Compact groups, integration & averaging

① Integration:

Def. • An integral on a compact group is a linear functional $\int : C(G) \to \mathbb{C}$ (where $C(G)$ = continuous functions $G \to \mathbb{C}$)

that is
- Positive, i.e. $f \geq 0 \Rightarrow \int f \geq 0$
- Continuous, i.e.

if f_n and g_n are continuous functions on G, and
$$\sup_{h \in G} |f_n(h) - g_n(h)| \underset{n\to\infty}{\longrightarrow} 0$$
then $\int f_n - \int g_n \to 0$.

["If the functions f and g are close together, then $\int f$ and $\int g$ will be close together."]

• Group multiplication defines left and right actions on $C(G)$ by $(h.f)(g) = f(h^{-1}g)$ [Left action]

and $(f.h)(g) = f(gh)$ [Right action]

• An integral $\int$ is said to be left (right) - invariant if for all $h \in G$ and $f \in C(G)$,
$$\int h.f = \int f \quad \left(\text{or } \int f.h = \int f\right)$$
in the right-inv. case.

Note: Thinking in terms of the dual action on the functional $\int \in C(G)^*$, this says that $(h.)^*(\int) = \int \ \forall h \in G$ or equivalently, $\int$ is a fixed vector of the G-action on $C(G)^*$.

Thm. (Haar) For every (locally) compact topological group G, there exists a non-zero left-inv. integral $\int : C(G) \to \mathbb{C}$, and furthermore it is unique up to scalar multiplication by some $\lambda > 0$.

(i.e. if $\int$ and $\int'$ are two left-inv non-zero integrals on G, then $\int = \lambda \int'$ for some $\lambda > 0$).

Rem: If G is compact, then
a) $\int$ is automatically right-inv. as well.
b) Integrating the constant func. 1 gives a finite $\neq 0$ value i.e. $\int 1 = M \in (0, \infty)$,

and WLOG we may normalize $\int$ so that $\int 1 = 1$, i.e. $\int$ is a probability distribution on G.

Ex. ① G is a finite group,
$$\int f = \frac{1}{|G|} \sum_{g \in G} f(g)$$ is the left + right - inv. positive integral of total mass 1.

② $G = S^1 \subseteq \mathbb{C}$ the unit norm complex numbers,
$$\int f = \int f(\theta)\, d\theta = \int_0^{2\pi} f(t)\, dt$$
is a left + right - inv. positive integral with total mass 2π
$$\left(\int 1\, d\theta = \int_0^{2\pi} dt = 2\pi\right).$$

[Being left-invariant means that $d\theta$ is rotation invariant, "$R_\theta^*(d\theta) = d\theta$"]

② We now extend the definition of integration to apply to continuous functions into a vector space.

Def. $C(G;V) = \{F : G \to V : F \text{ is continuous}\}$ where $V \cong \mathbb{R}^n$ has the standard topology.

Given an integral $\int : C(G) \to \mathbb{C}$, we can "extend" it to

$$\int^V : C(G;V) \to V$$

i.e. $(F(g) \in V) \rightsquigarrow \int^V F(g)\,dg \in V$.

Pick a basis $v_1, \dots, v_n \in V$ and expand $F : G \to V$ as a sum

$F(g) = \sum_{i=1}^n \lambda_i(g) v_i$, where the $\mathbb{C}$-valued functions λ_i are continuous.

Define. $\int^V F(g)\,dg = \sum_{i=1}^n \Big[\underbrace{\int \lambda_i(g)\,dg}_{\in \mathbb{C}}\Big] \cdot v_i$.

Claim: $\int^V$ is independent of the choice of basis, and if $\int$ was left (right)-inv. then so will $\int^V$ be.

Pf. Suppose $w_1, \dots, w_n$ is another basis to V and

$$v_i = \sum_{j=1}^n P_{ji} w_j .$$

Then $F(g) = \sum_{i=1}^n \lambda_i(g) v_i = \sum_{i,j} \lambda_i(g) P_{ji} w_j$

$$= \sum_{j=1}^n \Big(\underbrace{\sum_{i=1}^n P_{ji} \lambda_i(g)}_{= \delta_j(g)} \Big) w_j .$$

Computing $\int^V F$ w.r.t. the basis $\{w_j\}_{j=1}^n$

$$\int^V F(g)\,dg = \sum_{j=1}^n \Big[\int \delta_j(g)\,dg\Big] w_j$$

$$= \sum_{j=1}^n \Big[\int \sum_{i=1}^n P_{ji} \lambda_i(g)\,dg\Big] w_j$$

and by linearity of the integral,

$$= \sum_{i,j} P_{ji} \Big[\int \lambda_i(g)\,dg\Big] w_j$$

$$= \sum_{i=1}^n \Big[\int \lambda_i(g)\,dg\Big] \sum_{j=1}^n P_{ji} w_j = \sum_i \Big[\int \lambda_i\Big] \cdot v_i$$

and this is indeed well-defined.

Now, suppose $\int$ is left-inv and fix $h \in G$

$$(h.F)_{(g)} := F(h^{-1}g).$$

Expanding w.r.t. the basis $v_1, \dots, v_n$

$$(h.F)_{(g)} = F(h^{-1}g) = \sum_{i=1}^n \lambda_i(h^{-1}g) v_i$$

$$= \sum_{i=1}^n (h.\lambda_i)_{(g)} v_i ,$$

i.e. $(h.F) = \sum (h.\lambda_i) v_i$.

Integrating,

$$\int (h.F) = \sum_i \Big[\int (h.\lambda_i)\Big] v_i = \sum_i \Big[\int \lambda_i\Big] v_i = \int F$$

[by left-inv. of $\int$]

and indeed $\int^V h.F = \int^V F \quad \forall h \in G$. ▨

Note: Given a continuous function $F : G \to \mathrm{Hom}(V,W)$, we may integrate over F to get the "average" map

$$\int F(g)\,dg : V \to W \quad (\in \mathrm{Hom}(V,W))$$

an average of $F(g) : V \to W$, $\forall g \in G$.

Prop. Given vector spaces V, W, U and the following data -

$$V \xrightarrow{T(g)} W \xrightarrow{S} U$$

$$v \mapsto T(g)v \mapsto S \circ T(g) v$$

with $T : G \to \mathrm{Hom}(V,W)$ cts. $v \in V$ and $S \in \mathrm{Hom}(W,U)$ fixed.

Then, $S \circ \Big(\int T(g)\,dg\Big)(v) =$

$$= \int (S \circ T(g) v)\,dg$$

i.e. integration commutes with standard linear algebra operations.

To prove this, we will use the following trivial observations:

<u>Lemma 1</u>: $\int^V$ is linear for every vector space V.

<u>Pf.</u> If $F, F' : G \to V$ are given and $\delta \in \mathbb{C}$, then $\forall g \in G$

$$(\delta F + F')_{(g)} = \delta F_{(g)} + F'_{(g)}$$
$$= \delta \sum \lambda_i(g) v_i + \sum \lambda_i'(g) v_i$$
$$= \sum (\delta \lambda_i + \lambda_i')_{(g)} v_i$$

$$\Longrightarrow \int^V \delta F + F' = \sum \left[\int (\delta \lambda_i + \lambda_i')\right] v_i$$
$$= \sum \left[\delta \int \lambda_i + \int \lambda_i'\right] v_i = \delta \int^V F + \int^V F'.$$

linearity of the integral $\int$ ☒

<u>Lemma 2</u>: If $v \in V$ is fixed and $f : G \to \mathbb{C}$ is continuous, then we can define $F : G \to V$ by

$F = f \cdot v$ (scalar multiplication).

Integrating, $\int^V F = \int^V (f \cdot v) dg$ is equal to $(\int f dg) \cdot v$.

<u>Pf.</u> Choose a basis $v = v_1, \dots, v_n \in V$.

Then $F = f \cdot v = f \cdot v_1 + 0 \cdot v_2 + \dots + 0 \cdot v_n$

$$\longrightarrow \int^V F(g) dg = \int f(g) dg \cdot v_1 + 0$$
$$= \left(\int f(g) dg\right) \cdot v.$$

☒

Now we can prove the proposition.

<u>Pf.</u> Pick some basis $E_{ij} \in \mathrm{Hom}(V, W)$.

Then $T_{(g)} = \sum \lambda_{ij}(g) E_{ij}$ for $g \in G$.

$$S \circ T_{(g)}(v) = S \circ \left(\sum_{ij} \lambda_{ij}(g) E_{ij}\right)(v)$$
$$= \sum_{ij} \lambda_{ij}(g)\, S \circ E_{ij}(v).$$

i.e. $S \circ T(v) = \sum_{ij} \underset{\in C(G)}{\lambda_{ij}} \cdot \underbrace{S E_{ij} v}_{\in U}$

$\Longrightarrow$ By lemmas 1+2,

$$\int S \circ T_{(g)} v \, dg = \sum_{ij} \left(\int \lambda_{ij}(g) dg\right) S E_{ij} v$$
$$= S\left(\sum_{ij} \left(\int \lambda_{ij}(g) dg\right) E_{ij}\right) v$$

by linearity of S

$$= S \circ \left[\int T_{(g)} dg\right] v$$

☒

<u>Rem:</u> Choosing V cleverly also shows that

$$S \circ \int T_{(g)} dg \circ R = \int (S \circ T_{(g)} \circ R) dg$$

for linear maps S, R and $T_{(g)}$ $\forall g$.

③ Fourier transforms:

Given a representation $\rho : G \to GL(V)$, we get a natural map

$$C(G) \to C(G ; \mathrm{End}(V))$$
$$f \longmapsto f \cdot \rho$$

i.e. $(f \cdot \rho)_{(g)} = \underset{\in \mathbb{C}}{f(g)} \cdot \underset{\in \mathrm{End}(V)}{\rho(g)}$.

Applying the integral to this yields the Fourier transform:

$$\hat{f}(\rho) = \int^{\mathrm{End}(V)} (f(g) \cdot \rho(g)) dg$$

[Compare with $\hat{f}(\rho) = \sum_{g \in G} f(g) \rho(g)$ in the finite group case.]

We will say more on this in the future.

Much of the familiar properties are still true in this case.

e.g. define $f * f'(g) = \int f(gh^{-1}) g(h) dh$

then $\widehat{f * f'}(\rho) = \hat{f}(\rho) \circ \hat{f'}(\rho)$.

④ <u>Averaging</u>: In the finite G case we had the projection map

$$P_{triv} : V \to V^G, \quad P = \frac{1}{|G|}\sum_{g\in G} \rho(g).$$

The analog of this in the compact case will be

$$P = \int \rho(g)\, dg$$ (recall that $\int$ was take to be normalized)

<u>Prop.</u> P is the projection from V onto the fixed vectors V^G.

<u>Pf.</u> Let $v \in V$ be any vector.

$\forall h \in G$, $\rho(h)(Pv) = \rho(h)\left[\int \rho(g)\, dg\right] v$

$= \left[\int \rho(h)\rho(g)\, dg\right] v =$ [ρ is a group hom.]

$= \left[\int \rho(hg)\, dg\right] v =$

by left invariance

$= \left[\int \rho(g)\, dg\right] v = Pv$

so $\rho(h) Pv = Pv$.

And if $\rho(h)v = v \ \forall h \in G$ then

$Pv = \left(\int \rho(g)\, dg\right) v = \int \rho(g) v\, dg$

$= \int v\, dg = \left(\int 1\, dg\right) v$

$= 1 \cdot v = v$

so indeed $P : V \to V^G$ is the projection. ☒