A free differential Lie algebra model of the 2-cell

Nir Gadish

Einstein Institute of Mathematics, Hebrew University of Jerusalem

Email: `nir.gadish@mail.huji.ac.il`

Abstract

By extending the Lawrence-Sullivan model of the free-DGLA of the interval we will proceed in constructing a similar model for the 2-cell, known to exist by a general recursive argument of Sullivan. We will use a recursive technique to show the existence of such a model with special properties. The goal is to find an explicit formula for the differential of the 2-cell. Although this task is still far from being finished, we do have several results. In particular, we will write an explicit formula for δ_n when $0 \leq n \leq 4$ and show that one can always choose $\delta_n = 0$ for all odd $n > 1$. As for even n values, we find an explicit form for a part of the solution and suggest a way in which this could be extended to a full solution.

AMS Classification numbers Primary: 55U15

Secondary: 16E45 55P35

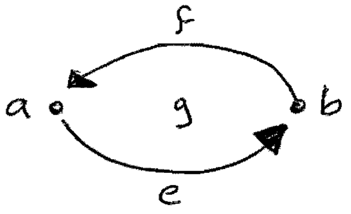
Keywords: infinity structure

1 Introduction

This project is intended to further the search for a free-DGLA model of the 2-cell with appropriate symmetries and differential that extends the (unique) model that was found for the interval [LS]. Sullivan [TZS] gave a general iterative procedure by which a model of a complex can be constructed; it is even "canonical" in some sense given a little extra structure, although very difficult to practically compute. Without the additional structure, there are many choices available at each stage, but it is known that for any choice of differential exact up to order n, there exists an extension to all higher orders making it exact. In this project we stipulate additional requirements for the differential. Specifically we are trying to find an explicit formula for the differential of the 2-cell which:

- represents the boundary of the cell to first order;
- displays the inherent symmetries of the 2-cell;
- is exact, meaning that its differential is 0, so that it can be extended to a differential on the whole Lie algebra with $\delta^2 = 0$.

All these demands will be properly defined in the following sections.



We will work with a very simple 2-cell model (see the bi-gon drawn above) which is simple enough for calculation, but at the same time general enough so that it could be extended to every 2-cell complex using gluing and splitting formulae. The connection between the original complex and the simpler complex will be given by the BCH formula [LS].

2 Preliminary definitions and terms

2.1 General DGLAs

We say that L is a **graded Lie algebra** if it is a $\mathbb{Z}$ graded vector space endowed with a bilinear operation (**Lie bracket**) $[.,.]\colon L \times L \longrightarrow L$ satisfying:

- (1) (Graded symmetry) $[b,a] = -(-1)^{|a||b|}[a,b]$;
- (2) (Jacobi identity)

$$(-1)^{|a||c|}[a,[b,c]] + (-1)^{|b||a|}[b,[c,a]] + (-1)^{|c||b|}[c,[a,b]] = 0$$

The bracket needs also to be additive with respect grading. That is, if a and b are homogeneous elements of L then $|[a,b]| = |a| + |b|$, where $|x|$ is the grading of a homogeneous element x.

A **differential** on L is a linear mapping $\partial\colon L \longrightarrow L$ which satisfies:

- (3) (Leibnitz rule) $\partial[a,b] = [\partial a, b] + (-1)^{|a|}[a, \partial b]$.
- (4) (Exactness) $\partial^2 = 0$.

Additionally, we require that the differential reduces the grading by 1, meaning that for a homogeneous element $a \in L$ there holds $|\partial a| = |a| - 1$.

A space L endowed with the structure described will be called a **differential graded Lie algebra** or DGLA for short.

2.2 The Adjoint operation

For an element $a \in L$, the **adjoint** ad_a acts by $ad_a(b) = [a,b]$, for all $b \in L$. A very useful notation which will be used extensively is a capital letter A for ad_a. In this notation we write $Ab = [a,b]$ and apply from right to left as with the usual composition: $ABc = A(Bc)$. For example, graded symmetry in this notation is $Ab = (-1)^{|a|+|b|}Ba$ for all $a, b \in L$.

Some operators $X\colon L \longrightarrow L$ have a well-defined grading, by which is meant the grading shift induced by this operator: $|X(a)| = |X| + |a|$ for all homogeneous $a \in L$. Such operators will be called **homogeneous**. It is easy to see that for a homogeneous $a \in L$ there holds $|A| = |a|$ and also that $|\partial| = -1$. With the Lie bracket of homogeneous operators defined by the signed commutator $[X,Y] = XY - (-1)^{|X||Y|}YX$, another graded Lie algebra can be generated by ∂ and operators $A = ad_a$ for $a \in L$.

We can now express (2) and (3) in a more compact form: for all $a, b, c \in L$:

- (2′) (Jacobi) $[Ab, c] = [A, B]c$ $\qquad \big(ad_{[a,b]} = [ad_a, ad_b]\big)$
- (3′) (Leibnitz) $[\partial b, c] = [\partial, B]c$ $\qquad (ad_{\partial b} = [\partial, ad_b])$

2.3 Bases for free Lie algebras

Definition 1 Suppose X a finite set of symbols with $\mathbb{Z}$ grading. Denote by $Lie(X)$ the free Lie algebra generated by X, and by $Lie_n(X)$ the span of all brackets of n (nt necessarily distinct) symbols from X (with $n-1$ brackets).

We will construct a way using which we can describe all elements in $Lie(X)$.

Definition 2 An element $x \in Lie(X)$ will be called **simple** is it can be written as a single iterated bracket expression of generators. This presentation will be called **a simple presentation**.

In essence, we say that an element is simple if it can be written as an expression with no addition symbols. For example $\Big[[x_1, x_2], [[x_2, x_3], [x_1, x_4]]\Big]$ is simple while $x_1 + [x_2, x_3]$ and $[x_1, x_2 + x_3]$ are not.

Note that it is always possible to express $x \in Lie(X)$ as a linear combination of simple elements: simply use the linearity of the Lie bracket until there are no addition symbols appearing in the scope of any bracket.

Further note that the set of simple elements could be defined recursively as the set generated from X using only the Lie bracket, i.e. the smallest subset that includes X and is closed under Lie multiplication. This observation will enable us to prove statements on simple elements using induction.

Definition 3 For a simple element x we define its **length**, denoted by $l(x)$, to be the number of generators appearing in a simple presentation of x. Equivalently $l(x)$ is the unique natural number n for which $x \in Lie_n(X)$.

Another way of defining the length (which clarifies that length is well defined for free Lie algebras) is by recursion:

Set $l(g) = 1$ for all generators $g \in X$ and define $l([x, y]) = l(x) + l(y)$.

It is clear that the three definitions are the equivalent.

Definition 4 Elements of the form

$$a = [g_1, [g_2, [\dots [g_{n-1}, g_n]] \dots]]$$

with $a_1, \dots, a_n \in X$ will be called **canonical** elements. This includes the case of $n = 1$ where $a \in X$ itself and thus all generators are also considered canonical.

It is clear that a canonical element is also simple and has $l(x) = n$.

- The generators g_{n-1} and g_n will be said to be **in the center** of the canonical element a. These are the elements which lay in the inner most bracket when a is written as a single nested bracket. If a is a generator itself, we will say that it has the generator a at it's center.
- A linear combination of canonical elements will be called a **canonical sum**.

Lemma 5 *For a simple element $x \in Lie_n(X)$ one can choose the following presentations:*

- *One can express x as a canonical sum of elements, all of which have the same length n.*
- *If a generator g appears in x, then one can choose a canonical sum in which g is at the center of all the canonical elements.*

We will give the general proof after the following example. Say $X = \{x_1, \dots, x_5\}$ all with grading 0.

$$\begin{aligned}
& [[[x_1, x_2], x_3], [x_4, x_5]] \\
= \;& [[X_1, X_2], X_3][x_4, x_5] = ([X_1, X_2]X_3 - X_3[X_1, X_2])\,[x_4, x_5] \\
= \;& ((X_1X_2 - X_2X_1)X_3 - X_3(X_1X_2 - X_2X_1))\,[x_4, x_5] \\
= \;& (X_1X_2X_3 - X_2X_1X_3 - X_3X_1X_2 + X_3X_2X_1))\,[x_4, x_5] \\
= \;& [x_1, [x_2, [x_3, [x_4, x_5]]]] - [x_2, [x_1, [x_3, [x_4, x_5]]]] - [x_3, [x_1, [x_2, [x_4, x_5]]]] \\
& \quad + [x_3, [x_2, [x_1, [x_4, x_5]]]]
\end{aligned}$$

which is a canonical sum of elements with x_5 at their center.

Proof Let $x \in Lie_n(X)$ be a simple element. If there isn't a particular generator g that we wish to have in the center, just pick g to be any generator appearing in x. We use induction on n.

If $n = 1$ then $x = g$ is canonical and with g in its middle.

Assume by induction that all simple elements of length $k < n$ containing g have a representation as a canonical sum of elements with length k and with g at their center.

For $n > 1$ one has $x = [b, c]$ with b and c simple elements and $l(b), l(c) \leq n-1$. WLOG g appears in c, otherwise we will use the commutation rule of the Lie-bracket.

b is a simple element and can therefore be written as an iterated bracket of $l(b) = m$ symbols, say $b_1, \ldots, b_m$. By the Jacobi relation (applied repeatedly), the operator $B = ad_b$ can be written as a similar iterated bracket of the operators $B_1, \ldots, B_m$. Since the bracket of operators is defined by signed commutators, $[A, B] = AB - (-1)^{|A||B|}BA$, this iterated bracket can be expanded as a signed sum of 2^{m-1} products of $B_1, \ldots, B_m$; in each product all the m symbols appear but in different orders. Thus x has been expressed as a signed sum of 2^{m-1} elements of the form

$$Y_1 \ldots Y_m c$$

By the induction hypothesis, c can be expressed as a linear combination of canonical elements with length $l(c)$ and g at their center. By linearity of the bracket, it remains only to note that if z is a canonical element with g at its center, then $Y_1 \ldots Y_m z$ is also a canonical element with g at its center and with length $l(z) + m$. Lastly note that $n = l(b) + l(c) = m + l(c)$ and thus all the elements in our canonical sum have length n. □

Remark 6 Since every element in $Lie(X)$ can be represented as a sum of simple elements, the lemma shows that it can also be expressed as a canonical sum. One can thus choose a basis for $Lie_n(X)$ consisting only of canonical elements with length n.

Example 1: $X = \{x\}$ with $|x| = 0$. $\forall n > 1$ $Lie_n(X) = 0$ as $[x, x] = 0$.

Example 2: $X = \{x\}$ with $|x| = 1$. $Lie_2(X)$ has basis $[x, x]$. $Lie_3(X) = 0$ as by Jacobi $[x, [x, x]] = 0$ and in fact $\forall n > 2$ $Lie_n(X) = 0$.

Example 3: $X = \{x, y\}$ with $|x| = 1$ and $|y| = 0$. $Lie_2(X) = \langle [x, x], [x, y] \rangle$, $Lie_3(X) = \langle [x, [x, y]], [y, [x, y]] \rangle$ by taking y at the center. As for the term $[y, [x, x]]$ one has $[y, [x, x]] = -[[x, x], y] = -[X, X]y = -2X^2 y = -2[x, [x, y]]$.

2.4 Relations

Even in a free Lie algebra, there are relations amongst simple elements generated by graded symmetry and Jacobi; some of them are at first sight non-trivial.

The few ideas presented below can be extended to higher lengths. For a, $b \in L$ one has:

(1) If a is of even grading then $[a, a] = 0$.

(2) If $|a| \equiv |b|_{mod\,2}$ then $ABAb = -BABa$.

(3) If b is of odd grading then $ABb = -2BBa$

Proof (1) Using graded symmetry: $[a,a] = -[a,a] \implies [a,a] = 0$

(2) Since $|[a,b]| = |a| + |b| \equiv 0_{mod\,2}$ and using (1) and Jacobi one gets

$$0 = [Ab, Ab] = [A,B]Ab \tag{2.1}$$

For evenly graded elements this is

$$0 = (AB - BA)Ab = ABAb - BAAb = ABAb + BABa \tag{2.2}$$

and for odds $0 = (AB + BA)Ab = ABAb + BAAb = ABAb + BABa$.

(3) The element Bb is evenly graded and therefore, regardless of a's grading, one gets

$$ABb = -[Bb, a] = -(BB + BB)a = -2BBa \tag{2.3}$$

□

2.5 Projections

Let L be a graded Lie algebra and $A \oplus B = L$ (as vector spaces). We will define the projection onto A with zero space B and go over some of its properties.

Definition 7 For every $x \in L$ there is a unique representation $x = x_A + x_B$ where $x_A \in A$ and $x_B \in B$. Define the projection $P_A \colon L \longrightarrow A$ by:

$$P_A(x) = x_A \tag{2.4}$$

We call P_A the **projection of** L **onto** A and B is its kernel. Note that even though the kernel B does not appear in the notation, it is important to remember what is this space. Whenever we will use projections we will be sure to specify what is B, unless it is clear from the context.

As is expected of a projection, P_A is linear and $P_A = P_A^2$.

Conversely, a linear projection P naturally divides L into the direct sum $A \oplus B$ where $B = Ker(P)$, $A = Im(P)$ and $P|_A = id_A$.

Lemma 8 *If A is a sub Lie algebra and B is an ideal then the projection P_A is a Lie algebra homomorphism.*

Proof By definition, P_A is linear. Now let's look at $P_A([x,y])$. Express x as a sum $x = x_A + x_B$ where $x_A = P_A(x) \in A$ and $x_B \in B$ and do the same for y. Using bilinearity

$$[x,y] = [x_A, y_A] + [x_B, y_A] + [x_A, y_B] + [x_B, y_B] \tag{2.5}$$

and since B is an ideal we have

$$[x_B, y_A] + [x_A, y_B] + [x_B, y_B] \in B = Ker(P_A) \tag{2.6}$$

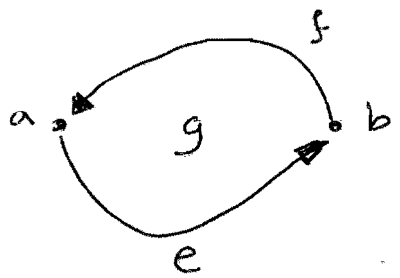
Also, since A is closed under Lie-multiplication we know that $[x_A, y_A] \in A$ and therefore

$$P_A([x,y]) = P_A([x_A, y_A]) = [x_A, y_A] = [P_A(x), P_A(y)] \tag{2.7}$$

□

3 Model of the bi-gon 2-cell

We will attempt to construct a free DGLA model of the bi-gon 2-cell.



Namely, as a free Lie algebra, it will have one generator for each different $0-$, $1-$ and $2-$cell, a generator corresponding to a d-cell having grading $d-1$.

We want to define a differential on the generators such that the first order term (the length-1 term) is the geometric boundary:

$$\begin{array}{lll} \delta_0 a = 0 & \delta_0 e = b - a & \delta_0 g = e + f \\ \delta_0 b = 0 & \delta_0 f = a - b & \end{array}$$

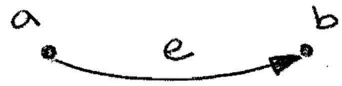
Furthermore, we want the model to be compatible with that of the interval found in [LS]. That is, each edge of the bi-gon should give a natural sub-DGLA which is the model associated with that edge as an interval.

The model for the interval had three generators a, b and e with gradings -1, -1 and 0 respectively corresponding to the three geometric cells. The boundary satisfied

$$\partial a = -\frac{1}{2}Aa\,, \quad \partial b = -\frac{1}{2}Bb\,, \quad \partial e = Eb + \frac{E}{e^E - 1}(b - a)$$

which first order term had

$$\delta_0 a = \delta_0 b = 0\,, \quad \delta_0 e = b - a$$



3.1 Description/Specifications

Our free DGLA L is generated by 5 generators:

- vertices a and b with -1 grading;
- edges e and f with 0 grading;
- 2-cell g with 1 grading.

This is our generating set $\tilde{X}$ and we will denote $L = Lie(\tilde{X})$ as our model.

From compatibility with the differential on the two embedded intervals in the bi-gon, the differential on the first four generators is given by:

$$\partial a = -\frac{1}{2}[a, a]; \quad \partial b = -\frac{1}{2}[b, b]$$

representing the flatness of the vertices, and

$$\partial e = \sum_{n=0}^{\infty} \frac{B_n}{n!} E^n(b - a) + Eb; \quad \partial f = \sum_{n=0}^{\infty} \frac{B_n}{n!} F^n(a - b) + Fa$$

where B_n are the Bernoulli numbers whose generating function is

$$\sum_{n=0}^{\infty} \frac{B_n x^n}{n!} = \frac{x}{e^x - 1} = 1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \dots$$

with $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_3 = \frac{1}{6}$, $B_4 = -\frac{1}{30}, \dots$ and $B_n = 0$ for all odd $n > 1$.

In order to have a complete model for the 2-cell DGLA we need only to find a differential for g (meaning a suitable element $x = \partial g \in Lie(X)$) satisfying the three demands:

- ∂g represents the boundary of the cell to first order (see §3.2);
- ∂g displays the inherent symmetries of the 2-cell (see §3.3);
- ∂g is exact, meaning that given a choice of $x = \partial g$ with grading 0, ∂ can be uniquely extended to a linear map on $Lie(X)$ satisfying the Leibnitz rule, and that as such, we require $\partial x = 0$ – this condition suffices to ensure that $\partial^2 = 0$ on all of $Lie(X)$ (see §3.2).

3.2 The differential

Using projections one can break ∂ down to simpler pieces:

$$\partial = \delta_0 + \delta_1 + \delta_2 + \dots \tag{3.1}$$

Definition 9 We look at the partition $Lie(X) = \bigoplus_{n=1}^{\infty} Lie_n(X)$. Given a differential ∂ define the mapping δ_n on the generating set X as $P_{Lie_{n+1}(X)} \circ \partial$. Extend δ_n's definition to the rest of $Lie(X)$ by recursion using the Leibnitz rule. We call these maps the **partial differentials of ∂.**

One should note the following:

- δ_n is a mapping $Lie_m(X) \longrightarrow Lie_{m+n}(X)$.
- A sum of mappings that obey Leibnitz, itself obeys the rule.
- By definition, $\partial = \sum_{n \in \mathbb{N}} \delta_n$ on a generating set. The previous fact ensures that the sum of δ's and ∂ are indeed equal on all of $Lie(X)$.

Exactness requires that $\partial^2 = 0$. By linearity it is sufficient that $\partial^2 x = 0$ for simple elements $x \in Lie(X)$. Furthermore, the Leibnitz rule applied twice gives

$$\begin{aligned}
\partial^2[x,y] &= \partial\Big([\partial x, y] + (-1)^{|x|}[x, \partial y]\Big) \\
&= [\partial^2 x, y] + (-1)^{|\partial x|}[\partial x, \partial y] + (-1)^{|x|}\Big([\partial x, \partial y] + (-1)^{|x|}[x, \partial^2 y]\Big) \\
&= [\partial^2 x, y] + [x, \partial^2 y]
\end{aligned}$$

so that exactness is guaranteed by $\partial^2 x = 0$ on generators.

One can describe exactness in an equivalent manner using the decomposition of ∂ above. On a generator $x \in X$, exactness requires

$$(\delta_0 + \delta_1 + \dots)^2 x = \partial^2 x = 0 \tag{3.2}$$

Now by projecting the two sides on the space $Lie_n(X)$ one gets the equivalent system of equations

$$(\delta_0\delta_n + \delta_1\delta_{n-1} + \dots + \delta_n\delta_0)x = 0 \tag{3.3}$$

Another way to write the latter equation is

$$\delta_0\delta_n x = -\delta_1\delta_{n-1}x - \ldots - \delta_n\delta_0 x \tag{3.4}$$

which gives us a recursive relation that we will use to find the partial differential $\delta_n x$ of x given that we know $\delta_0 x$ and $\delta_n\delta_0 x$ for all n.

Since we already know the existence of the model for the interval [LS], the problem reduces to finding ∂g for which $\partial^2 g = 0$. Note that $\delta_0 g = e + f$ so that $\delta_n\delta_0 g = \delta_n(e+f)$ is known for all n, from the model of the interval. Thus iterative construction is in principle possible – see §3.5 for a discussion of the existence of a solution $\delta_n g$ at each step.

3.3 Symmetries

We wish our model of the 2-cell to preserve certain symmetries of the original 2-cell. In particular, the differential of g should be invariant under symmetry transformations, such as replacing the names of the vertices in a structure preserving manner.

It is clear that the symmetry group of our 2-cell is $\mathbb{Z}_2 \times \mathbb{Z}_2$ and that it is generated by the following automorphisms:

- Horizontal flip: fix a and b; exchange $e \leftrightarrow (-f)$; and $g \mapsto (-g)$. Name this transformation ϕ.
- Rotation 180: exchange $a \leftrightarrow b$; $e \leftrightarrow f$; and fix g. Name this ρ.

There is of course a Vertical flip also, but it can be acquired by the composition of the two former ones. This means that it is enough if we make sure that invariance under ϕ and ρ is maintained.

Invariance of ∂g under the automorphism implies

$$\phi(\delta_n g) = \delta_n(\phi(g)) = -\delta_n g$$

and $\rho(\delta_n g) = \delta_n(\rho(g)) = \delta_n g$. These demands will prove to exclude a number of previously possible basis elements from appearing in $\delta_n g$.

3.4 Alternative basis

Since our result must be symmetric in the various generators, we would benefit from using an alternative and equivalent formulation.

The alternative generating set X is the set containing

$$\begin{array}{ccc} s = b - a & , & t = b + a \\ n = e - f & , & p = e + f \\ & g & \end{array}$$

which clearly generates the same Lie algebra L. The symmetries then act as:

$$\begin{array}{lcl} \phi & : & \text{fix } t,\ s \text{ and } n;\ \text{negate } p \text{ and } g. \\ \rho & : & \text{fix } t,\ p \text{ and } g;\ \text{negate } s \text{ and } n. \end{array}$$

This further means that all simple elements are eigenvectors of ϕ, ρ with eigenvalue ± 1.

$$\phi(x) = (-1)^{\#g(x)+\#p(x)}x, \quad \rho(x) = (-1)^{\#n(x)+\#s(x)}x$$

where $\#g(x)$ is the number of g symbols appearing in x and so on. One can use the simple action of ϕ, ρ to eliminate certain possibilities for $\delta_n g$. We will find that the only simple elements appearing in $\delta_n g$ are such that they have the same eigenvalue as $\delta_n g$ itself. This follows from the next lemma.

Lemma 10 *Let ψ be a symmetry transformation and $x \in Lie(X)$ such that $\psi(x) = (-1)^m x$. If $a_1, \ldots, a_n$ are linearly independent eigenvectors of ψ with eigenvalue ± 1 and λ_i, $i = 1, \ldots, n$ scalars such that $x = \sum_{i=1}^n \lambda_i a_i$ then*

$$\lambda_i \neq 0 \implies \psi(a_i) = (-1)^m a_i$$

Proof Denote m_i such that $\psi(a_i) = (-1)^{m_i} a_i$.

$$0 = \psi(x) - (-1)^m x = \sum_{i=1}^n \lambda_i(\psi(a_i) - (-1)^m a_i) = \sum_{i=1}^n ((-1)^{m_i} - (-1)^m)\lambda_i a_i$$

By linear independence one has $((-1)^{m_i} - (-1)^m)\lambda_i = 0$ and therefore if $\lambda_i \neq 0$ we must have

$$(-1)^{m_i} = (-1)^m$$

which completes the proof. □

Corollary 11 All canonical elements (in the alternative basis) appearing in a representation of $\delta_n g$ must have odd $\#g + \#p$ and even $\#n + \#s$.

Proof This is follows directly from the previous lemma and the fact that $\phi(\delta_n g) = \delta_n \phi(g) = -\delta_n g$ and $\rho(\delta_n g) = \delta_n \rho(g) = \delta_n g$. □

Remark 12 The differential δ_m on the generating set X is given by

$$\begin{aligned}
\delta_0 s = 0 \quad &, \quad \delta_0 t = 0\\
\delta_0 n = 2s \quad &, \quad \delta_0 p = 0\\
\delta_0 g &= p\\
\delta_1 s = -\frac{1}{2}Ts \quad &, \quad \delta_1 t = -\frac{1}{4}(Tt+Ss)\\
\delta_1 n = -\frac{1}{2}Tn \quad &, \quad \delta_1 p = -\frac{1}{2}Tp
\end{aligned}$$

and as for higher partial differentials, for all $m \geq 2$

$$\delta_m s = \delta_m t = 0$$

$$\delta_m n = \frac{2B_m}{2^m m!} \sum_{\{\pi\colon [m]\mapsto\{N,P\}\colon \, even \, \#N(\pi)\}} \pi(1)\ldots\pi(m)s$$

$$\delta_m p = \frac{2B_m}{2^m m!} \sum_{\{\pi\colon [m]\mapsto\{N,P\}\colon \, odd \, \#N(\pi)\}} \pi(1)\ldots\pi(m)s$$

where $[m] = \{1, 2, \ldots, m\}$ and $\#N(\pi)$ denotes the number of times that N appears in π.

In fact there is no need to specify the formulae for $m = 0$ directly since for all generators one can see that this is just the case of $m = 0$ in the general formulae above.

Proof Firstly one has $\delta_0 g$, by definition.

For s and t one can use graded symmetry (commutativity in this case) to get

$$\partial s = \partial b - \partial a = -\frac{1}{2}([b,b]-[a,a]) = -\frac{1}{2}[b+a, b-a] = -\frac{1}{2}Ts$$

$$\partial t = \partial b + \partial a = -\frac{1}{2}([b,b]+[a,a]) = -\frac{1}{4}([b+a,b+a]+[b-a,b-a]) = -\frac{1}{4}(Tt+Ss)$$

and, by projecting these results on the respective Lie_m space, one gets the aforementioned formulae.

Since $\delta_0 e = -\delta_0 f = b - a = s$, one gets the result for $\delta_0 n$ and $\delta_0 p$, using linearity.

Now for $n = 1$ one can check the definitions and find that

$$\delta_1 e = \frac{1}{2}E(b+a) = -\frac{1}{2}Te$$

and similarly $\delta_1 f = -\frac{1}{2}Tf$. Therefore linearity of both δ_1 and $-\frac{1}{2}T$ shows that

$$\delta_1(e \pm f) = -\frac{1}{2}T(e \pm f)$$

Substituting for the alternative basis of n and p completes the proof.

Finally for $n \geq 2$. Calculating for e and f first, one finds

$$\begin{aligned}
\delta_m e &= \frac{B_m}{m!}E^m s = \frac{B_m}{m!}\left(\frac{P+N}{2}\right)^m s = \frac{B_m}{2^m m!}(P+N)^m s \\
&= \frac{B_m}{2^m m!}\sum_{\pi \in \{N,P\}^m} \pi(1)\dots\pi(m)s
\end{aligned}$$

and

$$\begin{aligned}
\delta_m f &= -\frac{B_m}{m!}F^m s = -\frac{B_m}{m!}\left(\frac{P-N}{2}\right)^m s = -\frac{B_m}{2^m m!}(P-N)^m s \\
&= \frac{B_m}{2^m m!}\sum_{\pi \in \{N,P\}^m} (-1)^{\#N(\pi)+1}\pi(1)\dots\pi(m)s
\end{aligned}$$

Now taking the sum (or difference) of these two expressions we get only the elements with odd (or even) number of N terms in them and an additional factor 2. □

3.5 Existence

For the rest of this section we will use the grading to further decompose $Lie_n(X)$, writing $Lie_n^m(X)$ for the part graded by m. Note that since $|g| = 1$ and $|\partial| = -1$, thus $|\partial g| = 0$.

Following Sullivan [TZS], we will now explain why it is always possible to extend a differential exact to order n iteratively to one exact to all orders. In particular, given $\delta_i g \in Lie_{i+1}^0(X)$ for all $(i < n)$ for which

$$(\delta_0\delta_m + \delta_1\delta_{m-1} + \dots + \delta_m\delta_0)\, g = 0 \quad \forall m < n$$

that it is always possible to find $\delta_n g \in Lie_{n+1}^0(X)$ for which

$$\delta_0\delta_n g = y := -(\delta_1\delta_{n-1} + \dots + \delta_{n-1}\delta_1)\, g - \delta_n(e+f)$$

In other words, we must explain why the element on the right hand side, which we will denote by y, lies in $Im(\delta_0)$ considering $\delta_0\colon Lie_{n+1}^0(X) \longrightarrow Lie_{n+1}^{-1}(X)$. This follows from the following two lemmas.

Lemma 13 $y \in Ker(\delta_0)$

Proof Consider the operator $D = \delta_0 + \ldots + \delta_{n-1}$. This has grading -1 and D, D^2 commute as operators. By construction of $\delta_0, \ldots, \delta_{n-1}$,

$$D^2 = \sum_{i,j=0}^{n-1} \delta_i \delta_j = (\delta_1 \delta_{n-1} + \ldots + \delta_{n-1}\delta_1) + \sum_{i+j>n} \delta_i \delta_j$$

so that $[D, D^2]g = 0$ becomes

$$[\delta_0, \delta_1\delta_{n-1} + \ldots + \delta_{n-1}\delta_1]g + \sum_{k=1}^{n-1} \sum_{\substack{i,j=0 \\ i+j=n}}^{n-1} [\delta_k, \delta_i\delta_j]g + \sum_{\substack{i,j,k=0 \\ i+j>n}}^{n-1} [\delta_k, \delta_i\delta_j]g = 0$$

As before, every element of $Lie(X)$ can be decomposed by length, so that $[\delta_k, \delta_i\delta_j]g$ is a combination of simple elements of length $i+j+k+1 > n+1$ for all terms coming from the two sums. Those terms of length at most $n+1$ leave

$$[\delta_0, \delta_1\delta_{n-1} + \ldots + \delta_{n-1}\delta_1]g = 0$$

In other words,

$$\begin{aligned} \delta_0 \left(\delta_1\delta_{n-1} + \ldots + \delta_{n-1}\delta_1\right) g &= \left(\delta_1\delta_{n-1} + \ldots + \delta_{n-1}\delta_1\right) \delta_0 g \\ &= \left(\delta_1\delta_{n-1} + \ldots + \delta_{n-1}\delta_1\right)(e+f) \end{aligned}$$

By exactness of the models on the two intervals contained in the bi-gon, namely $\langle a, b, e\rangle$ and $\langle b, a, f\rangle$,

$$(\delta_1\delta_{n-1} + \ldots + \delta_{n-1}\delta_1)e = -(\delta_0\delta_n + \delta_n\delta_0)e$$

and similarly for f, so that

$$\delta_0 \left(\delta_1\delta_{n-1} + \ldots + \delta_{n-1}\delta_1\right) g = -(\delta_0\delta_n + \delta_n\delta_0)(e+f) = -\delta_0\delta_n(e+f)$$

since $\delta_0(e+f) = 0$, as required. $\square$

Lemma 14 *The short sequence* $Lie^0_n(X) \xrightarrow{\delta_0} Lie^{-1}_n(X) \xrightarrow{\delta_0} Lie^{-2}_n(X)$ *is exact for* $n > 1$.

In fact, more generally, for any fixed $n > 1$, the complex $Lie^m_n(X)$ with differential induced by the boundary δ_0, has trivial homology; see [TZS].

Since we have shown that $y \in Ker(\delta_0)$ there exists $x \in Lie^0_{n+1}$ such that $\delta_0 x = y$ as desired. Lastly, we are looking for models that respect the symmetries described above. Therefore we wish to find a solution that is compatible with the above requirements.

Lemma 15 *Suppose that for all $m < n$, $\delta_m g$ was compatible with the symmetries of the cell, that is, $\rho(\delta_m(g)) = \delta_m(g)$ and $\phi(\delta_m(g)) = -\delta_m(g)$. Then $\rho(y) = y$ and $\phi(y) = -y$. Furthermore, there exists $z \in Lie^0_{n+1}$ such that $\delta_0 z = y$ and*

$$\rho(z) = z\,, \quad \phi(z) = -z$$

Proof Let ψ be one of the symmetries of the 2-cell. It is easy to see that for every generator $x \neq g$ one has $\psi\delta_m x = \delta_m\psi x$, for all $m \in \mathbb{N}$. For $m < n$, by hypothesis $\psi\delta_m g = \delta_m\psi g$ and so ψ and δ_m commute on all generators.

We will show that the two compositions agree on all $Lie(X)$. Both are linear, and thus it is sufficient to check the equality on simple elements. Suppose by induction that z and w are such that make δ_m and ψ commute. Since ψ is a homomorphism and does not change grading

$$\psi\delta_m[z,w] = \psi([\delta_m z, w] + (-1)^{|z|}[z, \delta_m w]) = [\psi\delta_m z, \psi(w)] + (-1)^{|\psi(z)|}[\psi(z), \psi\delta_m w]$$

using the induction hypothesis

$$= [\delta_m\psi(z), \psi(w)] + (-1)^{|\psi(z)|}[\psi(z), \delta_m\psi(w)] = \delta_m[\psi(z), \psi(w)] = \delta_m\psi([z,w])$$

Now that we know δ_m and ψ commute for all $m < n$, and also $\psi(\delta_n p) = \delta_n\psi(p)$ one gets

$$\psi(y) = -\sum_{m=1}^{n-1} \psi\delta_m\delta_{n-m}g - \psi(\delta_n p) = -\sum_{m=1}^{n-1} \delta_m\delta_{n-m}\psi(g) - \delta_n\psi(p)$$

Since p and g change signs in the same manner under all symmetries, we get that $\psi(y) = \pm y$ and the change in sign is exactly that occurring in g: $\rho(y) = y$ and $\phi(y) = -y$.

Take any $x \in Lie^0_{n+1}$ such that $\delta_0 x = y$ and define

$$z = \frac{1}{4}\left(x - \phi(x) + \rho(x) - \phi\rho(x)\right)$$

z is of course still in Lie^0_{n+1} and since δ_0 commutes with all symmetries

$$\delta_0 z = \frac{1}{4}\left(y - \phi(y) + \rho(y) - \phi\rho(y)\right) = y$$

Knowing that the symmetry group is $\mathbb{Z}_2 \times \mathbb{Z}_2$ we know that ϕ and ρ commute and are of order 2. Calculating the operation on z:

$$\phi(z) = \frac{1}{4}\left(\phi(x) - \phi^2(x) + \phi\rho(x) - \phi^2\rho(x)\right) = \frac{1}{4}\left(\phi(x) - x + \phi\rho(x) - \rho(x)\right) = -z$$

and

$$\rho(z) = \frac{1}{4}\left(\rho(x) - \rho\phi(x) + \rho^2(x) - \phi\rho^2(x)\right) = \frac{1}{4}\left(\rho(x) - \rho\phi(x) + x - \phi(x)\right) = z$$

as desired. □

Now that we know that extension is always possible, we can proceed in building a model using a recursive technique. We do have some demands on the resulting differential which will guide us in selecting the elements appearing in it.

4 The recursive construction method

This next section describes a process using which one should be able to find all of the possibilities for $\delta_n g$.

Assuming that we have already found $\delta_k g$ for all $k < n$, we will find an element of length $n+1$ which will serve as $\delta_n g$. As we search for possible candidates we will require the expressions to be of a form that abides by certain restrictions:

- The grading of the element must be 0, as it is $|g| - 1$.
- The length must be $n+1$, as this is $l(g) + n$.
- The element must be symmetric, as was discussed in §3.3 and Corollary 11.

The final and most difficult condition to uphold is that the element must produce an exact differential in the sense of (3.4), meaning

$$\delta_0\delta_n g = -\delta_1\delta_{n-1}g - \ldots - \delta_n\delta_0 g = -\delta_1\delta_{n-1}g - \ldots - \delta_{n-1}\delta_1 g - \delta_n p \qquad (4.1)$$

Since we know what $\delta_n p$ is, and we are assuming that $\delta_m g$ is known for all $m < n$, this equation is well defined. δ_0 is a linear transformation and one can solve this linear equation using elementary linear algebra. First one must select a basis and calculate δ_0's operation on this basis, then by simple Gaussian elimination one will find all possible solutions.

To simplify the problem as much as possible, we will choose our basis according to the following guidelines. Let T_{n+1} be the set of all canonical elements with:

- length of $n+1$;
- grading of 0;
- even $\#n + \#s$ and odd $\#g + \#p$.

These three restrictions make certain that any linear combination will meet the restrictions imposed on $\delta_n g$ and at the same time they are imperative for any canonical element appearing in the expression of $\delta_n g$ as a canonical sum. Its span $\langle T_{n+1}\rangle \subset Lie^0_{n+a}(X)$ would include all possible solutions $\delta_n g$ to (4.1) in which we are interested.

4.1 Divisions into smaller subspaces

We can further subdivide the solution space into smaller subspaces.

Lemma 16 *Suppose that $V = W \oplus U$ are vector spaces and T is a linear transformation on V such that $T(W) = \tilde{W}$, $T(U) = \tilde{U}$ and $\tilde{W} \cap \tilde{U} = \{0\}$. For $X \in \{W, U\}$ there holds $T \circ P_X = P_{\tilde{X}} \circ T$.*

Proof Noting that $id = P_W + P_U$, one finds that $P_U = id - P_W$. From the definition of $\tilde{U}$ we find that $Im(T \circ (id - P_W)) = \tilde{U}$ and thus trivially intersects $\tilde{W}$. Applying $P_{\tilde{W}}$ one would get the zero function

$$0 = P_{\tilde{W}} \circ T \circ (id - P_W) = P_{\tilde{W}} \circ T - P_{\tilde{W}} \circ T \circ P_W$$

Now since $T(W) = \tilde{W}$ one can drop the latter projection and get

$$0 = P_{\tilde{W}} \circ T - T \circ P_W$$

□

Corollary 17 In the case described above, x is a solution to the equation $Tx = b$ iff $T \circ P_W(x) = P_{\tilde{W}}(b)$ and $T \circ P_U(x) = P_{\tilde{U}}(b)$.

Proof (if) We use the equalities $id_V = P_W + P_U$ and $id_{T(V)} = P_{\tilde{W}} + P_{\tilde{U}}$.

$$Tx = T \circ P_W(x) + T \circ P_U(x) = P_{\tilde{W}}(b) + P_{\tilde{U}}(b) = b$$

(only if) Conversely, the result is an immediate consequence of applying $P_{\tilde{X}}$ to both sides of $Tx = b$ and using the above lemma. □

Using this corollary we note that for a partition $\langle T_{n+1}\rangle = W \oplus U$ such that $\delta_0 W \cap \delta_0 U = \{0\}$ it is sufficient to find solutions to

$$\begin{aligned} \delta_0(w) &= P_{\delta_0(W)}\big(-\sum_{m=1}^{n} \delta_m \delta_{n-m} g\big) \\ \delta_0(u) &= P_{\delta_0(U)}\big(-\sum_{m=1}^{n} \delta_m \delta_{n-m} g\big) \end{aligned}$$

where $w \in W$ and $u \in U$. The resulting sum $w+u$ will be $\delta_n g$ satisfying (4.1). By induction this could be extended to a partition into multiple subspaces.

One of the most useful partitions in our case is to take

$$T_{n+1}^{(k)} = \{x \in T_{n+1} | \#p(x) + \#g(x) = k\}; \quad 1 \le k \le n+1$$

It is easy to see that $\langle T_{n+1}^{(k)} \rangle$ are vector spaces, that their sum is all of $\langle T_{n+1} \rangle$ and that every one of them trivially intersects the sum of the rest. Furthermore, since the number $\#p + \#g$ is preserved under δ_0 we have also that $\delta_0 \langle T_{n+1}^{(i)} \rangle \cap \sum_{j \ne i} \delta_0 \langle T_{n+1}^{(j)} \rangle = \{0\}$. Thus it is sufficient to examine equation (4.1) for each k separately.

Note the following:

- For even k values, $T_n^{(k)} = \emptyset$ since there are no elements with even $\#(g) + \#(p)$ in T_n.
- For all $n \in \mathbb{N}$ note that $\langle T_n^{(n)} \rangle = \{0\}$. Indeed an element of $T_n^{(n)}$ must be a canonical element involving only g's and p's. Grading demands that all are p's and one finds only $[p,p] = 0$ at the center of all basis elements.

4.2 Vanishing of the odd $\delta_n g$

Taking after the form of ∂p and ∂n in which all δ_n's vanish for odd $n > 1$ (the corresponding Bernoulli number being zero) one might expect to see $\delta_n g$ vanishing as well. Indeed we shall see that one can choose $\delta_n g = 0$ for odd $n \ne 1$ as a part of an exact differential (i.e. $\partial^2 = 0$).

Define a subset of generators $X_t = X \backslash \{t\}$ and denote the Lie- algebra generated by it as $L_t = Lie(X_t)$. Complementing L_t is the ideal generated by t, meaning the span of all canonical elements in which t is one of the generators. As proven in Lemma 6 of the section about projections, the projection $P_t := P_{L_t}$ is a homomorphism.

Lemma 18 *If ∂ respects the symmetries of the 2-cell, $\partial^2 g = 0$ and $\delta_0 g = p$ then $\delta_1 g = -\frac{1}{2} T g$.*

Proof Using (4.1) for $n = 1$ one gets $\delta_0 \delta_1 g = -\delta_1 \delta_0 g = \frac{1}{2} T p$. Since $\delta_0 t = 0$ one gets

$$\delta_0(-\frac{1}{2} T g) = -\frac{1}{2}([\delta_0 t, g] - [t, \delta_0 g]) = \frac{1}{2} T p$$

which implies $-\frac{1}{2}Tg - \delta_1 g \in Ker(\delta_0) \cap S_2$. Take a canonical element $x \in \langle T_2 \rangle$. Since $\#t$ is odd and the length of x is 2 there necessarily is exactly one t. The other generator in x must be g to achieve zero grading. Thus S_2 is one dimensional and spanned by Tg. Since $\delta_0 Tg \neq 0$ there are no non-trivial elements in the intersection above and $\delta_1 g = -\frac{1}{2}Tg$. □

Lemma 19 *If $\partial = \delta_0 + \delta_1 + \ldots$ extends the [LS] model of the interval and has $\delta_1 g = -\frac{1}{2}Tg$ then $\delta_1 = -\frac{1}{2}T$ on the set L_t. An equivalent formulation of this is $\delta_1 \circ P_t = -\frac{1}{2}T \circ P_t$.*

Proof First notice that like all the adjoint operators with odd grading, $-\frac{1}{2}T$ is a derivation on L, meaning that it is linear and satisfies the Leibnitz relation. To see this directly use the linearity of the Lie-bracket to get linearity and Jacobi to get the Leibnitz relation. We will show the latter:

$$T[x,y] - (-1)^{|x||t|}[x,Ty] = [Tx,y] \Rightarrow T[x,y] = [Tx,y] + (-1)^{|x|}[x,Ty]$$

multiply by $-\frac{1}{2}$ and we find the desired result.

Now that we have seen that necessarily $\delta_1 g = -\frac{1}{2}Tg$ one sees that δ_1 agrees with $-\frac{1}{2}T$ on the generating set of L_t. Hence, by the uniqueness of a recursively defined function on a generating set one gets that the two functions are the same. □

Lemma 20 *If δ is a derivation on L and $\delta x \in L_t$ for every generator $x \neq t$ then $\delta(L_t) \subset L_t$ i.e. the set L_t is closed under the operation of δ.*

Proof For all the generators of L_t we have $\delta x \in L_t$ by hypothesis.

Since L_t is a linear subspace, it is sufficient the check only for simple elements. Suppose by induction that $x, y \in L_t$ have $\delta x, \delta y \in L_t$, then

$$\delta[x,y] = [\delta x, y] + (-1)^{|x|}[x, \delta y] \in L_t$$

since L_t is a sub-Lie algebra and closed under Lie multiplication.

The induction gives the Lemma for all simple elements as desired. □

Lemma 21 *If $x \in \langle T_n \rangle$ then $P_t(x) \in \langle T_n \rangle$ as well.*

Proof Write x as a sum $\sum_{k=1}^{N} \lambda_k v_k$ where $v_k \in T_n$ for all $1 \leq k \leq N$.

Note that P_t is nothing more than scalar multiplication on each v_k since

$$P_t(v_k) = \begin{cases} 0 & \text{if } t \text{ appears in } v_k \\ v_k & \text{otherwise} \end{cases}$$

Thus applying P_t to the sum above results in another sum of elements from T_n and thus $P_t(x) \in \langle T_n \rangle$ as well. $\square$

Lemma 22 $P_t \circ \delta_0 = \delta_0 \circ P_t$

Proof Since the two compositions are linear, it is sufficient to check for homogeneous elements. First for the generators: $P_t \circ \delta_0 t = 0 = \delta_0 \circ P_t t$ and for all others the composition of P_t has no effect since none have t as their δ_0.

Assuming by induction that $P_t \circ \delta_0 = \delta_0 \circ P_t$ on the homogeneous x and y and using the fact that P_t is a homomorphism,

$$\begin{aligned} P_t \circ \delta_0[x,y] &= P_t([\delta_0 x, y]) - (-1)^{|x|} P_t([x, \delta_0 y]) \\ &= [P_t \circ \delta_0 x, P_t(y)] - (-1)^{|x|} [P_t(x), P_t \circ \delta_0 y] \\ &= [\delta_0 \circ P_t(x), P_t(y)] - (-1)^{|x|} [P_t(x), \delta_0 \circ P_t y] \\ &= \delta_0 [P_t(x), P_t(y)]) = \delta_0 \circ P_t([x,y]) \end{aligned}$$

where in the last equality we used the fact that P_t does not change the grading of x as can be easily confirmed. $\square$

Corollary 23 If $\delta_0 x \in L_t$ then $\delta_0 \circ P_t(x) = \delta_0 x$.

Proof By the commutation of δ_0 and P_t one gets

$$\delta_0 x \in L_t \Rightarrow \delta_0 x = P_t(\delta_0 x) = \delta_0 \circ P_t(x)$$

$\square$

Corollary 24 If $x = \delta_n g$ is a solution to (4.1) i.e.

$$\delta_0 x = - \sum_{m=1}^{n} \delta_m \delta_{n-m} g$$

and $\sum_{m=1}^{n} \delta_m \delta_{n-m} g \in L_t$ then $P_t(x)$ is also a solution and is in L_t.

Theorem 25 *There exists a differential $\partial = \delta_0 + \delta_1 + \ldots$ on L such that:*

- *∂ respects the symmetries of the 2-cell.*
- *∂ reduces to the differential found in [LS] on the edges and vertices.*
- $\delta_0 g = p$
- $\delta_1 g = -\frac{1}{2} T g$
- *$\delta_n g \in L_t$ for all $n \neq 1$, i.e. there are no t symbols in $\delta_n g$.*

Proof First take ∂x to be as stated in §3.4 for all generators $x \neq g$. This definition ensures that ∂ reduces to the one dimensional model on the edges and vertices. From [LS] we know that $\partial^2 x = 0$ for those generators and the problem of defining ∂ reduces to defining it on g such that $\partial^2 g = 0$.

From §3.5 we know that one can recursively construct an exact symmetry-respecting ∂g such that $\delta_0 g = p$, and moreover that any symmetry-respecting ∂g up to n^{th} order with $\delta_0 g = p$ can be extended to the $(n+1)^{th}$ order while preserving symmetry. From Lemma 18 we know that any such must satisfy $\delta_1 g = -\frac{1}{2}Tg$, and so we can dispose of $\delta_n g$ for all $n > 1$ and reconstruct these terms recursively starting with $n = 2$.

By looking at $\delta_m x$ for $x = s, n, p$ one sees that these are all expressions in L_t for $m \neq 1$. We will use this fact in the course of our proof.

<u>For $n = 2$</u>, writing equation (4.1) we get

$$\delta_0 x = -\delta_1\delta_1 g - \delta_2\delta_0 g = -\delta_1\delta_1 g - \delta_2 p$$

By §3.5 there exists a solution $x \in Lie_3^0(X)$ with the correct symmetry ($\rho(x) = x$ and $\phi(x) = -x$), then by the discussion at the start of §4, $x \in \langle T_3 \rangle$. Looking at the first term,

$$\begin{aligned}\delta_1\delta_1 g \;&=\; \delta_1(-\frac{1}{2}Tg) = -\frac{1}{2}([\delta_1 t, g] - T\delta_1 g) = -\frac{1}{2}\left(-\frac{1}{4}[Tt, g] - \frac{1}{4}[Ss, g] + \frac{1}{2}TTg\right) \\ &= -\frac{1}{2}\left(-\frac{1}{2}TTg - \frac{1}{2}SSg + \frac{1}{2}TTg\right) = \frac{1}{4}SSg \in L_t\end{aligned}$$

Now since $\delta_2 p \in L_t$ as well, thus from Corollary 23, $P_t(x)$ is also a solution and in $\langle T_3 \rangle$ by Lemma 21. By §3.5, we can replace $\delta_2 g = x$ by $P_t(x)$ and there exists an extension to a fully symmetric exact ∂g starting with $\delta_2 g$.

<u>For $n > 2$</u>, suppose by induction that we have a symmetric differential ∂ such that for all $1 \neq k < n$ one has $\delta_k g \in L_t$. By Lemma 20 $\delta_k(L_t) \subset L_t$ for all such k values. We write (4.1) as an equation for $x = \delta_n g$,

$$\delta_0 x = -\sum_{m=1}^{n-1} \delta_m \delta_{n-m} g - \delta_n p$$

For all $1 < m < n-1$ one has $1 \neq m, (n-m)$ and both are smaller than n, thus by the induction hypothesis we know that $\delta_m(L_t), \delta_{n-m}(L_t) \subset L_t$. In particular

$$g \in L_t \Rightarrow \delta_{n-m} g \in L_t \Rightarrow \delta_m \delta_{n-m} g \in L_t$$

As for $m = 1$ and $m = n-1$ one has $\delta_{n-1}g \in L_t$, so by Lemma 19, $\delta_1\delta_{n-1}g = -\frac{1}{2}T\delta_{n-1}g$ and using the Leibnitz rule

$$\delta_1\delta_{n-1}g + \delta_{n-1}\delta_1 g = -\frac{1}{2}(T\delta_{n-1} + \delta_{n-1}T)g = -\frac{1}{2}[\delta_{n-1}t, g] = 0$$

since $n-1 > 1 \Rightarrow \delta_{n-1}t = 0$.

Putting it all together we get

$$\delta_0 x = -\sum_{m=1}^{n-1} \delta_m\delta_{n-m}g - \delta_n p = -\sum_{m=2}^{n-2} \delta_m\delta_{n-m}g - \delta_n p$$

which we know to be in L_t. As before let $x \in \langle T_{n+1}\rangle$ be a solution, then by Corollary 24 $P_t(x) \in \langle T_{n+1}\rangle$ is a solution as well and we can alter the n^{th} stage of ∂ so that now $\delta_n g = P_t(x)$.

The differential ∂ resulting from this process clearly has all the properties listed above. □

Lemma 26 *For all odd n, every canonical element in T_{n+1} has the symbol t in it. Equivalently, $P_t(x) = 0$ for all canonical elements in T_{n+1}.*

Proof Symmetry requirements in T_{n+1} (see definition at the start of §4) demand an odd $\#p + \#g$ and an even $\#s + \#n$. Thus the number of all symbols other than t is odd. Since the length of a canonical element in T_{n+1} is $n+1$ (even), there has to be an odd $\#t$. In particular $\#t \neq 0$. □

Corollary 27 For the differential defined above $\delta_n g = 0$ for all odd $n \neq 1$.

From now on will restrict our search to a differential of this form.

4.3 Using the form of ∂p

There is a way of guessing a differential on a generator x if one knows what is $\partial(\delta_0 x)$. This method will be now described.

Let X be a generating set for the free $Lie(X)$ equipped with a differential defined on $X \setminus \{x\}$ for some $x \in X$. We wish to define a differential on x such that $\delta_0 x = y$, where $y \in X \setminus \{x\}$ is a generator which for any $n \in \mathbb{N}$ has $\delta_n y \in Lie_{n+1}(X)$. Suppose further that $\delta_n y$ is a canonical sum all of whose terms contain y, so that

$$\forall n \in \mathbb{N} \quad \delta_n y = \sum_{i=1}^{k_n} G_1^i \dots G_n^i y$$

where for all i and j one has $g^i_j \in X \setminus \{x\}$ and $\forall n\, \delta_n g^i_j \in Lie(X \setminus \{x\})$. In such a case $\delta_n x$ may take a similar form.

Theorem 28 *If one defines $\delta_n x = \sum_{i=1}^{k_n} G^i_1 \dots G^i_n x$ in the case described above, then the resulting differential on x satisfies the equation*

$$\sum_{m=0}^{n} \delta_m \delta_{n-m} x = \sum_{j=1}^{\tilde{k}} \tilde{G}^j_1 \dots \tilde{G}^j_n x$$

while $\sum_{j=1}^{\tilde{k}} \tilde{G}^j_1 \dots \tilde{G}^j_n y = 0$.

An equivalent formulation involves defining a mapping $\eta_x \colon L \longrightarrow L$. We define it recursively by defining it's operation on the generators of L. For x take $\eta_x(x) = y$ and for every other generator g take $\eta_x(g) = g$. This mapping simply replaces all instances of x by a y. The theorem states that the sum $\sum_{j=1}^{\tilde{k}} \tilde{G}^j_1 \dots \tilde{G}^j_n x$ is in $Ker(\eta_x)$.

Proof For short we will denote for all $k \in \mathbb{N}$

$$\delta_k(G_1 \dots G_n) = \sum_{d=1}^{n} (-1)^{|G_1 \dots G_{d-1}|} G_1 \dots ad_{\delta_k g_d} \dots G_n$$

using which one can write Leibnitz as

$$\delta_k(G_1 \dots G_n x) = \delta_k(G_1 \dots G_n)x + (-1)^{|G_1 \dots G_n|} G_1 \dots G_n(\delta_k x)$$

Since we assume that $\delta_0 x = y$ is part of a differential on x then $\delta_0 y = \delta_0^2 x = 0$ from (4.1). Also, since δ_n decreases the grading of y by 1 then necessarily $\forall i\, |G^i_1 \dots G^i_n| = (-1)$.

With this in mind one finds that for all i

$$\delta_0(G^i_1 \dots G^i_n x) = \delta_0(G^i_1 \dots G^i_n)x - G^i_1 \dots G^i_n y$$

Summing that terms for all i one gets

$$\delta_0 \delta_n x = \delta_0 \sum_{i=1}^{k_n} G^i_1 \dots G^i_n x = \sum_{i=1}^{k_n} \delta_0(G^i_1 \dots G^i_n)x - \delta_n y$$

and noting that the right most term is exactly $\delta_n \delta_0 x$ we find

$$(\delta_0 \delta_n + \delta_n \delta_0)x = \sum_{i=1}^{k_n} \delta_0(G^i_1 \dots G^i_n)x \tag{4.2}$$

We have defined $\delta_k x$ so that applying η_x to it results in $\delta_k y$. Furthermore, since $\delta_m g^i_j \in Lie(X \setminus \{x\})$ for all j, i and m one also gets that $\eta_x(\delta_m \delta_{n-m} x) = \delta_m \delta_{n-m} y$ for $m > 0$. As for the case of $m = 0$, applying η_x on both sides of equation (4.1)

$$\eta_x\big((\delta_0\delta_n + \delta_n\delta_0)x\big) = \sum_{i=1}^{k_n} \delta_0(G^i_1 \dots G^i_n)y = (\delta_0\delta_n + \delta_n\delta_0)y \tag{4.3}$$

Putting all these facts together and using the exactness of ∂y one gets

$$\eta_x\big(\sum_{m=0}^{n} \delta_m\delta_{n-m}x\big) = \sum_{m=0}^{n} \delta_m\delta_{n-m}y = 0 \tag{4.4}$$

$\square$

The meaning of this theorem is that defining $\delta_n x$ so that it follows the form of $\delta_n\delta_0 x$ is close to being exact, differing only by a member of $Ker(\eta_x)$. Note however that this does not generally mean the one can find an exact differential that follows from this construction. Take for example the pair n and $2b$ in our Lie algebra model. Using the above process one will define $\partial n = 2b - \frac{1}{2}Tn$ which satisfies

$$\begin{aligned}\partial^2 n &= -Tb - \frac{1}{2}([\partial t, n] - T(2b - \frac{1}{2}Tn)) \\ &= Tb - Tb - \frac{1}{2}(-\frac{1}{2}TTn - \frac{1}{2}SSn + \frac{1}{2}TTn) = \frac{1}{4}SSn\end{aligned}$$

This is of course in $Ker(\eta_n)$ since $SSs = 0$ but it is definitely not exact.

Although we haven't seen proof that there is an exact differential ∂g of this particular form, we will first look for such solutions, believing that some exist. We will seek solutions in which a single p symbol in every canonical element from $\delta_n p$ has been replaced with a g. This replacement would be a solution is the corresponding $Ker(\eta_g)$ elements is zero. Note that there are many possible replacements if there are several p symbols, for instance $PPPNb$ can be replaced with $tPPGNb + sPGPNb + (1-t-s)GPPNb$ for all t and s values.

In order to make use of the above theorem, we must first change the way we express $\delta_n p$. Noting that for all $n \in \mathbb{N}$ there is a p symbol in all the canonical expressions of $\delta_n p$, we will simply choose to move this p symbol and express the canonical element as a canonical sum with p at the center of all the elements.

Remark 29 When checking (4.1) on expressions of this form one can think of $\delta_0 g$ as being zero, since as we have seen, the expressions in which the derivation acts on g are canceled with $\delta_n p$. Therefore formally replacing all $\delta_0 g$ expression in (4.1) with zeros (including that of $\delta_n\delta_0 g = \delta_n p$) does not change the equation or it's solutions in any way, apart from leaving us with less calculations.

Corollary 30 The part of $\delta_n g$ that is in $\langle T^{(1)}_{n+1}\rangle$ can be chosen to be exactly like $\delta_n p$ with the inner most p symbol replaced with g. That is for $n \neq 2$

$$P_{\langle T^{(1)}_{n+1}\rangle}(\delta_n g) = -\frac{2B_{m+1}}{2^m m!}\sum_{k=0}^{n} N^k G N^{n-1-k} s$$

and $P_{\langle T^{(1)}_{n+1}\rangle}(\delta_1 g) = -\frac{1}{2}Tg$

Proof It is sufficient to note that $Ker(\eta_g) \cap \langle T^{(1)}_{n+1}\rangle = \{0\}$. This is because we are dealing with a free Lie-algebra and the only relations on it are those of graded symmetry and Jacobi.

The former demands more than one p symbol to exist in order to have a non-zero bracket sum to become zero with the replacement of g with p. However since we are allowing only one p or g symbols in the space $\angle T^{(1)}_{n+1}\rangle$, there are no such elements.

The latter, combined with the fact that there is only one g or p element, demands that the some elements include p that is not the center element of the canonical expression. Since we have chosen only elements that have g at their center, replacing it with p could not produce a trivial sum.

Finally, for all $n \neq 2$ take the form of $P_{\langle T^{(1)}_{n+1}\rangle}(\delta_n p)$ and use graded symmetry so that the p is at the center.

$$\frac{2B_{m+1}}{2^m m!}\sum_{k=0}^{n} N^k P N^{n-1-k} s = -\frac{2B_{m+1}}{2^m m!}\sum_{k=0}^{n} N^k [N^{n-1-k} s, p]$$

Replace the p with a g. The theorem ensures that this is a solution to (4.1) projected onto $\langle T^{(1)}_{n+1}\rangle$. Now use graded symmetry again to bring the g back to where the p used to be. This time the sign does not change and thus, the expression is left with the minus sign.

As for $n = 1$ simply replace the p with a g and we are done. $\square$

Note that for all $n \leq 2$ there are no elements in $\langle T_{n+1} \rangle$ other than $\langle T_{n+1}^{(1)} \rangle$. This implies that the following definitions of $\delta_n g$ satisfy equation (4.1).

$$\delta_0 g = p \tag{4.5}$$
$$\delta_1 g = -\frac{1}{2} T g \tag{4.6}$$
$$\delta_2 g = -\frac{1}{24}(NG + GN)s \tag{4.7}$$

5 Applying the method for $\delta_4 g$

We already have the part of $\delta_4 g$ in $\langle T_5^{(1)} \rangle$ and we also know that $\langle T_5^{(i)} \rangle = \{0\}$ for $i = 2, 4, 5$. Thus the only part we still have to find is the part in $\langle T_5^{(3)} \rangle$.

The form of $\delta_4 p$ is

$$\frac{2B_5}{2^4 4!}(PPPNb + PPNPb + PNPPb + NPPPb)$$

We will replace every one of the four canonical elements with the sum of it's three possible replacements such that applying η_g would result in the original term. For instance

$$PPPNb \mapsto tPPGNb + sPGPNb + (1 - t - s)GPPNb$$

Applying δ_0 on the sum and using $\delta_0 g = 0$ we will find how every combination is mapped and search for combinations that result in an expression satisfying (4.1) exactly.

5.1 Understanding δ_0's operation

We start by picking a basis for $Lie_5^{-1}(X)$:

(1) PPSSg

(2) SSPPg

(3) PSSPg

(4) SPPSg

(5) PSPSg

(6) SPSPg

They are all clearly independent since all have only one g and this symbol is at the center. In order to get from one such element to a sum of different elements one must use either Jacobi or graded symmetry and thus loosing g's central position.

Using all the relations and identities we know on L we will now understand the operation of δ_0. It would be very useful to prepare the following:

- $ad_{Ss} = [S, S] = 2SS$
- $ad_{PPx} = [P, Px] = [P, [P, X]] = [P, PX - XP] = PPX - 2PXP + XPP$
- $ad_{PSs} = [P, Ss] = [P, 2SS] = 2PSS - 2SSP$
- $ad_{SPs} = -ad_{ssp} = -\frac{1}{2}ad_{PSs} = SSP - PSS$
- $ad_{PSPs} = -ad_{PSSp} = \frac{1}{2}ad_{PPSs} = PPSS - 2PSSP + SSPP$
- $ad_{PPs} = PPS - 2PSP + SPP$
- $ad_{SPPs} = [B, PPS - 2PSP + SPP]$

$$= PPSS + SSPP + 2SPPS - 2SPSP - 2PSPS$$

In all following relations we will use $r_i = 1 - t_i - s_i$.

$$
\begin{aligned}
(1)\quad \delta_0 \quad & \big(t_1PPGNs + s_1PGPNs + r_1GPPNs\big)\\
= \quad & -2\big(t_1PPGSS + s_1PGPSS + r_1GPPSs\big)\\
= \quad & -2\big(-2t_1PPSSg - s_1P[PSs,g] - r_1[PPSs,g]\big)\\
= \quad & -2\big(-2t_1PPSS - s_1P(2PSS - 2SSP)\\
& \qquad -r_1(2PPSS - 4PSSP + 2SSPP)\big)g\\
= \quad & 4(t_1+s_1+r_1)PPSSg + 4r_1SSPPg + 4(-s_1-2r_1)PSSPg\\
= \quad & 4PPSSg + 4r_1SSPPg - 4(s_1+2r_1)PSSPg
\end{aligned}
$$

$$
\begin{aligned}
(2)\quad \delta_0 \quad & \big(t_2PPNGs + s_2PGNPs + r_2GPNPs\big)\\
= \quad & 2\big(t_2PPSGs - s_2PGSPs - r_2GPSPs\big)\\
= \quad & 2\big(t_2PPSSg + s_2P[SPs,g] + r_2[PSPs,g]\big)\\
= \quad & 2\big(t_2PPSS + s_2P(SSP - PSS) + r_2(PPSS - 2PSSP + SSPP)\big)g\\
= \quad & 2PPSSg + 2r_2SSPPg - 2(s_2+2r_2)PSSPg
\end{aligned}
$$

$$
\begin{aligned}
(3)\quad \delta_0 \quad & \big(t_3PNPGs + s_3PNGPs + r_3GNPPs\big)\\
= \quad & 2\big(t_3PSPGs + s_3PSGPs - r_3GSPPs\big)\\
= \quad & 2\big(t_3PSPSg + s_3PS[Ps,g] - r_3[SPPs,g]\big)\\
= \quad & 2\big(t_3PSPS + s_3PS(PS - SP)\\
& \qquad +r_3(PPSS + SSPP + 2SPPS - 2SPSP - 2PSPS)\big)g\\
= \quad & 2r_3PPSSg + 2r_3SSPPg - 2s_3PSSPg + 4r_3SPPSg\\
& \qquad +2(t_3+s_3-2r_3)PSPSg - 4r_3SPSPg\\
= \quad & 2r_3PPSSg + 2r_3SSPPg - 2s_3PSSPg + 4r_3SPPSg\\
& \qquad +2(1-3r_3)PSPSg - 4r_3SPSPg
\end{aligned}
$$

$$
\begin{aligned}
(4)\quad \delta_0 \quad & \big(t_4NPPGs + s_4NPGPs + r_4NGPPb\big)\\
= \quad & 2\big(t_4SPPGs + s_4SPGPs + r_4SGPPs\big)\\
= \quad & 2\big(t_4SPPSg + s_4SP[Ps,g] + r_4S[PPs,g]\big)\\
= \quad & 2\big(t_4SPPS + s_4SP(PS - SP) + r_4S(PPS - 2PSP + SPP)\big)g\\
= \quad & 2r_4SSPPg + 2(t_4+s_4+r4)SPPSg - 2(s_4+2r_4)SPSPg\\
= \quad & 2r_4SSPPg + 2SPPSg - 2(s_4+2r_4)SPSPg
\end{aligned}
$$

The sum of all these elements is the affine transformation $Ax + b$ where

$$\begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 4 & 2 & 2 & 2 & 0 & 0 & 0 & 0 \\ -8 & -4 & 0 & 0 & -4 & -2 & -2 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & -4 & 0 & 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix} + \begin{pmatrix} 6 \\ 0 \\ 0 \\ 2 \\ 2 \\ 0 \end{pmatrix}$$

First we will calculate the kernel of the matrix. Using the Gaussian elimination process, one can transform the matrix into

$$\begin{pmatrix} 4 & 2 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and one can easily find that the kernel is

$$span \left\{ \begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \\ -2 \\ 0 \\ 0 \\ 4 \end{pmatrix} \right\}$$

Now we will find a particular solution. Remembering that we are looking only for solutions that take after $\delta_4 p$, Remark 29 shows that one can formally take "$\delta_0 g = 0$" to simplify our calculations and without changing the resulting equation. With this in mind our equation takes the form $P_{\langle T_5^{(3)} \rangle} \delta_0 \delta_4 g = -P_{\langle T_5^{(3)} \rangle} \delta_2 \delta_2 g$. Calculating the right-hand side

$$\begin{aligned} \delta_2 \delta_2 g &= -\frac{1}{24} \delta_2 (NGs + GNs) = -\frac{1}{24} \delta_2 ([N,S]g + NSg) = \\ &= -\frac{1}{24} \delta_2 (2NSg - SNg) \end{aligned}$$

Taking only elements with $\#p + \#g = 3$ and using Leibnitz one gets

$$\begin{aligned}
P_{\langle T_5^{(3)}\rangle}\delta_2\delta_2 g &= -\frac{1}{24^2}(2[PPs, Sg] + S[PPs, g]) \\
&= -\frac{1}{24^2}\big(2(PPS - 2PSP + SPP)Sg + S(PPS - 2PSP + SPP)g\big) \\
&= -\frac{1}{24^2}(2PPSS - 4PSPS + 2SPPS + SPPS - 2SPSP + SSPP)g \\
&= -\frac{1}{24^2}(2PPSSg + SSPPg + 3SPPSg - 4PSPSg - 2SPSPg)
\end{aligned}$$

Putting this in the language of the equations above and dividing by $\frac{2B_5}{2^4 4!}$

$$Ax + b = -\frac{-5760}{24^2}\begin{pmatrix} 2 \\ 1 \\ 0 \\ 3 \\ -4 \\ -2 \end{pmatrix} \Rightarrow Ax = 10\begin{pmatrix} 2 \\ 1 \\ 0 \\ 3 \\ -4 \\ -2 \end{pmatrix} - \begin{pmatrix} 6 \\ 0 \\ 0 \\ 2 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 14 \\ 10 \\ 0 \\ 28 \\ -42 \\ -20 \end{pmatrix}$$

It is easily verifiable that a particular solution to this equation is given by the vector $(0, 0, 7, -2, 0, 0, 0, 0)^t$. The actual form of this solution is

$$\delta_4 g = (\text{the part with \#p=1 found in Cor 30})$$

$$+\frac{2B_5}{2^4 4!}(PPGNs + PPNGs - 6PNPGs + 7GNPPs + 3NPPGs - 2NGPPb)$$

As a conclusion we find that replacing p with a g does yield exact solutions for all $n \leq 4$ at the least.

Before proceeding to greater n values one has to pick one of the above solutions. There are many possible choices to be made (four dimensions worth) and most will not provide much insight to the possible form of $\delta_n g$ for $n \geq 4$. One can also choose to add elements from $Ker(\delta_0)$ that have the proper length, grading and symmetries such as $\delta_0(NNNNg) = \delta_0(N^4)g + NNNNp$. This will add new elements from $Lie(\{n, p\})$ that we managed to prove unnecessary.

I will not go any further for now, but I would like to offer my guess regarding the form of ∂g. I would imagine that one can continue the procedure that was used above:

- Work in the context of a single $\langle T_{n+1}^{(k)}\rangle$ at a time.
- For $k = 1$ use the form $-\frac{2B_{m+1}}{2^m m!}\sum_{k=0}^{n} N^k G N^{n-1-k} s$.
- For even k values and for $k = n$ take 0.

- For odd $k > 1$ start with $P_{\langle T^{(k)}_{n+1} \rangle}(\delta_n p)$ and extract a from it a set A of canonical elements that sum to it.
- From this set extract a basis B that includes all possible replacements in which a single p is replaced with a g.
- Take a general sum of elements from B such that applying η_g gives $P_{\langle T^{(k)}_{n+1} \rangle}(\delta_n p)$ back.
- Describe the operation of δ_0 on the coefficients of the general sum in terms of A. This gives a set of linear equations.
- Find the solution space (assuming that such solutions exist) and look for solutions that provide special insight and allow for more educated guessing.

My guess is that it would be possible to find solutions in this manner for all n since using the form of $\delta_n p$ almost ensures exactness and there are plenty of degrees of freedom to play with in order to find exact solutions. I especially believe that there will be no need for $Lie(\{n, p\})$ elements in ∂g and using this technique eliminates them from the sum completely.

6 Summary

In looking for a free DGLA model of the 2-cell, we used the recursive technique for finding the differential of g (the 2-cell). We showed that a model exists preserving the symmetry of the 2-cell, and in addition that one can choose to set $\delta_n g = 0$ for every odd $n \neq 1$, as well as there being no $t = a + b$ symbols (using the symmetrized basis) in any $\delta_n g$ for $n \neq 1$. This makes the task of finding possible solutions a bit easier and shows the resemblance between the differentials of all generators, as they all have this property. Taking this resemblance one step further, we suggested taking ∂g to be of the same form as $\partial(\delta_0 g)$ which is known. This allowed us to write an explicit expression for the part of ∂g with only one $\#p + \#g$ in it. There are still choices to be made as to what presentation to take, and there is the question of whether such a presentation even exists. However we do know that such presentations exist up to $n = 5$ and we have provided a technique using which one can find them.

References

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