COHOMOLOGY OF CONFIGURATION SPACES

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ABSTRACT. The configuration space of n distinct points in the complex plane is a complex n-manifold. This complex structure allows us to define holomorphic 1-forms and use their nice behavior under integration to understand the cohomology of the manifold.

Next we prove that a configuration space is a covering of a variety of polynomials with no repeated roots, and interpret the same 1-forms in terms of the discriminant of the polynomials.

In low dimensions there exists a morphism from configuration space of 4 points to that of 3 points, the Ferrari map. We will describe it's origins and compute it's action on the previously described 1-forms.

1. COHOMOLOGY OF CONFIGURATION SPACE OVER $\mathbb{C}$

Let $X_n = Conf_n(\mathbb{C})$ be the configuration space of n distinct points in the complex plane. This space is the open set of $\mathbb{C}^n$ defined by the inequalities $z_i \neq z_j$ for all $i \neq j$.

We have the natural coordinate functions $z_1, \ldots, z_n$ induces by restricting the n projections $\mathbb{C}^n \to \mathbb{C}$. Use these to define holomorphic 1-forms

$$w_{ij} = \frac{d(z_i - z_j)}{z_i - z_j}$$

for all $i \neq j$. In [1] Arnol'd states that the forms with $i < j$ induce linearly independent cohomology classes. Moreover, these in fact generate the cohomology ring as an alternating algebra, subject only to the relations $w_{ij} = w_{ji}$ and

$$w_{ij} \wedge w_{jk} + w_{jk} \wedge w_{ki} + w_{ki} \wedge w_{ij} = 0$$

This may be proved inductively using the fibration

$$\mathbb{C} \setminus \{\text{n points}\} \to Conf_{n+1}(\mathbb{C}) \to Conf_n(\mathbb{C})$$

and the Serre spectral sequence, but we will not cover this here. For a proof see T. Church's notes at [2].

In what follows we will verify directly that these forms induce non-trivial elements in $H^1(Conf_n(\mathbb{C}); \mathbb{C})$ and that these elements are linearly independent. After which, we proceed to proving that the above relations hold and that the appropriate k-forms

$$w_{i_1,j_1} \wedge \ldots \wedge w_{i_k,j_k}$$

are non-trivial and linearly independent in H^k for all $k \leq n$.

Proposition 1.1. *The forms w_{ij} defined above are closed and define cohomology classes $[w_{ij}] \neq 0$ that are all linearly independent in H^1.*

Proof. Consider the following maps $P_{ij} : X_n \to \mathbb{C} \setminus \{0\}$

$$P_{ij}(z_1, \dots, z_n) = z_i - z_j$$

It is now clear that for all $i < j$

$$w_{ij} = \frac{d(z_i - z_j)}{z_i - z_j} = P_{ij}^* \left(\frac{dz}{z} \right)$$

where $\frac{dz}{z}$ represents a generator of the cohomology $H^1(\mathbb{C} \setminus \{0\}; \mathbb{C})$. Since $\frac{dz}{z}$ is a closed form, the pullback w_{ij} is also closed.

Pairing the form w_{ij} with a 1-cycle γ we get

$$\oint_\gamma w_{ij} = \oint_\gamma P_{ij}^* \left(\frac{dz}{z} \right) = \oint_{P_{ij} \circ \gamma} \frac{dz}{z} = 2\pi i \cdot n\left(P_{ij} \circ \gamma, 0\right)$$

where $n(\delta, 0)$ is the winding number of a 1-cycle δ around the origin $0 \in \mathbb{C}$.

Define 1-cycles by the closed loops $\gamma_{kl} : [0,1] \to X$ for $k < l$

$$\gamma_{kl}(t) = (a_1, a_2, \dots, a_{l-1}, a_k - e^{2\pi i t}, a_{l+1}, \dots, a_n)$$

for fixed distinct complex numbers $a_1, \dots, a_n \in \mathbb{C}$ that are far apart, say $|a_i - a_j| > 3$ for all $i \neq j$.

Applying P_{ij} to the path γ_{kl} we get one of the following cycles in $\mathbb{C} \setminus \{0\}$

$$P_{ij} \circ \gamma_{kl}(t) = \begin{cases} a_i - a_j & \text{if } l \neq i, j \\ a_k - a_j - e^{2\pi i t} & \text{if } l = i \\ a_i - a_k + e^{2\pi i t} & \text{if } l = j \end{cases}$$

in particular if $(i, j) = (k, l)$ then the cycle becomes $e^{2\pi i t}$ and in any other case the modulus is bounded away from zero. Thus pairing w_{ij} with γ_{kl} results in the complex number $2\pi i \delta_{ik} \delta_{jl}$.

This proves that the cohomology classes $[w_{ij}]$ are linearly independent, and in particular they are non-trivial. □

Note that this gives us a geometric interpretation of the form w_{ij}: it measures the winding number around the (real codimension 2) hyperplane $z_i = z_j$. Alternatively, if we think of points in $Conf_n$ as n distinct points in the plane, then wij measures the winding of z_i around z_j.

Proposition 1.2. *The forms w_{ij} satisfy $w_{ij} = w_{ji}$ and the cyclic relation*

$$w_{ij} \wedge w_{jk} + w_{jk} \wedge w_{ki} + w_{ki} \wedge w_{ij} = 0$$

Proof. These are elementary calculation:

$$w_{ji} = \frac{dz_j - dz_i}{z_j - z_i} = \frac{dz_i - dz_j}{z_i - z_j} = w_{ij}$$

As for the cyclic relation,

$$w_{ij} \wedge w_{jk} = \frac{dz_i - dz_j}{z_i - z_j} \wedge \frac{dz_j - dz_k}{z_j - z_k} = \frac{dz_i \wedge dz_j + dz_j \wedge dz_k + dz_k \wedge dz_i}{(z_i - z_j)(z_j - z_k)}$$

Collecting the three terms together, we form the sum over the three cyclic permutations.

$$\frac{dz_i \wedge dz_j + dz_j \wedge dz_k + dz_k \wedge dz_i}{(z_i - z_j)(z_j - z_k)(z_k - z_i)} \left[(z_k - z_i) + (z_i - z_j) + (z_j - z_k)\right] = 0$$

and indeed the relation holds. □

Now we proceed to the higher cohomology.

Proposition 1.3. *Let $I = [(i_1, j_1), \ldots, (i_k, j_k)]$ be a double multi-index with $i_r < j_r$ for all r and $j_1 < j_2 < \ldots < j_k$. The k-forms $w_I = w_{i_1,j_1} \wedge \ldots \wedge w_{i_k,j_k}$ with I as above are closed and linearly independent in the cohomology $H^k(X_n; \mathbb{C})$.*

Proof. The wedge of closed forms is closed, so in particular w_I is closed.

Denote $P^I = P_{i_1,j_1} \times \ldots \times P_{i_k,j_k} : (X_n)^k \to (\mathbb{C} \setminus \{0\})^k$. Map X_n into $(\mathbb{C} \setminus \{0\})^k$ via the diagonal:

$$P_I : X_n \overset{\Delta}{\hookrightarrow} (X_n)^k \xrightarrow{P^I} (\mathbb{C} \setminus \{0\})^k$$

Clearly the projection of this map into the l-th coordinate coincides with P_{i_l,j_l}. Thus by naturality of the wedge product and the pullbacks, we get a characterization of w_I

$$w_I = {P_I}^* \left({\pi_1}^* \frac{dz}{z} \wedge \ldots \wedge {\pi_k}^* \frac{dz}{z} \right)$$

When paired with a k-chain γ,

$$\int_\gamma w_I = \int_{P_I \circ \gamma} {\pi_1}^* \frac{dz}{z} \wedge \ldots \wedge {\pi_k}^* \frac{dz}{z}$$

Now if the k-chain $P_I \circ \gamma$ happens to be homologous to a product of 1-chains $\gamma_1 \times \ldots \times \gamma_k$ where

$$\gamma_i : [0,1] \to \mathbb{C} \setminus \{0\}$$

then by Fubini's theorem, the integral $\int_\gamma w_I$ breaks up as the product of the 1-dimensional integrals

$$\int_{\gamma_1} \frac{dz}{z} \cdot \ldots \cdot \int_{\gamma_k} \frac{dz}{z} = (2\pi i)^k n(\gamma_1, 0) \cdot \ldots \cdot n(\gamma_k, 0)$$

If on the other hand $P_I \circ \gamma$ is homologous to a cycle $c : [0,1]^k \to \mathbb{C} \setminus \{0\}^k$ s.t.

$$c = (c'_{(s')}, c_{k(s',s_k)}) \text{ where } s' = (s_1, \ldots, s_{k-1})$$

and for every fixed s' the 1-cycle $c_k(s', \cdot)$ does not wind around 0, then again by Fubini's theorem

$$\int_\gamma w_I = \int_{s' \in c'} w' \int_{c_k(s',\cdot)} \frac{dz}{z} = 2\pi i \int_{s'} w' \cdot n(c_k(s', \cdot), 0) = 0$$

Thus proving that the cohomology classes $[w_I]$ are linearly independent reduces to finding k-cycles γ_I s.t. for every I the push-forward cycle $P_I \circ \gamma_I$ is a product of loops around the origin, like the torus $S^1 \times \ldots \times S^1 \subset (\mathbb{C} \setminus \{0\})^k$, and for $I' \neq I$ there is a presentation $c = (c', c_k)$ with $c_k(s', \cdot)$ not winding around 0 for every fixed s'.

Define the k-cycle $\gamma_I : [0,1]^k \to Conf_n(\mathbb{C})$ in the following way. First fix complex numbers $(a_1, \ldots, a_n)$ with $|a_j - a_i| \geq 3$ for all $i \neq j$. Define

$$(s_1, \ldots, s_k) \mapsto (f_1(\mathbf{s}), \ldots f_n(\mathbf{s}))$$

where f_l is defined recursively in l:

- If $l \neq j_1, \ldots, j_k$, set $f_l \equiv a_l$ as in the $1D$ case.
- If $l = j_r$, set $f_l(\mathbf{s}) = f_{i_r}(\mathbf{s}) + \frac{1}{2^r} e^{2\pi i s_r}$ (imagine the trajectories of moons in the solar system).

This definition ensure that for every $r' < r$ the distance between $f_{r'}$ and f_r is at least $\frac{1}{2^r}$, and thus $\gamma_I(\mathbf{s})$ indeed defines n distinct points in $\mathbb{C}$. Moreover,

$$f_{j_r} - f_{i_r} = \frac{1}{2^r} e^{2\pi i s_r}$$

while if $I' \neq I$, there is a pair $(i,j) \in I'$ that does not belong to I and for which $f_j - f_i$ is a function of $s_1, \dots, s_r$ alone with $(f_j - f_i)(s', \cdot)$ **not** winding around the origin for every fixed $s' = (s_1, \dots, s_{r-1})$.

Thus pairing $w_{I'}$ with γ_I is non-zero iff $I = I'$ proving that the forms $\{w_I\}_I$ induce linearly independent (and in particular non-trivial) cohomology classes. □

2. Connection with the space of polynomials

There is a natural (algebraic) map from the configuration space to the space of polynomials

$$\varphi : Conf_n(\mathbb{C}) \to \mathbb{C}_n[T] \cong \mathbb{C}^n$$

assigning for every n-tuple $(a_1, \dots, a_n)$ the unique monic polynomial of degree n whose roots are precisely $a_1, \dots, a_n$.

The image of this map is precisely those monic polynomials $p(T)$ with no repeated roots. This property can be described in terms of the coefficients of $p(T)$ by the non-vanishing of the discriminant $\Delta(p)$, and thus the image is a (Zariski) open set of $\mathbb{C}^n$.

Proposition 2.1. *The map φ is a local diffeomorphism.*

Proof. Note that φ sends $\mathbf{x} = (x_1, \dots, x_n)$ to the coefficients of the polynomial

$$P_{\mathbf{x}}(T) = \prod_{i=1}^{n} (T - x_i) = T^n + f_1(\mathbf{x}) T^{n-1} + \dots f_n(\mathbf{x})$$

Computing the partial derivatives of the map $\mathbf{x} \mapsto (f_1, \dots, f_n)$, we find that they form a new polynomial

$$\partial_j f_1 T^{n-1} + \dots + \partial_j f_n = \partial_j P_{\mathbf{x}}(T) = \partial_j \prod_{i=1}^{n} (T - x_i) = - \prod_{i \neq j} (T - x_i) =: Q_j(T)$$

This polynomial vanishes at x_i for all $i \neq j$. We use this observation to compute the Jacobian of φ.

$$\begin{pmatrix} \partial_1 f_1 & \dots & \partial_1 f_n \\ \vdots & & \vdots \\ \partial_n f_1 & \dots & \partial_n f_n \end{pmatrix} \begin{pmatrix} x_1^{n-1} & x_2^{n-2} & \dots & x_n^{n-1} \\ x_1^{n-2} & x_2^{n-2} & \dots & x_n^{n-2} \\ \vdots & & & \vdots \\ 1 & & \dots & 1 \end{pmatrix} = \begin{pmatrix} Q_1(x_1) & Q_1(x_2) & \dots & Q_1(x_n) \\ \vdots & & & \vdots \\ Q_1(x_n) & Q_n(x_2) & \dots & Q_n(x_n) \end{pmatrix}$$

so using the vanishing property of the Q_j's, the RHS is a diagonal matrix. Computing it's determinant,

$$\det \begin{pmatrix} Q_1(x_1) & 0 & \dots & 0 \\ 0 & Q_2(x_2) & \dots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & Q_n(x_n) \end{pmatrix} = Q_1(x_1) \cdot \dots \cdot Q_n(x_n) = (-1)^n \prod_{j=1}^{n} \prod_{i \neq j} (x_j - x_i)$$

By the multiplicativity of the determinant and the Vandermonde formula, we conclude that the Jacobian of φ is

$$\frac{(-1)^n \prod_{i\neq j}(x_i - x_j)}{\prod_{i<j}(x_i - x_j)} = (-1)^n \prod_{i>j}(x_i - x_j) \neq 0$$

If fact, this Jacobian is a square root of the discriminant of P_x (up to sign). By the inverse function theorem we conclude that $\varphi : Conf_n \to \mathbb{C}_n[T]$ is a local diffeomorphism. □

Consider the natural S_n action on $Conf_n(\mathbb{C})$ permuting the coordinates. The map φ is invariant under the S_n action, and thus there is an induced map

$$UConf_n(\mathbb{C}) := Conf_n(\mathbb{C})/S_n \to \{p \in \mathbb{C}_n[T] \mid \text{monic and } \Delta(p) \neq 0\} \subset \mathbb{C}_n[T]$$

which now becomes bijective. Thus the two spaces are naturally diffeomorphic.

Corollary 2.2. *The map φ is in fact a normal covering map with deck group S_n.*

Remark 2.3. Recall that the fundamental group of $Conf_n(\mathbb{C})$ is the pure braid group P_n while that of $UConf_n$ is the full braid group B_n. The S_n-covering just described corresponds to the short exact sequence (of non-abelian groups)

$$1 \to P_n \to B_n \to S_n \to 1$$

Remark 2.4. Consider the monodromy action of this covering. A path in the space of polynomials lifts to a path in the configuration space of their roots. However, as we move along a closed loop of polynomials and return to where we started, the roots must also return to the original configuration as an unordered set, but might get permuted in the process.

This nice observation may be used to give a geometric proof of the insolvability by radicals of quintic equations. For more on this, see the wonderful video [3].

It is interesting to consider the forms w_{ij} in light of the covering φ. Of course, they are not invariant under the S_n-action and so do not descend to forms on the space of polynomials, but there are invariant combinations of them and these do descend.

Example 2.5. If $n = 2$ then we have $w_{12} = w_{21}$ and so there is an induced form on the space of polynomials.

Explicitly, z_1 and z_2 are the two roots of the polynomial $P_z(T) = T^2 + bT + c$, and their difference $z_1 - z_2$ is expressed as a square root of the discriminant $b^2 - 4c$. The two choices of square root correspond to the two labelings of the points z_1 and z_2, which in turn correspond to the two point in the fiber above P_z. By the chain rule,

$$\frac{dz_1 - dz_2}{z_1 - z_2} = \frac{d(\sqrt{b^2 - 4c})}{\sqrt{b^2 - 4c}} = \frac{1}{2(b^2 - 4c)} d(b^2 - 4c) = \frac{1}{2} \frac{d(b^2 - 4c)}{(b^2 - 4c)}$$

This expresses a remarkable connection between the mutual winding of the two roots of $P(T) = T^2 + bT + c$ in the plane and the winding of it's discriminant $b^2 - 4c$ around the origin (the discriminant hits the origin precisely when P has a doubled root). The change in the argument of the difference $z_2 - z_1$ is precisely half the change in argument of the discriminant $b^2 - 4c$.

Example 2.6. If $n > 2$, to get an invariant form we must take the sum $\sum_{i<j} w_{ij}$. This descends to a form on $UConf_n$ or equivalently on polynomials.

By the properties of the logarithmic derivative

$$\frac{d(fg)}{fg} = \frac{df}{f} + \frac{dg}{g}$$

which in this case implies

$$\sum_{i<j} w_{ij} = \sum_{i<j} \frac{d(z_i - z_j)}{z_i - z_j} = \frac{d\left(\prod_{i<j}(z_i - z_j)\right)}{\prod_{i<j}(z_i - z_j)}$$

As above, this is the square root of the discriminant Δ of the polynomial $P_z(T)$, i.e.

$$\sum_{i<j} w_{ij} = \frac{1}{2}\frac{d\Delta}{\Delta}$$

and again the logarithmic derivative of the discriminant, measuring the change of argument of the Δ around 0, generates the cohomology and is naturally related to the forms w_{ij}.

3. Resolvent cubics and the Ferrari map

In the 16th century, Italian mathematicians managed to solve polynomial equations of degrees 3 and 4. The solution of the quartic is credited to L. Ferrari, who reduced solving the quartic equation to solving a cubic equation (now called **the resolvent cubic**).

Since then, several different methods were found for solving the quartic, each reducing the problem to the solution of a different cubic equation [4]. In what follows we use one of these techniques - the Euler resolvent - to construct a polynomial map between $Conf_4(\mathbb{C})$ and $Conf_3(\mathbb{C})$.

3.1. **Euler's resolvent cubic.** This technique may be described as follows (see [5]). Let

$$P(T) = T^4 + aT^3 + bT^2 + cT + d$$

be a monic quartic equation. Suppose it's roots are x_1, x_2, x_3, x_4. Instead of computing the roots directly, we consider the following involution J

$$\mathbf{s} = J\mathbf{x} \longleftrightarrow \begin{array}{l} s_0 = \frac{1}{2}(x_1 + x_2 + x_3 + x_4) \\ s_1 = \frac{1}{2}(x_1 - x_2 + x_3 - x_4) \\ s_2 = \frac{1}{2}(x_1 + x_2 - x_3 - x_4) \\ s_3 = \frac{1}{2}(x_1 - x_2 - x_3 + x_4) \end{array}$$

From $\mathbf{s}$ we may recover the roots, since $J^2 = Id$. One might recognize in s_1 the first symmetric polynomial in $\mathbf{x}$, which we know is equal to $(-a)$. Thus computing $\mathbf{x}$ reduces to finding s_1, s_2 and s_3.

The degree 6 polynomial

$$R(s^2) = (s^2 - {s_1}^2)(s^2 - {s_2}^2)(s^2 - {s_3}^2)$$

is symmetric as a function of the x_i's and thus may be expanded and expressed in terms of the coefficients of P. R a polynomial of degree 3 called **the Euler resolvent cubic of** P.

Finding the three roots of R and choosing square roots $\pm s_1, \pm s_2, \pm s_3$ will provide us with the four solutions of P. Different choices of signs may give rise to a different labeling of x_1, x_2, x_3, x_4.

Definition 3.1. Let the Ferrari map $fe : \mathbb{C}_4[T] \to \mathbb{C}_3[T]$ be the map that sends a quartic P to it's resolvent cubic R.

We can use the involution J to construct an explicit lifting of fe to the configuration spaces.

Definition 3.2. Let the Ferrari map $Fe : Conf_4(\mathbb{C}) \to Conf_3(\mathbb{C})$ be the map that sends the set of 4 roots of a quartic P to the three roots of the resolvent cubic R as expressed by the involution J.

Explicitly, this is the following polynomial map:

$$\begin{array}{c} z_1 = s_1^2 = \frac{1}{4}(x_1 - x_2 + x_3 - x_4)^2 \\ z_2 = s_2^2 = \frac{1}{4}(x_1 + x_2 - x_3 - x_4)^2 \\ z_2 = s_2^2 = \frac{1}{4}(x_1 - x_2 - x_3 + x_4)^2 \end{array}$$

If the four roots $\mathbf{x}$ of P are distinct, so will the roots $\mathbf{z}$ of R be distinct. Therefore this indeed defines a map between the configuration spaces. Moreover, permuting the four roots $\mathbf{x}$ induces a permutation of the three roots $\mathbf{z}$ of R, so this map descends to a map on the unordered configurations

$$fe : UConf_4(\mathbb{C}) \to UConf_3(\mathbb{C})$$

which, in terms of the isomorphism φ, is the same Ferrari map previously defined on the space of polynomials.

Remark 3.3. Notice the following implication of the existence of this map: fe and Fe induce homomorphisms on fundamental groups

$$\begin{array}{ccc} P_4 & \xrightarrow{Fe_*} & P_3 \\ \downarrow & & \downarrow \\ B_4 & \xrightarrow{fe_*} & B_3 \\ \downarrow & & \downarrow \\ S_4 & \xrightarrow{\overline{fe_*}} & S_3 \end{array}$$

The bottom most map is simple and yet interesting. Explicitly, every permutation $\sigma \in S_4$ acts on the four roots x_1, x_2, x_3, x_4 by changing labels. This action permutes the three roots z_1, z_2, z_3, and so defines $\overline{fe_*}(\sigma) \in S_3$. $\overline{fe_*}$ is interesting since a surjective homomorphism

$$S_n \to S_m \quad \text{where } n > m > 2$$

exists only when $n = 4$ and $m = 3$ (for $m = 2$ there is always the sign homomorphism). This is because for $n \geq 5$ the alternating group $A_n \leq S_n$ is simple and has cardinality greater than that of S_m so it must map to 1, and the image thus has cardinality ≤ 2.

This demonstrates that the Ferrari map is unique in that there are no analogous maps for higher dimensions. Since fe is used in the solution of the quartic, the nonexistence of higher dimensional analogs is again related to the insolvability of equations of degree ≥ 5.

3.2. **The induced map on cohomology.** We have explicit generators for the cohomology of configuration spaces represented by the 1-forms w_{ij}. Let's compute the action of the Ferrari map on these forms.

Proposition 3.4. *The action Fe^* on the forms w_{ij} is given by*

$$\begin{aligned} Fe^*(w_{12}) &= w_{14} + w_{23} \\ Fe^*(w_{13}) &= w_{12} + w_{34} \\ Fe^*(w_{23}) &= w_{13} + w_{24} \end{aligned}$$

Proof.

$$\begin{aligned} Fe^*(z_1 - z_2) &= \frac{1}{4}\left((x_1 - x_2 + x_3 - x_4)^2 - (x_1 + x_2 - x_3 - x_4)^2\right) \\ &= \frac{1}{4}(2x_1 - 2x_4)(-2x_2 + 2x_3) = (x_1 - x_4)(x_3 - x_2) \end{aligned}$$

and by the properties of the logarithmic derivative, the pullback form is

$$\begin{aligned} Fe^*(w_{12}) &= Fe^*\left(\frac{d(z_1 - z_2)}{z_1 - z_2}\right) = \frac{d\left((x_1 - x_4)(x_3 - x_2)\right)}{(x_1 - x_4)(x_3 - x_2)} \\ &= \frac{d(x_1 - x_4)}{x_1 - x_4} + \frac{d(x_3 - x_2)}{x_3 - x_2} = w_{14} + w_{23} \end{aligned}$$

Similarly, the other two forms w_{13} and w_{23} pullback as stated. □

Remark 3.5. These pullbacks are related to the homomorphism $S_4 \to S_3$ in the following way: the two transpositions (14) and (23) in S_4 map to the transposition (12) in S_3, and similarly for the other two forms.

Geometrically, this implies that if the transposition $x_i \leftrightarrow x_j$ gives rise to the transposition $z_k \leftrightarrow z_l$, then loops winding once around the hyperplane $x_i = x_j$ will be mapped to loops winding once around $z_k = z_l$

At the level of unordered configuration spaces, transfer shows that the cohomology is generated by the sum $\sum_{i<j} w_{ij}$. Pulling back this sum from $Conf_3$ to $Conf_4$ we get again the sum over all the forms. Thus fe^* maps $H^1(UConf_3;\mathbb{C})$ naturally onto $H^1(UConf_4;\mathbb{C})$

$$(w_{12} + w_{13} + w_{23}) \mapsto (w_{12} + w_{13} + w_{14} + w_{23} + w_{24} + w_{34})$$

sending the natural generator of $H^1(UConf_3;\mathbb{C})$ to that of $H^1(UConf_4;\mathbb{C})$.

Equivalently in terms of the discriminant on the space of polynomials,

$$fe^*\left(\frac{d\Delta_3}{\Delta_3}\right) = \frac{d\Delta_4}{\Delta_4}$$

References

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