue K_t.
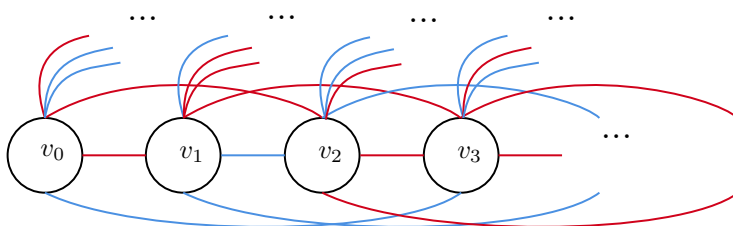
By Infinite Ramsey's Theorem for 2 colors, any 2-colored $K_{\mathbb{N}}$ must contain a red or blue monochromatic countably infinite complete subgraph. This subgraph then contains a red K_s or blue K_t, but the coloring that we've constructed for $K_{\mathbb{N}}$ does not contain either. We have reached a contradiction, which concludes the proof. □

3.3 Infinite Ramsey's Theorem for colored hypergraphs

We can further generalize Infinite Ramsey's Theorem for ordinary graphs to hypergraphs.

Definition 3.3.1. A **hypergraph** is a pair $H = (V, E)$, where V is a set of vertices and E is a set of subsets of V. Each **hyperedge** in E can connect any number of vertices in V.

Example 3.3.1. The following hypergraph H contains vertices $V = \{v_0, v_1, v_2, v_3, v_4\}$ and hyperedges $E = \{\{v_0, v_1\}, \{v_1, v_3, v_4\}, \{v_0, v_2\}\}$.
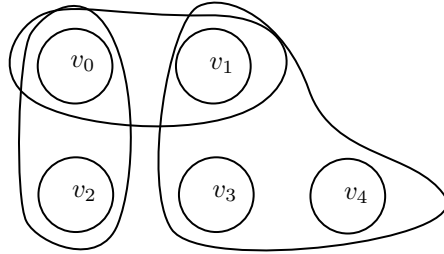
Example 3.3.2. An ordinary graph is a hypergraph where each hyperedge connects exactly 2 vertices.
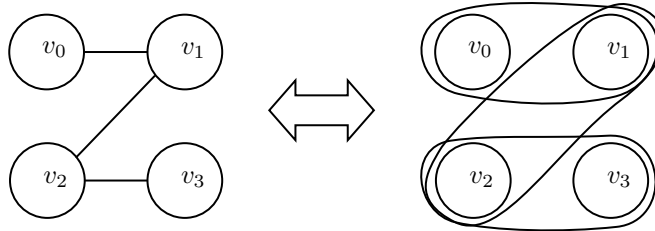
Definition 3.3.2. Let $H^{(k)}$ denote the **set of all size-k hyperedges** of a hypergraph H.

Example 3.3.3. In Example 2.4.1, $H^{(2)} = \{\{v_0, v_1\}, \{v_0, v_2\}\}$ and $H^{(3)} = \{\{v_1, v_3, v_4\}\}$.

As with ordinary graphs, there is a notion of r-colored sets of hyperedges, where each hyperedge is colored with one of r colors.

Example 3.3.4. The following is a 2-coloring of the hypergraph in Example 2.4.1: $H^{(2)} = \{\{v_0, v_1\}, \{v_0, v_2\}\}$ and $H^{(3)} = \{\{v_1, v_3, v_4\}\}$.

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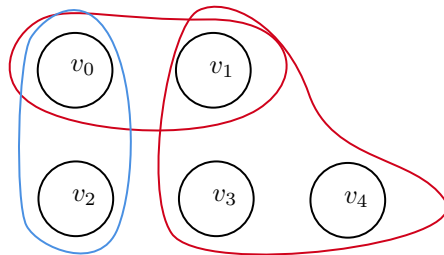
Definition 3.3.3. Let $H_\mathbb{N}$ denote the **countably infinite complete hypergraph**. Its vertices can be enumerated using the natural numbers, and it contains every possible hyperedge.

Theorem 3.3.1 (**Infinite Ramsey's for Hypergraphs**)**.** *If $H_\mathbb{N}^{(k)}$ is r-colored, where $1 \le r < \infty$ and $1 \le k < \infty$, then $H_\mathbb{N}$ contains a countably infinite subhypergraph $H'_\mathbb{N}$ such that ${H'_\mathbb{N}}^{(k)}$ is monochromatic.*

Proof. Proof by induction on the size of hyperedges, k. Note that this differs from the proof of Infinite Ramsey's for ordinary graphs, which uses induction on the number of colors r.

<u>Base cases</u>:

- $k = 1$: $H_\mathbb{N}^{(1)}$ is simply the infinite set of vertices of $H_\mathbb{N}$, which is colored with a finite number of colors r. By the infinite pigeonhole principle, $H_\mathbb{N}^{(1)}$ must contain a monochromatic infinite set of vertices ${H'_\mathbb{N}}^{(1)}$, where $H'_\mathbb{N} \in H_\mathbb{N}$ is the complete hypergraph induced by those vertices. The theorem then holds for $k = 1$.

- $k = 2$: $H_\mathbb{N}^{(2)}$ is the set of size-2 hyperedges of $H_\mathbb{N}$. As shown in Example 2.4.2, these edges are precisely the ordinary edges that make up the countably infinite complete graph, $K_\mathbb{N}$. Thus, this case is equivalent to Infinite Ramsey's Theorem for ordinary graphs for r colors, which states that $K_\mathbb{N}$ must contain a monochromatic countably infinite complete subgraph. Hence, $H_\mathbb{N}^{(2)}$ must contain a monochromatic infinite set of edges ${H'_\mathbb{N}}^{(2)}$, where $H'_\mathbb{N} \in H_\mathbb{N}$ is the complete hypergraph induced by those edges. The theorem then holds for $k = 2$.

Inductive Hypothesis: Assume for some k that if $H_\mathbb{N}^{(k)}$ is r-colored, then $H_\mathbb{N}$ contains a countably infinite subhypergraph $H'_\mathbb{N}$ such that ${H'_\mathbb{N}}^{(k)}$ is monochromatic. We show that this statement holds for $k + 1$.

Fix an r-coloring $C_0(e)$ of the hyperedges $e \in H_\mathbb{N}^{(k+1)}$, and set $H_0 := H_\mathbb{N}$. Pick some vertex v_0 in H_0, and define a new hypergraph $G_0 := H_0 - v_0$ (the hypergraph H_0 with one vertex removed). Color $G_0^{(k)}$ with r colors using the following mapping: for each size-k hyperedge $e \in G_0^{(k)}$,

$$C_1(e) = C_0(e \cup \{v_0\})$$

Thus, the color of e is defined by the color of the size-$(k+1)$ hyperedge connecting the vertices in e with vertex v_0 in H_0. By the inductive hypothesis, since $G_0^{(k)}$ is r-colored, then G_0 must contain a countably infinite subhypergraph H_1 such that $H_1^{(k)}$ is monochromatic in the coloring C_1.

Next, pick a vertex v_1 in H_1, and define a new hypergraph $G_1 := H_1 - v_1$. Color $G_1^{(k)}$ with r colors using the following mapping: for each edge $e \in G_1^{(k)}$,

$$C_2(e) = C_0(e \cup \{v_1\})$$

Again, by the inductive hypothesis, G_1 must contain a countably infinite sub-hypergraph H_2 such that $H_2^{(k)}$ is monochromatic in the coloring C_2.

This procedure can be repeated indefinitely since each hypergraph is infinite, resulting in an infinite sequence of r-colored vertices $V = \{v_0, v_1, v_2, \ldots\}$ and an infinite chain of subhypergraphs $H_0 \supset H_1 \supset H_2 \supset \ldots$ where each v_i is in the hypergraphs H_j where $j \leq i$. The color assigned to each vertex v_i is the same as the color of the monochromaic set $H_{i+1}^{(k)}$, which was generated using v_i.

There are infinitely many vertices in the set V, and finitely many possible colors for each vertex. By the infinite pigeonhole principle, there must be a monochromatic infinite subset $V' \subset V$.

Observe that by the definition of each coloring C_i, all size-$(k+1)$ hyperedges in the original hypergraph $H_\mathbb{N}$ whose only vertex outside of H_i is v_{i-1} must have the same color. For instance, if v_{i-1} is red, then any size-$(k+1)$ hyperedge in H_i consisting of vertex v_{i-1} with k vertices from the set $\{v_i, v_{i+1}, v_{i+2}, \ldots\}$ must also be colored red. Thus, since V' is monochromatic, every size-$(k+1)$ hyperedge connecting vertices in V' must have the same color. In other words, the countably infinite hypergraph $H'_\mathbb{N}$ formed on the set of vertices V', $H'^{(k+1)}_\mathbb{N}$, is monochromatic, as desired. $\square$

The infinite case of Ramsey's Theorem for hypergraphs can be used to prove the case for complete hypergraphs of finite size. We do not show the proof here, as it is very similar to the second proof of Ramsey's for finite ordinary graphs shown in section 2.3.

4 Van der Waerden's Theorem

A year before Ramsey published his theorem, Bart Leendert van der Waerden published his seminal paper in which he proved what would soon become known as Van der Waerden's Theorem.

Theorem 4.0.1 (**Van der Waerden's Theorem**). *For all positive integers, k and r, there exists a natural number W(k,r) such that, if the set of natural numbers 1,2,...,W(k,r) is r-colored, then it must contain at least one monochromatic k-term arithmetic progression.*

First, we must define **color focusing**, as it will be used frequently in proofs of different Van der Waerden's cases.

Definition 4.0.1. In an r-coloring of the natural numbers, t different k-term arithmetic progressions are **color focused** if

- they are monochromatic,

- none have the same color,

- their $(k+1)^{\text{th}}$ terms are equal. This term is called the **color focus**.

We use color focusing when one number in a sequence must be colored a certain color to prevent a monochromatic progression.

Example 4.0.1. Suppose we are trying to prevent a monochromatic progression of length 4 in the following 2-colored sequence: {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13}. Then 13 is the color focus because it is the fourth term of each monochromatic arithmetic progression 1, 5, 9 and 4, 7, 10. Hence, coloring 13 either red or blue forces a 4-term monochromatic progression.

Before proceeding with the proof of the general case of Van der Waerden's, we consider the following the example.

Example 4.0.2. $W(3,2) = 9$
To find $W(3,2)$ we must find a set of natural numbers $\{1, 2, ..., W(3,2)\}$ which when 2-colored must contain a monochromatic 3-term arithmetic progression.

We first set a lower bound: $W(3,2) \geq 8$ because we can find a coloring of $\{1,2,3,4,5,6,7,8\}$ without a 3-term arithmetic progression: $\{1,2,3,4,5,6,7,8\}$.

Then we must find an upper bound: we need to prove that $W(3,2) \leq 9$. If we can do that, then we know $8 < W(3,2) \leq 9$, so $W(3,2) = 9$. We can do this by looking at all the 2-colorings of $\{1,2,3,4,5,6,7,8,9\}$, as Taylor did [1]. All these 2-colorings can be reduced to a smaller amount of cases by focusing on the first four numbers: $\{1,2,3,4\}$. $\{1,2,3,4\}$ can be colored in sixteen ways, half of which color 1 red and the other half of which color 1 blue:

$\{1,2,3,4\}$	$\{1,2,3,4\}$
$\{1,2,3,4\}$	$\{1,2,3,4\}$
$\{1,2,3,4\}$	$\{1,2,3,4\}$
$\{1,2,3,4\}$	$\{1,2,3,4\}$
$\{1,2,3,4\}$	$\{1,2,3,4\}$
$\{1,2,3,4\}$	$\{1,2,3,4\}$
$\{1,2,3,4\}$	$\{1,2,3,4\}$
$\{1,2,3,4\}$	$\{1,2,3,4\}$

Only look at the cases where 1 is colored blue, as the other half are symmetric. We may also ignore the colorings of $\{1,2,3,4\}$ that already contain a monochromatic 3-term arithmetic progression. This leaves us with five cases:

$\{1, 2, 3, 4\}$ ~~$\{1, 2, 3, 4\}$~~

$\{1, 2, 3, 4\}$ ~~$\{1, 2, 3, 4\}$~~

~~$\{1, 2, 3, 4\}$~~ ~~$\{1, 2, 3, 4\}$~~

$\{1, 2, 3, 4\}$ ~~$\{1, 2, 3, 4\}$~~

$\{1, 2, 3, 4\}$ ~~$\{1, 2, 3, 4\}$~~

$\{1, 2, 3, 4\}$ ~~$\{1, 2, 3, 4\}$~~

$\{1, 2, 3, 4\}$ ~~$\{1, 2, 3, 4\}$~~

~~$\{1, 2, 3, 4\}$~~ ~~$\{1, 2, 3, 4\}$~~

Looking at the first remaining case, $\{1, 2, 3, 4\}$, we can now look at the full sequence:

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

Since coloring 5 red would create a red 3-term arithmetic progression ($\{3, 4, 5\}$), we must color it blue:

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

Then, we color 8 red to prevent a blue 3-term arithmetic progression ($\{2, 5, 8\}$):

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

.

Similarly, 6 must be colored blue and 7 must be colored red to avoid a red ($\{4, 6, 8\}$) and blue ($\{5, 6, 7\}$) 3-term arithmetic progression respectively:

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

Now we see no matter what we color 9 it will create a monochromatic 3-term progression ($\{7, 8, 9\}$ or $\{1, 5, 9\}$). The other four cases follow similarly:

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9\} \text{ forces } \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9\} \text{ forces } \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9\} \text{ forces } \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9\} \text{ forces } \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

Therefore, we have shown that $8 < W(3, 2) \leq 9$, so $W(3, 2) = 9$.

We now provide a proof of Van der Waerden's Theorem.

Proof. We will prove that $W(k,r)$ exists by showing it is bounded using induction on k, as Taylor did in his paper on Ramsey Theory [1]. We already know that we can find a natural number $W(1,r)$. We now assume that for any $q \leq k$ and any l we can find $W(q,l)$. We now show that $W(k+1,r)$ exists for every r.

Claim. For any t, such that $t \leq r$, there exists a natural number $W(t,k,r)$ such that whenever the set, $\{1,2,...,W(t,k,r)\}$ is r-colored, it must contain either a monochromatic $(k+1)$-term arithmetic progression or t color focused monochromatic k-term arithmetic progressions together with their color focus.

We prove this claim by induction on t. We have previously assumed that we can find a natural number $W(k,r)$. Since there exists one monochromatic k-term arithmetic progression in $\{1,2,...,W(k,r)\}$ it must be color focused and its focus must be its $(k+1)^{th}$-term. The arithmetic progression's $(k+1)^{th}$-term must be less than or equal to $2W(k,r)$. Therefore, $\{1,2,...,2W(k,r)\}$ must contain a color focused monochromatic k-term arithmetic progression together with its color focus. So $W(1,k,r) = 2W(k,r)$.

Now, we assume that $W(t,k,r)$ exists and we must prove the existence of $W(t+1,k,r)$. Begin by taking the natural number $X = 2W(t,k,r)W(k,r^{W(t,k,r)})$. We may then split the interval $[1,X]$ into blocks, each of order $W(t,k,r)$. We label each block B_i where i denotes the blocks position in $[1,X]$. Now we have:

$$[1,X] = \{1,2,...,W(t,k,r)\} \cup \{W(t,k,r)+1, W(t,k,r)+2, ..., 2W(t,k,r)\} \cup$$
$$... \cup \{X-(W(t,k,r)-1), X-(W(t,k,r)-2), ..., X\}$$
$$= B_1 \cup B_2 \cup ... \cup B_{2W(k,r^{W(t,k,r)})-1} \cup B_{2W(k,r^{W(t,k,r)})}$$

Now consider an r-coloring of $\{1,2,...,X\}$. There are $r^{W(t,k,t)}$ ways in which a set of order $W(t,k,r)$ can be r-colored, so each block, B_i, must be colored in one of these $r^{W(t,k,t)}$ ways.

If, when the interval $[1,X]$ is r-colored, and of the blocks of order $W(t,k,r)$ contain a monochromatic $(k+1)$-term arithmetic progression, we are done. So we assume that each block contains a t color focused monochromatic k-term arithmetic progression.

From the definition of $W(k,r^{W(t,k,r)})$, the set of natural numbers $\{1,2,...,W(k,r^{W(t,k,r)})\}$ must contain a monochromatic k-term arithmetic progression when $r^{W(t,k,r)}$-colored. Our r-coloring of $\{1,2,...,W(t,k,r)W(k,r^{W(t,k,r)})\}$ induces an $r^{W(t,k,r)}$-coloring of the set of blocks, $B_1, B_2, ..., B_{2W(k,r^{W(t,k,r)})}$, since each block has size $W(r,k,r)$ and thus is r-colored in one of $r^{W(t,k,r)}$ ways. Therefore, the first $W(k,r^{W(t,k,r)})$ blocks must contain a monochromatic k-block arithmetic progression. That is, there must exist k identically colored blocks: $B_a, B_{a+d}, B_{a+2d}, ..., B_{a+(k-1)d}$, whose

indices form an arithmetic progression. Since each block is of order $W(t, k, r)$ we may assume that they all contain t color focus, since otherwise once of the blocks must contain a monochromatic $(k + 1)$-term arithmetic progression and we would be done.

Now label each element of $\{1, 2, ..., X\}, b_{x,y}$ where x denotes the index of the block the element is in and y denotes that elements position in B_x. We denote the t color focused monochromatic k-term arithmetic progression in B_a as:

$$P_{a,1} = b_{a,\alpha}, b_{a,\alpha+\delta}, b_{a,\alpha+2\delta}, ..., b_{a,\alpha+(k-1)\delta},$$
$$P_{a,2} = b_{a,\mu}, b_{a,\mu+\nu}, b_{a,\mu+2\nu}, ..., b_{a,\mu+(k-1)\nu},$$
$$\vdots$$
$$P_{a,t} = b_{a,\phi}, b_{a,\phi+\psi}, b_{a,\phi+2\psi}, ..., b_{a,\phi+(k-1)\psi}.$$

These progressions each have their color focus at $b_{a,f}$. That is,

$$b_{a,\alpha+k\delta} = b_{a,\mu+k\nu} = \cdots = b_{a,\phi+k\psi} = b_{a,f}$$

Since all of the k blocks $B_a, B_{a+d}, B_{a+2d}, ..., B_{a+(k-1)d}$, are identically colored there must exist k-term arithmetic progressions:

$$P_{a,1} = b_{a,\alpha}, b_{a,\alpha+\delta}, b_{a,\alpha+2\delta}, ..., b_{a,\alpha+(k-1)\delta},$$
$$P_{a+d,1} = b_{a+d,\alpha}, b_{a+d,\alpha+\delta}, b_{a+d,\alpha+2\delta}, ..., b_{a+d,\alpha+(k-1)\delta},$$
$$\vdots$$
$$P_{a+(k-1)d,1} = b_{a+(k-1)d,\alpha}, b_{a+(k-1)d,\alpha+\delta}, b_{a+(k-1)d,\alpha+2\delta}, ..., b_{a+(k-1)d,\alpha+(k-1)\delta},$$
$$P_{a,2} = b_{a,\mu}, b_{a,\mu+\nu}, b_{a,\mu+2\nu}, ..., b_{a,\mu+(k-1)\nu},$$
$$\vdots$$
$$P_{a+(k-1)d,2} = b_{a+(k-1)d,\mu}, b_{a+(k-1)d,\mu+\nu}, b_{a+(k-1)d,\mu+2\nu}, ..., b_{a+(k-1)d,\mu+(k-1)\nu},$$
$$\vdots$$
$$P_{a+(k-1)d,1} = b_{a+(k-1)d,\phi}, b_{a+(k-1)d,\phi+\psi}, b_{a+(k-1)d,\phi+2\psi}, ..., b_{a+(k-1)d,\phi+(k-1)\psi}.$$

such that

$$\chi(P_{a,1}) = \chi(P_{a+d,1}) = \cdots = \chi(P_{a+(k-1)d,1}),$$
$$\chi(P_{a,2}) = \chi(P_{a+d,2}) = \cdots = \chi(P_{a+(k-1)d,2}),$$
$$\vdots$$
$$\chi(P_{a,t}) = \chi(P_{a+d,t}) = \cdots = \chi(P_{a+(k-1)d,t})$$

where $\chi(P_{i,j})$ denotes the color of the elements of the progression $P_{i,j}$. Together, each of the t progressions in each of the k blocks produce $t + 1$ color focused monochromatic k-term arithmetic progression. Indeed, consider the following k-term arithmetic progressions,

$$F_1 = b_{a,\alpha}, b_{a+d,\alpha+\delta}, b_{a+2d,\alpha+2\delta}, ..., b_{a+(k-1)d,\alpha+(k-1)\delta}$$
$$F_1 = b_{a,\mu}, b_{a+d,\mu+\nu}, b_{a+2d,\mu+2\nu}, ..., b_{a+(k-1)d,\mu+(k-1)\nu}$$
$$\vdots$$
$$F_1 = b_{a,\phi}, b_{a+d,\phi+\psi}, b_{a+2d,\phi+2\psi}, ..., b_{a+(k-1)d,\phi+(k-1)\psi}$$

Since each of the terms in F_i were taken from $P_{j,i}$ where $j \in \{a, a + d, ..., a + (k - 1)d\}$, each F_i must be monochromatic. That is,

$$b_{a+kd,\alpha+k\delta} = b_{a+kd,\mu+k\nu} = \cdots = b_{a+kd,\phi+k\psi}$$

This element is in X since $X = 2W(t, k, r)W(k, r^{W(t,k,r)})$ and each element we have used so far we have taken from the first $W(t, k, r)W(k, r^{W(t,k,r)})$ elements. Thus, each of the t monochromatic k-term arithmetic progressions we have produced, $F_1, F_2, ..., F_t$ have their color focus at $b_{a+kd,\alpha+k\delta} = b_{a+kd,\mu+k\nu} = \cdots = b_{a+kd,\phi+k\psi} = b_{a+kd,f}$. Clearly $b_{i,f}$ must be the same color in every block in the monochromatic k-term arithmetic progression. Therefore, the color focuses of the blocks, $B_a, B_{a+d}, ..., B_{a+(k-1)d}$, also form a monochromatic k-term arithmetic progression. These terms, along with other t monochromatic k-term arithmetic progressions, $b_{a,f}, b_{a+d,f}, ...b_{a+(k-1)d,f}$, must have a different color to each $F_1, F_2, ..., F_t$, since otherwise a monochromatic $k + 1$ term arithmetic progression would have been formed in one of the $B_a, B_{a+d}, ..., B_{a+(k-1)d}$, from the definition of a color focus. Thus, $b_{a+kd,f}$ is the color focus for the $t + 1$ monochromatic k-term arithmetic progressions. Therefore, $X = W(t + 1, k, r)$ and our claim is proved.

Since we have that $W(t, k, r)$ must exist for all $t \leq r$ we have that $W(r, k, r)$ must exist. That is, we can always find r color focused k-term arithmetic progressions or a monochromatic $(k + 1)$-term arithmetic progression in the r-colored set of natural numbers $\{1, 2, ..., W(r, k, r)\}$. If there exists a monochromatic $(k + 1)$-term arithmetic progression in this set we are done, so we assume a monochromatic $(k + 1)$-term arithmetic progression does not exist. Since we have only used r colors to color this set of natural numbers, the color focus of all the r arithmetic progressions must be colored with one of the r colors. Therefore the color focus must have the same color as one of the r k-term arithmetic progressions. Together with the color focus this arithmetic progression then forms a monochromatic $(k + 1)$-term arithmetic progression. Therefore, by induction for all positive integers, k and r, there exists a natural number $W(k, r)$ so that, if the set of natural numbers $\{1, 2, ..., W(k, r)\}$ is r-colored, there is at least one monochromatic k-term arithmetic progression. $\square$

5 Rado's Theorem

As discussed above, Van Der Waeden's Theorem concerns monochromatic arithmetic progressions. We can re-express these arithmetic progressions as systems of equations. Suppose $x_1, x_2, ..., x_n$ are an arithmetic progression; this can be represented by the system

$$x_2 - x_1 = x_3 - x_2$$

$$x_3 - x_2 = x_4 - x_3$$

$$...$$

$$x_{n-1} - x_{n-2} = x_n - x_{n-1}$$

which, in matrix form, is equivalently

$$\begin{bmatrix} 1 & -2 & 1 & 0 & 0 & ... \\ 0 & 1 & -2 & 1 & 0 & ... \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ 0 & 0 & ... & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ . \\ . \\ x_n \end{bmatrix} = \vec{0}$$

In other words, we see that Van Der Waerden's Theorem concerns itself with a particular matrices C such that $C\vec{x} = \vec{0}$. We now seek to characterize all matrices C such that $C\vec{x} = 0$ has a monochromatic solution. Intuitively, systems of

equations impose restrictions upon numbers, and we want to see what restrictions are necessary in order to find "enough order" for monochromatic sets to emerge. Rado's Theorem, which is the focus of this section, answers the question: which systems of homogeneous linear equations have a monochromatic solution no matter the coloring?

5.1 Regularity and the Columns Condition

Before we can state Rado's Theorem, we must introduce some terminology to describe systems of equations under colorings. Since Rado's Theorem concerns itself with monochromatic solutions, we need a property which succinctly describes the existence of such:

Definition 5.1.1. A system of equations with is **r-regular** if, for any r-coloring (coloring with r colors) of its solution space, it has a monochromatic solution. A system is **regular** if it is regular for all r.

Example 5.1.1. For instance, the system $\begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [0]$ is not regular.

To prove a system is not regular, it suffices to find a coloring of the integers such that no solution to the system is monochrome.

Consider the 2-coloring where odd numbers are colored blue and even numbers are colored the opposite of half their value. Formally, if 2^k is the largest power of 2 that divides into an integer n, color n blue if k even and red if k odd. Note that any solution to the system satisfies $x_1 = 2x_2$, meaning x_1 and x_2 will have opposite colors by definition, precluding a monochromatic solution. Since the system is not 2-regular, it therefore is not regular.

The point of Rado's Theorem is to determine *when* a system always has a monochromatic solution, i.e. *what conditions a system must satisfy to be regular*. We therefore introduce the Columns Condition, a condition which Rado's Theorem will show is both necessary and sufficient for regularity.

Definition 5.1.2. An *mxn* matrix $C = [c_1...c_n]$, where the c_i are column vectors, satisfies the **Columns Condition** if its columns can be partitioned into $C_1 \cup C_2 \cup ... \cup C_k$ with all C_i nonempty such that:

1. The elements (column vectors) in C_1 sum to 0; i.e. $\sum_{c_i \in C_1} c_i = \vec{0}$

2. for all $j > 1$ the sum of the elements in C_j (i.e. $\sum_{c_i \in C_j} c_i$) is expressible as a linear combination of elements in $C_1 \cup ... \cup C_{j-1}$.

Example 5.1.2. We take the simple example of $C = \begin{bmatrix} 1 & -2 \end{bmatrix}$ from before and show it does not satisfy the Columns Condition.

Immediately, it is evident that there does not exist C_1 such that $\sum_{c_i \in C_1} c_i = \vec{0}$; $[1] \neq [0], [-2] \neq [0], [1] + [-2] = [-1] \neq [0]$. Therefore, C does not satisfy the Columns Condition.

5.2 General Rado's Theorem

We have now introduced both concepts necessary for Rado's Theorem: regularity and the Columns Condition. We now formally state Rado's Theorem.

Theorem 5.2.1 (**Rado's Theorem**). *A system $C\vec{x} = \vec{0}$ is regular if and only if it satisfies the columns condition.*

The above examples, Examples 4.0.1 and 4.0.2, demonstrate Rado's Theorem. Since the system $\begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [0]$ is not regular, it cannot satisfy the Columns Condition; and indeed, as we showed, it does not. It is often more useful to think about satisfaction of the Columns Condition implying regularity since that is more easily verifiable; i.e. since the system does not satisfy the Columns Condition, it cannot be regular.

5.3 One Dimensional Rado's Theorem

We will not prove the above general version of Rado's Theorem. Instead, we will state and then prove a special case, where the system involves only one equation; i.e. C is a 1xn matrix for some n. It is simpler, but still underscores

the same proof techniques and concepts.

We first note that for a 1xn matrix, the Columns Condition can be simplified. Let such a matrix be $[c_1...c_n]$ where c_i are integers and $[c_i]$ are the 1x1 column vectors. As explicated above, the Condition requires the existence of a partitioning $C_1 \cup ... \cup C_n$ of the column vectors such that:

1. There exists a non empty subset C_0 of the column vectors which sums to the zero vector. Since the column vectors here are 1x1 vectors, this is equivalent to a non empty set of the c_i summing to 0.

2. For all $j > 1$, the elements of C_j are expressible as a linear combination of elements in $C_1 \cup ... \cup C_{j-1}$. Note that this is trivially always the case; any element of a C_j is a 1x1 vector $[c_i]$ which can be written as the linear combination of any group of other 1x1 vectors.

Thus, the Columns Condition for a 1xn matrix $[c_1...c_n]$ reduces to the existence of a nonempty set of the c_i summing to 0. We can now state the special 1D case of Rado's Theorem.

Theorem 5.3.1 (1D Case of Rado's Theorem). *Let $c_1, c_2, ...c_n$ be nonzero integers. Then the equation*

$$c_1 x_1 + c_2 x_2 + ... c_n x_n = 0$$

is regular if and only if some nonempty subset of the c_i sums to zero.

In order to prove this theorem, we need two lemmas.

Lemma 1. Let $[x]$ denote the set $1, 2, ...x$. Take any integers k, r, s. There exists an integer $n(k, r, s)$ such that for any r-coloring of $[n(k, r, s)]$, there exist integers a and d with $\{a, a+d, ...a+d(k-1)\} \cup \{sd\} \subseteq [n(k, r, s)]$ monochromatic.

Proof. Before beginning the proof, we take a moment to digest the statement of Lemma 1. Lemma 1 is a strengthening of Van Der Waerden; for any r-coloring, length k, and integer s, there exists an arithmetic sequence of length k such that it and s times its common difference are the same color. We now proceed with the proof.

As directed, take any integers k, r, s. We will prove Lemma 1, i.e. the existence of $n(k, r, s)$, by induction on r.

<u>Base Case:</u> r=1. By definition of a 1-coloring, all the integers are colored the same color. $a = 1, d = 1$ produces $\{1, 1 + 1, ..., 1 + 1(k - 1) = k\} \cup s * 1 \subseteq [\max(k, s)]$ which must be monochrome since there is only once color. Hence, $n(k, 1, s) = \max(k, s)$ suffices and therefore such a $n(k, 1, s)$ exists.

<u>Inductive Hypothesis:</u> Assume the Lemma holds for $r-1$. We show that it holds for r.

For simplicity, let $n = n(k, r-1, s)$, which we know exists by induction. Take any r-coloring of the integers. By Van Der Waerden's Theorem, we know there exists a and d such that

$$\{a, a+d, ...a + d(kn)\}$$

is monochrome. Call its color red. We have two cases:

1. There exists red jsd for $1 \leq j \leq n$. If this is the case, then consider $\{a, a+jd, ...a + (k-1)jd\}$. Each element is of the form $a + ijd$ for $i \leq k-1$; hence, $a + ijd < a + knd$ and so each element is contained in our original red set $\{a, a+d, ...a + d(kn)\}$ and is therefore red. Hence, $\{a, a+jd, ...a + (k-1)jd\} \cup \{sjd\}$ is monochromatic, which proves this case (for instance, one can take $n(k, r, s) = W(kn+1, r)$, where we recall $W(kn+1, r)$ is the Van Der Waerden Number of $kn+1$ and r.

2. Alternatively, there does not exist red jsd for any $1 \leq j \leq n$. Therefore consider the set $\{sd, 2sd, ..., nsd\} = sd[n]$. Since none of these elements are red, this set is $r-1$-colored. We're now going to create a $r-1$-coloring of $[n]$ by coloring $x \in [n]$ with the color of $xsd \in sd[n]$. Recalling that $n = n(k, r-1, s)$, we know there exists a, d such that $\{a, a+d, ..., a+d(k-1)\} \cup \{sd\} \subseteq [n]$ is monochrome; for any r-1 coloring, a and d must exist by the definition of $n(k, r-1, s)$. Therefore, $\{s(a), sd(a+d), ..., sd(a+d(k-1))\} \cup \{sd(sd)\} \subseteq sd[n]$ is the same color by definition of our coloring. Hence, we have proved this case.

Thus, we have concluded our proof by induction. $\square$

We use this Lemma 1 to prove a second, and final, lemma.

Lemma 2. For all nonzero integers s and t,

$$sx + ty = sz$$

is regular.

Proof. Take any nonzero s and t. Consider any r and any r-coloring. By Lemma 1, we know there exists $n(t+1, r, s)$ such that there exists a and d with $\{a, a+d, ...a + td\} \cup \{sd\} \subseteq [n(t+1, r, s)]$ monochrome. Thus, $a, a+td, sd$ all the same color. Noting that $s(a) + t(sd) = s(a+td)$, we see that $x = a, y = sd, z = a + td$ is a monochromatic solution. Hence the system $sx + ty = sz$ is regular. $\square$

Now we are ready to prove the 1D Case of Rado's Theorem. Recall that we must show that a system $c_1 x_1 + ... + c_n x_n = 0$ is regular iff there is a nonempty set of the c_i summing to 0.

Proof. Our proof has two parts: showing that the Columns Condition (the existence of a nonempty set of c_i summing to 0) is sufficient and showing it is

necessary for regularity. We begin with sufficient.

Suppose the Columns Condition holds. There exists a set of c_i summing to 0; WLOG, we take $c_1 + ... + c_k = 0$ for some $k \geq 1$.

Note that if $k = n$, then letting $x_1 = ... = x_n = 1$ gives $c_1 x_1 + ... + c_n x_n = c_1 * 1 + ... + c_n * 1 = 0$; assigning all x_i to 1 is a solution, and moreover, a monochromatic one.

Now we consider $k < n$. We construct a monochrome solution; let $x_1 = x$ and $x_2 = ... = x_k = z$ and $x_{k+1} = ... = x_n = y$. Then our system becomes

$$c_1 x + (c_2 + ... + c_k)z + (c_{k+1} + ... + c_n)y = 0.$$

Since $c_1 + (c_2 + ... + c_k) = 0$, the system is equivalently

$$c_1 x - c_1 z + (c_{k+1} + ... + c_n)y = 0$$

Rearranging terms yields

$$c_1 x + (c_{k+1} + ... + c_n)y = c_1 z$$

By Lemma 2, we know that there exists a monochrome solution in x, y, z to the system; thus, by definition of x, y, z, we have a monochrome solution in $x_1, ..., x_n$ to the system and it is thus regular, since the r-coloring was arbitrary. Thus, we have shown the Columns Condition is sufficient for regularity. Now we prove the converse.

Suppose the system $c_1 x_1 + ... + c_n x_n = 0$ is regular. Now take a prime p s.t. $p > \sum |c_i|$ and, using $p - 1$ colors, color $\mathbb{N}$ such that $x \in \mathbb{N}$ is colored by the last non-zero digit in the base p representation of x.

It is obvious to note that if d is the last nonzero digit base p of some integer x, then $x = d$ or $x = 0$ mod p; if p divides x then $x = 0$ mod p by definition, and if not, then d is the last digit of x base p and $x = d$ mod p.

Since by supposition the system is regular, we know there must exist a monochromatic solution $x_1 = x_1', ...x_n = x_n'$ (all colored with color d'). We can divide through by p^l where p^l is the greatest power dividing into each x_i', resulting in x_i'' such that $x_i' = p^l x_i''$. The x_i'' are all the same color still; the division only removes 0's base p and leaves the same last nonzero digit d'. They also still comprise a solution since $c_1 x_1'' + ... + c_n x_n'' = 0/p^l = 0$. WLOG, we can assume that p does not divide into $x_1'', ...x_k''$ and does into $x_{k+1}'', ..., x_n''$; we know that $k \geq 1$, i.e. that there exist x_i'' such that p does not divide into x_i'' since otherwise we could have divided out a larger power of p.

Since the x_i'' are solutions to the system, we have

$$c_1 x_1'' + ... + c_n x_n'' = 0 \text{ mod p}$$

We also know $x_1'' = ... = x_k'' = d' \mod p$ and $x_{k+1}'' = ... = x_n'' = 0 \mod p$, so

$$d'(c_1 + ... + c_k) + 0 + ... + 0 = 0 \mod \text{p}$$

meaning that p divides $c_1 + ... + c_k$. However, we selected p such that $p > \sum |c_i|$; thus $p > c_1 + ... + c_k$ and in order for p to divide into the sum, the sum must equal zero. Thus we have $c_1 + ... + c_k = 0$ as desired; the Columns Condition must hold, which concludes our proof. $\square$

References

[1] Gregory E. W. Taylor. Ramsey Theory (2006).
    http://web.mat.bham.ac.uk/D.Kuehn/RamseyGreg.pdf.

[2] Ronald L. Graham   Joel H. Spencer. Ramsey Theory. Scientific American (July 1990).