THE UNIVERSITY OF CHICAGO

A GENERAL FRAMEWORK FOR REPRESENTATION STABILITY, WITH

APPLICATIONS TO ARRANGEMENTS AND ARITHMETIC

A DISSERTATION SUBMITTED TO

THE FACULTY OF THE DIVISION OF THE PHYSICAL SCIENCES

IN CANDIDACY FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

BY

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CHICAGO, ILLINOIS

JUNE 2019

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ACKNOWLEDGMENTS

ABSTRACT

This thesis sets out to explore and implement the paradigm of representation stability, specifically in the study of sequences of linear subspace arrangements, their stable combinatorics, topology and arithmetic.

In a traditional sense, the sequences of arrangements in question do not exhibit any form of stability, e.g. the Betti numbers of their complements grow to infinity. But when one considers the symmetries at play and the various maps between the arrangements, a new notion of stability presents itself: representation stability, where a finite collection of patterns is merely translated around by increasingly large groups. In this sense, stability is understood as a notion of finite generation.

One way to encode the structure of various symmetry groups and intertwining maps between them is using the language of diagrams. These are functors from a fixed category into the category of arrangements, vector spaces or any other target category. Thus a system of intertwined group actions can be treated as a single mathematical object, and finite generation gets a precise meaning. Representation stability therefore consists of two main aspects: identify finitely generated diagrams and operations that preserve finite generation (e.g. Noetherianity theorems), then extract stable invariants from a finitely generated diagram (e.g. polynomial characters).

This work addresses both of the above aspects in a general axiomatic framework. We define and study finitely generated diagrams of linear subspace arrangements – these occur in many natural examples from algebraic geometry and combinatorics, such as colored configuration spaces, k-equals arrangements and covers of moduli spaces of rational maps. Such collections of arrangements exhibit finite generation in their intersection posets, and this in turn leads to finite generation in the cohomology of their complements. We then study the reprecautions of this finite generation using an adaptation of character theory to the study of diagrams. Lastly, we adapt the

Grothendieck-Lefschetz fixed point formula to provide a bridge between cohomological representation stability results and asymptotic factorization statistics of orbits over finite fields.

CHAPTER 1

INTRODUCTION

In a foundational piece of recent work, Church-Ellenberg-Farb [CEF2] related representation stability in the cohomology of configuration spaces to splitting statistics of square-free polynomials over finite fields and discovered that these statistics converge as the degree $\to \infty$. Attempting to apply the same set of ideas to study rational maps poses several conceptual challenges, which this thesis seeks to overcome.

The main players in Church-Ellenberg-Farb's argument are: representations of symmetric groups and **FI**-modules, hyperplane arrangements and their cohomology algebra, and an equivariant Grothendick-Lefschetz fixed-point formula for free actions. All three elements break when passing to the case of rational maps:

- The automorhpism groups are products of several symmetric groups;

- The topological spaces are complements of linear subspace arrangements of high codimension, and the combinatorics at play has not been studied;

- The group actions have nontrivial stabilizers.

This thesis is organized around addressing the three points above, as follows. In §2 we introduce an axiomatic framework for working with categories similar to **FI** – the category of finite sets and injections – and thus generalize the framework of representation stability to collections of finite groups other than the symmetric groups. This approach formally and generally explains much of the seemingly ad-hoc structure of representation stability for symmetric groups, as well as other specific generalizations that exist in the literature. The central tool is a form of character theory for representations of categories, via character polynomials (Definition 2.2.5 below), that reveals

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stability in the representation theory as well as in statistics of the automorphism groups themselves (see §2.7).

In §3 we discuss diagrams of linear subspace arrangements, and identify new stability patterns in their combinatorics and in the cohomology of their complements. The results presented here apply fairly generally, in particular to many variants of colored configuration spaces and covers of moduli spaces of rational maps. Yet, at the same time, the results here improve on the previously known cases of representation stability for hyperplane arrangements with symmetric group actions as discussed in [CEF2] – we show that the cohomology modules are in fact free. The consequence of this work is that representation stability is far more ubiquitous than was previously observed.

Lastly, in §4 we adapt the Grothendieck-Lefschetz fixed point formula to the case of group actions with stabilizers. With this, cohomological representation stability translates to asymptotic statistics in the factorization of orbits into Galois-cycles over finite fields. The formula can thus be applied to computing splitting statistics of general varieties of polynomials over finite fields, not necessarily square-free ones. Weighted point-counts for general polynomials are worked out explicitly in §4.3.

The chapters are mostly self contained and could be read independently.

CHAPTER 2

GENERALIZING REPRESENTATION STABILITY

Representation stability is a theory describing a way in which a sequence of representations of different groups is related, and essentially contains a finite amount of information. Starting with Church-Ellenberg-Farb's theory of **FI**-modules describing sequences of representations of the symmetric groups, the literature now contains good theories for describing representations of other collections of groups such as finite general linear groups, classical Weyl groups, and Wreath products $S_n \wr G$ for a fixed finite group G. This chapter attempts to uncover the mechanism that makes the various examples work, and offers an axiomatic approach that generates the essentials of such a theory: character polynomials and free modules that exhibit stabilization.

We give sufficient conditions on a category $\mathbf{C}$ to admit such structure via the notion of categories of **FI** type. This class of categories includes the examples listed above, and extends further to new types of categories such as the categorical power $\mathbf{FI}^m$, whose modules encode sequences of representations of m-fold products of symmetric groups. The theory is applied in §3 to give homological and arithmetic stability theorems for various moduli spaces, e.g. the moduli space of degree n rational maps $\mathbb{P}^1 \to \mathbb{P}^m$.

2.1 Introduction

The purpose of this chapter is to describe a categorical structure that is responsible for the existence of representation stability phenomena. Our approach is centered around free modules[1] and character polynomials (defined below). We show that our proposed categorical structure gives rise to free modules which satisfy the fundamental properties that produce representation stability, and in particular the Noetherian property. We

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1. These are commonly called #-modules in the context of the category **FI**.

take an axiomatic approach that applies in a broad context, generalizing many of the known examples.

2.1.1 Motivation

Let **FI** be the category of finite sets and injections. An **FI**-*module* is a functor from **FI** to the category of modules over some fixed ring R. An **FI**-module $M_\bullet$ is a single object that packages together a sequence of representations of the symmetric groups S_n for every $n \in \mathbb{N}$ (see e.g. [CEF1]). Objects of this form arise naturally in topology and representation theory, for example:

- Cohomology of configuration spaces $\{\mathrm{PConf}^n(X)\}_{n \in \mathbb{N}}$ for a manifold X.

- Diagonal coinvariant algebras $\{\mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]/\mathcal{I}_n\}_{n \in N}$ (see [CEF1]).

A fundamental result of Church-Ellenberg-Farb [CEF1] is that an **FI**-module over $\mathbb{Q}$ is finitely-generated, i.e. there exists a finite set of elements not contained in any proper submodule, if and only if the sequence of S_n-representations stabilizes in a precise sense (see [CEF1] for details). This phenomenon was named *representation stability*. In particular, if one defines class functions

$$X_k(\sigma) = \# \text{ of } k\text{-cycles in } \sigma$$

simultaneously on all S_n, then [CEF1] show that for every finitely-generated **FI**-module $M_\bullet$ then there exists a single polynomial $P \in \mathbb{Q}[X_1, X_2, \ldots]$ – a *character polynomial* – that describes the characters of the S_n-representations M_n independent of n for all $n \gg 1$.

The uniform description of the characters in terms of a single character polynomial accounts for the most direct applications of the theory, for example:

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- For every manifold X and $i \geq 0$, the dimensions of $\{\mathrm{H}^i(\mathrm{PConf}^n(X); \mathbb{Q})\}_{n \in \mathbb{N}}$ are given by a single polynomial in n for all $n \gg 1$.[2]

- Every polynomial statistic, regarding the irreducible decomposition of degree-n polynomials over $\mathbb{F}_q$, tends to an asymptotic limit as $n \to \infty$.[3]

However, the above logic could be reversed: as first suggested by Gan-Li in [GL2], Nagpal showed in [Na, Theorem A] that if $M_\bullet$ is a finitely generated **FI**-module, then in some range $n \gg 1$ it admits a finite resolution by free **FI**-modules (see below) and these have characters given by character polynomials. It follows that for every $n \gg 1$ the character of the M_n is itself given by a character polynomial. One can then get stabilization of the decomposition of M_n into irreducible representations as a corollary of this fact! We assert that the key property of character polynomials – responsible for all representation stability phenomena and applications – is the following.

Fact 2.1.1 ([CEF2, Theorem 3.9]). *The S_n-inner product of two character polynomials P and Q becomes independent of n for all $n \geq \deg(P) + \deg(Q)$.*

The benefit of Gan-Li's and Nagpal's approach is that free **FI**-modules and character polynomials readily generalize to a wide class of categories similar to **FI**, and do not require any understanding of the representation theory of the individual automorphism groups. Thus representation stability extends whenever these structures exist.

2.1.2   Generalization to other categories

Work on generalizing representation stability to other contexts has proceeded in several partially overlapping directions. A major direction on which we will be focused is that

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2. See [Ch].

3. See [CEF2].

of modules over other categories $\mathbf{C}$ of injections, whose automorphism groups are of interest. Let $\mathbf{C}$ be a category.

Definition 2.1.2 (**C-modules**). *A* **C***-module over a ring R is a covariant functor*

$$M_\bullet : \mathbf{C} \longrightarrow R - \mathbf{Mod}.$$

For every object c, the evaluation M_c is naturally a representation of the group $\mathrm{Aut}_{\mathbf{C}}(c)$ in R-modules, and these representation are related by the morphisms of $\mathbf{C}$.

One then studies this category of representations, describes the simultaneous class functions that generalize character polynomials, and proves the analog of Fact 2.1.1. For example:

1. Putman-Sam [PS] considered the category $\mathbf{C} = \mathbf{VI}_q$ of finite dimensional vector spaces over $\mathbb{F}_q$ and injective linear maps, whose representations encode sequences of $\mathrm{Gl}_n(\mathbb{F}_q)$-representations.

2. Wilson [Wi1] studied $\mathbf{C} = \mathbf{FI}_{\mathcal{W}}$ whose automorphism groups are the classical Weyl groups $\mathcal{W}_n$ of type B/C or D.

3. Sam-Snowden [SS3] and Gan-Li [GL2] considered categories $\mathbf{C} = \mathbf{FI}_G$ for some group G, encoding representations of Wreath products $S_n \wr G$. Casto [Ca] extended their treatment, and defined character polynomials in this context.

4. Barter [Ba] considered the category $\mathbf{C} = \mathbf{T}$ of rooted trees with root-preserving embeddings.

This approach has been further applied to topology, arithmetic and classical representation theory (see the respective citations).

Other generalizations considered categories of dimension zero, studied by Wiltshire-Gordon and Ellenberg (see [WG] with applications in [EWG]); homogeneous categories, studied by Randal-Williams and Wahl (see [RWW]); and modules over twisted commutative algebras, studied by Sam-Snowden (see [SS4]). We will not discuss these ideas here.

In this chapter we attempt to generalize and unify the treatments in Examples 1-4 and ask:

Question 2.1.3. *What structure do these categories possess that supports the existence of a representation stability theory?*

Here we offer an answer by fitting Examples 1-3 and others into the context of a broader theory: representation of categories *of* **FI** *type*, i.e. categories that have structural properties similar to those of **FI** (see Definition 2.1.6 below). This approach is intended to subsume the individual treatments and eliminate the need to introduce a new theory in each specific case. At the same time, it allows one to consider new types of categories, such as the next example.

Example 2.1.4 (**The categorical power $\mathbf{FI}^m$**)**.** As a first nontrivial example, and the original motivation behind this generalization, we consider the categorical powers $\mathbf{FI}^m$. These have objects that are (essentially) m-tuples $(n_1, \ldots, n_m) \in \mathbb{N}^m$ with automorphism groups the products $S_{n_1} \times \ldots \times S_{n_m}$. Such categories are the natural indexing category for various collections of linear subspace arrangements, to which our theory is applied in §3. To see this at work consider the following example.

Fix $m \geq 1$ and let $\mathrm{Rat}^n(\mathbb{P}^1, \mathbb{P}^{m-1})$ be the space of based, degree n rational maps $\mathbb{P}^1 \longrightarrow \mathbb{P}^{m-1}$ that send ∞ to $[1 : \ldots : 1]$. This space admits an $(S_n \times \ldots \times S_n)$-covering $\mathrm{PRat}^n(\mathbb{P}^1, \mathbb{P}^{m-1})$ by picking an ordering on the zeros of the restrictions to

the standard homogeneous coordinates functions on $\mathbb{P}^{m-1}$. The coverings fit naturally into a (contravariant) $\mathbf{FI}^m$-diagram of spaces, and their cohomology is an $\mathbf{FI}^m$-module.

The groups $\mathrm{H}^i(\mathrm{Rat}^n(\mathbb{P}^1, \mathbb{P}^{m-1}); \mathbb{Q})$ can then be computed from the invariant part of the $(S_n \times \ldots \times S_n)$-representation $\mathrm{H}^i(\mathrm{PRat}^n(\mathbb{P}^1, \mathbb{P}^{m-1}); \mathbb{Q})$ by transfer. Representation stability for $\mathbf{FI}^m$-modules then gives the following.

Theorem 2.1.5 (Homological stability for Rat^n**).** *For every $i \geq 0$ the i-th Betti number of $\mathrm{Rat}^n(\mathbb{P}^1, \mathbb{P}^{m-1})$ does not depend on n for all $n \geq i$.*

In §2.6 we discuss representation stability for $\mathbf{FI}^m$, which allows one to make such claims as Theorem 2.1.5. We remark that similar treatment could be applied to any product of categories whose representation stability is understood, but we do not pursue other examples here.

2.1.3   Categories of **FI** type and free modules

As outlined above, we are looking for categorical structure that gives rise to character polynomials satisfying Fact 2.1.1. We propose the following.

Definition 2.1.6 (Categories of FI type). *We say that a category $\mathbf{C}$ is of $\mathbf{FI}$ type if it satisfies the following axioms.*

1. *$\mathbf{C}$ is locally finite, i.e. all hom-sets are finite[4].*

2. *Every morphisms is a monomorphism, and every endomorphisms is an isomorphism.*

3. *For every pair of objects c and d, the group of automorphisms $\mathrm{Aut}_{\mathbf{C}}(d)$ acts transitively on the set $\mathrm{Hom}_{\mathbf{C}}(c, d)$.*

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4. Finiteness is not strictly necessary for many of the definitions and subsequent results. The author will be very interested to see how far one can push this theory with infinite automorphism groups

4. For every object d there exist only finitely many isomorphism classes of objects c for which $\mathrm{Hom}_{\mathbf{C}}(c, d) \neq \varnothing$ (we denote this by $c \leq d$).

5. $\mathbf{C}$ has pullbacks and (weak) pushouts[5].

Remark 2.1.7. Categories that satisfy the second half of condition 2 – where every endomorphism is an isomorphism – are called **EI** categories. The representation stability of such categories satisfying additional combinatorial conditions was studied by Gan-Li in [GL1].

We will denote the automorphism group of an object c by G_c.

In §2.2.1 we define the collection of *character polynomials* for a general category $\mathbf{C}$ of **FI** type - these are certain $\mathbb{C}$-valued class functions simultaneously defined on all automorphism groups G_c. Briefly, character polynomials are linear combinations of functions of the form $\binom{X}{\lambda}$ where $\lambda \subset G_c$ is some fixed conjugacy class. $\binom{X}{\lambda}$ evaluates on $g_d \in G_d$ to give the number of ways g_d can be restricted to an element $g_c \in \lambda$, i.e. via morphisms $c \xrightarrow{f} d$ for which $g_d \circ f = f \circ g_c$ with $g_c \in \lambda$.

However, it is not at all clear that these functions satisfy the analog of Fact 2.1.1, or even that they can be reasonably thought of as polynomials, i.e. closed under taking products. To demonstrate these fundamental properties we propose a categorification of character polynomials, similar to the way in which group representations categorify class functions. Our categorification takes the form of *free $\mathbf{C}$-modules*, introduced in Section §2.3.

Definition 2.1.8 (**Free C-modules**)**.** *A $\mathbf{C}$-module is said to be* free *if it is a direct sum of modules of the form $\mathrm{Ind}_c(V)$, where Ind_c is the left-adjoint functor to the restriction $M_\bullet \mapsto M_c$.*

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5. Pushouts are not quite what we want here, as these typically do not exists when one insists that all morphisms be injective. We replace this notion by *weak pushouts*, defined below.

Note 2.1.9. Since we are only discussing finitely-generated **C**-modules, free modules will always be taken to be *finite* direct sums. Over the field of complex numbers these **C**-modules are precisely the finitely-generated projectives.

This choice of categorification is justified by the following observation.

Theorem 2.1.10 (**Categorification of character polynomials**). *If $M_\bullet$ is a free* **C**-*module over* $\mathbb{C}$, *then there exists a character polynomial P whose restriction to G_c coincides with the character of M_c for every object c.*

Conversely, the character polynomials that arise in this way span the space of all character polynomials on **C**, *defined in §2.2.1 below.*

The structure of **FI** type then ensures that the class of free **C**-modules, and subsequently character polynomials, has the properties that ultimately produce representation stability.

Theorem 2.1.11 (**The class of free C-modules**). *If* **C** *is a category of* **FI** *type, then the class of (finitely-generated) free* **C**-*modules over* $\mathbb{C}$ *has the following properties:*

1. *The tensor product of two free* **C**-*modules is again free.*

2. *There is a* degree *filtration on the category of free* **C**-*modules, taking values in the objects of* **C**. *Direct sums and tensor products act on this degree in the usual way with respect to an order relation $\leq$ on* **C** *and object addition $+$ defined below.*

3. *Every free* **C**-*module $M_\bullet$ has a dual* **C**-*module $M_\bullet^* : c \mapsto \mathrm{Hom}_{\mathbb{C}}(M_c, \mathbb{C})$, which is again free of the same degree.*

4. *If $M_\bullet$ is a free* **C**-*module of degree $\leq c$, then for every object $d \geq c$ the coinvariants $(M_d)/G_d$ are canonically isomorphic.*

This statement – especially closure under tensor products – is nontrivial and depends critically on the structure of **FI** type. For example, the specialization to **FI**-modules was proved in [CEF1] using the projectivity of **FI#**-modules, and is related to the fact that products of binomial coefficients $\binom{n}{k}\binom{n}{l}$ can be expressed as linear combinations of $\binom{n}{r}$ with $r \leq k + l$.

Remark 2.1.12 (**Working over different fields**). Most of the results in Theorem 2.1.11 are set-theoretic in nature and follow from combinatorial properties of **C**-sets. They thus hold in greater generality with the base field $\mathbb{C}$ replaced with an arbitrary commutative ring R. However, when trying to decategorify and conclude character-theoretic results, the assumption of characteristic 0 becomes necessary. To simplify our exposition, we will phrase the results only for **C**-modules over $\mathbb{C}$.

Theorem 2.1.11 in particular gives the categorified analog of Fact 2.1.1. This fact captures the stabilization of the sequence of representations, as we shall see in the Application 1 below.

Corollary 2.1.13 (**Hom stabilization**). *If $M_\bullet$ and $N_\bullet$ are free **C**-modules of respective degrees $\leq c_1$ and $\leq c_2$, then the spaces*

$$\mathrm{Hom}_{G_d}(M_d, N_d) \cong (M^* \otimes N)_d/G_d \qquad (2.1.1)$$

are canonically isomorphic for all $d \geq c_1 + c_2$.

When the objects of **C** are parameterized by natural numbers, the addition $c_1 + c_2$ is the usual addition operations. For the general definition of addition on objects, see Definition 2.3.7 below. Note that the identification of the two sides in Equation 2.1.1 is where characteristic 0 assumption is used.

Decategorifying back to characters, one obtains the following.

Corollary 2.1.14 (**Inner product stabilization**). *If P and Q are character polynomials of respective degrees $\leq c_1$ and $\leq c_2$, then the inner products*

$$\langle P, Q \rangle_{G_d} = \frac{1}{|G_d|} \sum_{g \in G_d} \bar{P}(g) Q(g) \qquad (2.1.2)$$

become independent of d for all $d \geq c_1 + c_2$.

These claims will be proved in §2.3 and §2.4.

2.1.4 Application 1: Stabilization of irreducible multiplicities

Let G be a finite group. Recall that over $\mathbb{C}$ the irreducible decomposition of a G-representation can be detected by G-intertwiners. Explicitly, if V is a G-representation and W is an irreducible representation, then the multiplicity at which W appears in V is $\dim \mathrm{Hom}_G(W, V)$. Similarly, if $V = \oplus_i W_i^{r_i}$ is an irreducible decomposition then

$$\dim \mathrm{Hom}_G(V, V) = \sum_i r_i^2.$$

Corollary 2.1.13 then demonstrates that these dimensions stabilize in the case of free $\mathbf{C}$-modules.

Corollary 2.1.15 (**Stabilization of irreducible decomposition**). *Let $M_\bullet$ be a free $\mathbf{C}$-module of degree $\leq c$. At every object d let*

$$M_d = \oplus_i W(d)_i^{r(d)_i}.$$

be an irreducible decomposition. Then the sums $\sum_i r(d)_i^2$ do not depend on d for $d \geq c + c$. More generally, if $N_\bullet$ is any other free $\mathbf{C}$-module of degree $\leq c'$ with

irreducible decompositions

$$N_d = \oplus_i W(d)_i^{s(d)_i}$$

then the sums $\sum_i r(d)_i \cdot s(d)_i$ do not depend on d for all $d \geq c + c'$.

By choosing the test module $N_\bullet$ carefully, one can gain more information as to the individual multiplicities $r(d)_i$. In particular, it is often possible to relate the irreducible representations of the different groups G_d and show that the individual multiplicities in fact stabilize for all $d \geq c + c$.

2.1.5   Application 2: The category of **C**-modules is Noetherian

One of the most important themes in representation stability is the Noetherian property: the subcategory of finitely-generated modules is closed under taking submodules. This allows one to apply tools from homological algebra to finitely-generated modules, with far reaching applications (see e.g. [CEF1] and [Fa]).

Example 2.1.16 (Configuration spaces of manifolds [CEF1]). Let M be an orientable manifold. For every finite set S the space of S-configurations on M, $\mathrm{PConf}^S(M)$, is the space of injections from S to M. The functor $S \mapsto \mathrm{PConf}^S(M)$ is an **FI**$^{\mathrm{op}}$-space, and $S \mapsto \mathrm{H}^i(\mathrm{PConf}^S(M); \mathbb{Q})$ is an **FI**-module.

Totaro [To] proved that there is a spectral sequence converging to $\mathrm{H}^i(\mathrm{PConf}^S(M))$ and [CEF1] showed that the every $E_2^{p,q}$-term of this sequence is a finitely-generated **FI**-module. [CEF1] also prove that **FI**-modules over $\mathbb{Q}$ is Noetherian, and therefore finite-generation persists to the E_∞-page, and subsequently to H^i. Therefore the sequence $\mathrm{H}^i(\mathrm{PConf}^S(M); \mathbb{Q})$ exhibits representation stability. One direct result is that the i-th Betti number of $\mathrm{PConf}^S(M)$ is eventually polynomial in $|S|$.

A corollary of the theory developed here is that the same Noetherian property holds in general.

Theorem 2.1.17 (The category C−Mod is Noetherian). *If* **C** *is a category of* **FI** *type, then the category of* **C***-modules over* $\mathbb{C}$ *is Noetherian. That is, every submodule of a finitely generated* **C***-module is itself finitely generated.*

Theorem 2.1.17, proved below in §2.5, simultaneously generalizes the results by Church-Ellenberg-Farb [CEF1, Theorem 1.3] and independently by Sam-Snowden [SS1, Theorem 1.3.2], who proved that the category of **FI**-modules is Neotherian; Putman-Sam [PS], who proved the same for the category of **VI**-modules; and Wilson [Wi1], who proved this for **FI**$_\mathcal{W}$-modules.

Gan-Li [GL1] generalized all of these Noetherian results and found that they hold for every category with a skeleton whose objects are parameterized by $\mathbb{N}$ satisfying certain combinatorial conditions see ([GL1, Theorem 1.1]). However, their theory does not address categories whose objects are not parameterized by $\mathbb{N}$, such as **FI**^{m} treated in §2.6 below.

One reason Noetherian results are important in our context is that they ensure that finitely-generated **C**-modules exhibit the same stabilization phenomena as with free **C**-module discussed in Application 1 (although without the effective bounds on stable range).

Theorem 2.1.18 (Stabilization of finitely-generated C-modules). *If* $M_\bullet$ *is a finitely-generated* **C***-module, then the sequence of coinvariants* M_c/G_c *is eventually constant, i.e. all induced maps are* $M_c/G_c \longrightarrow M_d/G_d$ *are isomorphisms for sufficiently large objects c.*

More generally, for every free **C***-module* $F_\bullet$ *the sequence of spaces* $\mathrm{Hom}_{G_c}(F_c, M_c)$ *is eventually constant in the same sense.*

2.1.6   Application 3: Free modules in topology

Beyond the applications of free **C**-modules to the representation theory of the category **C**, they also appear explicitly in topology. In §3 we consider the cohomology of **C**-diagrams of linear subspace arrangements, for which we show that the induced cohomology **C**-module is free. An immediate consequence, stated here somewhat informally, is the following (see §3 for the precise definitions and statements).

Corollary 2.1.19 (Stability of C-diagrams of linear subspace arrangements).
*Every (contravariant) **C**-diagram $X^\bullet$ of linear subspace arrangements, that is generated by a finite collection of subspaces, exhibits cohomological representation stability. That is, for every $i \geq 0$ the **C**-module $\mathrm{H}^i(X^\bullet; \mathbb{C})$ is free. In particular, there exists a single character polynomial P_i of **C** that uniformly describes the G_c-representation $\mathrm{H}^i(X^c; \mathbb{C})$ for every object c.*

Moreover, the respective quotients X^c / G_c exhibit homological stability for $\mathbb{C}$-coefficients, and for various systems of constructible sheaves.

2.2   Preliminaries

Let **C** be a category. Objects of **C** will typically be denoted by c, d, and so on.

The categories with which we shall be working will have only injective morphisms. This typically precludes the possibility of having push-out objects. The following definition provides a means for salvaging some notion of a push-out diagram subject to this constraint.

Definition 2.2.1 (**Weak push-out**). *A* weak push-out *diagram is a pullback diagram*

$$
\begin{array}{ccc}
p & \xrightarrow{\tilde{f_1}} & c_1 \\
\tilde{f_2}\downarrow & & \downarrow f_1 \\
c_2 & \xrightarrow{f_2} & d
\end{array}
$$

with the following universal property: for every other pullback diagram

$$
\begin{array}{ccc}
p & \xrightarrow{\tilde{f_1}} & c_1 \\
\tilde{f_2}\downarrow & & \downarrow h_1 \\
c_2 & \xrightarrow{h_2} & z
\end{array}
$$

there exists a unique morphism $d \xrightarrow{h} z$ that makes all the relevant diagrams commute. We call d the weak push-out object *and denote it by $c_1 \coprod_p c_2$. The unique map h induced from a pair of maps $c_i \xrightarrow{h_i} z$ is denoted by $h_1 \coprod_p h_2$.*

This is similar to a usual push-out, but with "all" commutative squares replaced by only pullback squares. When starting from a category that has push-outs, such as **Set** and **Vect**$_k$, and passing to the subcategory that includes only injective maps, we lose the push-out structure. However, weak push-outs persist, and retain most of the same function.

A standard notation that we will use throughout is the following.

Definition 2.2.2. *We say that $c \leq d$ if there exists morphisms $c \longrightarrow d$.*

In categories of **FI** type (see Definition 2.1.6 above) this preorder relation between objects is essentially an order, i.e. if $c \leq d$ and $d \leq c$ then every morphism $c \longrightarrow d$ is invertible. However, as noted in part (5) of Definition 2.1.6, push-outs typically don't

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exist in categories of **FI**-type and we adjust the definition by demanding the following property instead.

Definition 2.2.3 (**Categories of FI type**). *A category* **C** *is said to be of* **FI** *type is it satisfies axioms (1)-(4) from Definition 2.1.6, and in addition:*

5. **C** *has pullbacks and weak push-outs, i.e. for every pair of morphisms* $p \xrightarrow{f_i} c_i$ *there exists a weak push-out* $c_1 \coprod_p c_2$; *and for every pair* $c_i \xrightarrow{g_i} d$ *there exists a pullback* $c_1 \times_d c_2$.

It seems possible that some of the theory should carry over to compact groups or even to semi-simple groups, but this direction will not be perused here.

The primary objects of study are the representation of **C**. These are the **C**-modules defined in Definition 2.1.2. Our goal is to understand the category of **C**-modules and relate it to the categories of representations of the individual automorphism groups G_c.

2.2.1 Binomial sets and Character Polynomials

The character polynomials for the symmetric groups are class functions simultaneously defined on S_n for all n. These objects are closely linked with the phenomenon of representation stability in that the character of a representation-stable sequence is eventually given by a single character polynomial (see [CEF1]). We will now define character polynomials for a general category of **FI** type.

The following notion generalizes the collection of subset of size k inside a set of n elements. It will be used below in the definition of character polynomials.

Definition 2.2.4 (**Binomial set**). *Let c and d be two objects of* **C**. *The group of automorphisms G_c acts on* $\mathrm{Hom}_{\mathbf{C}}(c, d)$ *on the right by precomposition. Denote the*

quotient $\mathrm{Hom}_{\mathbf{C}}(c, d)/G_c$ by $\binom{d}{c}$. *We will call this the* binomial set, d *choose* c. *If* $c \xrightarrow{f} d$ *is a morphism, we denote its class in* $\binom{d}{c}$ *by* $[f]$.

Since the set $\mathrm{Hom}_{\mathbf{C}}(c, d)$ *admits a left action by* G_d, *and this action commutes with the right action of* G_c, *the binomial set* $\binom{d}{c}$ *acquires a* G_d *action naturally by* $\sigma([f]) = [\sigma \circ f]$.

Note that in the case of $\mathbf{C} = \mathbf{FI}$, the category of finite sets and injections, the binomial set $\binom{n}{k}$ is naturally in bijection with the collection of size k subsets of n (hence the terminology). Replacing $\mathbf{FI}$ by $\mathbf{VI}_{\mathbb{F}}$, the category of finite dimensional $\mathbb{F}$-vector spaces and injective linear functions, the binomial set $\binom{n}{k}$ is naturally the Grassmanian of k-planes in $\mathbb{F}^n$.

Definition 2.2.5 (**Character polynomial**). *Let c be an object of $\mathbf{C}$ and $\mu \subseteq G_c$ a conjugacy class. In this case we will denote $|\mu| = c$. The* indicator character polynomial *of μ is the $\mathbb{C}$-valued class function $\binom{X}{\mu}$ simultaneously defined on all G_d by*

$$\binom{X}{\mu} : (\sigma \in G_d) \mapsto \left| \left\{ [f] \in \binom{d}{c} \mid \exists \psi \in \mu \ s.t. \ \sigma \circ f = f \circ \psi \right\} \right|. \qquad (2.2.1)$$

The degree *of $\binom{X}{\mu}$ is defined to be $\deg(\binom{X}{\mu}) := |\mu|$.*

A character polynomial *P is a $\mathbb{C}$-linear combination of such simultaneous class functions. We say that the* degree *of P is $\leq d$ for an object d if for every indicator $\binom{X}{\mu}$ that appears in P nontrivially we have $|\mu| \leq d$. We denote this by $\deg(P) \leq d$.*

The following lemma shows that the above definition indeed gives rise to well-defined class functions.

Lemma 2.2.6. *The function $\binom{X}{\mu}$ is a class function of every group G_d. Furthermore, its definition in Equation 2.2.1 does not depend on the choice of representative $f \in [f]$.*

Proof. First to see that Equation 2.2.1 does not depend on the choice of f, suppose f' is another representative of $[f] \in \binom{d}{c}$. Then there exists some $g \in G_c$ such that $f' = f \circ g$. Then for every $\sigma \in G_d$ and $\psi \in \mu$

$$\sigma \circ f = f \circ \psi \iff \sigma \circ f' = (f \circ \psi) \circ g = f' \circ (g^{-1}\psi g)$$

and $g^{-1}\psi g$ belongs to μ as well.

Lastly, to see that we get a class function take $\sigma' = h\sigma h^{-1}$. Then $[f] \in \binom{d}{c}$ satisfies $\sigma \circ f = f \circ \psi$ if and only in $[h \circ f]$ satisfies

$$\sigma' \circ (h \circ f) = h\sigma h^{-1}(h \circ f) = h \circ (f \circ \psi) = (h \circ f) \circ \psi.$$

If we denote by $U_\mu(\sigma)$ the set of classes $[f] \in \binom{d}{c}$ which is counted in Equation 2.2.1 for σ, then we see that $U_\mu(h\sigma h^{-1}) = h(U\mu(\sigma))$, and in particular these sets have equal cardinality. $\qquad \square$

Example 2.2.7 (**FI character polynomials**). For $\mathbf{C} = \mathbf{FI}$ the automorphism group of the object $n = \{0, 1, \ldots n-1\}$ is the symmetric group S_n. For any $k \in \mathbb{N}$ a conjugacy class in S_k is described by a cycle type, $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ where μ_i is the number of i-cycles. For any other $n \in \mathbb{N}$, if we denote by X_i the class function on S_n

$$X_i(\sigma) = \text{\# of } i\text{-cycles in } \sigma$$

then we claim that

$$\binom{X}{\mu}(\sigma) = \binom{X_1(\sigma)}{\mu_1} \cdots \binom{X_k(\sigma)}{\mu_k}. \tag{2.2.2}$$

Indeed, the class $[f] \in \binom{n}{k}$ of an injection $k \xrightarrow{f} n$ corresponds to the subset $Im(f) \subseteq n$. The condition that $\sigma \circ f = f \circ \psi$ translates to saying that $Im(f)$ is

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invariant under σ and that the induced permutation on this subset has cycle type μ. Then for a given $\sigma \in S_n$ the right-hand side of Equation 2.2.2 counts the number of ways to assemble such an invariant subset from the cycles of σ.

[CEF1] give Equation 2.2.2 as the definition of $\binom{X}{\mu}$. Thus our definition of character polynomials extends the classical notion of character polynomials for the symmetric groups to other classes of groups.

Example 2.2.8 (VI character polynomials). For $\mathbf{C} = \mathbf{VI}_{\mathbb{F}}$ the automorphism group of the object $[n] = \mathbb{F}^n$ is $\mathrm{GL}_n(\mathbb{F})$. We describe the degree 1 indicators. A conjugacy class in $\mathrm{GL}_1(\mathbb{F}) = \mathbb{F}^{\times}$ is just a non-zero element $\mu \in \mathbb{F}$. For every $n \in \mathbb{N}$ the function $\binom{X}{\mu}$ on a matrix $A \in \mathrm{GL}_n(\mathbb{F})$ is given by

$$\binom{X}{\mu}(A) = \# \text{ of 1D eigenspaces of } A \text{ with eigenvalue } \mu. \tag{2.2.3}$$

These are the **VI** analogs of X_1 on S_n, which counts the number of fixed points of a permutation.

2.3 Free C-modules

This section is devoted to defining free **C**-modules and proving that, when **C** is of **FI** type, these modules satisfy the fundamental properties stated in Theorem 2.1.11. Note that the statements and proofs in this section are essentially set-theoretic in nature, and therefore hold in a more general setting of **C**-modules over any ring R. For concreteness we will only describe here the results with $R = \mathbb{C}$.

Free **C**-modules are defined using a collection of left-adjoint functors. For every object c there is a natural restriction functor

$$\mathbf{C} - \mathbf{Mod} \xrightarrow{\mathrm{Res}_c} \mathbb{C}[G_c] - \mathbf{Mod}, \; M_\bullet \mapsto M_c \tag{2.3.1}$$

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Following [tD], this functor admits a left-adjoint as follows.

Definition 2.3.1 (**Induction C-modules**). *Let* $\text{Ind}_c : \mathbb{C}[G_c] - \mathbf{Mod} \longrightarrow \mathbf{C} - \mathbf{Mod}$
be the functor that sends a G_c-representation V to the $\mathbf{C}$-module

$$\text{Ind}_c(V)_\bullet = \mathbb{C}[\text{Hom}(d, \bullet)] \otimes_{G_d} V \tag{2.3.2}$$

where morphisms in $\mathbf{C}$ act on these spaces naturally through their action on $\text{Hom}(c, \bullet)$.

We call a $\mathbf{C}$-module of this form an induction module *of degree c, and denote*

$$\deg(\text{Ind}_c(V)) = c.$$

[tD] shows that the functor Ind_c is a left adjoint to Res_c. Recall that in Definition 2.1.8 we called direct sum of induction modules free. The following additional terminology will also be useful.

Definition 2.3.2 (**Degree of a free module**). *We say that a free $\mathbf{C}$-module $M_\bullet$ has degree $\leq d$ if for every induction module $\text{Ind}_c(V)$ that appears in $M_\bullet$ nontrivially we have $c \leq d$. In this case we denote $\deg(M_\bullet) \leq d$.*

A virtual *free $\mathbf{C}$-module is a formal $\mathbb{C}$-linear combination of induction modules, e.g.*

$$\oplus_{i=1}^n \lambda_i \text{Ind}_{c_i}(V_i) \text{ where } \lambda_i \in \mathbb{C}.$$

We extend the induction functors Ind_c linearly to virtual G_c-representations, i.e.

$$\text{Ind}_c(\oplus \lambda_i V_i) := \oplus \lambda_i \text{Ind}_c(V_i).$$

We propose that (virtual) free $\mathbf{C}$-modules are a categorification of character polynomials, much like the case for any finite group G where (virtual) G-representations

categorify class functions on G.

Definition 2.3.3 (The character of a C-module). *If $M_\bullet$ is a **C**-module, its character is the simultaneous class function*

$$\chi_M : \coprod_c G_c \longrightarrow \mathbb{C}$$

that for every object c sends the group G_c to the character of the G_c-representation M_c.

One can express the character of induction modules in terms of indicator character polynomials, as follows.

Lemma 2.3.4 (Character of induction modules). *If V is any G_c-representation whose character is χ_V, then the character of $\mathrm{Ind}_c(V)$ is given by*

$$\chi_{\mathrm{Ind}_c(V)} = \sum_{\mu \in \mathrm{conj}(G_c)} \chi_V(\mu) \binom{X}{\mu} \tag{2.3.3}$$

where $\mathrm{conj}(G_c)$ is the set of conjugacy classes of G_c, and $\chi_V(\mu)$ is the value χ_V takes on any $g \in \mu$. In particular we see that the character of $\mathrm{Ind}_c(V)$ is a character polynomial of degree c.

Proof. Since all morphisms in **C** are monomorphisms, it follows for every object d the equivalence class $f \circ G_c = [f] \in \binom{d}{c}$ is a right G_c-torsor. Thus there is an isomorphism of vector spaces

$$\mathrm{Ind}_c(V)_d = \mathbb{C}[\mathrm{Hom}_{\mathbf{C}}(c,d)] \underset{G_c}{\otimes} V = \underset{[f] \in \binom{d}{c}}{\oplus} \mathbb{C}([f]) \underset{G_c}{\otimes} V \cong \underset{[f] \in \binom{d}{c}}{\oplus} V \tag{2.3.4}$$

where the group G_d permutes the summands through its action on $\binom{d}{c}$. It follows that the trace of $\sigma \in G_d$ gets a contribution from the summand $\mathbb{C}([f]) \otimes_{G_c} V$ if and only

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if $\sigma([f]) = [f]$. Consider such $[f] \in \mathrm{Fix}(\sigma)$, i.e. there exists some $\psi \in G_c$ such that $\sigma \circ f = f \circ \psi$. We get a commutative diagram

$$
\begin{array}{ccc}
\mathbb{C}([f]) \otimes_{G_c} V & \xrightarrow{\ \sigma\ } & \mathbb{C}([f]) \otimes_{G_c} V \\
\cong \downarrow & & \downarrow \cong \\
V & \xrightarrow[\ \psi\ ]{} & V
\end{array}
$$

so the trace of $\sigma|_{\mathbb{C}([f]) \otimes_{G_c} V}$ is precisely $\chi_V(\psi)$. We get a formula for the character

$$
\chi_{\mathrm{Ind}_c(V)}(\sigma) = \sum_{\substack{[f] \in \binom{d}{c} \\ \exists \psi (\sigma \circ f = f \circ \psi)}} \chi_V(\psi). \tag{2.3.5}
$$

Arranging this sum according to the conjugacy class of ψ we get the equality claimed by Equation 2.3.3. $\qquad\square$

A corollary or Lemma 2.3.4 is that free **C**-modules indeed categorify character polynomials.

Theorem 2.3.5 (**Categoricifaction of character polynomials**). *Character polynomials of degree $\leq d$ are precisely the characters of virtual free **C**-modules of degree $\leq c$.*

Proof. It is sufficient to show that every $\binom{X}{\mu}$ is the character of some virtual free **C**-module of degree $\leq |\mu|$. Denote $c = |\mu|$ and consider the indicator class function on G_c

$$
\chi_\mu(\psi) = \begin{cases} 1 & \psi \in \mu \\ 0 & \psi \notin \mu. \end{cases}
$$

Since the characters of G_c-representations form a basis for the class functions on G_c, there exist G_c-representations $V_1, \ldots, V_n$ and complex numbers $\lambda_1, \ldots, \lambda_n$ such that

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the virtual representation

$$V_\mu = \oplus_{i=1}^n \lambda_i V_i$$

has character χ_μ. Then by Lemma 2.3.4 and linearity it follows that

$$\chi_{\mathrm{Ind}_c(V_\mu)} = \binom{X}{\mu}.$$

$\square$

2.3.1   Tensor products

The categorification of pointwise products of character polynomials is the tensor product of free **C**-modules. The goal of this subsection is to show that the product of two free modules is itself free.

Definition 2.3.6 (**Tensor product of C-modules**). *If $M_\bullet$ and $N_\bullet$ are two **C**-module, their tensor product $(M \otimes N)_\bullet$ is the **C**-module*

$$(M \otimes N)_d = M_d \otimes N_d \tag{2.3.6}$$

where a morphism $c \xrightarrow{f} d$ acts naturally by $M(f) \otimes N(f)$.

At the level of characters, the tensor product corresponds to pointwise multiplication:

$$\chi_{M \otimes N} = \chi_M \cdot \chi_N. \tag{2.3.7}$$

The main result of this subsection is the parts (1) and (2) of Theorem 2.1.11. The following definition gives meaning to addition of objects so as to make the degree additive.

Definition 2.3.7 (Sum of objects). *If c_1 and c_2 are two object of $\mathbf{C}$, then $c_1 + c_2$ denotes the collection of objects d that satisfy*

$$c_1 \coprod_p c_2 \leq d$$

for every weak push-out of c_1 and c_2. If d belongs to the collection $c_1 + c_2$ we denote $d \geq c_1 + c_2$, i.e.

$$d \geq c_1 + c_2 \iff d \in c_1 + c_2.$$

If $M_\bullet$ is a free $\mathbf{C}$-module, we say that $\deg(M) \leq c_1 + c_2$ if the degree is $\leq d$ for every $d \in c_1 + c_2$.

Note 2.3.8. If the collection $c_1 + c_2$ contains an essential minimum object d_0, then we can identify $c_1 + c_2$ with this minimum. In this case saying that $\deg(M) \leq c_1 + c_2$ is equivalent to saying $\deg(M) \leq d_0$. In all the examples we currently know, the essential minimum object of $c_1 + c_2$ is the weak coproduct $c_1 \coprod_\varnothing c_2$. In particular, when $\mathbf{C}$ has a skeleton whose objects are parameterized naturally by $\mathbb{N}$ then the object "$n_1 + n_2$" coincides with the standard addition $n_1 + n_2$ (hence the notation).

At the level of character polynomials Theorem 2.1.11(1) translates into the following result.

Corollary 2.3.9 (Closure under products). *The collection of character polynomials forms an algebra under pointwise products, and the degree is additive with respect products. Namely, if P and Q are character polynomials of respective degrees $\leq c_1$ and $\leq c_2$, then their product $P \cdot Q$ is a character polynomial of degree $\leq c_1 + c_2$.*

Note 2.3.10. It is not immediately clear that the product of two expressions $\binom{X}{\mu}$ and $\binom{X}{\nu}$ can be expanded in terms of other such expressions, but we now see they can. To demonstrate the nontriviality of this statement consider the standard binomial

coefficients: for $X = \binom{X}{1}$ we have an expansion

$$X \cdot \binom{X}{k} = (k+1)\binom{X}{k+1} + k\binom{X}{k}$$

Problem 2.3.11. Find general formula for the expansion of $\binom{X}{k_1}\binom{X}{k_2}$ in terms of $\binom{X}{k}$'s.

The proof of Theorem 2.1.11(1) will use the following definitions and lemmas. First we need an easy technical observation.

Lemma 2.3.12 (**Pull-back invariance**). *Suppose*

$$\begin{array}{ccc} a & \xrightarrow{f_1} & b_1 \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle g_1} \\ b_2 & \xrightarrow{g_2} & c \end{array}$$

is some commutative diagram in **C** *and* $c \xrightarrow{h} d$ *is some monomorphism. By composing with h we get another diagram*

$$\begin{array}{ccc} a & \xrightarrow{f_1} & b_1 \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle h \circ g_1} \\ b_2 & \xrightarrow{h \circ g_2} & d \end{array}$$

If one of these diagrams is a pull-back, then so is the other.

Second we define the *push-out set* of three objects.

Definition 2.3.13 (**Push-out set**). *Let c_1 and c_2 be two objects of* **C**. *For any object d we define the* push-out set PO $\binom{d}{c_1,c_2}$ *to be the set of pairs of morphisms $(c_i \xrightarrow{g_i} d \mid i = 1, 2)$ that present d as a weak push-out of c_1 and c_2. That is to say that*

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the pullback diagram

$$
\begin{array}{ccc}
c_1 \times_d c_2 & \longrightarrow & c_1 \\
\downarrow & & \downarrow {\scriptstyle g_1} \\
c_2 & \xrightarrow{\ g_2\ } & d
\end{array}
$$

is a weak push-out diagram.

Remark 2.3.14. It is straightforward to verify that the procedure of replacing d by an isomorphic object d' and mapping c_1 and c_2 into d' through any isomorphism $d \xrightarrow{\sim} d'$ preserves weak push-out diagrams. Therefore any such isomorphism induces a natural bijection of sets

$$
\mathrm{PO}\begin{pmatrix} d \\ c_1, c_2 \end{pmatrix} \xrightarrow{\sim} \mathrm{PO}\begin{pmatrix} d' \\ c_1, c_2 \end{pmatrix}
$$

by left-composition. In particular, the group of automorphisms G_d acts on $\mathrm{PO}\begin{pmatrix} d \\ c_1, c_2 \end{pmatrix}$ on the left. Similarly, the group $G_{c_1} \times G_{c_2}$ acts naturally on the right by precomposition.

The general philosophy of this thesis is the following: statements about representation stability (of which Theorem 2.1.11(1) is one) are reflected by statement about **C**-sets. Therefore closure under tensor products should be a consequence of a set-theoretic observation. This is the content of the next lemma.

Lemma 2.3.15 (Tensor products: set version). *Let* **C** *be a category of* **FI** *type. There is a natural isomorphism between the product functor*

$$
\mathrm{Hom}(c_1, \bullet) \times \mathrm{Hom}(c_2, \bullet)
$$

and the disjoint union functor

$$
\coprod_{[d]} \mathrm{Hom}\,(d, \bullet) \times_{G_d} \mathrm{PO}\begin{pmatrix} d \\ c_1, c_2 \end{pmatrix}
$$

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*where [d] ranges over the isomorphism classes of **C** and d is some representative of [d].
Furthermore this natural isomorphism respects the right $(G_{c_1} \times G_{c_2})$-action on the two
functors.*

Proof. For any object x and a representative d of the isomorphism class $[d]$ we define
a function

$$\operatorname{Hom}(d, x) \times_{G_d} \operatorname{PO}\begin{pmatrix} d \\ c_1, c_2 \end{pmatrix} \xrightarrow{\Psi_x^d} \operatorname{Hom}(c_1, x) \times \operatorname{Hom}(c_2, x) \qquad (2.3.8)$$

by composition, i.e.

$$\left[d \xrightarrow{f} x, (c_i \xrightarrow{r_i} d) \right] \mapsto \left(c_i \xrightarrow{f \circ r_i} x \right) \qquad (2.3.9)$$

By the associativity of composition, this is well-defined on the product over G_d. More-
over, $\Psi_\bullet^d$ is clearly natural in x and respects the right action of $G_{c_1} \times G_{c_2}$ given by
precomposition.

Letting d range over all isomorphism classes we get a natural transformation from
the union

$$\coprod_{[d]} \operatorname{Hom}(d, \bullet) \times_{G_d} \operatorname{PO}\begin{pmatrix} d \\ c_1, c_2 \end{pmatrix} \xrightarrow{\Psi_\bullet} \operatorname{Hom}(c_1, \bullet) \times \operatorname{Hom}(c_2, \bullet) \qquad (2.3.10)$$

which respects the right $G_{c_1} \times G_{c_2}$-action.

In the other direction, let x again be any object. We define a function

$$\operatorname{Hom}(c_1, x) \times \operatorname{Hom}(c_2, x) \xrightarrow{\Phi_x^d} \operatorname{Hom}(d, x) \times_{G_d} \operatorname{PO}\begin{pmatrix} d \\ c_1, c_2 \end{pmatrix} \qquad (2.3.11)$$

as follows. Let $(c_i \xrightarrow{f_i} x \mid i = 1, 2)$ be a pair of morphisms. Construct their pull-back

$$
\begin{array}{ccc}
p & \xrightarrow{\alpha_1} & c_1 \\
\alpha_2 \downarrow & & \downarrow f_1 \\
c_2 & \xrightarrow{f_2} & x
\end{array}
$$

and form the weak push-out for $p \xrightarrow{\alpha_i} c_i$

$$
\begin{array}{ccc}
p & \xrightarrow{\alpha_1} & c_1 \\
\alpha_2 \downarrow & & \downarrow r_1 \\
c_2 & \xrightarrow{r_2} & c_1 \coprod_p c_2 =: d
\end{array}
$$

The universal property of the weak push-out then implies that there exists a unique morphism $d \xrightarrow{f} x$ such that $f \circ r_i = f_i$. We define Φ_x^d by

$$
(c_i \xrightarrow{f_i} x) \mapsto \left[(d \xrightarrow{f} x), c_i \xrightarrow{r_i} d \right]. \tag{2.3.12}
$$

To see that Φ_x^d is well-defined, suppose

$$
\begin{array}{ccc}
p' & \xrightarrow{\alpha_1'} & c_1 \\
\alpha_2' \downarrow & & \downarrow r_1' \\
c_2 & \xrightarrow{r_2'} & d'
\end{array}
$$

is another weak push-out diagram produced by the same procedure and $d' \xrightarrow{f'} x$ is the corresponding induced map. First we observe that since p and p' are both pull-backs of the pair (f_1, f_2) there exists an isomorphism $p \xrightarrow{\tau} p'$ for which $\alpha_i' \circ \tau = \alpha_i$ for $i = 1, 2$. Second, we replace p' by p in the weak push-out diagram, mapping it though

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τ, i.e.

$$
\begin{array}{ccc}
p & \xrightarrow{\alpha_1' \circ \tau} & c_1 \\
{\scriptstyle \alpha_2' \circ \tau} \downarrow & & \downarrow {\scriptstyle r_1'} \\
c_2 & \xrightarrow{r_2'} & d'
\end{array}
$$

and this is again a weak push-out diagram. Therefore, by the universal property of the weak push-out, there exists a unique morphism $d \xrightarrow{\psi} d'$ for which $\psi \circ r_i = r_i'$. The same reasoning applied in reverse shows that ψ admits a unique inverse, and therefore $d \cong d'$. Since we picked d to be the representative for the isomorphism class $[d]$, it follows that $d = d'$ and that $\psi \in G_d$. The induced map f is characterized by the property that $f \circ r_i = f_i$, and similarly for f' and r_i'. Therefore we find that

$$
f_i = f' \circ r_i' = f' \circ \psi \circ r_i
$$

which by the universal property of d shows that in fact $f = f' \circ \psi$. Our function Φ_x^d is defined as to send the pair (f_1, f_2) to

$$
\left[f, (c_i \xrightarrow{g_i} d) \right] = \left[f' \circ \psi, (c_i \xrightarrow{g_i} d) \right] = \left[f', (c_i \xrightarrow{\psi \circ g_i} d) \right] = \left[f', (c_i \xrightarrow{g_i'} d) \right]
$$

which we now see that is uniquely defined.

The two functions Ψ_x^d and Φ_x^d are clearly inverse, and therefor they together form a natural isomorphism between the two functors. As stated above, this isomorphism respects the right $G_{c_1} \times G_{c_2}$-action. $\square$

Now we can prove that free **C**-modules are indeed closed under tensor products.

Proof of Theorem 2.1.11(1). By the distributivity of tensor products, it is enough to verify the claim for induction modules of respective degrees $\leq c_1$ and c_2 respectively. Moreover, by the transitivity of the order relation between objects, it will suffice if

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we assume that the degrees are precisely c_1 and c_2 respectively. Let $\mathrm{Ind}_{c_1}(V)$ and $\mathrm{Ind}_{c_2}(W)$ be two such **C**-modules.

We apply an easy-to-verify equality of tensor products,

$$
\begin{aligned}
\mathrm{Ind}_{c_1}(V)_\bullet \otimes_k \mathrm{Ind}_{c_2}(W)_\bullet &= \left(\mathbb{C}[\mathrm{Hom}\, c_1, \bullet] \otimes_{G_{c_1}} V \right) \otimes_k \left(\mathbb{C}[\mathrm{Hom}\, c_2, \bullet] \otimes_{G_{c_2}} W \right) \\
&\cong \left(\mathbb{C}[\mathrm{Hom}(c_1, \bullet)] \otimes_k \mathbb{C}[\mathrm{Hom}(c_2, \bullet)] \right) \bigotimes_{G_{c_1} \times G_{c_2}} (V \boxtimes W)
\end{aligned}
$$

and to this we can apply the natural isomorphism

$$
\mathbb{C}[\mathrm{Hom}(c_1, \bullet)] \otimes \mathbb{C}[\mathrm{Hom}(c_2, \bullet)] \cong \mathbb{C}\left[\mathrm{Hom}(c_1, \bullet) \times \mathrm{Hom}(c_2, \bullet)\right]. \qquad (2.3.13)
$$

In Lemma 2.3.15 we found a natural isomorphism between the product

$$
\mathrm{Hom}(c_1, \bullet) \times \mathrm{Hom}(c_2, \bullet)
$$

and the union

$$
\coprod_{[d]} \mathrm{Hom}\,(d, \bullet) \times_{G_d} \mathrm{PO}\begin{pmatrix} d \\ c_1, c_2 \end{pmatrix}
$$

which when composed with the permutation representation functor $X \mapsto \mathbb{C}[X]$ yields a natural isomorphism

$$
\mathbb{C}\left[\mathrm{Hom}(c_1, \bullet) \times \mathrm{Hom}(c_2, \bullet)\right] \cong \bigoplus_{[d]} \mathbb{C}[\mathrm{Hom}(d, \bullet)] \otimes_{G_d} \mathbb{C}[\mathrm{PO}\begin{pmatrix} d \\ c_1, c_2 \end{pmatrix}] \qquad (2.3.14)
$$

By the associativity of the tensor product, we get a natural isomorphism

$$\operatorname{Ind}_{c_1}(V)_\bullet \otimes_k \operatorname{Ind}_{c_2}(W)_\bullet \;\cong\; \bigoplus_{[d]} \mathbb{C}[\operatorname{Hom}(d, \bullet)] \underset{G_d}{\otimes} \mathbb{C}\!\left[\operatorname{PO}\binom{d}{c_1, c_2}\right] \underset{G_{c_1} \times G_{c_2}}{\bigotimes} (V \boxtimes W)$$

$$= \bigoplus_{[d]} \operatorname{Ind}_d \left(\mathbb{C}\!\left[\operatorname{PO}\binom{d}{c_1, c_2}\right] \underset{G_{c_1} \times G_{c_2}}{\bigotimes} (V \boxtimes W) \right)_\bullet.$$

as claimed.

Note that for d to have a non-zero contribution to this direct sum, the set $\operatorname{PO}\binom{d}{c_1, c_2}$ must be non-empty. In particular, there exists a decomposition $d = c_1 \coprod_p c_2$. This proves the claim regarding the degree of terms in the sum. Since there are only finitely many isomorphism classes of objects with such a presentation, the above direct sum decomposition is finite. $\qquad\square$

2.3.2 Dualization

One would like to define the dual of a **C**-module $M_\bullet$ by $(M^*)_c = (M_c)^*$. Unfortunately, this will not be a **C**-module in general (it will be a $\mathbf{C}^{\mathrm{op}}$-module). In this subsection we show that when dealing with free **C**-modules there is a good notion of dualization.

Definition 2.3.16 (**Dual C-module**). *For an induction module $\operatorname{Ind}_c(V)$ we define its dual* **C**-*modules by*

$$\operatorname{Ind}_c(V)^* = \operatorname{Ind}_c(V^*) \tag{2.3.15}$$

where V^* is the G_c-representation dual to V. Extend this definition linearly to all (virtual) free **C**-*modules.*

We claim that this indeed gives a good notion of duals.

Theorem 2.3.17. *If $M_\bullet$ is a free* **C**-*module then there is a homomorphism of* **C**-

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modules

$$M_\bullet^* \otimes M_\bullet \xrightarrow{\text{ev}} \mathbb{C}_\bullet \qquad\qquad (2.3.16)$$

where $\mathbb{C}_\bullet$ is the trivial $\mathbf{C}$-module with $\mathbb{C}_d = \mathbb{C}$ for every object. This pairing is non-degenerate and thus defines an isomorphism of G_d-representations $(M^*)_d \cong (M_d)^*$ for every object d.

We conclude that the dual of a free $\mathbf{C}$-module of degree $\le c$ is again a $\mathbf{C}$-module of degree $\le c$.

Proof. Suppose $M = \oplus_i \operatorname{Ind}_{c_i}(V_i)$. Then there is a decomposition of $\mathbf{C}$-modules

$$M^* \otimes M = \bigoplus_{i,j} \operatorname{Ind}_{c_i}(V_i^*) \otimes \operatorname{Ind}_{c_j}(V_j).$$

We define the pairing to be 0 for all $i \ne j$. For $i = j$ consider a single induction module $\operatorname{Ind}_c(V)$ and decompose it using Equation 2.3.4

$$\operatorname{Ind}_c(V)_d = \oplus_{[f]\in\binom{d}{c}} V \implies (\operatorname{Ind}_c(V^*) \otimes \operatorname{Ind}_c(V^*))_d = \oplus_{[f],[g]\in\binom{d}{c}} V^* \otimes V.$$

Set the pairing to be 0 on all $[f] \ne [g]$, and for $[f] = [g]$ use the natural contraction on $V^* \otimes V$. This produces a map

$$\oplus_{[f],[g]\in\binom{d}{c}} V^* \otimes V \longrightarrow \oplus_{[f]\in\binom{d}{c}} \mathbb{C} \xrightarrow{+} \mathbb{C}$$

which is the pairing we sought.

Explicitly, the pairing on $\operatorname{Ind}_c(V^*) \otimes \operatorname{Ind}_c(V)$ is given by

$$\langle f \otimes v^*, g \otimes v \rangle = \sum_{\substack{\psi \in G_c \\ f\circ\psi=g}} \langle v^*, \psi(v) \rangle = \begin{cases} \langle v^*, \psi(v) \rangle & f \circ \psi = g \\ 0 & [f] \ne [g] \end{cases} \qquad (2.3.17)$$

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It is straightforward to check that the above pairing is invariant under the action of morphisms in **C**. It is thus a morphism of **C**-modules, as claimed. One can also check that the pairing is non-degenerate, and thus defines the claimed G_d-equivariant isomorphism

$$(M^*)_d \xrightarrow{\sim} (M_d)^*.$$

$\square$

Corollary 2.3.18 (The Hom **C-module).** *If $M_\bullet$ is a free **C**-modules and $N_\bullet$ is any **C**-module, then there exists a **C**-module $\mathrm{Hom}(M,N)_\bullet$ whose value at d is the G_d-representation $\mathrm{Hom}_{\mathbb{C}}(M_d, N_d)$.*

A morphism $d \xrightarrow{f} e$ induces a function $\mathrm{Hom}_{\mathbb{C}}(M_d, N_d) \xrightarrow{f_*} \mathrm{Hom}_{\mathbb{C}}(M_e, N_e)$ satisfying the following naturality property: if $M_d \xrightarrow{T} N_d$ is any linear function, then there is a commutative diagram

$$
\begin{array}{ccc}
M_d & \xrightarrow{\ T\ } & N_d \\
{\scriptstyle M(f)}\big\downarrow & & \big\downarrow{\scriptstyle N(f)} \\
M_e & \xrightarrow[f_*(T)]{} & N_e
\end{array}
$$

Furthermore, if $N_\bullet$ is itself free, and the degrees of $M_\bullet$ and $N_\bullet$ are $\leq c_1$ and $\leq c_2$ respectively, then $\mathrm{Hom}(M,N)_\bullet$ is also free and has degree $\leq c_1 + c_2$.

Proof. The desired **C**-module is the tensor product $M^* \otimes N$. All other claims follow for the properties of tensor products and duals proved above. $\square$

2.4 The Coinvariant quotient and Stabilization

When G is a finite group, the coinvariants of a G-representation are the categorified analog of averaging over class function: if χ is the character of a G-representation V,

then

$$\dim V_G = \frac{1}{|G|} \sum_{g \in G} \chi(g). \tag{2.4.1}$$

Such averages appear in G-inner products, which we want to relate for the various automorphism groups G_c of our category $\mathbf{C}$. This section will therefore analyze the behavior of free $\mathbf{C}$-modules under taking their coinvariants. Recall that the coinvariant quotient of a G-representation V is its maximal invariant quotient, namely

$$V_G = V/\langle v - gv \mid v \in V, g \in G \rangle.$$

We will also denote this quotient by V/G.

In the context of a $\mathbf{C}$-module $M_\bullet$ we can form the G_c-coinvariant quotient of M_c for every object c. If $c \xrightarrow{f} d$ is any morphism and $M_c \xrightarrow{M(f)} M_d$ the induced map, then it descends to a well-defined map on the coinvariants. Indeed, this follows from the assumptions that G_d acts transitively on $\mathrm{Hom}_{\mathbf{C}}(c, d)$: if $g \in G_c$ is any automorphism, then f and $f \circ g$ are two morphisms from c to d and thus there exists some $\tilde{g} \in G_d$ for which $\tilde{g} \circ f = f \circ g$. This shows that for every $v \in M_c$

$$v - g(v) \xrightarrow{M(f)} f(v) - f \circ g(v) = f(v) - \tilde{g}\left(f(v)\right)$$

and indeed $v - gv$ gets mapped to zero in the coinvariant quotient of M_d.

Definition 2.4.1 (The coinvariant quotient). *We call the resulting $\mathbf{C}$-module of coinvariant quotients the coinvariant $\mathbf{C}$-module of $M_\bullet$ and denote it by $(M/G)_\bullet$.*

Note 2.4.2. Every two morphisms $c \xrightarrow{f} d$ and $c \xrightarrow{f'} d$ give rise to the same map between coinvariants. This is because there exists some $\tilde{g} \in G_d$ for which $\tilde{g} \circ f = f'$ and this $\tilde{g}$ acts trivially on $(M/G)_d$. Thus for every pair $c \leq d$ there is a well-defined map between the coinvariants $(M/G)_c \longrightarrow (M/G)_d$.

The coinvariant quotient forms an endofunctor on **C**-modules. In this subsection we study the action of this functor on free **C**-modules and demonstrate that they exhibit stability under its operation.

Lemma 2.4.3. *Let V be any G_c-representation and $\operatorname{Ind}_c(V)$ the corresponding induction module. The G_d-coinvariants of $\operatorname{Ind}_c(V)_d$ are given by*

$$(\operatorname{Ind}_c(V)/G)_d \cong \begin{cases} V/G_c & \text{if } c \le d \\ 0 & \text{otherwise} \end{cases} \tag{2.4.2}$$

with all morphisms $c \le d \xrightarrow{f} d'$ inducing the identity map.

Remark 2.4.4. This again reflects a statement about **C**-sets. Namely, that the set of orbits $G_d \backslash \operatorname{Hom}_{\mathbf{C}}(c, d)$ is either a singleton if $c \le d$ or empty otherwise.

Proof of Lemma 2.4.3. Recall that the coinvariant quotient of a G-representation W can be defined as the tensor product

$$(W)_G \cong \mathbb{C} \otimes_G W \tag{2.4.3}$$

where $\mathbb{C}$ denotes the trivial G-representation.

Using the associativity of tensor products, and the presentation of $\operatorname{Ind}_c(V)$ as one, we get

$$(\operatorname{Ind}_c(V)/G)_d \cong \mathbb{C} \otimes_{G_d} \mathbb{C}[\operatorname{Hom}_{\mathbf{C}}(c, d)] \otimes_{G_c} V \cong \mathbb{C}[G_d \backslash \operatorname{Hom}_{\mathbf{C}}(c, d)] \otimes_{G_c} V \quad (2.4.4)$$

By hypothesis the G_d action on $\operatorname{Hom}_{\mathbf{C}}(c, d)$ is transitive. Therefore if $c \le d$ then $\operatorname{Hom}(c, d) \ne \varnothing$ and this set forms a single orbit. Furthermore, a morphism $d \xrightarrow{f} d'$ carries this single orbit corresponding to d to the one corresponding to d'. In the case

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where there are no morphisms $c \longrightarrow d$ we have the empty set. In other words we have

$$\mathbb{C}[G_d \backslash \operatorname{Hom}_{\mathbf{C}}(c, d)] \cong \begin{cases} \mathbb{C} & \text{if } c \leq d \\ 0 & \text{otherwise} \end{cases} \tag{2.4.5}$$

and a morphism $c \leq d \xrightarrow{f} d'$ induces the identity map on $\mathbb{C}$. Tensoring with V over G_c we get V/G_c when $c \leq d$, zero otherwise, and morphisms as stated. $\qquad \square$

Applying this result to direct sums of induction $\mathbf{C}$-modules, we can formulate what happens to free $\mathbf{C}$-modules when we take their coinvariants.

Theorem 2.4.5 (Coinvariant stabilization). *When the coinvariants functor is applied to any free module of degree $\leq c$, all maps induced by $\mathbf{C}$-morphisms are injections, and all maps induced by morphisms between objects $\geq c$ are isomorphisms.*

Explicitly, the stable isomorphism type of the coinvariant quotient of a free $\mathbf{C}$-module $\oplus_i \operatorname{Ind}_{c_i}(V_i)$ is given by

$$\lim_{\bullet \to \infty} (\oplus_i \operatorname{Ind}_{c_i}(V_i)/G)_{\bullet} = \oplus_i V_i/G_{c_i}. \tag{2.4.6}$$

This translates to the following result regarding character polynomials.

Corollary 2.4.6 (Stabilization of Expectation). *If P is a character polynomial of degree $\leq c$, then its G_d-expected number*

$$\mathbb{E}_{G_d}[P] := \frac{1}{|G_d|} \sum_{\sigma \in G_d} P(\sigma)$$

does not depend on d for $d \geq c$.

Furthermore, if P is the characters of free $\mathbf{C}$-modules then $\mathbb{E}_{G_d}[P]$ is a non-negative integer, monotonically increasing in d.

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Proof. Recall that for a G-representation V the expectation

$$\frac{1}{|G|} \sum_{g \in G} \mathrm{Tr}(g) = \mathrm{Tr}\left(\frac{1}{|G|} \sum_{g \in G} g\right)$$

is the trace of the projection $V \twoheadrightarrow V^G$, whose existence also demonstrates that $V^G = V/G$. The expectation is thus $\dim_{\mathbb{C}}(V/G)$. In particular it is a non-negative integer.

Suppose P is the character of the free $\mathbf{C}$-module $M_\bullet$ of degree $\leq c$. By Theorem 2.4.5 the coinvariants $(M/G)_\bullet$ is a $\mathbf{C}$-module, all of whose induced maps are injections, and isomorphisms for objects $\geq c$. Thus the sequence of dimensions $\dim_{\mathbb{C}}(M/G)_d$ is monotonic in d and becomes constant when $d \geq c$.

The general statement follows by linearity. $\qquad\square$

We are often interested in the G-inner product of characters:

$$\langle \chi_1, \chi_2 \rangle_G = \frac{1}{|G|} \sum_{g \in G} \chi_1(g)\bar{\chi}_2(g) = \mathbb{E}_G[\chi_1 \cdot \bar{\chi}_2]$$

which is central to character theory. For character polynomials the previous corollary gives the following immediate stability statement.

Corollary 2.4.7 (**Stabilization of inner products**). *If P and Q are character polynomials of respective degrees $\leq c_1$ and $\leq c_2$, then the G_d-inner products*

$$\langle P, Q \rangle_{G_d} = \frac{1}{|G_d|} \sum_{\sigma \in G_d} P(\sigma)\bar{Q}(\sigma) \tag{2.4.7}$$

does not depend on d for all $d \geq c_1 + c_2$.

Furthermore, if P and Q are the characters of free $\mathbf{C}$-modules then $\langle P, Q \rangle_{G_d}$ is a non-negative integer, monotonic in d.

Proof. The claim follows directly from the presentation

$$\langle P, Q\rangle_{G_d} = \mathbb{E}_{G_d}[P\bar{Q}]$$

and Corollary 2.4.6.

If P and Q are the characters of $M_\bullet$ and $N_\bullet$ then $P\bar{Q}$ is the character of the free **C**-module $M \otimes N^*$. Integrality and monotonicity follow. $\qquad\square$

2.5 Noetherian property

In this section we apply the theory developed in the previous sections to prove that the category of **C**-modules is Noetherian. Our proof strategy follows the argument made by Gan-Li in [GL1]. The main theorem of this section is the following.

Theorem 2.5.1 (C − Mod is a Noetherian category). *Every **C**-submodule of a finitely generated **C**-module is itself finitely generated.*

Theorem 2.5.1 will be proved at the end of this section. First we need some preliminary results. We start with an extension result for equivariant homomorphisms between free **C**-modules.

Lemma 2.5.2 (Equivariant extension). *Let $M_\bullet$ be a free **C**-module. There is a left-exact endofunctor on **C**-modules*

$$N_\bullet \mapsto \mathrm{Hom}_{G_\bullet}(M_\bullet, N_\bullet) \tag{2.5.1}$$

*whose image is contained in trivial **C**-modules. The value of the module $\mathrm{Hom}_{G_\bullet}(M_\bullet, N_\bullet)$ at an object d is the vector space $\mathrm{Hom}_{G_d}(M_d, N_d)$. In particular, for every $d \leq e$ there is a canonical map*

$$\mathrm{Hom}_{G_d}(M_d, N_d) \xrightarrow{\Psi_d^e} \mathrm{Hom}_{G_e}(M_e, N_e) \tag{2.5.2}$$

that promotes a G_d-linear map to a G_e-linear one.

Furthermore, if $N_\bullet$ is itself free, and the degrees of $M_\bullet$ and $N_\bullet$ are $\leq c_1$ and $\leq c_2$ respectively, then the extension map Ψ_d^e is an isomorphism whenever $d \geq c_1 + c_2$. In particular, equivariant morphisms extend uniquely in this range.

Proof. To get the proposed endofunctor we use dualization, tensor products and coinvariants:

$$N_\bullet \mapsto (M^* \otimes N)_\bullet \mapsto [(M^* \otimes N)/G]_\bullet \tag{2.5.3}$$

This gives rise to an endofunctor whose value at d is

$$\left(M_d^* \otimes N_d \right)/G_d.$$

The tensor product is naturally isomorphic to $\mathrm{Hom}_\mathbb{C}(M_d, N_d)$ and averaging over G_d gives a natural lift from coinvariants to invariants. Thus the value at d is naturally isomorphic to

$$\mathrm{Hom}_\mathbb{C}(M_d, N_d)^{G_d} = \mathrm{Hom}_{G_d}(M_d, N_d)$$

and indeed the desired functor exists. Left exactness follows from the general fact that the functor $\mathrm{Hom}(M, \bullet)$ is left exact.

Lastly, if $M_\bullet$ and $N_\bullet$ are free of respective degrees $\leq c_1$ and $\leq c_2$ then by Theorem 2.1.11(1) $M^* \otimes N$ is free of degree $\leq c_1 + c_2$. We then apply Theorem 2.4.5 and see that its coinvariants stabilize for all $d \geq c_1 + c_2$ in the sense that all induced maps Ψ_d^e are isomorphisms. $\qquad\square$

When the range $N_\bullet$ is not free we cannot guarantee that the extension maps Ψ_d^e be eventually isomorphisms. But in the case where the range is contained in a free module, we can at least salvage injectivity.

Corollary 2.5.3 (Injective extension). *If $M_\bullet$ and $N_\bullet$ are free $\mathbf{C}$-modules of respective degrees $\le c_1$ and $\le c_2$, and $X_\bullet \subseteq N_\bullet$ is any $\mathbf{C}$-submodule, then the extension maps*

$$\mathrm{Hom}_{G_d}(M_d, X_d) \xrightarrow{\Psi_d^e} \mathrm{Hom}_{G_e}(M_e, X_e) \qquad (2.5.4)$$

are injective for all $e \ge d \ge c_1 + c_2$.

Proof. For every $e \ge d$ we have a commutative square of extensions

$$
\begin{array}{ccc}
\mathrm{Hom}_{G_d}(M_d, X_d) & \xrightarrow{X_d \hookrightarrow N_d} & \mathrm{Hom}_{G_d}(M_d, N_d) \\
\Big\downarrow{\scriptstyle \Psi_d^e} & & \Big\downarrow{\scriptstyle \Psi_d^e} \\
\mathrm{Hom}_{G_e}(M_e, X_e) & \xrightarrow[X_e \hookrightarrow N_e]{} & \mathrm{Hom}_{G_e}(M_e, N_e)
\end{array}
$$

and since M and N are free of the given degrees, it follows that the rightmost extension map is an isomorphism when $d \ge c_1 + c_2$. Furthermore, the two horizontal maps are injective by left-exactness. Thus we have a square in which all but the leftmost map are injections. This implies that the leftmost map is injective as well. $\square$

We are now ready to prove that $\mathbf{C} - \mathbf{Mod}$ has the Noetherian property.

Proof of Theorem 2.5.1. Suppose that $M_\bullet$ is a finitely generated $\mathbf{C}$-module and $X_\bullet^0 \subseteq X_\bullet^1 \subseteq \ldots \subseteq M_\bullet$ is an ascending chain of submodules. We need to show that $X^N = X^{N+1} = \ldots$ for some $N \in \mathbb{N}$. As in the standard proofs of Hilbert's Basis Theorem, we divide the task into two parts: controlling growth in all large degrees, then handling lower degrees using Noetherian property of finite direct sums.

We assume without loss of generality that $M_\bullet$ is a free, finitely-generated $\mathbf{C}$-module of degree c, as every finitely-generated $\mathbf{C}$-module is a quotient of a finite sum of such.

For brevity we denote the functor $X_\bullet \mapsto \operatorname{Hom}_{G_\bullet}(M_\bullet, X_\bullet)$ by F, i.e.

$$F(X)_d := \operatorname{Hom}_{G_d}(M_d, X_d).$$

Since $M_\bullet$ is free of degree $\leq c$, it follows that all induced extension maps

$$F(M)_d \xrightarrow{\Psi_d^e} F(M)_e$$

are isomorphisms when $d \geq c + c$. Fix an object $d \geq c + c$. We get a collection of subspaces inside $F(M)_d$ by considering the images

$$\left\{ F(X^n)_e \hookrightarrow F(M)_e \xrightarrow{(\Psi_d^e)^{-1}} F(M)_d \right\}_{n \in \mathbb{N},\, e \geq d} \tag{2.5.5}$$

Since $F(M)_d$ is itself Noetherian (a finite dimension vector space), this collection of subspaces has a maximal element, say the image of $F(X^{N_0})_{e_0}$.

Claim 1. *For all $n \geq N_0$ and $e \geq e_0$ we have $X_e^n = X_e^{N_0}$.*

Proof. For every $n \geq N_0$ and objects $e \geq e_0$ we have a commutative diagram

$$
\begin{array}{ccccccc}
F(X^{N_0})_{e_0} & \hookrightarrow & F(X^n)_{e_0} & \hookrightarrow & F(M)_{e_0} & & \\
\Psi_{e_o}^e \downarrow & & \Psi_{e_o}^e \downarrow & & \Psi_{e_o}^e \downarrow & \searrow^{(\Psi_d^{e_o})^{-1}} & \\
F(X^{N_0})_e & \hookrightarrow & F(X^n)_e & \hookrightarrow & F(M)_e & \xrightarrow{(\Psi_d^e)^{-1}} & F(M)_d
\end{array}
$$

which by Corollary 2.5.3 all vertical extension maps are injective.

But we chose $F(X^{N_0})_{e_0}$ to be the subspace whose image inside $F(M)_d$ is maximal. It thus follows that all arrows in the above diagram are surjective. In particular, the injection $F(X^{N_0})_e \hookrightarrow F(X^n)_e$ is an isomorphism. Recalling the definition of F, we

found that the inclusion

$$\mathrm{Hom}_{G_e}(M_e, X_e^{N_0}) \hookrightarrow \mathrm{Hom}_{G_e}(M_e, X_e^n)$$

is an isomorphism, where $X_e^{N_0} \subseteq N_e^n \subseteq M_e$ are G_e-subrepresentations. By Mashke's theorem, this happens precisely when $X_e^{N_0} = X_e^n$ thus proving the claim. $\square$

It remains to show that we can find some $N_1 \geq N_0$ such that for all objects $e < e_0$ the term $X_e^{N_1}$ stabilized. Indeed, since $\mathbf{C}$ is of $\mathbf{FI}$ type, there are only finitely many isomorphism classes of objects $\leq e_0$. Pick representatives for them $e_1, \ldots, e_n$ and consider the direct sum

$$\bigoplus_{k=1}^{n} M_{e_k}.$$

Since each M_{e_k} is Noetherian (a finite dimensional vector space), this direct sum is Noetherian as well. We can therefore find $N_1 \geq N_0$ for which the sum

$$\bigoplus_{k=1}^{n} X_{e_k}^{N_1} \subseteq \bigoplus_{k=1}^{n} M_{e_k}$$

stabilized. Now for every $n \geq N_1$ and every object e we have $X_e^{N_1} = X_e^n$ thus showing that $X_\bullet^{N_1}$ is a maximal element of our chain. $\square$

Remark 2.5.4. Making contact with related work, we remark that in [GL1, Theorem 1.1] Gan-Li list a set of combinatorial condition on categories of a certain type, which are sufficient for proving the Noetherian property [GL1]. Their conditions are

- **Surjectivity**: The groups G_d act transitively on incoming morphisms $c \longrightarrow d$.

- **Bijectivity**: Some sequence of double-coset spaces $H_d \backslash G_d / H_d$ stabilizes as d get sufficiently large.

These conditions are related to the present context as follows. First, the Surjectivity condition is incorporated into our definition of categories of **FI** type. As for Bijectivity, it was explained to me by Kevin Casto that by choosing a compatible system of morphisms $c \longrightarrow d$ for every pair $c \leq d$ one gets a natural isomorphism

$$G_d \backslash (\mathrm{Hom}_{\mathbf{C}}(c,d) \times \mathrm{Hom}_{\mathbf{C}}(c,d)) \cong H_d \backslash G_d / H_d$$

where $H_d \backslash G_d / H_d$ is the double-coset space the appears in the Bijectivity condition. In this sense, the objects considered in this chapter are a coordinate-free interpretation of those the appeared in [GL1]. Arguing in this coordinate-free manner allows us to consider categories whose objects are not linearly ordered, avoid having to find a compatible system of morphisms, and show that the bijectivity condition holds for all categories of **FI** type. This is a direct result of Lemma 2.3.15.

The following stabilization result is a central motivation for one to be interested in the Noetherian property. It shows that finitely-generated **C**-modules exhibit the same representation stability phenomena as free **C**-module, only without the explicit stable range.

Theorem 2.5.5 (Stabilization of finitely-generated C-modules). *If $M_\bullet$ is a finitely-generated **C**-module, then all induced maps in the associated module of coinvariants are eventually isomorphisms. That is, there exists an upward-closed and cofinal set of objects X such that if $c \in X$ and $d \geq c$, then the induced map*

$$M_c / G_c \longrightarrow M_d / G_d$$

is an isomorphism.

*More generally, if $F_\bullet$ is any free **C**-module, then the coinvariants of $F \otimes M$ even-*

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tually stabilizes in the above sense. In particular, the spaces $\mathrm{Hom}_{G_c}(F_c, M_c)$ *stabilize as well.*

Proof. By the Noetherian property, a finitely-generated **C**-module is finitely-presented, i.e. there exist free **C**-modules $F_\bullet^i$ for $i = 0, 1$ and an exact sequence

$$F^1 \longrightarrow F^0 \longrightarrow M \longrightarrow 0$$

Since the functor of coinvariants is right-exact we get a similar sequence of coinvariants. But by Theorem 2.4.5 the coinvariants of a free **C**-module stabilize in the desired sense. The Five-Lemma then implies that the same stabilization occurs for M/G.

For the more general statement, suppose $F_\bullet$ is some free **C**-module. By the right-exactness of the tensor product it follows that

$$F \otimes F^1 \longrightarrow F \otimes F^0 \longrightarrow F \otimes M \longrightarrow 0$$

is itself exact. Theorem 2.1.11(1) shows that for $i = 0, 1$ the product $F \otimes F^i$ is free. Thus by the same reasoning as above stabilization follows.

Lastly, replacing F with its dual F^* (which is again free) and using the isomorphisms

$$(F_c^* \otimes M_c)/G_c \cong \mathrm{Hom}(F_c, M_c)^{G_c} = \mathrm{Hom}_{G_c}(F_c, M_c)$$

we find that the spaces on the right-hand side stabilize as well. □

2.6 Example: Representation stability for $\mathbf{FI}^m$

This section is devoted to the category $\mathbf{FI}^m$, its free modules, and representation stability in this context. We also give an explicit description of $\mathbf{FI}^m$-character polynomials in terms of cycle-counting functions. The results presented below generalize to

the category $(\mathbf{FI}_G)^m$ and its representation, where G is some finite group, using the technique presented in [SS3, Theorem 3.1.3].

Recall that we denote the category of finite sets and injective functions by $\mathbf{FI}$. Consider the categorical power $\mathbf{FI}^m$, whose objects are ordered m-tuples of finite sets $\bar{n} = (n^{(1)}, \ldots, n^{(m)})$, and whose morphisms $\bar{n} \xrightarrow{\bar{f}} \bar{n}'$ are ordered m-tuples of injections $\bar{f} = (f^{(1)}, \ldots, f^{(m)})$ where $n^{(i)} \xrightarrow{f^{(i)}} n'^{(i)}$. In everything that follows we denote the $\mathbf{FI}^m$ analog of notions from $\mathbf{FI}$ by an over-line. The ordering on objects in $\mathbf{FI}^m$ is the following: $\bar{n} \leq \bar{n}'$ if and only if for every $1 \leq i \leq m$ there is an inequality of sizes $|n^{(i)}| \leq |n'^{(i)}|$. The group of automorphisms of an object $\bar{n}$ is the product of symmetric groups $S_{n^{(1)}} \times \ldots \times S_{n^{(m)}}$, which we will denote by $S_{\bar{n}}$.

Many natural sequences of spaces and varieties are naturally parameterized by $\mathbf{FI}^m$. For example, fix some space X and consider the following generalization of the configurations spaces

$$\mathrm{PConf}^{(n_1, \ldots, n_m)}(X) := \{[(x_1^{(1)}, \ldots, x_{n_1}^{(1)}), \ldots, (x_1^{(m)}, \ldots, x_{n_m}^{(m)})] \mid \forall i \neq j (x_{k_i}^{(i)} \neq x_{k_j}^{(j)})\}$$

inside the product $X^{n_1} \times \ldots \times X^{n_m}$. Every inclusion $\bar{n} \hookrightarrow \bar{n}'$ induces a continuous map by forgetting coordinates, so this is naturally a contravariant $\mathbf{FI}^m$-diagram of spaces. Applying a cohomology functor to this diagram of spaces yields an $\mathbf{FI}^m$-module. The special case of based rational maps $\mathbb{P}^1 \longrightarrow \mathbb{P}^{m-1}$ was described in the introduction, to which theory below applies and gives Corollary 2.1.5.

The category $\mathbf{FI}^m$ fits in with our general framework, as the following demonstrates.

Proposition 2.6.1 ($\mathbf{FI}^m$ is of FI type). $\mathbf{FI}^m$ *is a locally finite category of* $\mathbf{FI}$ *type. Pullbacks and weak push-outs are given by the corresponding operations in* $\mathbf{FI}$ *applied coordinatewise.*

Proof. First we consider the case $m = 1$, i.e. we need to show that $\mathbf{FI}$ is indeed of

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FI type. **FI** is a subcategory of the category of finite sets, which has pullbacks and push-outs. The **Set**-pullback of two injections itself has injective structure maps, and is thus naturally a pullback in **FI**. Regarding weak push-outs, note that if

$$
\begin{array}{ccc}
p & \xrightarrow{\ f_1\ } & c_1 \\
{\scriptstyle f_2}\big\downarrow & & \big\downarrow{\scriptstyle g_2} \\
c_2 & \xrightarrow[\ g_2\ ]{} & d
\end{array}
$$

is a pullback diagram in **Set**, then the images of g_1 and g_2 intersect precisely in the image of the composition $g_1 \circ f_1 = g_2 \circ f_2$. Thus if all four maps are injections, the universal function from the **Set** push-out $c_1 \cup_p c_2$ into d is injective. We therefore see that the **Set** push-out is a weak push-out in **FI**. The other axioms of **FI** type are clear.

The case $m > 1$ follows easily from the previous paragraph when pullbacks and weak push-outs are computed coordinatewise. $\qquad\square$

We turn to the decomposition into irreducible subrepresentations. First we recall some of the terminology related to the case $m = 1$.

Definition 2.6.2 (Padded partitions and irreducible representations). *Recall that a* partition *of a natural number n is a sequence $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k)$ such that $\sum_{i=1}^{k} \lambda_i = n$. In this case we write $\lambda \vdash n$ and refer to n as the* degree *of λ. This degree will be denoted by $|\lambda|$.*

For every other natural number $d \geq |\lambda| + \lambda_1$ we define the padded partition

$$
\lambda(d) = (d - |\lambda| \geq \lambda_1 \ldots \geq \lambda_k) \vdash d \tag{2.6.1}
$$

By deleting the largest part of a partition, we see that every partition of d is of the

form $\lambda(d)$ for some partition $\lambda \vdash n < d$.

Recall that the partitions on d are in one-to-one correspondence with the irreducible representations of S_d. Denote the corresponding irreducible representation by $V_{\lambda(d)}$.

To consider the case $m > 1$ recall that the irreducible representations of a product of finite groups $G \times H$ are given exactly by the pairs $V \boxtimes W$ where V and W are irreducible representations of G and H respectively. The symbol $\boxtimes$ is the usual tensor product on the underlying vector spaces V and W and the action of $G \times H$ on this product is defined by $(g, h).(v \otimes w) = g(v) \otimes h(w)$.

Corollary 2.6.3. *The irreducible representations of $S_{\bar{n}} = S_{n^{(1)}} \times \ldots \times S_{n^{(m)}}$ are precisely external tensor products of the form*

$$V_{\bar{\lambda}(\bar{n})} := V_{\lambda^{(1)}(n_1)} \boxtimes \ldots \boxtimes V_{\lambda^{(m)}(n_m)}$$

where $|\lambda^{(i)}| + \lambda_1^{(i)} \leq n^{(i)}$ for every $1 \leq i \leq m$. Furthermore the character of such a product is given by the product of the individual characters.

Following this observation we define a $\boxtimes$ operation on **FI**-modules.

Definition 2.6.4 (External tensor product). *Let $(M^{(1)}, \ldots, M^{(m)})$ be an m-tuple of **FI**-modules. We define their external tensor product to be the $\mathbf{FI}^m$-module*

$$\bar{M} = M^{(1)} \boxtimes \ldots \boxtimes M^{(m)} \tag{2.6.2}$$

by composing the functor $(M^{(1)}, \ldots, M^{(m)}) : \mathbf{FI}^m \longrightarrow (R - \mathbf{Mod})^m$ with the m-fold tensor product functor on R-modules.

We then see that if $\bar{n}$ is any object, then the $S_{\bar{n}}$-representation $\bar{M}_{\bar{n}}$ is precisely the external tensor product $M^{(1)}_{n^{(1)}} \boxtimes \ldots \boxtimes M^{(m)}_{n^{(m)}}$. Consequently, the character of $\bar{M}$ is the product of the **FI**-characters of the factors.

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Remark 2.6.5. It is also important to note that the external tensor operation commutes with the Ind functors in the following sense:

$$\text{Ind}_{\bar{n}}(V^{(1)} \boxtimes \ldots \boxtimes V^{(m)}) \cong \text{Ind}_{n^{(1)}}(V^{(1)}) \boxtimes \ldots \boxtimes \text{Ind}_{n^{(m)}}(V^{(m)}). \qquad (2.6.3)$$

This can be verified e.g. by considering the definition of Ind in Definition 2.3.1, and using the associativity and commutativity of the tensor product.

Theorem 2.6.6 (**Relating $\mathbf{FI}^m$-modules to FI-modules**)**.** *The following relationships hold between the representation theory of $\mathbf{FI}^m$ and that of $\mathbf{FI}$.*

1. *Every free $\mathbf{FI}^m$ module of degree $\leq \bar{n}$ is the direct sum of external tensor products of free $\mathbf{FI}$-modules, where the i-th component is of degree $\leq n^{(i)}$.*

2. *Every $\mathbf{FI}^m$-character polynomial of degree $\leq \bar{n}$ decomposes as a sum of products of $\mathbf{FI}$-character polynomials, where the i-th factor has degree $\leq n^{(i)}$.*

Remark 2.6.7. In most related work on the representation theory of the category $\mathbf{FI}$, free modules are called projective or $\mathbf{FI}\#$-modules. See [CEF1] for the relevant definitions and a proof that these concepts are equivalent.

Proof. We start with the first assertion. Let $\bar{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(m)})$ be an m-tuple of partitions and $\bar{n}$ some m-tuple of natural numbers satisfying $n^{(i)} \geq |\lambda^{(i)}| + \lambda_1^{(i)}$ for all $i = 1, \ldots, m$. We apply the fact that Ind commutes with external tensor products to the irreducible $S_{\bar{n}}$-representation

$$V_{\bar{\lambda}(\bar{n})} = V_{\lambda^{(1)}(n^{(1)})} \boxtimes \ldots \boxtimes V_{\lambda^{(m)}(n^{(m)})}.$$

This gives a presentation

$$\text{Ind}_{\bar{n}}(V_{\bar{\lambda}}(\bar{n})) \cong \text{Ind}_{n^{(1)}}(V_{\lambda^{(1)}(n^{(1)})}) \boxtimes \ldots \boxtimes \text{Ind}_{n^{(m)}}(V_{\lambda^{(m)}(n^{(m)})})$$

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which proves the first assertion of the theorem for $\mathrm{Ind}_{\bar{n}}(V)$ when V is irreducible.

For a general $S_{\bar{n}}$-representation V, decompose V into irreducible subrepresentations $V = V_1 \oplus \ldots \oplus V_r$. Since Ind commutes with direct sums, the induction module $\mathrm{Ind}_{\bar{n}}(V)$ is a direct sum of external tensor products of induction **FI**-modules.

Lastly, the assertion applies to all free $\mathbf{FI}^m$-modules, since they are directs sum of induction modules of the form previously considered.

As for the second assertion, a character polynomials of degree $\leq \bar{n}$ is a k-linear combination of the characters of free $\mathbf{FI}^m$-modules of degree $\leq \bar{n}$. By the first statement such a free module is the sum of external tensor products of free **FI**-modules with the appropriate bounds on their degrees. But the character of an external tensor product is the product of the individual characters, which in the case of products of free **FI**-modules are by definition **FI**-character polynomials. Thus every $\mathbf{FI}^m$-character polynomial is indeed a k-linear combination of products of **FI**-character polynomials with the appropriate bound on degree. $\qquad\square$

Theorem 2.6.6 allows us to give an explicit description of the character polynomials of $\mathbf{FI}^m$ is terms of cycle counting functions.

Definition 2.6.8 (Cycle counting functions). *For every natural number k, let $X_k : \coprod_n S_n \longrightarrow \mathbb{N}$ be the simultaneous class function on the symmetric groups*

$$X_k(\sigma) = \#\ of\ k\text{-}cycles\ appearing\ in\ \sigma.$$

On the products $S_{n^{(1)}} \times \ldots \times S_{n^{(m)}}$ we define a similar function $X_k^{(i)}$ by

$$X_k^{(i)}(\sigma^{(1)}, \ldots, \sigma^{(m)}) = \#\ of\ k\text{-}cycles\ appearing\ in\ \sigma^{(i)}.$$

The study of polynomials in the class functions X_k dates back to Frobenius, and

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they are what is classically known as *character polynomials.* The following proposition shows that our definition of character polynomials generalizes this classical idea.

Theorem 2.6.9 (**Character polynomials of FI^m**). *The filtered $\mathbb{C}$-algebra of character polynomials of* **FI**^{m} *coincides with the polynomial ring*

$$R = \mathbb{C}[X_1^{(1)}, \ldots, X_1^{(m)}, X_2^{(1)}, \ldots, X_2^{(m)}, \ldots].$$

where we define $\deg(X_k^{(i)}) = (0, \ldots, k, \ldots, 0) = k\bar{e}^{(i)}$.

Proof. We first prove this when $m = 1$. For the inclusion $R \subseteq \text{Char}_{\mathbf{FI}}$ we show that for every k the function X_k is indeed a character polynomial. Recall that in Example 2.2.7 we showed that for every cycle type $\mu = (\mu_1, \ldots, \mu_k)$ the associated character polynomial satisfies

$$\binom{X}{\mu}(\sigma) = \binom{X_1(\sigma)}{\mu_1} \cdots \binom{X_k(\sigma)}{\mu_k}. \tag{2.6.4}$$

Thus by taking $\mu_k = 1$ and $\mu_j = 0$ for all $j \neq k$ we get a character polynomial

$$\binom{X}{\mu}(\sigma) = \binom{X_k(\sigma)}{1} = X_k(\sigma). \tag{2.6.5}$$

For the reverse inclusion, one can construct the right-hand side of

$$\binom{X}{\mu} = \binom{X_1}{\mu_1} \cdots \binom{X_k}{\mu_k} \tag{2.6.6}$$

in the algebra generated by $X_1, X_2, \ldots$, thus realizing every generator $\binom{X}{\mu}$ of $\text{Char}_{\mathbf{FI}}$. This concludes the proof in the case $m = 1$.

For $m > 1$, Theorem 2.6.6 states that every **FI**^{m}-character polynomial is a linear combination of external products of **FI**-character polynomials. We saw that the latter class of functions is precisely the ring of polynomials in $X_1, X_2, \ldots$. The function $X_k^{(i)}$

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is the external product of X_k in the i-th coordinate with 1's in all other coordinates, and thus polynomials in $X_k^{(i)}$ clearly generate all linear combinations of external products of $X_1, X_2, \ldots$. This proves the claim. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Our general theory of stabilization for inner products thus applies to expressions involving the functions $X_k^{(i)}$.

Corollary 2.6.10 (Stabilization of inner products). *The $S_{\bar{n}}$-inner product of two polynomials $P, Q \in \mathbb{C}[X_k^{(i)} : k \in \mathbb{N}, 1 \leq i \leq m]$ does not depend on $\bar{n}$ for all $\bar{n} \geq \deg(P) + \deg(Q)$, where the degree of $X_k^{(i)}$ is $k\bar{e}^{(i)}$ and addition of degrees is defined coordinatewise.*

In the case $\mathbf{C} = \mathbf{FI}$ this result is proved in [CEF2, Theorem 3.9] via a direct calculation of the S_n-inner products. The $\mathbf{C} = \mathbf{FI}_{\mathbb{Z}/2\mathbb{Z}}$-analog is proved in [Wi1]. When G is any other finite group, a non-effective analog of Corollary 2.6.10 for $\mathbf{C} = \mathbf{FI}_G$ is implicit in [SS3, Theorem 3.2.2].

We turn to discussing representation stability for $\mathbf{FI}^m$. First consider the case $m = 1$: the irreducible representations of symmetric groups of different orders are naturally related in the following sense.

Fact 2.6.11 (The modules $V_{\lambda(\bullet)}$). *For every partition $\lambda \vdash |\lambda|$, there exists an $\mathbf{FI}$-submodule of $\mathrm{Ind}_{|\lambda|}(V_\lambda)$, which we will denote by $V_{\lambda(\bullet)}$, whose value at every $d \geq |\lambda| + \lambda_1$ is isomorphic to the irreducible S_d-representation $V_{\lambda(d)}$. Moreover, for every partition λ there exists a character polynomial P_λ of degree $|\lambda|$ such that the character of $V_{\lambda(\bullet)}$ coincides with P_λ on S_d for all $d \geq |\lambda| + \lambda_1$.*

See [CEF1] for the existence of $V_{\lambda(\bullet)}$ and [Ma, Example I.7.14] for P_λ.

This fact extends to all $m > 1$ via the external tensor product.

Corollary 2.6.12 (The modules $V_{\bar{\lambda}(\bullet)}$). *For every m-tuple $\bar{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(m)})$ of partitions there exists an $\mathbf{FI}^m$-submodule of $\mathrm{Ind}_{|\bar{\lambda}|}(V_{\bar{\lambda}})$, which we will denote by $V_{\bar{\lambda}(\bullet)}$, whose value at $\bar{n}$ is the $S_{\bar{n}}$-irreducible representation*

$$V_{\bar{\lambda}(\bar{n})} := V_{\lambda^{(1)}(n^{(1)})} \boxtimes \ldots \boxtimes V_{\lambda^{(m)}(n^{(m)})}$$

for all $\bar{n} \geq |\bar{\lambda}| + \bar{\lambda}_1$. Here $|\bar{\lambda}|$ is the m-tuple $(|\lambda^{(1)}|, \ldots, |\lambda^{(m)}|)$, the expression $\bar{\lambda}_1$ is $(\lambda_1^{(1)}, \ldots, \lambda_1^{(m)})$ and $+$ coincides with coordinatewise addition.

Moreover, the character of $V_{\bar{\lambda}(\bullet)}$ is the following character polynomial of degree $|\bar{\lambda}|$

$$P_{\bar{\lambda}} := P_{\lambda^{(1)}} \cdot \ldots \cdot P_{\lambda^{(m)}}.$$

These sequences of irreducible representations allow us to formulate the notion of representation stability for free $\mathbf{FI}^m$-modules.

Theorem 2.6.13 (Representation stability for $\mathbf{FI}^m$). *Suppose $F_\bullet$ is a free $\mathbf{FI}^m$-module that is finitely-generated in degree $\leq \bar{n}$. Then there exist m-tuples of partitions $\bar{\lambda}_1, \ldots, \bar{\lambda}_k$, satisfying $|\bar{\lambda}_j| \leq \bar{n}$ for all $j = 1, \ldots, k$, such that for all $\bar{d} \geq 2 \times \bar{n} = (2n^{(1)}, \ldots, 2n^{(m)})$ the $S_{\bar{d}}$-module $F_{\bar{d}}$ decomposes into irreducibles as*

$$F_{\bar{d}} \cong (V_{\bar{\lambda}_1(\bar{d})})^{r_1} \oplus \ldots \oplus (V_{\bar{\lambda}_k(\bar{d})})^{r_k}$$

and the multiplicities $r_1, \ldots, r_k$ do not depend on $\bar{d}$.

Note 2.6.14. The original definition of Representation Stability given in [CF] includes additional injectivity and surjectivity conditions on top of the stabilization of irreducible decompositions. We will not discuss these aspects of the definition, although the reader familiar with them will readily notice that they are immediately satisfied by all free $\mathbf{C}$-modules.

Proof of Theorem 2.6.13. The case $m = 1$ asserts the representation stability of free finitely-generated **FI**-modules. This follows directly from the Branching rule for inducing representations of the symmetric group (see [FH]), and is proved in [CEF1, Theorem 1.13].

For $m > 1$ the statement follows from Theorem 2.6.6 using the corresponding statement in the case $m = 1$. Since every free $\mathbf{FI}^m$-module $M_\bullet$ is a sum of the external tensor products of free **FI**-modules, and each of those decomposes as a stabilizing direct sum of irreducibles, the same is true for $M_\bullet$. $\square$

At the level of character polynomials Theorem 2.6.13 translates to the following orthonormality statement.

Corollary 2.6.15 (Spectral orthonormal basis for character polynomials). *The character polynomials*

$$\left\{ P_{\bar{\lambda}} := P_{\lambda_1} \cdot \ldots \cdot P_{\lambda_m} \right\}_{|\bar{\lambda}| \leq \bar{n}}$$

form an orthonormal basis for all $\mathbf{FI}^m$-character polynomials of degree $\leq \bar{n}$ with respect to the inner product

$$\langle P, Q \rangle = \lim_{\bullet \to \infty} \langle P, Q \rangle_\bullet = \langle P, Q \rangle_{\deg(P) + \deg(Q)}. \tag{2.6.7}$$

2.7 Stable statistics for finite general linear groups

Let us now apply the combinatorics of character polynomials to study stable statistics of the groups themselves. As an example, we consider the general linear groups over finite fields as analogs of permutation groups.

Picking permutations at random, the expected number of d-cycles is known to be

$1/d$ and is, in particular, independent of the size of the permuted set. In this section we discuss similar size-independent statistics of finite general linear groups: ones that depend only on 'small minors'. The proof technique uses a twisted version of Burnside's Lemma, motivated by the combinatorics of character polynomials, and applies simultaneously to symmetric groups, finite linear groups and many other settings.

Statistics of finite matrix groups is a rich field with many successful techniques and applications to number theory, combinatorics and computer science (see e.g. [Fu] and the references therein). A typical question that one asks in this field is "what is the probability that the characteristic polynomial a randomly chosen matrix have a certain form?" and a typical answer is asymptotic in nature. The kind of question that this section considers is different: while the characteristic polynomial depends on the entire matrix, we will focus on more local properties – ones that depend only on small minors – and our answers will be exact.

The question of expected number of d-cycles in a random permutation σ could be rephrased as follows: it is the number of σ-invariant subsets of size d, to which σ restricts to a d-cycle. This count is exactly the evaluation $\binom{X}{C_d}(\sigma)$ of the **FI** character polynomial corresponding to the conjugacy class of the d-cycle $C_d \subseteq S_d$. A natural generalization to the setting of finite linear groups is given as follows. Fix a finite field $\mathbb{F} = \mathbb{F}_q$.

Theorem 2.7.1 (Stable statistics for Gl_n). *Fix a conjugacy class $C \subset \mathrm{Gl}_d(\mathbb{F})$. Then for a random $T \in \mathrm{Gl}_n(\mathbb{F})$, the expected number of d-dimensional subspaces $W \leq \mathbb{F}^n$ for which $T(W) = W$ and $T|_W \in C$ is independent of n once $n \geq d$. In particular, calculating the case $n = d$ gives that this expectation is precisely $\frac{|C|}{|\mathrm{Gl}_d(\mathbb{F})|}$.*

More generally, all joint higher moments of these random variables are eventually independent of n. Explicitly, number of subspaces $W \leq \mathbb{F}^n$ from the previous paragraph is precisely the evaluation of the character polynomial $\binom{X}{C}(T)$. Thinking of

these polynomials as random variables on $\mathrm{Gl}_n(\mathbb{F}_q)$ and considering conjugacy classes $C_i \subseteq \mathrm{Gl}_{d_i}(\mathbb{F}_q)$, the expectation $\mathbb{E}\left[\binom{X}{C_1} \cdot \ldots \cdot \binom{X}{C_r}\right]$ is the same for all $n \geq d_1 + \ldots + d_r$.

In particular, taking $C = \{1\} \subset \mathrm{Gl}_1(\mathbb{F})$ gives a count of the number of fixed vectors $\neq 0$ of a random $T \in \mathrm{Gl}_n$. As the expected number of such is independent of n, it can be computed with $n = 1$: picking $\lambda \in \mathbb{F}^\times$ randomly, there are $q - 1$ non-zero fixed points if $\lambda = 1$ and 0 otherwise.

Corollary 2.7.2. *The expected number of non-zero fixed vectors of a random $T \in \mathrm{Gl}_n(\mathbb{F})$ equals 1. The same is true when replacing "fixed vectors" by "eigenvectors with eigenvalue $\lambda \in \mathbb{F}^*$".*

The proof of Theorem 2.7.1 is a special case of the general theory of charter polynomials and thus revolves around basic category theory. Namely, studying the category of finite-dimensional $\mathbb{F}$-vector spaces, of which $\mathrm{Gl}_n(\mathbb{F})$ are the automorphism groups. The point here is that the approach has nothing to do with linear algebra, and could be used to prove analogous results in vastly different contexts. For example, the same technique gives the analogous:

Fact 2.7.3 ([CEF1, Proposition 3.9]). *Fix a conjugacy class $C \subset S_d$. Then for a random $\tau \in S_n$, the expected number of subsets $W \subset [n]$ of cardinality d such that $\tau(W) = W$ and $\tau|_W \in C$ does not depend on n once $n \geq d$. In particular, this expected number is $\frac{|C|}{d!}$.*

In [CEF1] this fact is proved combinatorially – counting the number of permutations of various kinds. It is somewhat comforting that a single argument produces the same result in the general linear setting as well as in the combinatorial one.

Proof of Theorem 2.7.1. Let **VI** be the category of finite-dimensional $\mathbb{F}$-**V**ector spaces and ***I**njective* linear transformations. The set of morphisms $\mathbb{F}^d \hookrightarrow \mathbb{F}^n$ will be denoted

by $\mathbf{VI}(\mathbb{F}^d, \mathbb{F}^n)$. Since a linear injection $\mathbb{F}^n \hookrightarrow \mathbb{F}^n$ is an isomorphism, the endomorphisms $\mathrm{End}_{VI}(\mathbb{F}^n)$ are precisely the group $Gl_n(\mathbb{F})$. It is easy to verify that $\mathbf{VI}$ is a category of $\mathbf{FI}$ type and thus the theory of character polynomials applies.

Applying Corollary 2.4.6 to the character polynomial $\binom{X}{C}$, it follows that the expected number of $\mathrm{Gl}_d(\mathbb{F})$-orbits of $f : \mathbb{F}^d \hookrightarrow \mathbb{F}^n$, on which $T \circ f = f \circ B$ with $B \in C$, does not depend on n. But an injection $f : \mathbb{F}^d \hookrightarrow \mathbb{F}^n$ determines a d-dimension subspace $W := \mathrm{Im}(f) \leq \mathbb{F}^n$, and two injections f and f' determine the same W if and only if they differ by some precomposition $f' = f \circ B$ for $B \in \mathrm{Gl}_d(\mathbb{F})$. Therefore the orbits $\mathbf{VI}(\mathbb{F}^d, \mathbb{F}^n)/\mathrm{Gl}_d(\mathbb{F})$ are in natural bijection with the set of d-dimensional subspaces $W \leq \mathbb{F}^n$. This completes the proof of the first statement.

For joint higher moments of these random variables, it is shown in Corollary 2.3.9 that if $\binom{X}{C_1}$ and $\binom{X}{C_2}$ are two such random variables corresponding to conjugacy classes $C_i \subset \mathrm{Gl}_{d_i}(\mathbb{F})$, then their product $\binom{X}{C_1} \cdot \binom{X}{C_2}$ is a linear combination of $\binom{X}{C'}$ with $C' \subset \mathrm{Gl}_{d'}(\mathbb{F})$ for $d' \leq d_1 + d_2$. By induction, the general statement now follows from the previous special case. $\qquad\square$

The same approach applies in the following generality: let $\mathcal{C}$ be a locally-finite category (i.e. hom-sets all are finite). For every two objects c and d denote the monomorphisms from c to d by $\mathrm{Mon}(c, d)$ and the automorphism group of d by $\mathrm{Aut}(d)$. A *c-shaped subobject* of d is an orbit in $\mathrm{Mon}(c, d)/\mathrm{Aut}(c)$.

Theorem 2.7.4 (Size independent statistics). *Suppose that for every two objects c and d the composition action $\mathrm{Aut}(d) \curvearrowright \mathrm{Mon}(c, d)$ is transitive. Fix a conjugacy class $C \subset \mathrm{Aut}(c)$. Then choosing $g \in \mathrm{Aut}(d)$ randomly, the expected number of c-shaped subobjects $[f]$ of d which are fixed by g and $g \circ f = f \circ h$ with $h \in C$ is precisely $|C|/\mathrm{Aut}(c)$. In particular this expected number does not depend on d.*

CHAPTER 3

DIAGRAMS OF LINEAR SUBSPACE ARRANGEMENTS

Church-Ellenberg-Farb [CEF2] used the language of **FI**-CHA to identify certain sequences of hyperplane arrangements with S_n-actions that satisfy cohomological representation stability. Here we vastly extend their results, and define when a collection of arrangements is "finitely generated". Using this notion we get stability results to:

- General linear subspace arrangements, not necessarily of hyperplanes.

- A wide class of group actions, replacing **FI** by a general category **C**.

We show that the cohomology of such collections of arrangements satisfies a strong form of representation stability, with many concrete applications. For example, this implies that their Betti numbers are always given by certain generalized polynomials.

For this purpose we use the theory of representation stability for quite general classes of groups, developed in a §2. We apply this theory to get classical cohomological stability of quotients of linear subspace arrangements with coefficients in certain constructible sheaves.

3.1   Introduction

A *linear subspace arrangement* is a finite collection $\mathcal{A}$ of linear subspaces in $\mathbb{C}^n$, all containing the origin and possibly of different dimensions. $\mathcal{A}$ determines:

Algebro-geometric data: An algebraic variety, the complement $M_{\mathcal{A}} = \mathbb{C}^n - \cup \mathcal{A}$.

Combinatorial data: A partially ordered set $P_{\mathcal{A}}$ of the intersections of subspaces in $\mathcal{A}$, ordered by reverse inclusion.

Representations: An Aut($\mathcal{A}$)-representation H$^*(M_{\mathcal{A}})$.

The interaction between these three viewpoints was studied by Arnol'd [Ar], and later by Goresky-MacPherson [GM], Orlik-Solomon [OS], Lehrer-Solomon [LS], and many others.

Many natural arrangements appear in families, along with actions of finite groups $G \le \mathrm{Aut}(\mathcal{A})$.

Example 3.1.1. The *braid arrangement* $\mathcal{B}_n = \{z_i = z_j\}_{1 \le i < j \le n}$ in $\mathbb{C}^n$, with actions of $S_n \le \mathrm{Aut}(\mathcal{B}_n)$, the symmetric group on n letters.

Example 3.1.2. The arrangement $\mathcal{C}_{(n_1,\dots,n_m)} = \{z_{i_1}^{(1)} = \dots = z_{i_m}^{(m)}\}_{\forall(1 \le j \le m), 1 \le i_j \le n_j}$ inside $\mathbb{C}^{n_1} \times \dots \times \mathbb{C}^{n_m}$, with $S_{n_1} \times \dots \times S_{n_m} \le \mathrm{Aut}(\mathcal{C}_{(n_1,\dots,n_m)})$.

Example 3.1.3. The arrangement $\mathcal{D}_n = \{v_i \ne g(v_j)\}_{1 \le i < j \le n, g \in G}$ in V^n, where G is some finite group acting on a complex vector space V. Here $G^n \rtimes S_n \le \mathrm{Aut}(\mathcal{D}_n)$.

In the special case of the first example $\mathcal{B}_n$, Church, Ellenberg and Farb ([CF] and [CEF1]) discovered patterns in the S_n-representations $\mathrm{H}^i(M_{\mathcal{B}_n}; \mathbb{Q})$: their characters can be expressed as a single "character polynomial" independent of n; and their irreducible decompositions stabilize in a precise sense. They named this *representation stability*. The theory of **FI**-modules, developed in [CEF1], gives a powerful viewpoint that explains this phenomenon as the *finite-generation* of a single object.

[CEF2] developed the framework of **FI**-CHA for discussing families of hyperplane arrangements similar to $\mathcal{B}_n$, which captures the sense in which the arrangements themselves are already "finitely-generated". They then show that the cohomology of such families always forms a finitely-generated **FI**-module, thus lifting finite-generation to the level of spaces in that case. The approach in [CEF2] does not naturally generalize to variations, e.g. to the arrangements in Examples 3.1.2 and 3.1.3 above. The purpose of the present chapter is to extend these results to include many families of linear subspace arrangements (A_n) (including Examples 3.1.2 and 3.1.3).

A major obstacle is that groups $G_n \leq \mathrm{Aut}(\mathcal{A}_n)$ can be quite general. The theory of **FI**-modules applies only for $G_n = S_n$; it also depends heavily on the specific naming of irreducible representations of S_n, which is not available in a more general context. An attempt to overcome this obstacle was proposed in §2, using the framework of *categories of* **FI** *type.* It turns out that such categories also provide a robust framework for analyzing families of subspace arrangements. Our goals in this work are therefore three-fold:

(I) We identify when a collection of linear subspace arrangements fits together to form a "natural family". If $(\mathcal{A}_c)$ is a family of arrangements indexed by some category **C**, this approach packages the family into a single object: a **C**-*arrangement*, defined in §3.2. Representation stability then reduces to combinatorial properties of this one object, namely *finite-generation* and *downward stability* (see §3.3.1).

(II) We extend the stability results of [CEF2] to linear subspace arrangements generated by arbitrary linear subspaces, not necessarily hyperplanes.

(III) We complete the project of *lifting* the "finite-generation" property, which characterizes representation stability of **FI**-modules, to the level of the arrangements themselves.

This aspect of the project began with a question from Benson Farb who, upon learning that the sequence of complements $M^n - \cup_{i \neq j}\{m_i = m_j\}$ for a manifold M exhibits representation stability when hit with both the cohomology functors H^i and with the homotopy group functors π_i[1], asked whether the spaces themselves are finitely-generated in some sense, and the observed finiteness results are a mere shadow of this fact. Here we offer an answer to this question in the case of linear subspace arrangements.

1. See e.g. Church [Ch] for H^i and Kupers-Miller [KM] for π_i.

3.1.1   Statement of the results

Let **C** be a small category. We say that **C** is *of* **FI** *type*, roughly, if every morphism is a monomorphism and **C** has pullbacks and push-outs (see Definition 2.1.6). Categories of **FI** type include many natural categories that have recently been studied in the context of representation stability. Among these:

1. The category **FI** itself, of finite sets and injective functions, with automorphism groups S_n for $n \in \mathbb{N}$ (see [CEF1]).

2. Finite powers $\mathbf{FI}^m$, with automorphism groups $S_{n_1} \times \ldots \times S_{n_m}$ for $n_i \in \mathbb{N}$.

3. The category $\mathbf{VI}_k$ of finite dimensional k-vector spaces and injective linear maps, with automorphism groups $\mathrm{Gl}_n(k)$ for $n \in \mathbb{N}$ (see Putman-Sam [PS]).

4. The class of categories $\mathbf{FI}_G$ defined by Sam-Snowden [SS3] where G is some group, with automorphism groups $G^n \rtimes S_n$ (see [Wi2] and [Ca] for naturally occurring families of arrangements with these symmetries).

A **C***-arrangement* $\mathcal{A}_\bullet$ is a functor from **C** into the category of linear subspace arrangements (see §3.2 for our definition of this category). Many infinite families of arrangements can be defined very succinctly and contain a finite amount of information, as illustrated using the following notion. We say that a **C**-arrangement $\mathcal{A}_\bullet$ is *finitely-generated* if there exist finitely many linear subspaces $\{L_i \in \mathcal{A}_{c_i}\}_{i=1}^m$, for $\{c_i\}_{i=1}^m$ objects of **C**, such that for every object d of **C** the arrangement $\mathcal{A}_d$ is generated by intersecting the images of $\{L_i\}_{i=1}^m$ under all morphisms $c_i \to d$. Our three examples above each fit into a **C**-arrangement generated by a single subspace:

Example	Category	Arrangement	Group	Generating Subspace
3.1.1 - $\mathcal{B}_\bullet$	**FI**	$\mathcal{B}_n$	S_n	$(z_1 = z_2) \subset \mathbb{C}^2$
3.1.2 - $\mathcal{C}_\bullet$	$\mathbf{FI}^m$	$\mathcal{C}_{(n_1,\ldots,n_m)}$	$S_{n_1} \times \ldots \times S_{n_m}$	$(z^{(1)} = \ldots = z^{(m)}) \subset \mathbb{C}^m$
3.1.3 - $\mathcal{D}_\bullet$	$\mathbf{FI}_G$	$\mathcal{D}_n$	$G^n \rtimes S_n$	$(v_1 = v_2) \subset V^2$

Consider a cohomology functor[2] H^i. By applying H^i to the complement varieties $M_{\mathcal{A}_d}$ we get a family of abelian groups parameterized by $\mathbf{C}$, i.e. a functor from $\mathbf{C}$ into abelian groups (or more generally into $R - \mathbf{Mod}$ for some ring R, which in our context will always be $\mathbb{Q}_\ell$). We call such a functor a $\mathbf{C}$-*module*. If $V_\bullet$ is a $\mathbf{C}$-module then at every object d of $\mathbf{C}$ the module V_d has an action of the group $\mathrm{Aut}_{\mathbf{C}}(d)$, and these representations are related by the morphisms of $\mathbf{C}$.

We say that a $\mathbf{C}$-module $V_\bullet$ is *finitely-generated* if there exists a finite collection of elements $\{v_i \in V_{c_i}\}_{i=1}^m$ not contained in any proper sub-$\mathbf{C}$-module. $V_\bullet$ is *free*[3] if it is the sum of $\mathbf{C}$-modules, each induced from some fixed $\mathrm{Aut}_{\mathbf{C}}(d)$-representation for some object d (see Definition 2.3.1). It is the finitely-generated and free $\mathbf{C}$-modules that exhibit what we call *representation stability*, i.e. their constituent representations stabilize in a precise sense. The theory of such objects is developed in §2 and applied here.

Our main result states that the cohomology of a finitely-generated $\mathbf{C}$-arrangement is a finitely-generated, free $\mathbf{C}$-module. The freeness assertion is the most surprising and consequential part of this statement. We will give concrete applications of this result below.

To avoid pathologies we assume that $\mathcal{A}_\bullet$ respects the structure of $\mathbf{C}$, as follows.

---

2. Since we will only be concerned with algebraic varieties, take $\mathrm{H}^*(M)$ to mean ℓ-adic cohomology $\mathrm{H}^*_{\acute{e}t}(M_{\bar{k}}; \mathbb{Q}_\ell)$ with its $\mathrm{Gal}(\bar{k}/k)$-action. However, when the base field is $\mathbb{C}$ one can simply take H^* to mean singular cohomology with coefficients in $\mathbb{Q}_\ell$ or even $\mathbb{Q}$ with inconsequential differences.

3. Free $\mathbf{FI}$-modules are precisely the $\mathbf{FI}\#$-modules presented in [CEF1].

We say that $\mathcal{A}_\bullet$ is *continuous* if it respects pullbacks in **C** (see Definition 3.2.5)[4]. $\mathcal{A}_\bullet$ is *normal*, roughly, if no subspace appears later than it could (see Definition 3.3.16). Many natural examples of arrangements satisfy these hypotheses including the ones in Examples 3.1.1, 3.1.2 and 3.1.3.

Theorem 3.1.4 (**Representation stability of C-arrangements**). *Let* **C** *be a category of* **FI** *type, and let* $\mathcal{A}_\bullet$ *be a continuous, normal, and finitely-generated* **C**-*arrangement. For all* $i \geq 0$ *the* **C**-*module of cohomology groups* $\mathrm{H}^i(\mathcal{M}_\mathcal{A})_\bullet$ *is finitely-generated and free.*

Theorem 3.1.4 will be proved in §3.3. We consider two sets of applications:

Application I (Representation stability). For simplicity of exposition we specialize the theory in this introduction to classes of arrangements indexed by m-tuples of natural numbers, i.e. $\mathbf{FI}^m$-arrangements and their cohomology. In Example 3.6.5 we consider an $\mathbf{FI}^m$-family of varieties as follows. Fix $r, k \in \mathbb{N}$, then for every m-tuple $(n_1, \ldots, n_m)$ we have the variety $\mathcal{M}_{m,k}^{(n_1,\ldots,n_m)}(\mathbb{C}^r)$ whose points are ordered tuples

$$[(v_1^{(1)}, \ldots, v_{n_1}^{(1)}), \ldots, (v_1^{(m)}, \ldots, v_{n_m}^{(m)})] \in (\mathbb{C}^r)^{n_1} \times \ldots \times (\mathbb{C}^r)^{n_m}$$

where there does not exist any vector $v \in \mathbb{C}^r$ that appears k times within the list $(v_1^{(j)}, \ldots, v_{n_j}^{(j)})$ for all $1 \leq j \leq m$. These varieties are the complements of an $\mathbf{FI}^m$-arrangement $\mathcal{A}_{(n_1,\ldots,n_m)}^{m,k}(\mathbb{C}^r)$, generalizing $\mathcal{C}_{(n_1,\ldots,n_m)}$ from Example 3.1.2 above.

Remark 3.1.5 (**Connection with configuration spaces**). More geometrically, the varieties $\mathcal{M}_{m,k}^{(n_1,\ldots,n_m)}$ parameterize m-tuples of ordered configurations in $\mathbb{C}^r$, where we allow points to collide, subject to the restriction that the configurations cannot all have k points in common including multiplicity. We will discuss special cases of these below.

---

4. Note that this definition is weaker than the standard definition of continuity of functors.

In §2 we show that for each category $\mathbf{C}$ of $\mathbf{FI}$ type we get an algebra of *character polynomials*. These are class functions, simultaneously defined on all groups $\mathrm{Aut}_{\mathbf{C}}(c)$, that uniformly describe the characters of finitely-generated free $\mathbf{C}$-modules. In the case $\mathbf{C} = \mathbf{FI}^m$ a *character polynomial* is any polynomial in the class functions $X_k^{(j)}$ for $1 \le j \le m$ and $k \ge 0$, defined simultaneously on all m-fold products $S_{n_1} \times \ldots \times S_{n_m}$ by

$$X_k^{(j)}(\sigma_1, \ldots, \sigma_m) = \# \text{ of } k\text{-cycles in } \sigma_j. \tag{3.1.1}$$

Theorem 3.1.6 (Representation stability of $\mathcal{M}_{m,k}^{\bullet}$). *For every triple of natural numbers (m, k, r), and for each $i \ge 0$, the cohomology*

$$\mathrm{H}^i(\mathcal{M}_{m,k}^{\bullet}(\mathbb{C}^r))$$

forms a finitely-generated, free $\mathbf{FI}^m$-module of multi-degree $\lfloor \frac{i}{r} \rfloor k(1, \ldots, 1)$.

In particular, there exists a single $\mathbf{FI}^m$-character polynomial

$$P_i \in \mathbb{Q}[X_d^{(j)} \mid 1 \le j \le m, d \ge 0]$$

of multi-degree $\lfloor \frac{i}{r} \rfloor k(1, \ldots, 1)$ such that for every $(n_1, \ldots, n_m) \in \mathbb{N}^m$ there is an equality of class functions

$$\chi_{\mathrm{H}^i(\mathcal{M}_{m,k}^{(n_1, \ldots, n_m)}(\mathbb{C}^r))} = P_i \tag{3.1.2}$$

The multi-degree of $X_d^{(j)}$ is defined to be $d \cdot \bar{e}^{(j)}$ (zero except for a d appearing in the j-th entry).

Example 3.1.7. When $m = 2$ and $k = r = 1$ the varieties $\mathcal{M}_{2,1}^{(n_1, n_2)}(\mathbb{C})$ are covers of the space of rational maps $\mathrm{Rat}_*^n(\mathbb{C})$ studied by Segal (see Example 3.6.12 below). In this case, the character polynomials P_1 and P_2 are $\chi_{\mathrm{H}^1(\mathcal{M}_{2,1}^{(n_1, n_2)}(\mathbb{C}))} = X_1^{(1)} \cdot X_1^{(2)}$ of

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multi-degree $(1, 1)$ and

$$
\chi_{\mathrm{H}^1(\mathcal{M}_{2,1}^{(n_1,n_2)}(\mathbb{C}))} = X_1^{(1)} \left(\binom{X_1^{(2)}}{2} - X_2^{(2)} \right) + X_1^{(2)} \left(\binom{X_1^{(1)}}{2} - X_2^{(1)} \right)
$$

$$
+ 2 \binom{X_1^{(1)}}{2} \binom{X_1^{(2)}}{2} - 2 X_2^{(1)} X_2^{(2)}
$$

of multi-degree $(2, 2)$, both independent from (n_1, n_2).

Other special cases to which Theorem 3.1.6 applies include:

1. The braid arrangements, i.e. the classifying spaces of Artin's braid groups, discussed in Example 3.6.9 below.

2. Spaces of configurations in $\mathbb{C}^r$, discussed in Example 3.6.10 below.

3. The k-equals arrangements, related to incomputability problems and classifying homotopy links, discussed in Example 3.6.11 below.

4. Covers of the spaces of based holomorphic maps $\mathbb{P}^1 \longrightarrow \mathbb{P}^m$, discussed in Example 3.6.12 below.

Theorem 3.1.6 shows that all of these examples exhibit representation stability with explicit stable ranges. Moreover, we get information regarding the Betti numbers of all these varieties. Applying Theorem 3.1.6 to $\sigma = id$ gives the following.

Corollary 3.1.8 (Polynomial Betti numbers for $\mathcal{M}_{m,k}^\bullet$). *For every $i \geq 0$ there exists a polynomial $p_i \in \mathbb{Q}[t_1, \ldots, t_m]$ of multi-degree $\lfloor \frac{i}{r} \rfloor k(1, \ldots, 1)$ such that*

$$
\dim_{\mathbb{Q}} \mathrm{H}^i(\mathcal{M}_{m,k}^{(n_1,\ldots,n_m)}(\mathbb{C}^r)) = p_i(n_1, \ldots, n_m)
$$

for all $(n_1, \ldots, n_m) \in \mathbb{N}^m$.

Application II (Classical cohomological stability). The quotient spaces $X_{\mathcal{A}_d} :=$ $M_{\mathcal{A}_d}/\operatorname{Aut}_{\mathbf{C}}(d)$ come up in multiple contexts:

1. For $\mathcal{B}_n$ from Example 3.1.1, $X_{\mathcal{B}_n}$ is the space $\operatorname{Poly}_n(\mathbb{C})$ of degree-n square-free polynomials, studied by Arnol'd [Ar].

2. For $\mathcal{C}_{(n,\dots,n)}$ from Example 3.1.2, $X_{\mathcal{C}_{(n,\dots,n)}}$ is the space $\operatorname{Hol}^n_*(\mathbb{P}^1, \mathbb{P}^m)$ of based degree-n holomorphic maps, studied by Segal [Se].

3. For $\mathcal{M}^{(n,\dots,n)}_{m,k}(\mathbb{C})$ discussed above, the quotient $X^n_{m,k}$ is the space $\operatorname{Poly}^n_{m,k}(\mathbb{C})$ of m-tuples of polynomials with restrictions on root coincidences, introduced by Farb-Wolfson [FW]. These generalize the two previous examples.

The cohomology of a quotient of some variety M by a finite group G is given by transfer

$$\operatorname{H}^i(M/G) = \operatorname{H}^i(M)^G. \tag{3.1.3}$$

Thus Theorem 3.1.4 applied to the trivial subrepresentation gives a classical cohomological stability statement.

Theorem 3.1.9 (Cohomological stability for arrangement quotients). *Suppose that $\mathbf{C}$ and $\mathcal{A}_\bullet$ satisfy the hypotheses of Theorem 3.1.4 and that $|\operatorname{Aut}_{\mathbf{C}}(d)| < \infty$ for every object d. Then the cohomology groups $\operatorname{H}^i(X_{\mathcal{A}_d})$ stabilize in the following sense: if any morphisms $c \to d$ exist in $\mathbf{C}$ then there is a canonical injection*

$$\operatorname{H}^i(X_{\mathcal{A}_c}) \hookrightarrow \operatorname{H}^i(X_{\mathcal{A}_d})$$

and these maps become isomorphisms when c is sufficiently large relative to i (an explicit stable range is given in §3.5).

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In some special cases, Theorem 3.1.9 was previously proved for integral cohomology using clever but ad-hoc techniques, see e.g. [Se]. Applying the theorem to $\mathcal{M}_{m,k}^{(n,\ldots,n)}$ we get a new proof of the cohomological stability proved in [FW] for the rational cohomology of $\mathrm{Poly}_{m,k}^n(\mathbb{C})$.

Theorem 3.1.9 considers only the trivial subrepresentation of $\mathrm{H}^i(\mathcal{M}_{\mathcal{A}_d})$. This is a very special case of the following more general version that considers the entire representation. Every $\mathbf{C}$-module $N_\bullet$ induces a natural constructible sheaf $\widetilde{N}_d$ on the quotient $X_{\mathcal{A}_d}$ whose stalk above the orbit $[x]$ satisfies

$$(\widetilde{N}_d)_{[x]} \cong N_d^{\mathrm{Stab}(x)}$$

where $\mathrm{Stab}(x)$ is the $\mathrm{Aut}_{\mathbf{C}}(d)$-stabilizer of $x \in \mathcal{M}_{\mathcal{A}_d}$ (see §3.5 for the construction).

Theorem 3.1.10 (**Twisted stability for arrangement quotients**). *Suppose that* $\mathbf{C}$ *and* $\mathcal{A}_\bullet$ *satisfy the hypotheses of Theorem 3.1.9. Let* $N_\bullet$ *be a finitely-generated, free* $\mathbf{C}$-*module. Then the sheaf cohomology groups* $\mathrm{H}^i(X_{\mathcal{A}_d}; \widetilde{N}_d)$ *stabilize in the sense of Theorem 3.1.9.*

In the context of ℓ-adic cohomology one needs $N_\bullet$ to take values in continuous $\mathbb{Q}_\ell$-modules.

By considering the trivial $\mathbf{C}$-module $N_\bullet \equiv \mathbb{Q}_\ell$ we recover Theorem 3.1.9. The proof of Theorem 3.1.10 will be presented in §3.5.

Lastly, through the Grothendieck-Lefschetz fixed-point theorem, the twisted stability result in Theorem 3.1.10 has implications for arithmetic statistics of varieties over finite fields. This direction will be developed further in §4.

3.2   Preliminaries

This section will introduce the necessary terminology and categories in which we will be working. We start with the fundamental object of interest, namely linear subspace arrangements.

Definition 3.2.1 (**Linear arrangements**). *The category of* linear subspace arrangements *over a field k, denoted by $\mathbf{Arr}_k$, consists of pairs $A = (V, L)$ where V is a finite dimensional vector space over k and L is a finite set of linear subspaces of V, all containing the origin[5], such that L is closed under intersections.* A morphism $(V_1, L_1) \xrightarrow{f} (V_2, L_2)$ *is a surjective linear map $V(f) : V_2 \longrightarrow V_1$ such that for every subspace $W \in L_1$ the preimage $V(f)^{-1}(W)$ belongs to L_2.*

We remark that traditionally, and particularly in the context of hyperplane arrangements, a distinction is made between an arrangement of distinguished subspaces and the collection of all intersections generated by the arrangement. In the above definition we chose to identify the two concepts, as we are already dealing with subspaces of arbitrary codimension and since this makes the definition of morphisms cleaner.

As the definition suggests, there are two natural functors from the category of arrangements:

Definition 3.2.2 (**Underlying vector space**). *The* underlying vector spaces *functor $V : \mathbf{Arr}_k^{op} \longrightarrow \mathbf{Vect}_k$ is defined by sending an arrangement (V, L) to the vector space V. Morphisms of arrangements are defined as being (contravariant) linear maps between the vector spaces, and these define the action of V on morphisms: $f \mapsto V(f)$.*

and secondly,

---

5. This is often called *a central arrangement.* All arrangements here are assumed to be central.

Definition 3.2.3 (Intersection poset). *The* intersection poset *functor*

$$L : \mathbf{Arr}_k \longrightarrow \mathbf{Pos}$$

is defined by sending an arrangement (V, L) to the ranked poset (L, cd) ordered by reverse inclusion of subspaces, where cd is the codimension function: $\mathrm{cd}(W) = \dim_k(V) - \dim_k(W)$ (see 3.3.1 later for the definition of the category $\mathbf{Pos}$ of ranked posets). A morphism $(V_1, L_1) \xrightarrow{f} (V_2, L_2)$ defines a set function $L_1 \xrightarrow{L(f)} L_2$ by $W \mapsto V(f)^{-1}(W)$, which preserves inclusions and respects intersection.

An arrangement naturally gives rise to an algebraic variety:

Definition 3.2.4 (The complement of an arrangement). *The* complement *functor $\mathcal{M} : \mathbf{Arr}_k^{op} \longrightarrow \mathbf{Var}_k$ from arrangements to the category of algebraic varieties over k (or when $k = \mathbb{C}$, to complex manifolds) is the contravariant functor that sends an arrangement (V, L) to the k-variety $V - \cup L$. The morphism of arrangements $(V_1, L_1) \xrightarrow{f} (V_2, L_2)$ induces a map $V_2 - \cup L_2 \xrightarrow{V(f)} V_1 - \cup L_1$ by restriction.*

Note that the preimage of $\cup L_1$ is contained inside $\cup L_2$, and thus the restriction is well-defined.

We are concerned with families of arrangements, and their complements, indexed by some category $\mathbf{C}$. Formally this is given by a diagram of arrangements, i.e. a functor. Throughout this chapter we denote a covariant (resp. contravariant) functor $\mathcal{F} : X \longrightarrow Y$ by $\mathcal{F}_\bullet$ (resp. $\mathcal{F}^\bullet$).

Definition 3.2.5 (C-arrangements). *Let $\mathbf{C}$ be a category. A $\mathbf{C}$-arrangement (over k) is a functor $\mathcal{A} : \mathbf{C} \longrightarrow \mathbf{Arr}_k$. We denote the compositions $V \circ \mathcal{A}$, $L \circ \mathcal{A}$ and $\mathcal{M} \circ \mathcal{A}$ by $V_{\mathcal{A}}^\bullet$, $L_\bullet^{\mathcal{A}}$ and $\mathcal{M}_{\mathcal{A}}^\bullet$ respectively. Note that the associations $c \mapsto \mathcal{A}_c \mapsto V(\mathcal{A}_c), \mathcal{M}(\mathcal{A}_c)$ are contravariant and are therefore denoted with an upper index, i.e. $\mathcal{A}_c = (V_{\mathcal{A}}^c, L_c^{\mathcal{A}})$.*

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If $\mathcal{A}$ is a $\mathbf{C}$-arrangement, we say that the underlying diagram of vector spaces $V_{\mathcal{A}}$ is continuous *if it takes pullback diagrams in $\mathbf{C}$ to push-out diagrams in $\mathbf{Vect}_k$. When this is the case, we also say that $\mathcal{A}_{\bullet}$ itself is* continuous.

We will often omit the superscript and subscript of $\mathcal{A}$ from $L^{\mathcal{A}}$ and $V_{\mathcal{A}}$ when there is no ambiguity as to which arrangement is involved.

By applying a cohomology functor H^i to the $\mathbf{C}^{op}$-variety $\mathcal{M}_{\mathcal{A}}$ we get a representation of $\mathbf{C}$, also called a $\mathbf{C}$-module, and these modules are the subject of Theorem 3.1.4. These objects are defined and studied in §2.

3.3 Steps towards representation stability

We now set out to prove Theorem 3.1.4. As discovered by Goresky-MacPherson, the cohomology of an arrangement complement is determined by the combinatorial data encoded in its partially ordered set of subspaces. We therefore start the proof by setting up the terminology involving these objects and the notion of their combinatorial stability.

3.3.1 Step 1 - Ranked posets and combinatorial stability

Definition 3.3.1 (**Ranked posets**). *The category of* finite ranked posets, *denoted by* **Pos**, *is described as follows. The objects are pairs (P, r) where P is a finite partially ordered set and r is a function $r : P \longrightarrow \mathbb{Z}$, called the* rank function, *that is strictly increasing. A morphism $(P_1, r_1) \xrightarrow{f} (P_2, r_2)$ is a set function $f : P_1 \longrightarrow P_2$ that preserves both ordering and rank, i.e. $x \leq_1 y \implies f(x) \leq_2 f(y)$ and $r_1(x) = r_2(f(x))$ for all $x, y \in P_1$.*

Note that one traditionally requires a rank function to satisfy $r(x) = r(y) + 1$ whenever x covers y, and functions as in the definition above are called a *generalized*

rank function. In our context it will be most convenient to adopt the generalized notion, as the rank will often be determined by codimension.

Recall that the homology of a poset P is defined to be the homology of its nerve, i.e. the simplicial set $\Delta(P)$ whose n-simplices are order-chains $x_0 \le x_1 \le \ldots \le x_n$ in P. Note that both the nerve $P \mapsto \Delta(P)$ and the poset homology are functors. The following concept explicitly appears in Goresky-MacPherson's formula for the cohomology of the complement of a linear subspace arrangement – see [Bj] for a discussion on the motivation for this definition.

Definition 3.3.2 (Ranked Whitney homology). *The* ranked Whitney homology *functors of a ranked poset (P, r) are defined by*

$$WH_n(P, r) := \bigoplus_{x \in P^n} \widetilde{\mathrm{H}}_{n-2}(P^{<x}; \mathbb{Z}) \qquad (3.3.1)$$

where $\widetilde{\mathrm{H}}$ stands for reduced integral homology, n is an integer, P^n is the subposet of elements with rank n, and $P^{<x} = \{y \in P \mid y < x\}$ with its induced ordering.

A morphism $(P_1, r_2) \xrightarrow{f} (P_2, r_2)$ sends every subposet $P_1^{<x}$ to $P_2^{<f(x)}$ and thus induces homomorphisms $\widetilde{\mathrm{H}}_{n-2}(P_1^{<x}) \xrightarrow{f_*} \widetilde{\mathrm{H}}_{n-2}(P_2^{<f(x)})$. The direct sum of these homomorphisms is the induced homomorphism $WH_n(P_1, r_1) \xrightarrow{f_*} WH_n(P_2, r_2)$.

Note that this definition is different from that of standard Whitney homology (appearing in e.g. [Bj]): for one, we are not assuming that our posets contain minimal and maximal elements, which are then to be removed from homology calculations. But more importantly, the direct sum of Equation 3.3.1 includes only terms of a given r-rank.

We now consider families of posets. Let $\mathbf{C}$ be some indexing category.

Definition 3.3.3. *A $\mathbf{C}$-poset is a functor $P_\bullet : \mathbf{C} \longrightarrow \mathbf{Pos}$.*

Combinatorial stability of such families of posets is defined by the following two properties.

Definition 3.3.4 (**Finite generation**). *We say that a* **C**-*poset* $P_\bullet$ *is* finitely-generated *if for every rank* $n \in \mathbb{Z}$ *there exist finitely many elements* $\{x_i \in P_{c_i}^n\}_{i=1}^k$ *whose orbits under* **C** *contain* P_d^n *for every object* d.

Note 3.3.5. If we change the rank function by composing it with an injective order preserving function $\mathbb{Z} \longrightarrow \mathbb{Z}$, the notion of being finitely-generated remains unchanged.

Definition 3.3.6 (**Downward stability**). *We say that a* **C**-*poset* $P_\bullet$ *is* downward stable *if for every morphism* $c \xrightarrow{f} d$ *in* **C** *and an element* $x \in P_c$ *the induced poset map* $P_c^{<x} \xrightarrow{f_*} P_d^{<f(x)}$ *is an isomorphism.*

We can now phrase our notion of stability for **C**-posets.

Definition 3.3.7 (**Combinatorial stability**). *The* **C**-*poset is said to exhibit* combinatorial stability *if it is both finitely-generated and downward stable.*

As the following theorem shows, combinatorial stability at the level of **C**-posets implies more familiar stability phenomena that occur in the context of representation stability: recall that an **FI**-module is representation stable in the sense of [CF] if and only if it is finitely-generated (see [CEF1]).

Theorem 3.3.8 (**Finitely-generated C-poset homology**). *If a* **C**-*poset* $(P, r)_\bullet$ *is combinatorially stable, then its Whitney homology* $WH_n(P, r)_\bullet$ *is a finitely-generated* **C**-*module for all* n.

Proof. Let n be a natural number. Since $P_\bullet$ is finitely-generated we can find a finite list of elements $x_i \in P_{c_i}^n$ for $i = 1, \ldots, l$ whose **C**-orbits contain all rank n elements of $P_\bullet$. Each of the Whitney homology groups $WH_n(P_{c_i}, r_{c_i})$ is finitely-generated, so it will suffice to show that their **C**-orbits span all other Whitney homology groups.

Let d be any object of $\mathbf{C}$ and $y \in P_d^n$. Suffice it to show that $\widetilde{\mathrm{H}}_{n-2}(P_d^{<y})$ is contained in the $\mathbf{C}$-orbits of the above groups. By our choice of $x_1, \ldots, x_l$, there exists some $1 \leq i \leq l$ and a morphism $c_i \xrightarrow{f} d$ such that $f_*(x_i) = y$. By the downward stability assumption, the induced map

$$P_{c_i}^{<x_i} \xrightarrow{f_*} P_d^{<y}$$

is an isomorphism, and therefore the induced homomorphism on homology is also an isomorphism. In particular it is surjective. $\qquad\square$

3.3.2   Step 2 - The cohomology of an arrangement complement

Goresky-MacPherson used Stratified Morse Theory to give a formula for the cohomology groups of real and complex linear subspace arrangement complements in terms of the associated intersection poset (see [GM]). Later, Björner-Ekedahl compute the ℓ-adic cohomology of the complement of a linear subspace arrangement defined over some arbitrary field k (see [BE]). Their formula coincides with the Goresky-MacPherson result for the case $k = \mathbb{C}$. The cohomology groups are given as follows.

Theorem 3.3.9 ([BE], Theorem 4.9)**.** *Let $A = (V, L)$ be a subspace arrangement and $\mathcal{M}_A$ its complement. If $\ell \neq char(k)$ is a prime number then the ℓ-adic cohomology of $\mathcal{M}_A$ is given by*

$$
\begin{aligned}
\widetilde{\mathrm{H}}_{\acute{e}t}^i(\mathcal{M}_{A/\bar{k}}; \mathbb{Q}_\ell) &\cong \bigoplus_{x \in L^A} \widetilde{\mathrm{H}}_{2\,\mathrm{cd}(x)-i-2}(\Delta(L^{<x})) \otimes \mathbb{Q}_\ell(-\mathrm{cd}(x)) && (3.3.2) \\
&= \bigoplus_{n \geq 0} W H_{2n-i}(L, 2\,\mathrm{cd} - i) \otimes \mathbb{Q}_\ell(-n) && (3.3.3)
\end{aligned}
$$

where $\mathrm{cd}(x)$ is the codimension of the subspace x in V, and $L^{<x}$ is the subposet of spaces in L that contain x. The term $\mathbb{Q}_\ell(n)$ is the n-fold tensor power of the ℓ-adic

cyclotomic character (see [Hu]).

Notation 3.3.10. In everything that follows we will abbreviate the functor $\widetilde{\mathrm{H}}^{i}_{\acute{e}t}(\cdot_{/\bar{k}};\mathbb{Q}_{\ell})$ to $\mathrm{H}^{i}(\cdot)$ – note that these always represent <u>reduced</u> cohomology groups!

The key feature of Isomorphism (3.3.2) is that it is natural, i.e. we can read off pullback maps between cohomology groups from the poset maps and the induced homomorphisms on Whitney homology. This can be seen e.g. by applying Poincaré duality to the spectral sequence described by Petersen in [Pe].

Implicit in Equation (3.3.3) is that the direct sum of Whitney homologies is finite. This observation is essential to establishing finite generation of **C**-modules that occur as cohomology of complements of **C**-arrangements. Indeed,

Lemma 3.3.11. *The direct sum decomposition for* $\mathrm{H}^{i}(\mathcal{M}_{A})$ *given in* (3.3.2) *and* (3.3.3) *includes contributions only from subspaces* $x \in L$ *of codimension* $\frac{i}{2} \leq \mathrm{cd}(x) \leq i$. *Equivalently, the only Whitney homology groups that contribute to the direct sum are the ones whose index* n *satisfies* $\frac{i}{2} \leq n \leq i$.

Proof. By Deligne's bounds [De1] the weights that occur in $\mathrm{H}^{i}_{\acute{e}t}(X;\mathbb{Q}_{\ell})$ are bounded between i and $2i$. The lemma now follows from the fact that the weight of the n-th summand of (3.3.2) is $2n$.

Alternatively, we can see this directly by elementary means, which enjoy the benefit of applying to real arrangements with successive (real) codimensions ≥ 2. For the lower bound notice that if $x \in L$ has codimension smaller than $\frac{i}{2}$, then the direct summand corresponding to x in (3.3.2) is

$$\widetilde{\mathrm{H}}_{2\,\mathrm{cd}(x)-i-2}(\Delta(L^{<x})) \otimes \mathbb{Q}_{\ell}(-\,\mathrm{cd}(x))$$

where $2\,\mathrm{cd}(x) - i - 2 \leq -2$. Since reduced homology groups are zero below degree -1,

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this summand is zero[6].

For the upper bound the claim will follow if we show that $\Delta(L^{<x})$ has non-degenerate $(2\,\mathrm{cd}(x) - i - 2)$-simplices only when $\mathrm{cd}(x) \leq i$. Suppose

$$x_0 < x_1 < \ldots < x_n\,(< x)$$

is a non-degenerate simplex in $\Delta(L^{<x})$. Then the strict monotonicity of codimension gives

$$1 \leq \mathrm{cd}(x_0) < \mathrm{cd}(x_1) < \ldots < \mathrm{cd}(x_n) < \mathrm{cd}(x)$$

and since these are all integers, $n + 1 < \mathrm{cd}(x)$. Thus the existence of a non-degenerate $(2\,\mathrm{cd}(x) - i - 2)$-simplex implies that $(2\,\mathrm{cd}(x) - i - 2) + 1 < \mathrm{cd}(x)$, or equivalently $\mathrm{cd}(x) < i + 1$. $\qquad\square$

Corollary 3.3.12 (**Combinatorially stable posets imply controlled cohomology**). *Let $\mathcal{A} : \mathbf{C} \longrightarrow \mathbf{Arr}_k$ be a $\mathbf{C}$-arrangement. If the intersection poset $L_\bullet^{\mathcal{A}}$ is combinatorially stable (i.e. finitely-generated and downward stable), then for each $i \geq 0$ the $\mathbf{C}$-module $\mathrm{H}^i(\mathcal{M}_{\mathcal{A}})_\bullet$ is finitely-generated.*

Proof. We have seen in Theorem 3.3.8 that a combinatorially stable $\mathbf{C}$-poset gives rise to finitely-generated Whitney homology groups (in every degree). Since $\mathrm{H}^i(\mathcal{M}_{\mathcal{A}})$ is naturally isomorphic to a **finite** direct sum of such homology groups, the resulting cohomology $\mathbf{C}$-module is finitely-generated. $\qquad\square$

---

6. Note that there could be a contribution in degree -1 since, by convention, $\widetilde{\mathrm{H}}_{-1}(\varnothing; \mathbb{Z}) = \mathbb{Z}$. This term occurs precisely when x is minimal in L, or equivalently not contained in any other subspace in L.

3.3.3   Step 3 - Criterion for stability

This subsection discusses properties of **C**-arrangements which, together with some structural assumptions on the category **C**, will ensure that the associated intersection poset will be combinatorially stable. First we define a notion of finite-generation for **C**-arrangements. It is this property of an arrangement that ensures the finite-generation of the associated intersection poset. Downward stability poses more of a challenge, and it will lead us to the notion of a normal **C**-arrangement.

Definition 3.3.13 (**Finitely-Generated C-arrangements**). *A* **C**-*arrangement* $\mathcal{A}_\bullet$ *is said to be* generated by the set of subspaces $\{x_\alpha \subset V^{c_\alpha}\}_{\alpha \in A}$ *if for every object d of* **C** *and every subspace $y \in L_d^{\mathcal{A}}$ there exists a finite list of morphisms $c_{\alpha_i} \xrightarrow{f_i} d$ where $1 \le i \le l$ such that*

$$y = L(f_1)x_{\alpha_1} \cap \ldots \cap L(f_l)x_{\alpha_l}. \tag{3.3.4}$$

Equivalently, if $\mathcal{A}_\bullet$ is the least **C**-*arrangement that contains all of the subspaces $\{x_\alpha\}_{\alpha \in A}$ among its chosen subspaces.*

When this is the case, we say that $\mathcal{A}_\bullet$ is generated in degrees $\{c_\alpha\}_{\alpha \in A}$. The **C**-*arrangement* $\mathcal{A}_\bullet$ *is* finitely-generated *if it is generated by some finite set of subspaces.*

Example 3.3.14. The braid **FI**-arrangement over k is generated in degree 2 by a single subspace: $\{z_1 = z_2\} \subset k^2$. See example 3.6.9 for an elaboration.

The following notation will prove useful.

Definition 3.3.15. *Let* **C** *be a category. If c and d are two objects, we say that $c \le d$ if $\mathrm{Hom}_{\mathbf{C}}(c, d) \ne \varnothing$. Moreover, we say that $c < d$ if $c \le d$ and $d \nleq c$.*

Downward stability turns out to be related to a notion of saturation of a **C**-arrangement. We make this connection precise using the following definition.

Definition 3.3.16 (Normality and primitive subspaces). *Let $\mathcal{A}_\bullet = (V^\bullet, L_\bullet)$ be a $\mathbf{C}$-arrangement.*

- $\mathcal{A}_\bullet$ *is* normal *if for every morphism $c \xrightarrow{f} d$, every subspace $x \in L_d^{\mathcal{A}}$ that contains $\ker V(f)$ is in the image of $L(f) : L_c^{\mathcal{A}} \longrightarrow L_d^{\mathcal{A}}$. Equivalently, when the direct image $V(f)x \subseteq V^c$ is a member of $L_c^{\mathcal{A}}$.*

- *A subspace $x \in L_d^{\mathcal{A}}$ is* primitive *if it does not contain the kernel of any linear map induced by a morphism $c \longrightarrow d$ where $c < d$. We define the* degree *of x to be the object d and denote $\deg(x) = d$.*

Normality and primitivity are well-behaved since we are assuming that all the subspaces in $L^{\mathcal{A}}$ contain the origin.

Example 3.3.17. To illustrate the meaning of (non)normality, consider the following example. Let $\mathbf{C} = \{0 \xrightarrow{f} 1\}$ be a category of two objects with a single morphism between them. Define a diagram of vector spaces by $V^0 = V^1 = \mathbb{C}$ with $V(f) = \mathrm{Id} : V^1 \longrightarrow V^0$ and construct a $\mathbf{C}$-arrangement $\mathcal{A}_\bullet = (V^\bullet, L_\bullet)$ by choosing

$$L_0 = \varnothing, \ L_1 = \{0\}.$$

The arrangement thus constructed is not normal, as $0 \in L_1$ is not in the image of $L(f)$ even though it could be there (and perhaps "morally should" be there). Had we chosen $L_0 = L_1$ we would have defined a normal arrangement, since then every element of L_1 that *could* lie in the image of $L(f)$ indeed appears there.

As previously declared, normality guarantees downward stability.

Lemma 3.3.18. *The intersection poset $L_\bullet^{\mathcal{A}}$ of a normal $\mathbf{C}$-arrangement $\mathcal{A}$ is downward stable.*

Proof. Throughout this proof we will denote the poset $L_c^{\mathcal{A}}$ by L_c. Suppose $c \xrightarrow{f} d$ and $x \in L_c$. We will construct an inverse to the induced poset morphism

$$L_c^{<x} \xrightarrow{L(f)} L_d^{<L(f)x}.$$

By definition of the order on L_d, every $y \in L_d^{<L(f)x}$ satisfies $L(f)(x) \subset y$ and thus

$$\ker V(f) \subseteq V(f)^{-1}(x) \subset y,$$

so by normality $V(f)y$ is a subspace in L_c. Since $V(f)$ is surjective we have $x \subset V(f)y$, so $V(f)y \in L_c^{<x}$. The inverse map to $L(f) = V(f)^{-1}$ is therefore the direct image $y \mapsto V(f)y$. $\square$

Next we show that under structural assumptions on **C** the properties defined above indeed ensure the combinatorial stability of the associated intersection poset.

Definition 3.3.19 (**Weakly filtering categories**). *A category* **C** *is* weakly filtering *if for every pair of objects c_1 and c_2 there exists a finite collection of objects $d_1, \ldots, d_k$ and morphisms $c_i \xrightarrow{f_{ij}} d_j$ where $i = 1, 2$ and $1 \leq j \leq k$, such that every pair of morphisms $c_i \xrightarrow{g_i} e$ for $i = 1, 2$ factors through one of the d_j's. Explicitly, for every pair of morphisms $c_i \xrightarrow{g_i} e$ there exists some $1 \leq j \leq k$ and a morphism $d_j \xrightarrow{g} e$ such that $g \circ f_i = g_i$ for $i = 1, 2$.*

This property is called property (F) *by Sam-Snowden in [SS2].*

Lemma 3.3.20. *If the category* **C** *is weakly filtering, then the intersection poset of every finitely-generated* **C**-*arrangement is a finitely-generated* **C**-*poset.*

Proof. Fix a finite set of generators $X = \{x_\alpha \subset V^{c_\alpha}\}_{\alpha \in A}$ and a codimension n. For every object e and a subspace $y \in L_e$ of codimension n, there exists a finite list of

morphisms $c_{\alpha_i} \xrightarrow{g_i} e$ such that

$$y = L(g_1)x_{\alpha_1} \cap \ldots \cap L(g_l)x_{\alpha_l}.$$

But since y is of codimension n, it can be written as the intersection of no more than n subspaces. Thus without loss of generality we can assume that $l \leq n$. We will show that all such intersections are in the **C**-orbits of $\bigcup_{d \in D_n} L_d$ for a fixed finite collection D_n of objects in **C**. Since every poset L_d is finite, the union $\bigcup_{d \in D_n} L_d$ has finitely many elements, and this will complete the proof.

We prove this by induction on n. For $n = 1$ it suffices to take the finite set $D_1 = \{c_\alpha\}_{\alpha \in A}$. For the induction step, suppose D_{n-1} is already defined. For every $c_1 \in D_{n-1}$ and $c_2 \in D_1$ find a finite collection of objects $d_1, \ldots, d_k$ through which every pair of morphisms factors. The set D_n will be defined to be the union of these finite lists as c_1 and c_2 range over the finite sets D_{n-1} and D_1 respectively.

We need to show that D_n satisfies the desired property. Suppose $y \in L_e$ is given by

$$y = L(g_1)x_{\alpha_1} \cap \ldots \cap L(g_l)x_{\alpha_l}$$

as above with $l \leq n$. By repeating the last term if necessary, we can assume that $l = n$. By the choice of D_{n-1} there exists some $c_1 \in D_{n-1}$ and a morphism $c_1 \xrightarrow{h_1} e$ such that the $(n-1)$-fold intersection $L(g_1)x_{\alpha_1} \cap \ldots \cap L(g_{n-1})x_{\alpha_{n-1}}$ is contained in the image of $L(h_1)$, say it is equal to $L(h_1)y_1$ for $y_1 \in L_{c_1}$. The remaining term $L(g_n)x_{\alpha_l}$ is contained in the image of $c_2 \xrightarrow{h_2} e$ for $c_2 \in D_1$, say it is equal to $L(h_2)y_2$ (explicitly, take $c_2 := c_{\alpha_n}$, $h_2 := g_n$ and $y_2 := x_n$). Thus there exists some $d \in D_n$ and morphisms $c_i \xrightarrow{f_i} d$ through which the two morphisms h_i factor, i.e. there exists $d \xrightarrow{h} e$ such that $h \circ f_i = h_i$. But then L_d contains the subspaces $L(f_1)y_1$ and $L(f_2)y_2$, so it contains their intersection, which maps to y under $L(h)$. Thus y is in the **C**-orbits of L_d, and

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D_n satisfies our assumption. □

In all of our applications, the indexing category $\mathbf{C}$ is closely related to the category $\mathbf{FI}$. In order to unify the treatment of these examples, we use the notion of a category of $\mathbf{FI}$ type from Definition 2.1.6. In short, these are categories that have pullbacks and a weakened version of push-outs (see Definition 2.2.1), and are weakly partially ordered by morphisms ($c \leq d \iff \exists c \longrightarrow d$) such that every lower interval is essentially finite. The latter finiteness condition has the following obvious but useful consequence:

Fact 3.3.21 (Descending chain condition). *Suppose that $\mathbf{C}$ is a category of $\mathbf{FI}$ type. Then every nonempty collection X of objects of $\mathbf{C}$ contains a least element, i.e. there exists some $c_0 \in X$ such that if $c \in X$ satisfies $c \leq c_0$ then $c_0 \cong c$.*

With this notion can now formulate a criterion for the combinatorial stability of the associated intersection poset.

Theorem 3.3.22 (Finite-generation implies poset stability). *Suppose $\mathbf{C}$ is a category of $\mathbf{FI}$ type and that $\mathcal{A}_\bullet$ is a finitely-generated, normal $\mathbf{C}$-arrangement. Then the induced intersection poset $L_\bullet^{\mathcal{A}}$ is combinatorially stable.*

Proof. We have already seen in Lemma 3.3.18 that a normal $\mathbf{C}$-arrangement gives rise to a downward-stable intersection poset. It thus remains to show that when $\mathbf{C}$ is of $\mathbf{FI}$ type, the $\mathbf{C}$-poset $L_\bullet^{\mathcal{A}}$ is finitely-generated.

By Lemma 3.3.20 it will suffice to show that a category of $\mathbf{FI}$ type is weakly filtering. For every triple of objects p, a_1, a_2 and a pair of maps $p \xrightarrow{\tilde{f}_i} a_i$, where $i = 1, 2$, we can form a weak push-out

$$
\begin{array}{ccc}
p & \xrightarrow{\tilde{f}_1} & a_1 \\
\tilde{f}_2 \downarrow & & \downarrow f_1 \\
a_2 & \xrightarrow{f_2} & a_1 \coprod_p a_2
\end{array}
$$

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Because **C** is of **FI** type, the automorphisms of a_i act transitively on incoming maps, therefore replacing $p \xrightarrow{\tilde{f}_i} a_i$ by some other morphism $\tilde{g}_i$ amounts to post-composing with some automorphism $\varphi_i \in G_{a_i}$. But such a replacement of $\tilde{f}_i$ with $\varphi_i \circ \tilde{f}_i$ for $i = 1, 2$ results in an isomorphic weak push-out object, with isomorphism given by the universal property as $\varphi_1 \coprod_p \varphi_2$. Thus the weak push-out $a_1 \coprod_p a_2$ is uniquely determined by p up to isomorphism.

Let c_1 and c_2 be two objects of **C**. By the definition of **FI** type, there exist only finitely many objects p that admit maps into c_1, up to isomorphism. Thus by the previous argument there are only finitely many isomorphism types of weak push-outs involving c_1 and c_2. Pick representatives for these isomorphism classes $d_j = c_1 \coprod_{p_j} c_2$ for $j = 1, \ldots, k$. Then for every pair of morphisms $c_i \xrightarrow{g_i} e$ we can form their pullback

$$
\begin{array}{ccc}
p & \xrightarrow{\tilde{f}_1} & c_1 \\
{\scriptstyle \tilde{f}_2}\big\downarrow & & \big\downarrow {\scriptstyle g_1} \\
c_2 & \xrightarrow[g_2]{} & e
\end{array}
$$

Now by the universal property of a weak push-out, there exists a (unique) morphism $c_1 \coprod_p c_2 \xrightarrow{g} e$ that satisfies $g \circ f_i = g_i$. Find $1 \leq j \leq k$ such that $d_j \cong c_1 \coprod_p c_2$, then the pair g_1 and g_2 factors through d_j via this isomorphism. We have thus shown that **C** is indeed weakly filtering, as every pair of morphisms $c_i \xrightarrow{f_i} e$ factors through one of the objects $d_1, \ldots, d_k$. This completes the proof. $\qquad \square$

Note 3.3.23 (**Explicit degrees of generators**). For the purpose of finding generators explicitly in the case of categories of **FI** type, we trace through the construction of the sets D_n from Lemma 3.3.20. D_n is constructed inductively from D_{n-1} and D_1 through the process of listing all possible weak push-outs. Thus if the **C**-arrangement $\mathcal{A}_\bullet$ is generated by subspaces of $V^{c_1}, \ldots, V^{c_n}$, the set D_n on which all codimension n

generators are obtained, is the collection of objects of the form

$$\left(\dots\left(c_{i_1}\coprod_{p_2}c_{i_2}\right)\coprod_{p_3}c_{i_3}\dots\right)\coprod_{p_n}c_{i_n}.\tag{3.3.5}$$

As discussed in Step 2 on $\mathbf{C}$-posets, a combinatorially stable intersection poset gives rise to cohomology groups that are finitely-generated as a $\mathbf{C}$-module. We have thus proved the following.

Theorem 3.3.24 (Cohomology preserves finite-generation). *If $\mathbf{C}$ is a category of $\mathbf{FI}$ type and $\mathcal{A}_\bullet$ is finitely-generated, normal $\mathbf{C}$-arrangement, then for every $i \geq 0$ the cohomology groups $\mathrm{H}^i(\mathcal{M}_\mathcal{A}; \mathbb{Q}_\ell)_\bullet$ form a finitely-generated $\mathbf{C}$-module.*

3.3.4   Step 4 - Freeness

Throughout this step we assume that $\mathbf{C}$ is a category of $\mathbf{FI}$ type and denote the group $\mathrm{Aut}(d)$ by G_d for every $\mathbf{C}$-object d. Recall Definition 2.3.1 of free and induced $\mathbf{C}$-modules over $\mathbb{Q}_\ell$: let d be an object of $\mathbf{C}$; a $\mathbf{C}$-modules is said to be an *induced module* if it is of the form

$$\mathrm{Ind}_d(V)_\bullet = \mathbb{Q}_\ell[\mathrm{Hom}(d, \bullet)] \otimes_{G_d} V \tag{3.3.6}$$

where V is some G_d-representation, and morphisms in $\mathbf{C}$ act on the tensor products naturally through their action on $\mathrm{Hom}(d, \bullet)$. The *degree* of such a $\mathbf{C}$-module is defined to be the isomorphism class $[d]$.

A *free* $\mathbf{C}$-module is a direct sum of induced modules, and its *degree* is the isomorphism class of a least object[7] d (if such exists) that is $\geq$ all the degrees of the induced modules that appear in its direct sum decomposition. If there is no such d we say that the degree is ∞.

---

7. Least with respect to the preordering between objects.

Theorem 3.3.25 (Free cohomology). *If $\mathcal{A}_\bullet$ is a continuous, normal $\mathbf{C}$-arrangement then the cohomology groups $\mathrm{H}^i(\mathcal{M}_\mathcal{A})$ form a free $\mathbf{C}$-module.*

The proof of this theorem proceeds in steps. First, we make explicit an observation that appears in the proof of §2[Lemma 3.4] and gives a more useful characterization of induced $\mathbf{C}$-modules. This uses binomial sets, whose definition is given in 2.2.4.

Lemma 3.3.26 (Structure of induced modules). *Suppose $M_\bullet$ is a $\mathbf{C}$-module of the form*

$$M_d = \oplus_{[f] \in \binom{d}{c}} V_{[f]}$$

for every object d, and that a morphism $d_1 \xrightarrow{g} d_2$ sends the factor $V_{[f]}$ to $V_{[g \circ f]}$ isomorphically. Then $M = \mathrm{Ind}_c(V)$ where $V := M_c$.

Proof. This appears as a step in the proof of Lemma 2.3.4. □

The key players in the decomposition of $\mathrm{H}^i(\mathcal{M}_\mathcal{A})_\bullet$ as a sum of induced $\mathbf{C}$-modules are primitive subspaces (see Definition 3.3.16). First we show that every normal $\mathbf{C}$-arrangement is generated by its primitive subspaces.

Lemma 3.3.27 (Primitive generators). *If $\mathcal{A}$ is a normal $\mathbf{C}$-arrangement then every subspace $x \in L_c^\mathcal{A}$ is the image of some primitive subspace.*

Proof. Let $x \in L_c$ be any subspace and define X to be the collection of all objects e for which L_e contains a preimage of x. Explicitly, $e \in X$ when there exists some $z \in L_e$ and a morphism $e \xrightarrow{f} x$ such that $L(f)z = x$.

By the descending chain condition for categories of $\mathbf{FI}$ type the set X contains a least object e_0. Choose $z_0 \in L_{e_0}$ and $e_0 \xrightarrow{f_0} c$ satisfying $L(f_0)z_0 = x$. We claim that z_0 is primitive. Indeed, if not then by definition there exists some $e_1 < e_0$ and a morphism $e_1 \xrightarrow{f_1} e_0$ for which $\ker(f_1) \subseteq z_0$. Since that arrangement $\mathcal{A}$ is assumed

to be normal, this implies that $z_1 = V(f_1)z_0 \in L_{e_1}$ is a preimage of z_0. But then z_1 is also a preimage of x, whereby we find that $e_1 \in X$. This is a contradiction to the minimality of e_0 and thus the subspace z_0 must indeed be primitive. $\qquad\square$

The crucial observation to make regarding primitive subspaces is that they cannot appear nontrivially in the image of any induced map of posets. For this reason they shed light onto the structure of $\mathbf{C}$, e.g. they detect isomorphisms. More generally, the following useful claim shows that two subspaces will never have equal images by way of a coincidence.

Lemma 3.3.28 (Subspaces with equal image). *Suppose that $\mathcal{A}_\bullet$ is a continuous $\mathbf{C}$-arrangement, $z \in L_{c_0}$ is any primitive subspace and $x \in L_{c_1}$ is any subspace. If there exists some object d and morphisms $c_i \xrightarrow{f_i} d$ such that $L(f_0)z = y = L(f_1)x$, then there exists a morphism $c_0 \xrightarrow{\varphi} c_1$ that sends z to x and satisfies $f_0 = f_1 \circ \varphi$. In particular, if x is itself primitive, the morphism φ is an isomorphism.*

The conclusion holds even when the subspace $x \subset V^{c_1}$ is not a priori assumed to belong to the arrangement.

Proof. Form the pullback of the two morphisms f_0 and f_1

$$
\begin{array}{ccc}
p & \xrightarrow{\ r_0\ } & c_0 \\
{\scriptstyle r_1}\downarrow & & \downarrow{\scriptstyle f_0} \\
c_1 & \xrightarrow[\ f_1\ ]{} & d
\end{array}
$$

and consider the corresponding diagram of vector spaces

$$
\begin{array}{ccc}
V^p & \xleftarrow{\ V(r_0)\ } & V^{c_0} \\
{\scriptstyle V(r_1)}\uparrow & & \uparrow{\scriptstyle V(f_0)} \\
V^{c_1} & \xleftarrow[\ V(f_1)\ ]{} & V^d
\end{array}
$$

By the continuity assumption on $\mathcal{A}_\bullet$ this is a push-out diagram of vector spaces.

If we can show that z contains $\ker V(r_0)$, then since z is primitive this would imply that $c_0 \leq p$ and in particular $p \xrightarrow{r_0} c_0$ is an isomorphism since morphisms weakly order objects. Using the inverse to r_0 we find a morphism $\varphi = r_1 \circ r_0^{-1} : c_0 \longrightarrow c_1$ that satisfies

$$f_1 \circ \varphi = f_1 \circ r_1 \circ r_0^{-1} = f_0.$$

In particular we have

$$L(f_1)x = y = L(f_0)z = L(f_1)L(\varphi)z.$$

The function $L(f_1)$ is injective, since it is defined to be $V(f_1)^{-1}$ for the surjective function $V(f_1)$. Thus we will see that $x = L(\varphi)z$, which will conclude the proof.

It remains to show that z contains $\ker V(r_0)$. Since all subspaces contain the origin, we have

$$\ker V(f_1) \subset V(f_1)^{-1}(x) = y = V(f_0)^{-1}(z)$$

thus it follows that z contains $V(f_0)\,(\ker V(f_1))$. The claim would then follow if we can prove that there is an inclusion

$$\ker V(r_0) \subseteq V(f_0)\,(\ker V(f_1))\,.$$

This follows from the universal property of V^p being a push-out: V^{c_1} admits a well-defined map into the quotient $V^{c_0}/V(f_0)\,(\ker V(f_1))$ by first lifting to V^d and then

mapping into V^{c_0} via $V(f_0)$, thus by the universal property there exists a map

$$V^{c_0} \longrightarrow V^{c_0}/V(f_0)\,(\ker V(f_1))$$

with $V(r_0)$ mapping down to V^p and $\exists!$

that makes the diagram commute. But this implies that $\ker V(r_0)$ maps to zero through the quotient map, so $\ker V(r_0) \subseteq V(f_0)\,(\ker V(f_1))$, as claimed.

Lastly, if x is primitive then the same argument applied in reverse shows that $c_1 \leq c_0$ as well. Thus by the weak order property, every morphism between c_0 and c_1 is an isomorphism. $\square$

We are now ready to begin proving the freeness statement. The general philosophy behind our approach is that representation stability phenomena are the linearized reflection of combinatorial stability. We will therefore demonstrate that freeness is already exhibited at the level of $\mathbf{C}$-sets. These sets are parameterized by primitive elements up to the following natural notion of equivalence.

Definition 3.3.29 (Equivalence of primitive subspaces). *If $z_i \in L_{c_i}$ are primitive subspace with $i = 1, 2$, we write $z_1 \sim z_2$ when their orbits in $L_\bullet$ under the action of $\mathbf{C}$ coincide, denoted $\mathbf{C}(z_1)_\bullet = \mathbf{C}(z_2)_\bullet$. Equivalently, $z_1 \sim z_2$ if there is a morphism $c_1 \xrightarrow{f} c_2$ for which $L(f)z_1 = z_2$.*

Denote the equivalence class of z_1 by $[z_1]$, and the set of all equivalence classes by $\mathcal{Z}$.

Lemma 3.3.30 (Freeness: set version). *Suppose $\mathcal{A}$ is a continuous, normal $\mathbf{C}$-arrangement. Then the intersection poset $L^{\mathcal{A}}$ decomposes as a disjoint union of $\mathbf{C}$-subposets corresponding to the equivalence classes of primitive subspaces of $\mathcal{A}$. Moreover, if $\mathbf{C}(z)_\bullet$ is the orbit of the primitive subspace $z \in L_c$, then $\mathbf{C}(z)_d$ decomposes*

as

$$\mathbf{C}(z)_d = \coprod_{[f] \in \binom{d}{c}} \mathbf{C}(z)_{[f]}$$

where $\mathbf{C}(z)_{[f]}$ is the set of images of z under maps induced by morphisms $c \xrightarrow{f} d$ in the equivalence class $[f]$. Lastly, every morphism $d_1 \xrightarrow{g} d_2$ induces a bijection $\mathbf{C}(z)_{[f]} \xrightarrow{L(g)} \mathbf{C}(z)_{[g \circ f]}$.

Proof. For every primitive subspace $z \in L_c$ we consider its $\mathbf{C}$-orbit, i.e. the $\mathbf{C}$-subposet of $L_{\mathcal{A}}$ described by

$$\mathbf{C}(z)_d = \{L(f)z \mid f \in \mathrm{Hom}_{\mathbf{C}}(c, d)\}.$$

This is clearly closed under the action of $\mathbf{C}$ on $L_\bullet^{\mathcal{A}}$. We decompose this further as

$$\mathbf{C}(z)_d = \bigcup_{[f] \in \binom{d}{c}} \mathbf{C}(z)_{[f]}$$

where $\mathbf{C}(z)_{[f]} = \{L(f)z \mid f \in [f]\}$. Again, it is clear that the poset map induced by $d \xrightarrow{g} e$ takes the set $\mathbf{C}(z)_{[f]}$ into $\mathbf{C}(z)_{[g \circ f]}$. We start by showing that this union is in fact disjoint. Suppose that there exists some $x \in \mathbf{C}(z)_{[f_0]} \cap \mathbf{C}(z)_{[f_1]}$. Then by Lemma 3.3.28 it follows that there exists a morphism $c \xrightarrow{\varphi} c$ that satisfies $f_1 = f_0 \circ \varphi$. But since $\varphi \in \mathrm{Hom}_{\mathbf{C}}(c, c) = G_c$ we find that $f_1 \sim f_0$.

Next we show that for every morphism $c \xrightarrow{f} d$ then induced map

$$\mathbf{C}(z)_{[\mathrm{Id}_c]} \xrightarrow{L(f)} \mathbf{C}(z)_{[f]}$$

is a bijection. This will prove that all morphisms $d \xrightarrow{g} e$ indeed induce bijections $\mathbf{C}(z)_{[f]} \xrightarrow{L(g)} \mathbf{C}(z)_{[g \circ f]}$. We define the inverse map as follows. Consider the induced linear surjection $V_d \xrightarrow{V(f)} V_c$. The inverse function to $L(f) = V(f)^{-1}$ is the direct

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image under $V(f)$. To see that this indeed provides an inverse, note that since $V(f)$ is surjective it follows that for all $x \in \mathbf{C}(z)_{[\mathrm{Id}_c]}$

$$ x = V(f)\left(V(f)^{-1}x\right) = V(f)\left(L(f)x\right) $$

and for the reverse composition, if $y \in \mathbf{C}(z)_{[f]}$ then there exists some morphism $f' \in [f]$ for which $L(f')z = y$. Since $f \sim f'$, there exists some $\varphi \in G_c$ such that $f' = f \circ \varphi$. Denote $x = L(\varphi)z$ and observe that $L(f)x = L(f')z = y$. It now follows that

$$ y = L(f)x = L(f)V(f)\left(L(f)x\right) = L(f)\left(V(f)y\right). $$

Suppose that there exists some $x \in \mathbf{C}(z_0)_{[f_0]} \cap \mathbf{C}(z_1)_{[f_1]}$, i.e. there exist two morphisms f_i for $i = 1, 2$ such that $L(f_0)z_0 = x = L(f_1)z_1$. By Lemma 3.3.28 there exists an isomorphism $c_0 \xrightarrow{\varphi} c_1$ taking z_0 to z_1. This shows that the two orbits $\mathbf{C}(z_0)$ and $\mathbf{C}(z_1)$ coincide.

Lastly, we need to show that $L_\bullet$ is a disjoint union of such $\mathbf{C}$-sets $\mathbf{C}(z)_\bullet$. Lemma 3.3.27 asserts that every subspace $x \in L_d$ is the image of some primitive subspace, and thus it belongs to one of the $\mathbf{C}$-sets $\mathbf{C}(z)_\bullet$ described here. For their disjointness, assume that $z_i \in L_{c_i}$ for $i = 1, 2$ are two primitive subspaces such that there exists some $x \in \mathbf{C}(z_0)_{[f_0]} \cap \mathbf{C}(z_1)_{[f_1]}$, i.e. there exist two morphisms f_i for $i = 1, 2$ such that $L(f_0)z_0 = x = L(f_1)z_1$. By Lemma 3.3.28 there exists an isomorphism $c_0 \xrightarrow{\varphi} c_1$ taking z_0 to z_1, demonstrating that the two orbits $\mathbf{C}(z_0)_\bullet$ and $\mathbf{C}(z_1)_\bullet$ coincide. $\square$

We can now proceed with the final step of the proof: showing that the cohomology groups of a normal $\mathbf{C}$-arrangement are free.

Proof of Theorem 3.3.25. Recall that Formula (3.3.2) states that

$$\mathrm{H}^i(\mathcal{M}_\mathcal{A})_c = \bigoplus_{x \in L_c} \widetilde{\mathrm{H}}_{2\,\mathrm{cd}(x)-i-2}(\Delta(L_c^{<x}))$$

and that a morphism $c \xrightarrow{f} d$ acts on these expressions through the induced iso-morphism of posets $L_c^{<x} \xrightarrow{L(f)} L_d^{<L(f)x}$. For the sake of brevity we denote the i-th cohomology group of $\mathcal{M}_\mathcal{A}^d$ by H_d^i and its summand $\widetilde{\mathrm{H}}_{2\,\mathrm{cd}(x)-i-2}(\Delta(L_c^{<x}))$ by $\mathrm{H}^i(x)$. In this notation, the map f_* on cohomology, induced by a morphism $c \xrightarrow{f} d$, maps $\mathrm{H}^i(x)$ isomorphically onto $\mathrm{H}^i(L(f)x)$.

Let $L_\bullet = \coprod_{[z]\in\mathcal{Z}} \mathbf{C}(z)_\bullet$ be the disjoint union decomposition described in the set version of the statement, Lemma 3.3.30. Then the cohomology decomposes as a direct sum of $\mathbf{C}$-submodules:

$$\mathrm{H}_d^i = \bigoplus_{[z]\in\mathcal{Z}} \left(\bigoplus_{x\in\mathbf{C}(z)_d} \mathrm{H}^i(x) \right) =: \bigoplus_{[z]\in\mathcal{Z}} M_d^{[z]} \tag{3.3.7}$$

We claim that for every $[z] \in \mathcal{Z}$ the corresponding direct summand $M_\bullet^{[z]}$ is an induced module, hence $\mathrm{H}_\bullet^i$ is free.

Indeed, the direct sum decomposes further as

$$M_d^{[z]} = \bigoplus_{x\in\mathbf{C}(z)_d} \mathrm{H}^i(x) = \bigoplus_{[f]\in\binom{d}{c}} \left(\bigoplus_{x\in\mathbf{C}(z)_{[f]}} \mathrm{H}^i(x) \right) =: \bigoplus_{[f]\in\binom{d}{c}} M_{[f]}^{[z]} \tag{3.3.8}$$

Every morphism $d \xrightarrow{g} e$ takes the set $\mathbf{C}(z)_{[f]}$ bijectively onto $\mathbf{C}(z)_{[g\circ f]}$, and for every element $x \in \mathbf{C}(z)_{[f]}$ the map $\mathrm{H}^i(x) \xrightarrow{g} \mathrm{H}^i(L(g)x)$ is an isomorphism. Thus g_* maps the summand $M_{[f]}^{[z]}$ isomorphically onto $M_{[g\circ f]}$. Thus according to the characterization of induced modules given in Lemma 3.3.26 this is indeed an induced $\mathbf{C}$-module. If

$z \in L_c$ is a representative primitive subspace of the class $[z]$ then $M_\bullet^{[z]}$ has degree c and is generated by the G_c-representation

$$M_c^{[z]} = \bigoplus_{x \in \mathbf{C}(z)_c} \mathrm{H}^i(x) = \mathrm{Ind}_{\mathrm{Stab}(z)}^{G_c} \mathrm{H}^i(z) \tag{3.3.9}$$

$\square$

For the purpose of keeping track of the degree of the free $\mathbf{C}$-module $\mathrm{H}^i(\mathcal{M}_\mathcal{A})_\bullet$, observe that the degrees of the induced modules that appear in its direct sum decomposition are the objects on which primitive subspaces are defined. Since only primitive subspaces z of codimension $\frac{i}{2} \leq \mathrm{cd}(z) \leq i$ contribute to the cohomology groups, the degrees range only over objects that carry primitive subspaces with codimension in this range. Furthermore, from Note 3.3.23 we know that if the $\mathbf{C}$-arrangement $\mathcal{A}$ is generated in degrees $c_1, \ldots, c_n$, then all codimension-($\leq i$) primitive subspaces are in the image of iterated weak push-outs of at most i many objects from this list. The following definition will make referring to the resulting degrees easier.

Definition 3.3.31. *If c and d are two objects of $\mathbf{C}$, let $c + d$ denote a minimal (isomorphism class of) object that satisfies*

$$c + d \geq c \coprod_p d$$

for every weak push-out of c and d.

Similarly if $i \in \mathbb{N}$, let $i \times c$ denote a minimal (isomorphism class of) object that satisfies

$$i \times c \geq c \coprod_{p_1} \cdots \coprod_{p_{i-1}} c$$

for every i-fold weak push-out of c.

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Such a minimal objects exist in a category of **FI** type because of the descending chain condition. In fact, in all of our examples these objects are uniquely determined and can be identified explicitly: it will be given by weak coproduct $c + d = c \coprod_\varnothing d$ and $i \times c = c \coprod_\varnothing \ldots \coprod_\varnothing c$.

Using this notation we succinctly bound the degree of the resulting **C**-modules.

Corollary 3.3.32 (Bound on degree). *If $\mathcal{A}_\bullet$ is a continuous, normal **C**-arrangement generated in degrees $\leq c$. Then the degree of the free **C**-module $\mathrm{H}^i(\mathcal{M}_\mathcal{A})$ is $\leq i \times c$.*

Proof. By the comment made in Note 3.3.23, if $\mathcal{A}_\bullet$ is generated in degrees

$$c_1, \ldots, c_n \leq c$$

then the **C**-module $\mathrm{H}^i(\mathcal{M}_\mathcal{A})$ is generated in degrees given by their i-fold iterated weak push-outs. It is easy to verify that the definition of weak push-outs implies that for all $p \leq c_i, c_j$ we have a relation

$$c_i \coprod_p c_j \ \leq \ c \coprod_p c \underset{\text{def.}}{\leq} \ c + c = 2 \times c$$

Then by induction we see that $i \times c$ is greater than all i-fold weak push-outs of the objects $c_1, \ldots, c_n$. In particular, all the primitive generators of $\mathrm{H}^i_\bullet$ must appear in degrees $\leq i \times c$. $\qquad\square$

This concludes the proof that the cohomology groups form a free **C**-module, and thus Theorem 3.1.4 is proved.

3.4   Normalization and a criterion for normality

The normality assumption in Theorem 3.1.4 is meant to exclude cases where subspaces that could appear early in the **C**-arrangement are omitted for some reason and only appear later. A normal **C**-arrangement is saturated in the sense that every subspace that "should" belong to it actually does.

We saw earlier in Lemma 3.3.27 that a normal **C**-arrangement is generated by primitive subspaces. The theorem we now state provides a converse. It also serves as an easily verifiable criterion for checking normality.

Theorem 3.4.1 (Primitive generators imply normality). *Suppose* **C** *has pullbacks and* $\mathcal{A}_\bullet$ *is a continuous* **C**-*arrangement. If* $\mathcal{A}_\bullet$ *is generated by primitive subspaces then it is a normal* **C**-*arrangement.*

Proof. Suppose $Z = \{z_\alpha \subset V^{c_\alpha}\}_{\alpha \in A}$ is a set of primitive subspaces that generates $\mathcal{A}_\bullet$. Let $d \xrightarrow{g} e$ be a morphism and let $y \in L_e$ be a subspace that contains $\ker V(g)$. We need to show that y is in the image of $L(g)$.

By assumption Z generates the **C**-arrangement, thus there exist morphisms $c_{\alpha_i} \xrightarrow{f_i} e$ where $1 \leq i \leq l$ such that

$$y = L(f_1)z_{\alpha_1} \cap \ldots \cap L(f_l)z_{\alpha_l}.$$

Note that for every $1 \leq i \leq l$ it follows that $\ker V(f_i) \subseteq L(f_i)z_{\alpha_i}$.

We prove the claim by induction on l. For $l = 1$ the subspace $y \in L_e$ is the image of a primitive subspace $z \in L_c$ under some morphism $c \xrightarrow{f} e$. Consider the direct image $x := V(g)y$. Since $\ker V(g) \subset y$, it follows that $V(g)^{-1}x = y$. Thus by Lemma 3.3.28 there exists a morphism $c \xrightarrow{\varphi} d$ that takes z to x. In particular $x \in L_d$ and $L(g)x = y$, as desired.

Now for the induction step. Assume that $y = y_1 \cap y_2$ where each is the intersection of less than l many images of primitive subspaces. Since $\ker V(g) \subset y_1, y_2$ the induction hypothesis implies that y_1 and y_2 are in the image of $L(g)$. But since $L(g)$ respects intersection y is also in the image of $L(g)$. This completes the proof. $\square$

The result stated in Theorem 3.1.4 does not apply to **C**-arrangements that are not normal. However, even in the general case the result holds in the limit as the objects c become sufficiently large. This follows from the following construction which we call *normalization.*

Theorem 3.4.2 (**Normalization of a C-arrangement**). *Suppose **C** is a category of **FI** type and $\mathcal{A}_\bullet$ is a continuous, finitely-generated **C**-arrangement, with underlying diagram of vector spaces $V_\mathcal{A}^\bullet$. Then there exists a unique finitely-generated, normal **C**-arrangement $\overline{\mathcal{A}}_\bullet$ also defined on $V_\mathcal{A}^\bullet$ that coincides with $\mathcal{A}_\bullet$ on a full subcategory that is upward closed and cofinal in **C**. This normal **C**-arrangement $\overline{\mathcal{A}}_\bullet$ will be called the* normalization of $\mathcal{A}$.

Proof. For the uniqueness statement, it suffices to show that any two normal **C**-arrangements, whose underlying diagram of vector spaces is $V^\bullet$, and that coincide on a cofinal subcategory, are equal. Indeed, suppose $\mathcal{B}$ and $\mathcal{B}'$ are two such arrangements with corresponding intersection poset $L_\bullet$ and $L_\bullet'$ resp. Let $x \in L_c$ be any subspace and $c \le d$ is an object such that $L_d = L_d'$. Pick a morphism $c \xrightarrow{f} d$, then $y = L(f)x \in L_d$ contains $\ker(V(f))$. Thus since $\mathcal{A}'$ is normal and $y \in L_d'$, the subspace $V(f)y = x$ belongs to L_c'. This shows that $L_c \subseteq L_c'$ for every object c. The same argument gives the opposite inclusion.

For existence suppose $K = \{x_i \subset V^{c_i}\}_{i=1}^n$ is a set of generators for $\mathcal{A}_\bullet$. We will define a new arrangement by specifying a generating set $\overline{K}$ as follows. For every i let X_i be the collection of objects e for which there exists a morphism $e \xrightarrow{f} c_i$ with

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$\ker(V(f)) \subseteq x_i$. By the descending chain condition X_i contains a least object e_i that possesses such a morphism $e_i \xrightarrow{f} c_i$. Denote the image $V(f)x_i \subset V^{e_i}$ by z_i and observe that since $\ker(V(f)) \subseteq x_i$ we get an equality

$$x_i = V(f)^{-1}V(f)x_i = V(f)^{-1}z_i.$$

We claim that z_i is primitive, i.e. it does not contain $\ker V(g)$ for any $e \xrightarrow{g} e_i$ with $e < e_i$. Indeed, if $e \xrightarrow{g} e_i$ is a morphism and $\ker V(g) \subseteq z_i$ then it follows that

$$x_i = V(f)^{-1}z_i \supseteq V(f)^{-1}\ker V(g) = \ker(V(g) \circ V(f)) = \ker V(f \circ g)$$

so by definition $e \in X_i$. But this is a contradiction to the minimality of e_i in X_i.

Define a new set of generators $\overline{K} = \{z_1, \ldots, z_n\}$ and let $\overline{\mathcal{A}}_\bullet$ be the **C**-arrangement generated by them: the underlying diagram of vector spaces is the same as that of $\mathcal{A}_\bullet$, and the intersection poset at an object d is made up of all the subspaces of the form

$$V(g_1)^{-1}(z_{i_1}) \cap \ldots \cap V(g_l)^{-1}(z_{i_l})$$

for an l-tuple of morphisms $e_{i_j} \xrightarrow{g_j} d$ and $l \in \mathbb{N}$. It is straightforward to check that this indeed produced a **C**-arrangement, and by construction it is generated by the set $\overline{K}$ of primitive subspaces. Lemma 3.4.1 then shows that this arrangement is normal.

We claim that $\mathcal{A}$ is a subarrangement of $\overline{\mathcal{A}}$ and that the two coincide on all objects $d \geq c_1, \ldots, c_n$ (this is clearly an upward-closed and cofinal subcategory). For the first claim, note that for every i there is a morphism $e_i \xrightarrow{f_i} c_i$ such that $L(f_i)z_i = x_i$. Thus $x_i \in L_{c_i}^{\overline{\mathcal{A}}}$, and since these subspaces generate $\mathcal{A}$ we have containment $L_d^{\mathcal{A}} \subseteq L_d^{\overline{\mathcal{A}}}$ for every object d. Conversely, suppose d admits maps from $c_1, \ldots, c_n$. It will suffice to show that every morphism $e_i \xrightarrow{g} d$ factors as $e_i \xrightarrow{f_i} c_i \xrightarrow{h} d$ for some h, for then the

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images of z_i coincide with the images of x_i in $L_d^{\overline{\mathcal{A}}}$. Indeed, pick any morphism $c_i \xrightarrow{h_0} d$ and consider the two morphisms $e_i \xrightarrow{h_0 \circ f_i} d$ and $e_i \xrightarrow{g} d$. Since $G_d := \mathrm{Aut}_{\mathbf{C}}(d)$ acts transitively on incoming morphisms we can find an automorphism $\varphi \in G_d$ such that $\varphi \circ h_0 \circ f_i = g$. Set $h = \varphi \circ h_0$, it satisfies $h \circ f_i = g$ as desired. $\qquad\square$

Note 3.4.3. For concreteness' sake we reiterate that if $\mathcal{A}$ is generated in degrees $\leq c$, then a full subcategory on which $\mathcal{A}$ coincides with its normalization is made up of objects d that satisfy $d \geq c$.

As immediate corollaries we find that the results of Theorem 3.1.4 apply to general finitely-generated arrangements in degrees larger than those of the generating subspaces. The combinatorial version of this observation is the following.

Theorem 3.4.4 (Limiting combinatorial stability). *Suppose $\mathbf{C}$ is a category of* **FI** *type and $\mathcal{A}$ is a continuous, finitely-generated $\mathbf{C}$-arrangement, generated in degrees $\leq c$. Then the intersection poset of $\mathcal{A}$ coincides with a combinatorially stable $\mathbf{C}$-poset on the full subcategory of objects $d \geq c$. Namely, it coincides with the intersection poset of the normalization $\overline{\mathcal{A}}$.*

The representation theoretic version is the following.

Theorem 3.4.5 (Limiting freeness of cohomology). *If $\mathbf{C}$ and $\mathcal{A}$ are as in Theorem 3.4.4, the cohomology groups $\mathrm{H}^i(\mathcal{M}_{\mathcal{A}})$ coincide with a finitely-generated, free $\mathbf{C}$-module on the full subcategory of objects $d \geq c$.*

3.5   Cohomological stability of arrangement quotients

Let $\mathbf{C}$ be a category of **FI** type and let $\mathcal{A}$ be a $\mathbf{C}$-arrangement. For every object c the group G_c acts on the variety $\mathcal{M}_{\mathcal{A}}^c$ and we can form the orbit space (or scheme)

$\mathcal{M}_\mathcal{A}^c/G_c$. We denote the collection of quotient spaces by $\mathcal{M}_\mathcal{A}^\bullet/G_\bullet$ (even though they do not in fact form a functor).

Let us consider families of sheaves that arise from $\mathbf{C}$-modules via the following process. Fix some $\mathbf{C}$-module $N_\bullet$. For every object d one can form the constant sheaf $\bar{N}_d$ on the space $\mathcal{M}_\mathcal{A}^d$. Now the pairs $(\mathcal{M}_\mathcal{A}^d, \bar{N}_d)$ fit together naturally into a $\mathbf{C}^{op}$-diagram of spaces plus a sheaf on each space. In particular one can apply the sheaf cohomology functor to this collection and get a $\mathbf{C}$-module.

Pushing the sheaf $\bar{N}_d$ forward to the quotient $\mathcal{M}_\mathcal{A}^d \xrightarrow{q^d} \mathcal{M}_\mathcal{A}^d/G_d$, the sheaf $q_*^d(\bar{N}_d)$ now admits a G_d-action.

Definition 3.5.1 (Twisted sheaf induced by a C-module). *The subsheaf of G_d-invariant sections $q_*^d(\bar{N}_d)^{G_d}$ will be called the twisted N_d-sheaf on $\mathcal{M}_\mathcal{A}^d/G_d$ and it will be denoted by $\widetilde{N}_d$.*

Remark 3.5.2. When G_d acts on the space $\mathcal{M}_\mathcal{A}^d$ freely, this construction yields (the sheaf analog of) the familiar Borel construction of a flat vector bundle $\mathcal{M}_\mathcal{A}^d \times_{G_d} N_d$. Otherwise $\widetilde{N}_d$ will be a constructible sheaf whose stalks might be smaller than N_d. One could check that for every point $x \in \mathcal{M}_\mathcal{A}^d/G_d$ there is an isomorphism

$$(\widetilde{N}_d)_x \cong N_d^{\mathrm{Stab}_{G_d}(\tilde{x})}$$

where $\tilde{x} \in (q^d)^{-1}(x)$ and the group $\mathrm{Stab}_{G_d}(\tilde{x}) \subset G_d$ is the stabilizer of $\tilde{x}$. Different choices of points $\tilde{x}$ will produce different isomorphisms.

We claim that when $N_\bullet$ is a free $\mathbf{C}$-module, cohomological stability holds with these systems of twisted coefficients.

Theorem 3.5.3 (Twisted cohomological stability). *Suppose $\mathbf{C}$ is a category of* **FI**-*type with $|\mathrm{Aut}_\mathbf{C}(d)| < \infty$ for every object d. Let $\mathcal{A}_\bullet$ be a continuous, normal $\mathbf{C}$-arrangement, generated in degree $\leq c$. Let $N_\bullet$ be a free $\mathbf{C}$-module over $\mathbb{Q}$ and suppose*

that it is generated in degree $\leq c'$. Then the sheaf cohomology groups

$$\mathrm{H}^i(\mathcal{M}_{\mathcal{A}}^d/G_d; \widetilde{N}_d)$$

exhibit cohomological stability in the following sense: if $d \leq e$ then there is a well-defined injective map

$$\mathrm{H}^i(\mathcal{M}_{\mathcal{A}}^d/G_d; \widetilde{N}_d) \hookrightarrow \mathrm{H}^i(\mathcal{M}_{\mathcal{A}}^e/G_e; \widetilde{N}_e)$$

and these maps become isomorphisms when $d \geq (i \times c) + c'$.

In particular, when considering the trivial $\mathbf{C}$-module $N_\bullet \equiv \mathbb{Q}_\ell$ (free of degree 0), this yields a classical cohomological stability statement for $\mathrm{H}^i(\mathcal{M}_{\mathcal{A}}^d/G_d)$ in the range $d \geq i \times c$.

In the context of ℓ-adic cohomology one needs $N_\bullet$ to take values in continuous $\mathbb{Q}_\ell$-modules.

Proof. For every object d let i_d denote the inclusion $\widetilde{N}_d = (q_*^d \bar{N}_d)^{G_d} \hookrightarrow q_*^d \bar{N}_d$. In the other direction define a transfer morphism $\widetilde{N}_d \xleftarrow{\tau_d} q_*^d \bar{N}_d$ by $\tau_d = \frac{1}{|G_d|} \sum_{g \in G_d} g(\cdot)$. Clearly the composition $\tau_d \circ i_d$ is the identity map on $\widetilde{N}_d$ and the reverse composition $i_d \circ \tau_d$ is the projection onto the G_d-invariants of $q_*^d \bar{N}_d$ which are also the G_d-coinvariants.

Consider the induced maps on the sheaf cohomology

$$\mathrm{H}^i(\mathcal{M}_{\mathcal{A}}^d/G_d; \widetilde{N}_d) \overset{\tau_d}{\underset{i_d}{\leftrightarrows}} \mathrm{H}^i(\mathcal{M}_{\mathcal{A}}^d/G_d; q_*^d \bar{N}_d) = \mathrm{H}^i(\mathcal{M}_{\mathcal{A}}^d; \bar{N}_d)$$

They induce a natural isomorphism $\mathrm{H}^i(\mathcal{M}_{\mathcal{A}}^d/G_d; \widetilde{N}_d) \cong \mathrm{H}^i(\mathcal{M}_{\mathcal{A}}^d; \bar{N}_d)_{G_d}$ where the latter is the coinvariant quotient.

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By the Universal Coefficients Theorem there is a natural isomorphism

$$\mathrm{H}^i(\mathcal{M}_{\mathcal{A}^d}; \bar{N}_d) \cong \mathrm{H}^i(\mathcal{M}_{\mathcal{A}^d}) \otimes N_d \tag{3.5.1}$$

and as d ranges over all objects this is the tensor product of two free $\mathbf{C}$-modules of respective degrees $\leq i \times c$ and c'. The analysis of free $\mathbf{C}$-modules, found in §2, applies to this exact case: Parts (1-2) of Theorem 2.1.11 implies that the tensor product in (3.5.1) is again free of degree $\leq (i \times c) + c'$, and Part (4) of Theorem 2.1.11 then shows that the coinvariant quotients stabilize in the desired sense. $\square$

Often, one is interested in cohomological stability with coefficients in the various sequences of representations, e.g. the irreducibles V_λ in the case of S_n. Such a sequence is a natural candidate for a homological stability statement e.g. if its characters are given by a single character polynomial (see Definition 2.2.5 for a general treatment). In this case, when one is interested only in the dimension of the sheaf cohomology groups a stronger stability statement can be phrased.

Theorem 3.5.4 (Stabilization of twisted Betti numbers). *If $\mathcal{A}$ is as in Theorem 3.5.3 and $N_\bullet$ is any $\mathbf{C}$-module (continuous over $\mathbb{Q}_\ell$) whose character coincides with a character polynomial of degree $\leq d$ ($N_\bullet$ need not be free), then the dimensions of the sheaf cohomology groups*

$$\dim_k \mathrm{H}^i(\mathcal{M}_{\mathcal{A}^e}/G_e; \widetilde{N}_e)$$

do not depend on e for all $e \geq (i \times c) + d$.

Remark 3.5.5. In Theorem 3.5.4 there is no reference to the structure of $N_\bullet$ other than its character. For example, all morphisms $d \xrightarrow{f} e$ for $d < e$ might induce the zero map, and this will not be detected by the character. In particular, even though the cohomology groups eventually have the same dimension, there is no hope

of finding natural isomorphisms $H^i(\mathcal{M}^d_{\mathcal{A}}/G_d; \widetilde{N}_d) \xrightarrow{f_*} H^i(\mathcal{M}^e_{\mathcal{A}}/G_e; \widetilde{N}_e)$ coming from the structure of $N_\bullet$.

Proof. The argument in the proof of Theorem 3.5.3 above shows that

$$H^i(\mathcal{M}^d_{\mathcal{A}}/G_d; \widetilde{N}_d) \cong \left(H^i(\mathcal{M}_{\mathcal{A}^d}) \otimes_k N_d \right)_{G_d} \tag{3.5.2}$$

and the dimension of this coinvariant quotient is given by the G_d-inner product of characters

$$\dim_k H^i(\mathcal{M}^d_{\mathcal{A}}/G_d; \widetilde{N}_d) = \langle H^i(\mathcal{M}_{\mathcal{A}^d})^*, N_d \rangle_{G_d}. \tag{3.5.3}$$

Part (3) of Theorem 2.1.11 shows that the pointwise dual of a free **C**-module can also be given the structure of a free **C**-module, and Theorem 2.1.10 shows that the character of such is given by a character polynomial of the same degree. With these facts we see that the character of $H^i(\mathcal{M}_{\mathcal{A}^d})^*$ is given by a character polynomial Q of degree $\leq i \times c$. By assumption there exists some character polynomial P of degree d that coincides with the character of $N_\bullet$. Therefore the above inner product of characters is given by the inner product $\langle Q, P \rangle_{G_e}$ which stabilizes for all $e \geq \deg(Q) + \deg(P^*) = (i \times c) + d$ by Corollary 2.4.7. $\qquad \square$

3.6 Applications

In all of the following examples we consider complex varieties, i.e. we take $k = \mathbb{C}$. However, it should be noted that the same results hold in positive characteristic as well. Moreover, if the subspace arrangements are defined over $\mathbb{Z}$, the **C**-modules we get are naturally isomorphic for any characteristic, so in this sense we might as well concentrate on the complex version of the statements.

Consider the category **FI** of finite sets and injections. Every finite set is isomorphic

to a unique set of the form

$$n := \{0, \ldots, n-1\}$$

and the endomorphisms of n are the symmetric group on n letters, S_n. Furthermore we consider the categorical power $\mathbf{FI}^m$ where $m \in \mathbb{N}$. Every object in this category is isomorphic to a unique m-tuple $\bar{n} := (n^{(1)}, \ldots, n^{(m)})$, and the automorphism group of $\bar{n}$ is the product of symmetric groups $S_{\bar{n}} := S_{n^{(1)}} \times \ldots \times S_{n^{(m)}}$. The $+$ and $\times$ operations on objects coincide in this case with coordinatewise addition and multiplication.

Definition 3.6.1 (The $\mathbf{FI}^{op}$-vector space $V^\bullet$). *Fix some finite dimensional complex vector space V. We consider the $\mathbf{Set}^{op}$-vector space $V^\bullet : c \mapsto \mathrm{Hom}_{\mathbf{Set}}(c, V)$, i.e. V^c is the $\mathbb{C}$-vector space of functions from c to V with pointwise operations.*

This is a continuous contravariant functor from sets to vector spaces. Moreover, since in $\mathbf{Set}$ every injection has a retraction, the functor $V^\bullet$ sends injections to surjective linear maps. Restricting to the subcategory $\mathbf{FI}$ we get an $\mathbf{FI}^{op}$-vector space with all induced maps surjective.

Definition 3.6.2 (The $\mathbf{FI}^m$-vector space $V^\bullet$). *Embedding $\mathbf{FI}^m$ into $\mathbf{Set}$ naturally by considering $(A_1, \ldots, A_m) \mapsto A_1 \coprod \ldots \coprod A_m$, we turn $V^\bullet$ into a $(\mathbf{FI}^m)^{op}$-vector space.*

The embedding $\mathbf{FI}^m \hookrightarrow \mathbf{Set}$ sends every morphism to an injective function, and furthermore takes pullbacks and (weak) push-outs in $\mathbf{FI}^m$ respectively to pullbacks and push-outs in $\mathbf{Set}$. In turn we find that $V^\bullet$ respects the pullbacks and weak push-outs of $\mathbf{FI}^m$ and sends every morphism to a surjective linear map of vector space. Thus $V^\bullet$ can serve as a continuous underlying diagram of vector spaces for $\mathbf{FI}^m$-arrangements.

Note 3.6.3. Unpacking the definition of $V^\bullet$ we see that its value at $(n^{(1)}, \ldots, n^{(m)})$ is

$$V^{n^{(1)} \coprod \cdots \coprod n^{(m)}} \cong V^{n^{(1)}} \times \ldots \times V^{n^{(m)}}.$$

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The automorphism group $S_{\bar{n}} = S_{n^{(1)}} \times \ldots \times S_{n^{(m)}}$ acts on this product through the action of every $S_{n^{(i)}}$ permuting the order of coordinates in $V^{n^{(i)}}$.

Using Criterion 3.4.1 for the normality of **C**-arrangements we get the following result.

Corollary 3.6.4 (Producing normal FI^m-arrangements). *Let $\mathcal{A}_\bullet$ be an FI^m-arrangement whose underlying diagram of vector spaces is $V^\bullet$ for some V and that is generated by primitive subspaces. Then $\mathcal{A}_\bullet$ is normal.*

Theorem 3.1.4 then implies that for every $i \geq 0$ the **FI**^{m}-module $\mathrm{H}^i(\mathcal{M}_\mathcal{A})_\bullet$ exhibits representation stability – see §2[Theorem 6.13] for concrete representation-theoretic consequences of this fact.

A general example to which this theory applies is the following. All later examples will be instances of this general case.

Example 3.6.5 (The arrangement $\mathcal{M}_{m,k}^\bullet$ and $X_{m,k}^\bullet$). Let V be a finite-dimensional complex vector space. Fix a pair of natural numbers (m, k) and consider the **FI**^{m}-arrangement $\mathcal{A}_\bullet^{m,k}(V)$ whose underlying diagram of vector spaces is $V^\bullet$ and is generated by the diagonal line in $V^k \times \ldots \times V^k = V^{km}$:

$$\Delta_{(k,\ldots,k)} = \left\{ \left((z_1^{(1)}, \ldots, z_k^{(1)}), \ldots, (z_1^{(m)}, \ldots, z_k^{(m)}) \right) \mid z_{j_1}^{(i_1)} = z_{j_2}^{(i_2)} \; \forall i_1, i_2, j_1, j_2 \right\}$$

The preimage of $\Delta_{(k,\ldots,k)}$ under an injection $(k, \ldots, k) \xrightarrow{\bar{f}} (n^{(1)}, \ldots, n^{(m)})$ is the subspace of $V^{n^{(1)}} \times \ldots \times V^{n^{(m)}}$ defined by the equations

$$z_{f_{i_1}(j_1)}^{(i_1)} = z_{f_{i_2}(j(2))}^{(i_2)}$$

for all $1 \leq i_1, i_2 \leq m$ and $1 \leq j_1, j_2 \leq k$. In other words, this is the subspace in which the coordinates specified by $\bar{f}$ are all equal. As we let $\bar{f}$ range over all injections, we

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see that the induced arrangement on $V^{\bar{n}}$ is made up of precisely the tuples in which there exists some $z \in V$ that <u>appears in every $V^{n^{(i)}}$ factor at least k times.</u>

The $S_{\bar{n}}$-quotient of this arrangement (resp. its complement) is formed by forgetting the ordering of the entries in each V^{n_i} factor, i.e. it is the space of unordered sets with multiplicities $U_1, \ldots, U_m \in \mathbb{N}[V]$ with $|U_i| = n_i$ and the intersection of all these sets contains a point with multiplicity $\geq k$ (resp. contains no point with multiplicity $\geq k$).

Definition 3.6.6 ($\mathcal{M}_{m,k}^{\bullet}$ **and** $X_{m,k}^{\bullet}$). *We denote the complement of $\mathcal{A}_{\bar{n}}^{m,k}(V)$ in $V^{\bar{n}}$ by $\mathcal{M}_{m,k}^{\bar{n}}(V)$ and its $S_{\bar{n}}$-quotient by $X_{m,k}^{\bar{n}}(V)$.*

As stated in the introduction, these varieties are various spaces of configuration of points in V. We will explore this geometric aspect below in specific examples.

Note 3.6.7. If $k = 2$ and $m = 1$ then (and only then) the S_n action is free. In this case the quotient map $\mathcal{M}_{1,2}^{\bullet} \longrightarrow X_{1,2}^{\bullet}$ is a normal S_n-cover. It also follows that if $N_{\bullet}$ is any **FI**-module then the twisted sheaf $\widetilde{N}_n$ on $X_{1,2}^n$ is actually a vector bundle isomorphic to $\mathcal{M}_{1,2}^n \times_{S_n} N_n$.

In other cases we get a branched cover $\mathcal{M}_{m,k}^{\bar{n}}(V) \to X_{m,k}^{\bar{n}}(V)$, and the twisted coefficient sheaf $\widetilde{N}_{\bar{n}}$ on $X_{m,k}^{\bar{n}}(V)$ has different stalks above different points. This phenomenon has a natural interpretation in our case. The following example illustrates this well.

Example 3.6.8. Think of $\mathrm{Poly}_k^n(\mathbb{C}) := X_{1,k}^n(\mathbb{C})$ as the space of degree n polynomials that have no roots of multiplicity $\geq k$ (see example 3.6.11 later), and construct the twisted coefficient sheaf corresponding to the permutation representation $S_n \curvearrowright \mathbb{Q}^n$.

Then the stalk over a polynomial $p \in \mathrm{Poly}_k^n(\mathbb{C})$ can be naturally described as the $\mathbb{Q}$-vector space spanned freely by the distinct roots of p. In particular, when p has multiple roots (a Zariski closed condition) this vector space will have dimension smaller than n.

We conclude the general discussion with a proof of Theorem 3.1.6.

Proof of Theorem 3.1.6. Observe that the generating subspace $\Delta_{(k,\dots,k)}$ is primitive, as it does not contain the kernel of any map induced by any proper injection $\bar{n} \hookrightarrow (k,\dots,k)$. Therefore by Corollary 3.6.4, the arrangement $\mathcal{A}_\bullet^{m,k}(V)$ is a normal $\mathbf{FI}^m$-arrangement generated in degree $(k,\dots,k)$.

The only part of the statement of Theorem 3.1.6 that does not immediately follow from this together Theorem 3.1.4 is the claimed bounds on generation degree. Let $r = \dim(V)$ and recall that by Lemma 3.3.11 a subspace $x \in L^{\mathcal{A}}$ contributes to H^i only if $\mathrm{cd}(x) \leq i$. Since the arrangement is generated by the diagonal line $\Delta_{(k,\dots,k)} \subset V^{km}$, the subspace x is the intersection of a certain number of preimages of this generating diagonal, say

$$x = \Delta^1 \cap \dots \cap \Delta^l$$

is such a presentation with l least. We have a sequence of proper inclusions

$$\Delta^1 \supset \Delta^1 \cap \Delta^2 \supset \dots \supset x$$

where at each step we find a certain number of copies of V. Thus the codimension of every successive pair is a positive multiple of r, in particular the successive codimensions are $\geq r$. This forces the codimension of x to be at least $r \cdot l$. The bound $\mathrm{cd}(x) \leq i$ then implies that $l \leq \frac{i}{r}$. In other words: x is already generated by an $\lfloor \frac{i}{r} \rfloor$-fold intersection of images of the generating diagonal. By the same argument as in Lemma 3.3.20, x is therefore in the image of some subspace defined in degree $\lfloor \frac{i}{r} \rfloor \times (k,\dots,k)$. This shows that the $\mathbf{FI}^m$-module H^i is generated in the stated degree.

All the other statement follow from the general theory of $\mathbf{FI}^m$-modules presented in §2.6 and Theorem 3.1.9. $\qquad\square$

3.6.1 Specializing to important examples

We now vary the parameters m, k and $r = \dim(V)$, and specialize to concrete cases of interest. All of the following examples exhibit the stability properties described in Theorem 3.1.6 with the specified stable ranges.

Example 3.6.9 (Configurations of points in the plane and square-free polynomials). Let $V = \mathbb{C}$ (i.e. $r = 1$), $m = 1$ and $k = 2$. The resulting spaces form the $\mathbf{FI}^{op}$-space of *ordered (or pure) configuration space* of distinct points in $\mathbb{C}$, denoted by $\mathrm{PConf}^{\bullet}(\mathbb{C})$. The quotients by the action of the symmetric groups S_n are the unordered configuration spaces, denoted by $\mathrm{Conf}^{\bullet}(\mathbb{C})$ which, by the fundamental theorem of algebra, are naturally isomorphic to the spaces of monic square-free polynomials of degree $\bullet$, denoted by $\mathrm{Poly}^{\bullet}(\mathbb{C})$. The isomorphism is explicitly given by sending an n-tuple of distinct points to the unique monic, degree n polynomial which vanishes precisely at these points.

For every natural number n, the space $\mathrm{Conf}^{n}(\mathbb{C})$ is aspherical and its fundamental group is Artin's braid group on n strands, denoted by B_n. By forgetting the ordering $\mathrm{PConf}^{n}(\mathbb{C}) \to \mathrm{Conf}^{n}(\mathbb{C})$ is a normal S_n-cover corresponding to the short exact sequence

$$1 \longrightarrow P_n \longrightarrow B_n \longrightarrow S_n \longrightarrow 1$$

where P_n is the pure braid group. Since both spaces are aspherical, they serve as classifying spaces for B_n and P_n respectively, and their cohomology coincides with the group cohomology.

This sequence of spaces has been intensely studies starting with Arnol'd ([Ar]) and Fuks ([Fu]), and more recently it served as the catalyst for the development of the theory of representation stability in [CF] and later of **FI**-modules in [CEF1].

In this context Theorem 3.1.6 reproves representation stability, first demonstrated

in [CF], using a generalization of the [CEF2] notion of **FI**-CHAs. At the level of S_n-quotient spaces we get cohomological stability with various systems of twisted coefficients. Every S_n-representation naturally becomes a B_n-representation via the natural projection $B_n \to S_n$. Thus we can consider the group cohomology of B_n with coefficients in S_n-representations, and the results of Theorems 3.5.3 and 3.5.4 specialize to the twisted cohomological stability – these results previously appeared in [CEF2].

A similar example arises by considering a vector space V of higher dimension.

Example 3.6.10 (**Configurations of points in even-dimensional Euclidean space**). Fix any $r \geq 1$ and consider $V = \mathbb{C}^r \cong \mathbb{R}^{2r}$. By taking $m = 1$ and $k = 2$ in Example 3.6.5 we get in degree n the space $\mathrm{PConf}^n(V)$ of ordered configurations of n distinct points in V. The S_n-quotient is the space $\mathrm{Conf}^n(V)$ of unordered configurations of n points in V.

The cohomology ring of $\mathrm{PConf}^n(V)$ was computed by F. Cohen in [Co] where it was shown to have a similar structure to that of $\mathrm{PConf}^n(\mathbb{C})$ with all degrees multiplied by r. These spaces arise as a local model for ordered and unordered configurations of points in smooth complex varieties and even dimensional orientable manifolds. In [To], Totaro uses the cohomology of $\mathrm{PConf}^n(V)$ to compute the cohomology of the space of ordered configurations $\mathrm{PConf}^n(M)$ where M is a complex smooth projective variety.

Specializing Theorems 3.1.6 to this case we get a new proof of representation stability and freeness for $\mathrm{H}^*(\mathrm{PConf}^n(V))$, previously proved in [CEF1] for the more general case of all connected, orientable open manifolds.

Example 3.6.11 (**The k-equals arrangement**). Let $V = \mathbb{C}$, $m = 1$ and $k \geq 2$ be arbitrary in Example 3.6.5. The resulting **FI**-arrangement is called the *k-equals arrangement* (see e.g. [BW]). The complement of this arrangement parametrizes all ordered configurations of points in the plane with possible coincidences, but where no

k points are allowed to coincide. Taking the quotient by the action of the symmetric group we get the unordered version of such configurations. By assigning a configuration of n points to the unique monic, degree n polynomial that vanishes on these points (with the specified multiplicity), we get an isomorphism from the n-th quotient $X_{1,k}^n(\mathbb{C})$ to the space $\mathrm{Poly}_k^n(\mathbb{C})$ of monic degree n polynomials that have no root of multiplicity $\geq k$. These spaces of polynomials are the complements of the natural stratification of the space of all monic, degree n polynomials ($\cong \mathbb{A}^n$), where we filter based on the maximal multiplicity of their roots.

The intersection poset of the k-equals arrangement is usually denoted by $\Pi_{n,k}$, and is isomorphic to the lattice of partitions of $\{1, \dots, n\}$ such that every non-singleton block has size at least k. These posets were studied in [FNRS] relating to complexes of disconnected k-graphs, and by Vassiliev in connection with homotopy classification of links.

The real version of these subspace arrangements comes up in problems of computational complexity. Consider the following problem:

Given real numbers $x_1, \dots, x_n$, decide whether at least k are equal.

Put in other words, we are asking whether the vector $(x_1, \dots, x_n) \in \mathbb{R}^n$ belongs to the k-equals arrangement. Björner-Lovász show in [BL] that if one tries to solve this problem using the computational model of a linear decision tree[8], then the size and depth of this tree can be bounded from below by expressions involving the Betti numbers of (the complement of) the real k-equals arrangement.

The Betti numbers of the real and complex arrangements and their complements are computed in [BW]. Among other things, our theory provides a new proof that the Betti numbers are polynomial in n of the correct degree. To the best of the author's

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8. See [BL] for a definition.

knowledge the representation stability results of Theorem 3.1.6, and the cohomological stability results of Theorem 3.1.9 in this context are new.

Example 3.6.12 (**Based rational maps** $\mathbb{P}^1 \longrightarrow \mathbb{P}^{m-1}$)**.** Let $V = \mathbb{C}$, $k = 1$ and $m \geq 2$ be arbitrary in Example 3.6.5. The resulting space at degree

$$\bar{n} = (n^{(0)}, \ldots, n^{(m-1)})$$

consists of m-tuples of ordered configurations of points in the plane (with possible coincidences) whose sizes are $\bar{n}$ and who do not all have a point in common. The $S_{\bar{n}}$-quotient is the unordered version which is naturally isomorphic to the space of m-tuples of monic polynomials $(p_0(t), \ldots, p_{m-1}(t))$, of degrees given by $\bar{n}$, such that the gcd of the polynomials in the tuple is 1. An equivalent description of an orbit is given by considering the algebraic function it defines

$$[p_0(t) : \ldots : p_{m-1}(t)] : \mathbb{P}^1 \longrightarrow \mathbb{P}^{m-1}.$$

When restricting to the case $m = 2$ and to objects of the form (n, n), the quotient space is naturally isomorphic to the space of rational maps $\mathbb{P}^1 \longrightarrow \mathbb{P}^1$ of degree n that are *based* in the sense that they send ∞ to 1. We denote the resulting space by $\mathrm{Rat}^n_*(\mathbb{C})$. This space is the key to understanding the space $\mathrm{Rat}^n(\mathbb{C})$ of all degree n rational maps, since there is a fibration sequence

$$\mathrm{Rat}^n_* \longrightarrow \mathrm{Rat}^n \xrightarrow{\mathrm{ev}_\infty} \mathbb{P}^1$$

where the latter map is evaluation at ∞ and the fiber over $1 = [1 : 1] \in \mathbb{P}^1$ is precisely Rat^n_*.

The sequence of spaces $\mathrm{Rat}^n_*(\mathbb{C})$ was studied by Segal (see [Se]), where its integral

cohomological stability was demonstrated. Our techniques shows that the sequence of $S_n \times S_n$-covers (obtained by choosing orderings on the zeros and poles) satisfies representation stability rationally, and in particular rational cohomological stability follows. In fact, here we extend the rational stability result to the 2-dimensional sequence in which we allow n_0 and n_1 to vary independently.

For the case $m > 2$, one considers a similar restriction to degrees of the form $(n, \ldots, n)$, in which case we get the space of degree n rational maps $\mathbb{P}^1 \longrightarrow \mathbb{P}^{m-1}$ that are based, i.e. send $\infty \in \mathbb{P}^1$ to $[1 : \ldots : 1]$. We denote this space by $\mathrm{Rat}^n_{m*}(\mathbb{C})$. As in the $m = 2$ case, this space is the key to understanding the space of all degree n rational maps from $\mathbb{P}^1$ to $\mathbb{P}^{m-1}$ through the fibration sequence

$$\mathrm{Rat}^n_{m*}(\mathbb{C}) \longrightarrow \mathrm{Rat}^n_m(\mathbb{C}) \xrightarrow{\mathrm{ev}_\infty} \mathbb{P}^{m-1}$$

where ev_∞ is the evaluation at ∞ function.

Specializing Theorems 3.1.6 and 3.1.9 to this case we get new cohomological stability results for spaces of (based) rational maps. A non-trivial example of a twisted coefficient sheaf (which is not a local system) is the sheaf whose stalks above a based rational map f is the $\mathbb{Q}$-vector space spanned freely by the <u>distinct</u> m-tuples $(a_0, \ldots, a_{m-1})$, where $f(a_i)$ is contained in the hyperplane $z_i = 0$. This is the sheaf associated to the free, degree-1 $\mathbf{FI}^m$-module $\mathrm{Ind}_1(\mathrm{Triv})$.

We can also consider all of the above examples with $\mathbb{C}$ replaces by $\mathbb{C}^r$. They all satisfy representation stability (Theorem 3.1.6 for the ordered version) and cohomological stability (Theorem 3.1.9 for the unordered version) with improved stability ranges as r grows.

CHAPTER 4

TRACE FORMULA WITH STABILIZERS

A standard observation in algebraic geometry and number theory is that a ramified cover of an algebraic variety $\widetilde{X} \to X$ over a finite field $\mathbb{F}_q$ furnishes the rational points $x \in X(\mathbb{F}_q)$ with additional arithmetic structure: the Frobenius action on the fiber over x. For example, in the case of the Vieta cover of polynomials over $\mathbb{F}_q$ this structure describes a polynomial's irreducible decomposition type.

Furthermore, the distribution of these Frobenius actions is encoded in the cohomology of $\widetilde{X}$ via the Grothendieck-Lefschetz trace formula. This chapter presents a version of the trace formula that is suited for studying the distribution in the context of representation stability: for certain sequences of varieties $(\widetilde{X}_n)$ the cohomology, and therefore the distribution of the Frobenius actions, stabilizes in a precise sense.

We conclude by fully working out the example of the Vieta cover of the variety of polynomials. The calculation includes the distribution of cycle decompositions on cosets of Young subgroups of the symmetric group, which might be of independent interest.

4.1   Introduction

Representation stability identifies sequences of spaces equipped with group actions $(G_n \curvearrowright X_n)_{n \in \mathbb{N}}$ whose cohomology groups exhibit a kind of stabilization as representations for $n \to \infty$. One then hopes to translate the observed cohomological stabilization into arithmetic results via the bridge provided by the Grothendieck-Lefschetz trace formula. This program was realized e.g. by Church-Ellenberg-Farb [CEF2] in the case of statistics of square-free polynomials and maximal tori in Gl_n over finite fields.

One difficulty that the program faces is the possible presence of nontrivial stabilizers of the group actions, and their effect on the trace formula. This chapter offers a treatment of actions with stabilizers and the adaptation of the trace formula to representation stability applications. The formula, presented in Theorem 4.1.2 below, is proved using standard methods and will not be considered new by algebraic-geometers[1]. Rather, it is presented as a 'ready for use' tool to be applied in the context of representation stability.

Using the approach presented here, we extend the project initiated in [CEF2] to include the statistics of polynomials with possible root multiplicities (see details in §4.1.3). Let us remark that much of the work that goes into polynomial statistics often passes through calculations on square-free polynomials and ignores the rest (the latter being relatively uncommon), see e.g. [ABR, Section 4]. The calculations below suggest a way to handle more general polynomials: we introduce an algebra of *division symbols* on the space of polynomials, and show that these give rise to functions that serve as a direct link between the statistics of polynomials and those of symmetric groups (see §4.1.3).

4.1.1   Distribution of rational orbits

Let $\widetilde{X}$ be an algebraic variety over a finite field $\mathbb{F}_q$, endowed with an action of a finite group G. Then the variety of orbits $X = \widetilde{X}/G$ acquires arithmetic information from $\widetilde{X}$: a rational point $x \in X(\mathbb{F}_q)$ corresponds to a G-orbit of $\widetilde{X}(\overline{\mathbb{F}}_q)$ that is stable under the Frobenius automorphism Fr_q. Thus for every $x \in X(\mathbb{F}_q)$ the Frobenius determines a G-equivariant permutation σ_x on a transitive G-set, and this additional information distinguishes rational points in X in a subtle way. For example, let X be the space of monic degree d polynomials. Ordering the roots of a polynomial gives that X is

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1. For example, the same ideas and definitions appear in Grothendieck's [Gr] and in Serre's [Se1].

the quotient $\mathbb{A}^d/S_d$ with S_d, the symmetric group on d letters, acting by permuting the entries of $\mathbb{A}^d$. Then for every polynomial $f(t) \in \mathbb{F}_q[t]$ the permutation σ_f encodes precisely the decomposition type of f into irreducible factors over $\mathbb{F}_q$.

On the other hand, following the philosophy of the Weil conjectures, it is known that the action of Fr_q on the étale cohomology groups $\mathrm{H}^*_{\acute{e}t}(\widetilde{X}_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)$ encodes arithmetic information. At the same time, an action $G \curvearrowright \widetilde{X}$ induces a G-representation on $\mathrm{H}^*(\widetilde{X})$, and it is natural to ask: What arithmetic information is encoded by the joint action of Fr_q and G on $\mathrm{H}^*(\widetilde{X})$?

One answer, given below, is that the information encoded in $\mathrm{H}^*(\widetilde{X})$ is in some sense the distribution of the permutations σ_x attached to rational points $x \in X(\mathbb{F}_q)$. However, it is not initially clear what one should mean by a "distribution" of permutations σ_x on abstract G-orbits. The Tanakian point of view tells us instead to examine how σ_x acts on G-representations, or equivalently: how it evaluates on G-characters.

We therefore detect the distribution of the permutations σ_x by evaluating them on class functions of G as follows. Let $\chi : G \longrightarrow \mathbb{C}$ be a class function. If the quotient map $p : \widetilde{X} \longrightarrow X$ is unramified at $x \in X(\mathbb{F}_q)$ (i.e. the stabilizer of a lift $\widetilde{x} \in p^{-1}(x)$ is trivial), then σ_x determines an element $g_x \in G$, unique up to conjugacy, by $\sigma_x(\widetilde{x}) = g_x.\widetilde{x}$ for a chosen lift $\widetilde{x} \in p^{-1}(x)$. Changing the lift $\widetilde{x}$ only amounts to conjugating g_x, so it is possible to unambiguously define $\chi(\sigma_x) := \chi(g_x)$.

However, when p is ramified at x, the permutation σ_x no longer determines a conjugacy class: if $H_{\widetilde{x}} \subseteq G$ is the stabilizer of a lift $\widetilde{x} \in p^{-1}(x)$, then the condition $\sigma_x(\widetilde{x}) = g_x.\widetilde{x}$ only determines a coset $g_x H_{\widetilde{x}}$. The best one can do in this situation is to average:

Definition 4.1.1 (Evaluating σ_x on class functions). *Let $\chi : G \longrightarrow \mathbb{C}$ be a class*

function. For $x \in X(\mathbb{F}_q)$ and a lift $\widetilde{x} \in p^{-1}(x)$ with stabilizer $H_{\widetilde{x}}$ define

$$\chi(\sigma_x) := \frac{1}{|H_{\widetilde{x}}|} \sum_{h \in H_{\widetilde{x}}} \chi(g_x h)$$

where $g_x \in G$ is any element satisfying $\sigma_x(\widetilde{x}) = g_x.\widetilde{x}$.

Theorem 4.1.2 below relates the sum $\displaystyle\sum_{x \in X(\mathbb{F}_q)} \chi(\sigma_x)$ to the representation $\mathrm{H}^*(\widetilde{X})$. As χ ranges over all class functions, these sums in some sense capture the distribution of σ_x.

Theorem 4.1.2 (Frobenius distribution trace formula). *Let G be a finite group, acting on an algebraic variety $\widetilde{X}$ over the finite field $\mathbb{F}_q$, and let $X = \widetilde{X}/G$ as above. Fix a prime $\ell \gg 1$ coprime to $q|G|$ and let $\mathrm{H}_c^i(\widetilde{X})$ denote the compactly supported ℓ-adic cohomology $\mathrm{H}_{c,\text{ét}}^i(\widetilde{X}_{\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)$. Decompose $\mathrm{H}_c^i(\widetilde{X}) \otimes \overline{\mathbb{Q}}_\ell$ into generalized eigenspaces[2] of the Fr_q action:*

$$\mathrm{H}_c^i(\widetilde{X}) \otimes \overline{\mathbb{Q}}_\ell = \bigoplus_{\lambda \in \overline{Q}_\ell} \mathrm{H}_c^i(\widetilde{X})_\lambda.$$

Note that this sum includes only finitely many nonzero summands, and that each $\mathrm{H}_c^i(\widetilde{X})_\lambda$ is a G-subrepresentation. Then for every class function $\chi : G \longrightarrow \overline{\mathbb{Q}}_\ell$,

$$\sum_{x \in X(\mathbb{F}_q)} \chi(\sigma_x) = \sum_{\lambda \in \overline{\mathbb{Q}}_\ell} \lambda \sum_{i=0}^{\infty} (-1)^i \langle \mathrm{H}_c^i(\widetilde{X})_\lambda^*, \chi \rangle_G \qquad (4.1.1)$$

where the inner product $\langle V, \chi \rangle_G$ for a G-representation V is the standard character inner product of χ with the character of V. Note again that this is really a finite sum.

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2. Here "generalized eigenspaces" means all possibly nontrivial Jordan blocks. This is to avoid questions of possible non-semisimplicity of the Galois action, which are not of interest to us in the current context.

When $\widetilde{X}$ is further taken to be smooth, Poincaré duality implies

$$\sum_{x \in X(\mathbb{F}_q)} \chi(\sigma_x) = q^{\dim(\widetilde{X})} \sum_{\lambda} \lambda^{-1} \sum_{i=0}^{\infty} (-1)^i \langle \mathrm{H}^i(\widetilde{X})_\lambda, \chi \rangle_G. \qquad (4.1.2)$$

4.1.2 Implications of representation stability

Representation stability provides examples of sequences of spaces $(\widetilde{X}_n)_{n \in \mathbb{N}}$ where the symmetric group S_n acts on $\widetilde{X}_n$, and where the induced representations $H^i(\widetilde{X}_n)$ stabilize in the following sense: if $(\chi_n : S_n \to \mathbb{Q})_{n \in \mathbb{N}}$ is a certain natural sequence of class functions (namely, given by a character polynomial, see 2.2.5) then the character inner products

$$\langle H^i(\widetilde{X}_n), \chi_n \rangle_{S_n} \qquad (4.1.3)$$

become independent of n for $n \gg 1$. In the algebraic setting, the same stabilization occurs within every eigenspace of Fr_q (this observation follows immediately from the Noetherian property of the category $\mathbf{FI}$, see [CEF1]). Denote the stable values of these inner products by

$$\langle H^i(\widetilde{X}_\infty)_\lambda, \chi_\infty \rangle.$$

This phenomenon was used in [CEF2], along with the Grothendieck-Lefschetz trace formula in the unramified context, to demonstrate that the factorization statistics of degree d square-free polynomials over $\mathbb{F}_q$ and maximal tori in $\mathrm{GL}_d(\mathbb{F}_q)$ tend to a limit as $d \to \infty$ (see [CEF2, Theorem 1 and 5.6 respectively].

A general type of result that one gets by combining representation stability and the trace formula of Theorem 4.1.2 is the following.

Corollary 4.1.3 (Limiting arithmetic statistics). *Let $(\widetilde{X}_n)_{n \in \mathbb{N}}$ be a sequence of smooth algebraic varieties over $\mathbb{F}_q$ where S_n acts on $\widetilde{X}_n$, and denote the quotients by $X_n = \widetilde{X}_n / S_n$. Suppose that the cohomology $\mathrm{H}^i(\widetilde{X}_n)$ exhibits representation stability*

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in the sense of [CEF1]. Further suppose that $\mathrm{H}^*(\widetilde{X}_\bullet)$ is convergent in the sense of [CEF2, Definition 3.12]. Then for every sequence of class functions $(\chi_n)_{n \in \mathbb{N}}$, given uniformly by a character polynomial, the following equality holds

$$\lim_{n \to \infty} q^{-\dim(\widetilde{X}_n)} \sum_{x \in X_n/S_n} \chi_n(\sigma_x) = \sum_{i \geq 0} \sum_\lambda \frac{(-1)^i}{\lambda} \langle \mathrm{H}^i(\widetilde{X}_\infty)_\lambda, \chi_\infty \rangle. \qquad (4.1.4)$$

In particular, the limit on the left exists, and its value is given by the value of the convergent sum on the right.

Remark 4.1.4 (**Other sequences of groups**). In §2 we discussed *categories of* **FI**-*type*, to which the theory of representation stability can be extended. In particular, Corollary 4.1.3 holds more generally for any diagram $\widetilde{X}_\bullet$ of varieties for which $\mathrm{H}^*(\widetilde{X}_\bullet)$ exhibits representation stability and has a subexponential bound on the growth of certain invariants.

The main example of such sequences of spaces is given in §3, where we found that many collections of complements of linear subspace arrangements indeed give rise to cohomology groups which exhibit representation stability. Therefore, if such a collection satisfies an additional subexponential growth condition as above, then an analog of Corollary 4.1.3 holds for them.

4.1.3 Example: Spaces of polynomials and Young cosets

We conclude in §4.3 by fully working out the example of the space of polynomials $\mathrm{Poly}^d = A^d/S_d$ mentioned in the first paragraph. In this case, as introduced in the beginning of 4.1.1, the permutation σ_f that the Frobenius induces on the roots of a polynomial $f \in \mathrm{Poly}^d(\mathbb{F}_q)$ records the irreducible decomposition of f. Thus Theorem 4.1.2 is concerned with the fundamental question of factorization statistics of polynomials over $\mathbb{F}_q$.

On the one hand the variety $\widetilde{X}$ in this case is $\mathbb{A}^d$ and has very simple cohomology. On the other hand, the quotient map $p : \mathbb{A}^d \longrightarrow \mathrm{Poly}^d$ is highly ramified and the associated permutations σ_f are very far from defining conjugacy classes of S_d. Thus §4.3 deals mainly with the combinatorial challenge of evaluating $\chi(\sigma_f)$ when f is a ramified point, i.e. computing averages of χ over cosets of Young subgroups of S_d. The same calculation is useful in many other contexts when the symmetric group acts by permutations.

Applying Theorem 4.1.2 to this case produces the following apparent coincidence (which will become obvious once the two sides of Equation 4.1.5 are evaluated):

Corollary 4.1.5 (**Equal expectations**). *Endow the two finite sets S_d and $\mathrm{Poly}^d(\mathbb{F}_q)$ with uniform probability measures. Then every S_d-class function χ simultaneously defines a random variable on both spaces (for a point $f \in \mathrm{Poly}^d(\mathbb{F}_q)$ define $\chi(f) := \chi(\sigma_f)$ as in Definition 4.1.1), and for every such χ*

$$\mathbb{E}_{\mathrm{Poly}^d(\mathbb{F}_q)}[\chi] = \mathbb{E}_{S_d}[\chi]. \tag{4.1.5}$$

To understand the left hand side of this equation, one has to interpret the value of $\chi(f)$ for every $f \in \mathrm{Poly}^d(\mathbb{F}_q)$. The explicit description of this value requires some notation, and the complete answer is given in Theorem 4.1.7 on the next page. For the necessary notation – since $\chi(f)$ is related to the divisors f, it will be most convenient to describe its value using the following natural structure of *division symbols*.

For every polynomial $g \in \mathbb{F}_q[t]$ define a function $\epsilon_g : \mathbb{F}_q[t] \longrightarrow \{0, 1\}$ by

$$\epsilon_g(f) = \begin{cases} 1 & \text{if } g \mid f \\ 0 & \text{otherwise.} \end{cases} \tag{4.1.6}$$

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Further define formal multiplication on the symbols ϵ_g by

$$\epsilon_g \epsilon_{g'} = \epsilon_{gg'}. \tag{4.1.7}$$

Note that this multiplication does not commute with the evaluation on a polynomial f, and in fact every ϵ_g evaluates on f nilpotently.

To evaluate $\chi(f)$ for every χ, it suffices to consider a spanning set of class functions. The most convenient in this context are given by character polynomials (see [CEF2]) which are described in §4.3.1 below. Briefly, for every k let X_k by the class function

$$X_k(\sigma) = \# \text{ of } k\text{-cycles in } \sigma$$

and for every multi-index $\mu = (\mu_1, \mu_2, \ldots)$ define

$$\binom{X}{\mu} = \binom{X_1}{\mu_1}\binom{X_2}{\mu_2}\ldots.$$

Explicitly, $\binom{X}{\mu}$ is the S_d-class function that counts the number of ways to choose μ_k many k-cycles in a permutation $\sigma \in S_d$. Instead of evaluating each $\binom{X}{\mu}$ separately, it is possible evaluate them all simultaneously be means of a generating function.

Definition 4.1.6. *Introduce indeterminants* $t_1, t_2, \ldots$, *and define a generating function*

$$F(\boldsymbol{t}) = \sum_{\mu=(\mu_1, \mu_2, \ldots)} \binom{X}{\mu} t_1^{\mu_1} t_2^{\mu_2} \ldots = (1 + t_1)^{X_1}(1 + t_2)^{X_2} \ldots$$

Evaluating $\binom{X}{\mu}(f)$ for every μ is the same as evaluating $F(\mathbf{t})$ on f.

Theorem 4.1.7 (Evaluation of $\binom{X}{\mu}$). *When evaluating* $f \in \mathrm{Poly}^d(\mathbb{F}_q)$ *on class*

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functions, there is an equality of generating functions

$$F(\boldsymbol{t}) = \exp\left(\sum_{k=1}^{\infty} \sum_{p \in \mathrm{Irr}_{|k}} \deg(p) \epsilon_p^{\frac{k}{\deg(p)}} \frac{t_k}{k}\right) \tag{4.1.8}$$

where $\mathrm{Irr}_{|k}$ *is the set of irreducible polynomials over* $\mathbb{F}_q$ *whose degree divides* k.

Unpacking Equation 4.1.8, for every multi-index $\mu = (\mu_1, \mu_2, \ldots)$ *there is an equality*

$$\binom{X}{\mu}(f) = \prod_{k=1}^{\infty} \frac{1}{k^{\mu_k} \mu_k!} \left(\sum_{p \in \mathrm{Irr}_{|k}} \deg(p) \epsilon_p^{\frac{k}{\deg(p)}}\right)^{\mu_k} (f) \tag{4.1.9}$$

where the evaluation of the right-hand side proceeds by first expanding into monomial terms in the ϵ_g *symbols using Equation 4.1.7, then evaluating according to Equation 4.1.6.*

With Theorem 4.1.7 it is possible explain the coincidence of the two expectations in Corollary 4.1.5:

Proposition 4.1.8 (Necklace relations). *The equalities* $\mathbb{E}_{\mathrm{Poly}^d(\mathbb{F}_q)}[\chi] = \mathbb{E}_{S_d}[\chi]$ *for every* χ *are equivalent to the Necklace relations*

$$\sum_{d|k} d N_d = q^k$$

where N_d *is the number of monic irreducible polynomials of degree* d *over* $\mathbb{F}_q$.

Proposition 4.1.8 shows that in fact Corollary 4.1.5 says nothing new about polynomials. However, the explicit evaluation of $\chi(f)$ in Theorem 4.1.7 contains much more information: one can impose any S_d-invariant restriction on the roots of polynomials (see examples in §4.3) and get an equality similar to Corollary 4.1.5 – relating the factorization statistics of those polynomials that satisfy the restriction with various

expectations calculated over S_n.

Remark 4.1.9 (**Statistics on cosets of Young subgroups**). The calculation involved in the evaluation $\chi(f)$ is entirely combinatorial, and can be considered independently from polynomial counting problems. In §4.3.3 we phrase this as an independent combinatorial result, which might be of interest in other contexts. Consider the following: let $H_\lambda \leq S_d$ be a Young subgroup and let gH_λ be a coset with $g \in N(H_\lambda)$.

Question 4.1.10. *What is the distribution of cycle types of permutations in gH?*

Theorem 4.3.15 below answers the question and provides additional statistics on gH_λ.

4.2 The Frobenius distribution trace formula

Let G be a finite group. As described above, one can extend the domain of G-class functions to equivariant permutations of G-orbits by averaging.

Definition 4.2.1 (**Evaluating class functions on permutations**). *Let S be a transitive G-set (possibly with non-trivial stabilizers) and let k be a field of characteristic 0. For every G-equivariant function $\sigma : S \longrightarrow S$ and every class function $\chi : G \longrightarrow k$ define*

$$\chi(\sigma) := \frac{1}{|\operatorname{Stab}(s)|} \sum_{\substack{g \in G \\ g.s = \sigma(s)}} \chi(g) \tag{4.2.1}$$

where $s \in S$ is any element and $\operatorname{Stab}(s)$ is its stabilizer subgroup.

Note that because χ is a class function, this definition does not depend on the choice of s, as any other choice reduces to conjugating all elements in the sum.

For the proof of Theorem 4.1.2 we recall the following definitions.

Definition 4.2.2 (**G-equivariant sheaf**). *A G-action on a sheaf $\mathcal{F}$ over a G-space $\widetilde{X}$ is a collections of sheaf morphisms $\varphi_g : (g^{-1})^* \mathcal{F} \to \mathcal{F}$ indexed by G that make the following diagrams commute for every $g, h \in G$:*

$$
\begin{array}{ccc}
(g^{-1})^*(h^{-1})^*\mathcal{F} & \xrightarrow{\;\sim\;} & (h^{-1}g^{-1})^*\mathcal{F} \\
{\scriptstyle (g^{-1})^*\varphi_h} \downarrow & & \downarrow {\scriptstyle \varphi_{gh}} \\
(g^{-1})^*\mathcal{F} & \xrightarrow[\;\varphi_g\;]{} & \mathcal{F}
\end{array}
$$

Example 4.2.3 (**Constant sheaf with a G-action**). Given a G-action on an abelian group A, construct a G-action on the constant sheaf $\overline{A}$ over a G-space $\widetilde{X}$ by defining $(g^{-1})^*\overline{A} \xrightarrow{\varphi_g} \overline{A}$ to act by g on all stalks, which are canonically isomorphic to A.

Now suppose $p : \widetilde{X} \longrightarrow X$ is the ramified G-cover discussed in Theorem 4.1.2 and let $\mathcal{F}$ be a sheaf on $\widetilde{X}$ equipped with a G-action. Then the push-forward $p_*\mathcal{F}$ on X acquires the G-action

$$
p_*\mathcal{F} = (p \circ g)_*\mathcal{F} \xrightarrow{\;\simeq\;} p_*g_*\mathcal{F} \xrightarrow{\;\simeq\;} p_*(g^{-1})^*\mathcal{F} \xrightarrow{\;p_*\varphi_g\;} p_*\mathcal{F}
$$

which is now acting by sheaf automorphisms.

Definition 4.2.4 (**Twisted coefficient sheaf**). *In the situation described in the previous paragraph, define the twisted coefficient sheaf corresponding to the G-sheaf $\mathcal{F}$ on $\widetilde{X}$ to be the subsheaf of invariants $(p_*\mathcal{F})^G$ on X. We denote this sheaf by $\mathcal{F}/G$ (corresponding to $X = \widetilde{X}/G$).*

This construction is the sheaf analog of the Borel construction in topology: giving rise to a (flat) twisted fiber bundle from the data of a π_1-action on the fiber.

Lemma 4.2.5 (**Transfer for $\mathcal{F}/G$**). *If multiplication by $|G|$ is an invertible transfor-*

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mation on $\mathcal{F}$, there is an isomorphism

$$\mathrm{H}^i_{c,\acute{e}t}(X_{\overline{\mathbb{F}}_q}; \mathcal{F}/G) \cong \mathrm{H}^i_{c,\acute{e}t}(\widetilde{X}_{\overline{\mathbb{F}}_q}; \mathcal{F})^G.$$

Furthermore, since the G-action is Galois-equivariant, so is this isomorphism.

Proof. Denote the inclusion $\mathcal{F}/G = (p_* \mathcal{F})^G \hookrightarrow p_* \mathcal{F}$ by ι and define a transfer morphism $(p_* \mathcal{F})^G \overset{\tau}{\longleftarrow} p_* \mathcal{F}$ by $\tau = \sum_{g \in G} g(\cdot)$. Clearly the composition $\tau \circ \iota$ is multiplication by $|G|$ on $(p_* \mathcal{F})^G$ and the reverse composition $\iota \circ \tau$ is $|G|$ times the projection onto the G-invariants of $p_* \mathcal{F}$.

Consider the induced maps on cohomology

$$\mathrm{H}^i_c(X_{\overline{\mathbb{F}}_q}; \mathcal{F}/G) \overset{\tau}{\underset{\iota}{\leftrightarrows}} \mathrm{H}^i_c(\widetilde{X}_{\overline{\mathbb{F}}_q}; \mathcal{F})$$

Assuming multiplication by $|G|$ is an invertible transformation on $\mathcal{F}$, these maps induce the desired isomorphism. Lastly, since the G-action on $\mathcal{F}$ is Galois equivariant, so is τ as a sum of group elements. $\qquad\square$

With this in hand, the proof of Theorem 4.1.2 follows.

Proof of Theorem 4.1.2. First, observe that by the linearity of the two sides of equation 4.1.2, it will suffice to prove the equality for a spanning set of class functions. In particular, it will suffice to consider only characters of G-representations.

If ξ is a $|G|$-primitive root of unity, then $\mathbb{Q}[\xi]$ is a splitting field for G, i.e. every G-representation in characteristic 0 is realized over $\mathbb{Q}[\xi]$ (see [Se2, §12.3, Corollary to Theroem 24]). Since there are only finitely many irreducible representations of G, and each one of those is represented by finitely many matrices with entries in $\mathbb{Q}[\xi]$, then for every ℓ excluding a finite set of primes every one of the matrix entries is an ℓ-adic

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integer. Fix any ℓ prime to $q|G|$ and large enough to have this property, i.e. that every G-representation in a $\mathbb{Q}_\ell$-vector space is defined over $\mathbb{Z}_\ell$.

Suppose the class function χ is the character of a G-representation in a n-dimensional $\mathbb{Q}_\ell$-vector space V. Let $\overline{V}$ denote the constant ℓ-adic sheaf of rank n on $\widetilde{X}_{\overline{\mathbb{F}}_q}$, and define a G-action on $\overline{V}$ as described in Example 4.2.3. Note that since Fr_q commutes with the G-action on $\widetilde{X}_{\overline{\mathbb{F}}_q}$, it also commutes with this G-action on $\overline{V}$.

The Grothendieck-Lefschetz trace formula [De2, Rapport, Theorem 3.2] applied to the twisted sheaf $\overline{V}/G$ tells us in this case that

$$\sum_{x \in X(\mathbb{F}_q)} \mathrm{Tr}\left((\mathrm{Fr}_q)_x \curvearrowright (\overline{V}/G)_x\right) = \sum_{i=0}^{\infty} (-1)^i \, \mathrm{Tr}\left(\mathrm{Fr}_q \curvearrowright \mathrm{H}^i_{c,\text{ét}}\left(X_{\overline{\mathbb{F}}_q}; \overline{V}/G\right)\right) \quad (4.2.2)$$

The rest of the proof is rewriting this equation in the form stated by the theorem.

Starting with the left-hand side, the stalk $(\overline{V}/G)_x$ is the vector space of G-invariant functions on $p^{-1}(x)$, and restricting to any choice of lift $\widetilde{x} \in p^{-1}(x)$ gives an isomorphism

$$\left(\overline{V}/G\right)_x \cong \left(V^{p^{-1}(x)}\right)^G \xrightarrow{r_{\widetilde{x}}} V^{H_{\widetilde{x}}} \quad (4.2.3)$$

where $H_{\widetilde{x}} = \mathrm{Stab}_{\widetilde{x}}$ and $V^{H_{\widetilde{x}}}$ is the subspace of $H_{\widetilde{x}}$-invariants. Indeed, this follows immediately from Frobenius reciprocity (since $p^{-1}(x)$ is a transitive G-set, the representation $V^{p^{-1}(x)}$ is the coinduced module $\mathrm{coInd}_{H_{\widetilde{x}}}^G V$).

Pick an element $g_0 \in G$ such that $\mathrm{Fr}_q(\widetilde{x}) = g_0(\widetilde{x})$. Then for every $v \in V^{H_{\widetilde{x}}}$ let $s = r_{\widetilde{x}}^{-1}(v)$ be the associated G-invariant section, i.e. $s_{g(\widetilde{x})} = g(v)$. The following equalities hold

$$v \xoverset{(r_{\widetilde{x}})^{-1}}{\longmapsto} s \xoverset{(\mathrm{Fr}_q)_x}{\longmapsto} s \circ \mathrm{Fr}_q \xoverset{r_{\widetilde{x}}}{\mapsto} \left(s \circ \mathrm{Fr}_q\right)_{\widetilde{x}} = s_{g_0(x)} = g_0(v)$$

and the trace of $(\mathrm{Fr}_q)_x$ coincides with that of g_0 acting on $V^{H_{\widetilde{x}}}$. To compute this trace

let

$$\mathrm{P}_{H_{\widetilde{x}}} = \frac{1}{|H_{\widetilde{x}}|} \sum_{h \in H_{\widetilde{x}}} h : V \longrightarrow V^{H_{\widetilde{x}}}$$

be the usual projection operator. Composing g_0 with $\mathrm{P}_{H_{\widetilde{x}}}$ gives a self-map on V that restricts to g_0 on $V^{H_{\widetilde{x}}}$ and is zero on the complementary representation. Thus the trace of $g_0 \circ \mathrm{P}_{H_{\widetilde{x}}}$ agrees with the trace of $g_0 \curvearrowright V^{H_{\widetilde{x}}}$. On the other hand,

$$\mathrm{Tr}(g_0 \circ \mathrm{P}_{H_{\widetilde{x}}}) = \frac{1}{|H_{\widetilde{x}}|} \sum_{h \in H_{\widetilde{x}}} \mathrm{Tr}(g_0 h) = \frac{1}{|H_{\widetilde{x}}|} \sum_{h \in H_{\widetilde{x}}} \chi(g_0 h) \qquad (4.2.4)$$

Denoting the induced permutation $\mathrm{Fr}_q \curvearrowright p^{-1}(x)$ by σ_x, the above average is, by definition, the evaluation $\chi(\sigma_x)$. This shows that the left-hand side of the Grothendieck-Lefchetz trace formula (Equation 4.2.2) coincides with that of the formula that we are proving.

Now for the right-hand side of Equation 4.2.2. The transfer isomorphism of Lemma 4.2.5 gives

$$\mathrm{H}_c^i(X_{\overline{\mathbb{F}}_q}; \overline{V}/G) = \mathrm{H}_c^i(\widetilde{X}_{\overline{\mathbb{F}}_q}; \overline{V})^G.$$

Recall that $\overline{V}$ is a constant sheaf, so the cohomology groups can be expressed as

$$\mathrm{H}_c^i(\widetilde{X}_{\overline{\mathbb{F}}_q}; \overline{V}) \cong \mathrm{H}_c^i(\widetilde{X}_{\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell) \otimes V.$$

Extend scalars to $\overline{\mathbb{Q}}_\ell$ and decompose $\mathrm{H}_c^i(\widetilde{X}_{\overline{\mathbb{F}}_q}) \otimes \overline{\mathbb{Q}}_\ell$ into generalized Fr_q-eigenspaces

$$\mathrm{H}_c^i(\widetilde{X}_{\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell) \otimes \overline{\mathbb{Q}}_\ell = \oplus_\lambda \mathrm{H}_c^i(\widetilde{X}_{\overline{\mathbb{F}}_q})_\lambda.$$

Since Fr_q commutes with the G-action, the same decomposition holds after tensoring

with V and restricting to the G-invariant subrepresentation:

$$(\mathrm{H}_c^i(\widetilde{X}_{\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell) \otimes V)^G = \oplus_\lambda (\mathrm{H}_c^i(\widetilde{X}_{\overline{\mathbb{F}}_q})_\lambda \otimes V)^G.$$

Therefore the trace of Fr_q is $\sum_\lambda \lambda \dim(\mathrm{H}_c^i(\widetilde{X}_{\overline{\mathbb{F}}_q})_\lambda \otimes V)^G$. The theorem follows since

$$\dim \left(\mathrm{H}_c^i(\widetilde{X}_{\overline{\mathbb{F}}_q})_\lambda \otimes V \right)^G = \langle \mathrm{H}_c^i(\widetilde{X}_{\overline{\mathbb{F}}_q})_\lambda^*, V \rangle_G$$

and V was chosen so that its character is χ. □

4.3 Example: the space of polynomials

Consider the Vieta map $v : \mathbb{A}^N \to \mathbb{A}^N$ given by sending an N-tuple $(z_1, \ldots, z_N)$ of geometric points in $\mathbb{A}^1$ to the (coefficients of the) unique monic polynomial that has precisely these roots including multiplicity, i.e.

$$v(z_1, \ldots, z_N) = f(t) = \prod_{i=1}^N (t - z_i).$$

Denote the space of monic degree N polynomials by Poly^N. This space is again $\mathbb{A}^N$, parametrized by the polynomials' coefficients, and the coordinates of the Vieta map v are given by the elementary symmetric polynomials. Thus the map $v : \mathbb{A}^N \longrightarrow \mathbb{A}^N$ is a ramified S_N cover befitting the context of Theorem 4.1.2, where S_N acts on the domain by permuting the coordinates.

By considering S_N-invariant subvarieties $\widetilde{X} \subset \mathbb{A}^N$ cut out by various constraints, Theorem 4.1.2 can be used to compute statistics of spaces of polynomials whose roots are subject to the same constraints. Example of such constraints include the space of polynomials with

- root multiplicity bounded by some fixed k;

- all roots colinear; etc.

However, to get concrete information out of Theorem 4.1.2 one must be able to evaluate $\chi(f)$ for every S_N-class function χ and every polynomial $f \in \mathrm{Poly}^N(\mathbb{F}_q)$. In particular, if $f(t)$ has multiple roots then it is a ramification point of the Vieta map, and the value $\chi(f)$ is an average over some coset in S_N. The current section is concerned with computing the evaluations $\chi(f)$ precisely.

4.3.1   Evaluation of $\chi(f)$

For computing $\chi(f)$ on every S_N-class function χ, it suffices to compute them on the following convenient spanning set of class functions.

Definition 4.3.1 (**Character polynomials**). *For every $k \in \mathbb{N}$ let $X_k : S_N \longrightarrow \mathbb{N}$ be the cycle-counting function*

$$X_k(\sigma) = \# \text{ of } k\text{-cycles in } \sigma.$$

A $\underline{\text{character polynomial}}$ is any $P \in \mathbb{Q}[X_1, X_2, \ldots]$. Every character polynomial P gives rise to class functions $P : S_N \to \mathbb{Q}$, which will also be denoted by P. Note that $X_k \equiv 0$ whenever $k > N$.

Furthermore, for every $k, \mu_k \in \mathbb{N}$ define a character polynomial

$$\binom{X_k}{\mu_k} = \frac{1}{\mu_k!} X_k(X_k - 1) \ldots (X_k - \mu_k + 1).$$

More generally, for every multi-index $\mu = (\mu_1, \mu_2, \ldots)$ define its norm $\|\mu\| = \sum_{i=k}^{\infty} k\mu_k$

and the character polynomial

$$\binom{X}{\mu} = \binom{X_1}{\mu_1}\binom{X_2}{\mu_2}\cdots.$$

Note that $\binom{X}{\mu} = 0$ unless $\mu_k = 0$ for all $k > N$, so a non-zero product of this form is necessarily finite.

Note 4.3.2. The class function $\binom{X}{\mu} : S_N \longrightarrow \mathbb{Q}$ is counting, for every $\sigma \in S_N$, the number of ways to choose μ_k disjoint k-cycles in σ for all k simultaneously. Note that when $\|\mu\| = N$ there is at most one way to arrange the cycles of σ in this way, so $\binom{X}{\mu}$ is the indicator function of the conjugacy class C_μ specified by having exactly μ_k many k-cycles for every k. For this reason it follows that the functions $\binom{X}{\mu}$ with $\|\mu\| = N$ form a basis for the class functions on S_N. Lastly, if $\|\mu\| > N$ then there are not enough disjoint cycles in σ, so $\binom{X}{\mu} \equiv 0$.

Theorem 4.1.7 describes the evaluation $\binom{X}{\mu}(f)$ for every polynomial $f \in \mathrm{Poly}^N(\mathbb{F}_q)$. The statement involves the divisibility of f by other polynomials g, and its formulation used the division symbols ϵ_g, recalled below.

Notation 4.3.3 (The algebra of division symbols). For every monic polynomial $g \in \mathbb{F}_q[t]$ introduce a formal symbol ϵ_g that measures divisibility by g in the following way: for every polynomial $f \in \mathbb{F}_q[t]$ set

$$\epsilon_g(f) = \begin{cases} 1 & g|f \\ 0 & \text{otherwise.} \end{cases} \tag{4.3.1}$$

Let R_q be the free $\mathbb{Q}$-vector space spanned by these ϵ_g symbols with g ranging over all monic polynomials in $\mathbb{F}_q[t]$. Extend the evaluation maps $f \mapsto \epsilon_g(f)$ linearly to R_q.

Furthermore define multiplication on the symbols ϵ_g by

$$\epsilon_g \cdot \epsilon_h = \epsilon_{gh}. \tag{4.3.2}$$

turning R_q into a $\mathbb{Q}$-algebra. One should take care and observe that the evaluation $f \mapsto \epsilon_g(f)$ is *not* multiplicative on R_q. In fact, with respect to the evaluation on any $f \in \mathbb{F}_q[t]$, every element ϵ_g is nilpotent. This nilpotence property turns out to be an essential part of Theorem 4.1.7.

One can evaluate all functions $\binom{X}{\mu}$ simultaneously using a generating function: set

$$F(\mathbf{t}) = F(t_1, t_2, \ldots) := \sum_{\mu = (\mu_1, \mu_2, \ldots)} \binom{X}{\mu} t_1^{\mu_1} t_2^{\mu_2} \ldots$$

This series evaluates on a polynomial $f \in \mathrm{Poly}^N(\mathbb{F}_q)$ term-wise, that is:

$$F(\mathbf{t})_{(f)} = \sum_{\mu} \binom{X}{\mu}_{(f)} t_1^{\mu_1} t_2^{\mu_2} \ldots$$

and the resulting series is generated by that coefficients that we wish to compute. Theorem 4.1.7 then provides an explicit description of all evaluations by

$$F(\mathbf{t})_{(f)} = \exp\left(\sum_{k=1}^{\infty} \sum_{p \in \mathrm{Irr}_{|k}} \deg(p) \epsilon_p^{\frac{k}{\deg(p)}} \frac{t_k}{k} \right)_{(f)} \tag{4.3.3}$$

with $\mathrm{Irr}_{|k}$ being the set of monic irreducible polynomials of degree dividing k. Recall that to evaluate an element of R_q on f one must first expand any product into monomials in the ϵ_g symbols, and then evaluate according to the divisibility of f by g.

We illustrate how to use this result in a couple of examples.

Example 4.3.4 (Square-free polynomials). Consider the special case where $f \in$

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Poly^N is square-free. This case is simple since such f are unramified points of the Vieta map, and furthermore one does not encounter the non-multiplicative behavior of the evaluation on ϵ_g symbols.

One the one hand, since f is an unramified point, the Frobenius permutation on the roots determines an element $g_f \in S_N$ unique up to conjugation, and by definition $\chi(f) = \chi(g_f)$. In particular, the evaluation is multiplicative in χ. Elementary Galois theory shows that for $\chi = X_k$ the value $X_k(f)$ is the number of degree k irreducible factors of f. Using these facts one can easily write down a formula for $\binom{X}{\mu}(f)$ that does not go though Theorem 4.1.7. However, the point of this example is to see how to use Equation 4.1.9 in calculations, so we shall ignore this argument.

For every irreducible polynomial $p(t)$, evaluating the symbol ϵ_{p^r} on f would give 0 whenever $r > 1$, as f will not be divisible by such powers of p. Thus in Equation 4.1.9, the only contributions to the sum over $p \in \mathrm{Irr}_{|k}$ come from p of degree k precisely. We can therefore simplify the expression and get that on the set of square-free polynomials the following two functions coincide:

$$\binom{X}{\mu} = \prod_{k=1}^{\infty} \frac{1}{\mu_k!} \Big(\sum_{p \in \mathrm{Irr}_{=k}(\mathbb{F}_q)} \epsilon_p \Big)^{\mu_k} \qquad (4.3.4)$$

where $\mathrm{Irr}_{=k}(\mathbb{F}_q)$ is the set of irreducible polynomials of degree *equal* to k over $\mathbb{F}_q$. Observe the following two properties of this product.

1. If p and q are coprime polynomials then $\epsilon_p(f) \cdot \epsilon_q(f) = \epsilon_{pq}(f)$ for every f. Therefore, since the k-th term of the product in Equation 4.3.4 involves only irreducible polynomials of degree k, the different terms evaluate on f multiplicatively, i.e.

$$\left[\prod_{k=1}^{\infty} \frac{1}{\mu_k!} \Big(\sum_{p \in \mathrm{Irr}_{=k}(\mathbb{F}_q)} \epsilon_p \Big)^{\mu_k} \right] (f) = \prod_{k=1}^{\infty} \left[\frac{1}{\mu_k!} \Big(\sum_{p \in \mathrm{Irr}_{=k}(\mathbb{F}_q)} \epsilon_p \Big)^{\mu_k} (f) \right]$$

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and thus each term may be evaluated separately.

2. For each k, use again the fact that any power $(\epsilon_p)^r = \epsilon_{p^r}$ evaluates to 0 on a square-free polynomial whenever $r > 1$. Expanding the μ_k-th power and eliminating high powers of ϵ_p's

$$\frac{1}{\mu_k!}\Big(\sum_{p \in \mathrm{Irr}_{=k}(\mathbb{F}_q)} \epsilon_p\Big)^{\mu_k} = \sum_{\{p_1,\ldots,p_{\mu_k}\} \subset \mathrm{Irr}_{=k}(\mathbb{F}_q)} \epsilon_{p_1 \ldots p_{\mu_k}}$$

where the sum goes over all sets of degree-k irreducibles of cardinality μ_k. Since the ϵ symbols evaluate to either 1 or 0 on f, it follows that the sum evaluates to *the number of ways to choose μ_k distinct irreducible factors of f with degree k.*

Corollary 4.3.5. *If f is a square-free polynomial then*

$$\binom{X}{\mu}(f) = \#\ \text{ways to choose } \mu_k \text{ many degree } k \text{ irreducible factors of } f,\ \forall k.$$

Example 4.3.6 (Degree 1 character polynomials). The evaluation of X_k is most straightforward. This is the expression $\binom{X}{\mu}$ with $\mu_k = 1$ and $\mu_j = 0$ for all $j \neq k$. In this case Equation 4.1.9 simplifies to

$$X_k = \frac{1}{k}\sum_{p \in \mathrm{Irr}_{|k}} \deg(p)\epsilon_p^{\frac{k}{\deg(p)}} = \sum_{d|k}\frac{d}{k}\sum_{p \in \mathrm{Irr}_{=d}} \epsilon_{p^{k/d}}.$$

When evaluating this expression on a polynomial f, since ϵ-symbols evaluate to either 0 or 1, the sum becomes a simple count:

$$X_k(f) = \sum_{d|k}\frac{d}{k}\#\{\ \text{irreducible degree } d \text{ factors that divide } f \text{ at least } k/d \text{ times }\}.$$

$$(4.3.5)$$

Considering the two extreme cases: if f is square-free this reduces back to the count

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of degree k irreducible factors; and if $f = p(t)^r$ with p irreducible of degree d then $X_{d \cdot \ell}(f) = \frac{1}{\ell}$ if $r \geq \ell$ and 0 otherwise.

Remark 4.3.7. The origin of the sum in Equation 4.3.5 is clear when one considers the stack quotient $[\mathbb{A}^N / S_N]$: the Vieta map factors though the universal map from the quotient stack to the quotient variety

$$[\mathbb{A}^N / S_N] \longrightarrow \mathbb{A}^N / S_N = \mathrm{Poly}^N$$

and the fiber of this map over f contains multiple points that each contribute a term to $X_k(f)$. At the same time, the points on the stack have automorphisms which account for the denominators.

Proof of Theorem 4.1.7. Let $f(t)$ be a monic polynomial over $\mathbb{F}_q$. Suppose f decomposes as

$$f = p_1(t)^{r_1} \dots p_n(t)^{r_n}$$

where the $p_i(t)$'s are the distinct irreducible factors of f and set $d_i = \deg(p_i)$. Over the algebraic closure $\overline{\mathbb{F}}_q$ every factor $p_i(t)$ decomposes further as a product of linear terms

$$p_i(t) = (t - \alpha_{i,1})(t - \alpha_{i,2}) \dots (t - \alpha_{i,d_i})$$

with all $\alpha_{i,k}$ distinct. Thus $f(t)$ is the product

$$f(t) = \prod_{\substack{1 \leq i \leq n \\ 1 \leq k \leq d_i}} (t - \alpha_{i,k})^{r_i}.$$

and the degree of f is $N = \sum_{i=1}^n d_i r_i$.

Under the Vieta map $v : \mathbb{A}^N \longrightarrow \mathbb{A}^N$

$$(z_1, \ldots, z_N) \mapsto p(t) = (t - z_1) \ldots (t - z_N)$$

the polynomial $f(t)$ is the image of the N-tuple

$$\alpha_f := (\underbrace{\alpha_{1,1}, \ldots, \alpha_{1,1}}_{r_1 \text{ times}}, \underbrace{\alpha_{1,2}, \ldots, \alpha_{1,2}}_{r_1 \text{ times}}, \ldots, \underbrace{\alpha_{1,d_1}, \ldots, \alpha_{1,d_1}}_{r_1 \text{ times}}, \underbrace{\alpha_{2,1}, \ldots, \alpha_{2,1}}_{r_2 \text{ times}}, \ldots).$$

Under the S_N-action of permuting the coordinates, the stabilizer of α_f is the Young subgroup

$$H_f := \underbrace{S_{r_1} \times \ldots \times S_{r_1}}_{d_1 \text{ times}} \times \ldots \times \underbrace{S_{r_n} \times \ldots \times S_{r_n}}_{d_n \text{ times}} = (S_{r_1})^{d_1} \times \ldots (S_{r_n})^{d_n}.$$

For clarity of notation, relabel the points of the set $\{1, \ldots, N\}$ by choosing a bijection

$$T_f = \{(i, j, k) : 1 \leq i \leq n,\ 1 \leq j \leq r_i,\ k \in \mathbb{Z}/d_i\mathbb{Z}\} \overset{\cong}{\longrightarrow} \{1, \ldots, N\}$$

and thinking of S_N as the symmetry group of T_f. One should think of the 3-tuple $(i, j, k) \in T_f$ as corresponding to the root $\alpha_{i,k}$ of the j-th copy of $p_i(t)$ (that is, i indexes the irreducible factor, j indexes which of the multiple copies of p_i one is considering, and k indexes the roots of p_i).

The Frobenius Fr_q acts on the roots of $f(t)$ by the cyclic permutation $\alpha_{i,k} \mapsto \alpha_{i,k+1}$, where the second subscript k belongs to $\mathbb{Z}/d_i\mathbb{Z}$. Thus the Fr_q-action is lifted by the permutation $\tau \in S_N$ given by $\tau(i, j, k) = (i, j, k + 1)$.

Our task is now to compute the value of $\chi(\sigma_f)$ for every class function χ, i.e. the

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average value of χ on the coset τH_f. Define a new function $\chi^{(\tau)} : S_N \longrightarrow \mathbb{Q}$ by

$$\chi^{(\tau)}(h) = \chi(\tau \cdot h).$$

Then it is clear that $\chi(f) = \dfrac{1}{|H_f|} \displaystyle\sum_{h \in H_f} \chi^{(\tau)}(h) = \mathbb{E}_{H_f}[\chi^{(\tau)}]$ and that the operation $\chi \mapsto \chi^{(\tau)}$ commutes with arithmetic operations on functions. Thus to evaluate f on arbitrary character polynomials, one can start by understanding the function $X_k^{(\tau)}$. For this purpose introduce the following notation.

Definition 4.3.8 (The projection operator m_{i_0}). *For every $1 \le i_0 \le n$ (i.e. for every irreducible factor $p_{i_0} | f$) define a projection operator $m_{i_0} : H_f \longrightarrow S_{r_{i_0}}$ as follows.*

An element of $H_f = \prod_{i,k} S_{r_i}$ is given by a sequence of permutations $(h_{i,k})_{i,k}$ where $h_{i,k} \in S_{r_i}$. Define

$$m_{i_0} : (h_{i,k})_{i,k} \mapsto h_{i_0,d_i} \cdot h_{i_0,d_i-1} \cdot \ldots \cdot h_{i_0,2} \cdot h_{i_0,1}.$$

Claim 2. *For every $r \in \mathbb{N}$ the function $X_r^{(\tau)}$ can be presented as a sum*

$$X_r^{(\tau)} = \sum_{\substack{1 \le i \le n \\ d_i | r}} X_{\frac{r}{d_i}} \circ m_i. \tag{4.3.6}$$

Proof. Fix an element $h = (h_{i,k})_{i,k} \in H_f$. The value $X_r(\tau h)$ is the number of r-cycles the permutation $\tau h \in \tau H_f$ has when acting on the set T_f. The action of this permutation on an element $(i, j, k) \in T_f$ is by

$$(i, j, k) \overset{h}{\mapsto} (i, h_{i,k}(j), k) \overset{\tau}{\mapsto} (i, h_{i,k}(j), k + 1).$$

Observe that the i-value is fixed by this action. Thus every cycle of τh has a well-

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defined i-value, and there is a decomposition

$$X_r^{(\tau)} = \sum_{i=1}^{n} X_r^{(i)}$$

where $X_r^{(i)}$ counts only the number of r-cycles with i-value equal to i. It therefore remains to show that $X_r^{(i)} = X_{\frac{r}{d_i}} \circ m_i$ if $d_i \mid r$ and 0 otherwise.

Fix i. An element $(i, j, 1) \in R_f$ belongs to an r-cycle if it is fixed by $(\tau h)^r$ and not by any smaller power of $\tau \cdot h$. Compute

$$(\tau h)^m(i, j, 1) = (i, (h_{i,m} \cdot \ldots \cdot h_{i,2} \cdot h_{i,1})(j), m + 1)$$

so for $(i, j, 1)$ to be fixed, demand that $m + 1 \equiv 1 \mod d_i$, i.e. that $d_i \mid m$. Restricting to this case, write $m = \ell \cdot d_i$. Now the j-value after applying $(\tau h)^m$ is

$$(h_{i,m} \cdot \ldots \cdot h_{i,1})(j) = (h_{i,d_i} \cdot \ldots \cdot h_{i,1})^{\ell}(j) = m_i(h)^{\ell}(j)$$

This shows that $(i, j, 1)$ belongs to an r cycle if and only if $d_i \mid r$ and j belongs to an $\frac{r}{d_i}$-cycle of $m_i(h)$.

Since every orbit of $\tau \cdot h$ includes an element of the form $(i, j, 1)$ (as any triple (i, j, k) goes to an element of the form $(i, j', 1)$ after $(d_i - k + 1)$ applications of τh), one only need to count the number of such elements that belong to r-cycles. By the previous paragraph, the number of j's for which $(i, j, 1)$ belong to r-cycle is equal to the number of $\frac{r}{d_i}$-cycles of $m_i(h)$ when $d_i \mid r$, and that otherwise there are none. Thus

the proclaimed equality follows

$$X_r^{(i)}(\tau h) = \begin{cases} X_{\frac{r}{d_i}}(m_i(h)) & d_i \mid r \\ 0 & \text{otherwise} \end{cases}.$$

$\square$

We are now ready to compute $\binom{X}{\mu}(f)$ for every multi-index μ. Our calculation proceeds using the generating function introduced above. Note that for every fixed N, the class functions X_k with $k > N$ are identically 0, so the function $F(\mathbf{t})$ is really a polynomial in the variables $t_1, \ldots, t_N$. Throughout the proceeding calculation one should remember that all expressions involved are really finite.

Apply the operation $\chi \mapsto \chi^{(\tau)}$ to F termwise: $F^{(\tau)}(\mathbf{t}) = \sum_\mu \binom{X}{\mu}^{(\tau)} \mathbf{t}^\mu$. Using the observation of Claim 2 it follows that

$$
\begin{aligned}
F^{(\tau)} &= \prod_{k=1}^\infty (1 + t_k)^{\sum_{\{i:d_i|k\}}^n X_{k/d_i} \circ m_i} \\
&= \prod_{k=1}^\infty \prod_{\{i:d_i|k\}}^n (1 + t_k)^{X_{k/d_i} \circ m_i} = \prod_{i=1}^\infty \prod_{\ell=1}^\infty (1 + t_{d_i \ell})^{X_\ell \circ m_i}.
\end{aligned}
$$

where in the last equality we relabeled $k = \ell \cdot d_i$ to include only pairs (i, k) in which $d_i | k$. One now notices that the function $F^{(\tau)}$ factors though the product of projections

$$m = (m_1, \ldots, m_n) : H_f \longrightarrow S_{r_1} \times \ldots \times S_{r_n}.$$

The next observation simplifies the problem greatly.

Claim 3. *The product of projections*

$$m = (m_1, \ldots, m_n) : H_f \longrightarrow S_{r_1} \times \ldots \times S_{r_n}$$

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is precisely an $\frac{|H_f|}{|S_{r_1} \times \ldots \times S_{r_n}|}$-to-1 function. In other words, the map m is measure preserving between the two uniform probability spaces.

Proof. Fix any element $(\sigma_1, \ldots, \sigma_n) \in S_{r_1} \times \ldots \times S_{r_n}$. Then for every i and every choice of elements $(h_{i,1}, \ldots, h_{i,d_i-1}) \in S_{r_i}^{d_i-1}$ there exists a unique h_{i,d_i} that satisfies the equality

$$h_{i,d_i} \cdot h_{i,d_i-1} \cdot \ldots \cdot h_{i,1} = \sigma_i$$

namely $h_{i,d_i} = \sigma_i \cdot (h_{i,d_i-1} \cdot \ldots \cdot h_{i,1})^{-1}$. Since the terms with different index i do not appear in this expression, they may be chosen independently.

It follows that there is a bijection $\prod_{i=1}^{n} S_{r_i}^{d_i-1} \cong m^{-1}(\sigma_1, \ldots, \sigma_n)$, thus demonstrating the claim. $\qquad\square$

The fact that $F^{(\tau)}$ factors through the measure preserving map m allows one to compute the H_f-expected value by computing it on $\prod_{i=1}^{n} S_{r_i}$ instead:

$$\mathbb{E}_{H_f}[F^{(\tau)} \circ m] = \mathbb{E}_{\prod_{i=1}^{n} S_{r_i}}[F^{(\tau)}] = \prod_{i=1}^{n} \mathbb{E}_{S_{r_i}}[\prod_{\ell=1}^{\infty}(1 + t_{d_i\ell})^{X_\ell}]. \qquad (4.3.7)$$

The derivation of our formula follows from the following key observation.

Theorem 4.3.9. *Fix $r \in \mathbb{N}$ and let ϵ be a formal nilpotent element such that $\epsilon^{r+1} = 0$ but $\epsilon^r \neq 0$. Then for every $d \in \mathbb{N}$ there is an equality*

$$G_{d,r}(\boldsymbol{t}, \epsilon) := \mathbb{E}_{S_r}[\prod_{\ell=1}^{\infty}(1 + \epsilon^\ell t_{d\ell})^{X_\ell}] = \exp\left(\sum_{\ell} \epsilon^\ell \frac{t_{d\ell}}{\ell}\right). \qquad (4.3.8)$$

Furthermore, the introduction of ϵ to the left-hand side of the equation does not cause any loss of information in the sense that replacing every nonzero power of ϵ by 1 recovers the expectations. Formally, denoting the ring $\mathbb{Q}[t_1, t_2, \ldots]$ by A, the A-module

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map $A[\epsilon]/(\epsilon^{r+1}) \xrightarrow{\phi_\epsilon} A$ defined by $\epsilon^j \mapsto 1$ for all $j \leq r$ sends

$$\phi_\epsilon : G_{d,r}(\boldsymbol{t}, \epsilon) \mapsto G_{d,r}(\boldsymbol{t}, 1).$$

The latter function is the generating function of the expectations $\mathbb{E}_{S_r}\left[\binom{X_{\ell/d}}{\mu}\right]$ (defined to be 0 unless $d|\ell$).

Proof. Expand the left-hand side

$$\prod_{\ell=1}^{\infty} (1 + \epsilon^\ell t_{d\ell})^{X_\ell} = \sum_\mu \binom{X}{\mu} \epsilon^{1\mu_1 + 2\mu_2 + \cdots} t_d^{\mu_1} t_{2d}^{\mu_2} \cdots = \sum_\mu \binom{X}{\mu} \epsilon^{\|\mu\|} t_d^{\mu_1} t_{2d}^{\mu_2} \cdots . \quad (4.3.9)$$

Recall that if $\|\mu\| > r$ then $\binom{X}{\mu} \equiv 0$, so setting $\epsilon^i = 1$ for all $i \leq r$ is the same as having 1's in place of ϵ everywhere.

Use the following calculation of [CEF2].

Fact 4.3.10 (In the proof of [CEF2, Proposition 3.9]). *If μ is a multi-index with $\|\mu\| \leq r$ then*

$$\mathbb{E}_{S_r}\left[\binom{X}{\mu}\right] = \prod_{\ell=1}^{\infty} \frac{1}{\ell^{\mu_\ell} \mu_\ell!}.$$

When multiplying this equation by $\epsilon^{\|\mu\|}$ one gets an equality that holds for all μ. Thus when evaluating the expectation on the generating function

$$\mathbb{E}_{S_r}\left[\sum_\mu \binom{X}{\mu} \epsilon^{\|\mu\|} t_d^{\mu_1} t_{2d}^{\mu_2} \cdots\right] = \sum_\mu \frac{(\epsilon t_d)\mu_1}{1^{\mu_1}\mu_1!} \frac{(\epsilon^2 t_{2d})\mu_2}{2^{\mu_2}\mu_2!} \cdots = \prod_{\ell=1}^{\infty} \sum_{\mu_\ell=1}^{\infty} \frac{1}{\mu_\ell!} \left(\frac{\epsilon^\ell t_{d\ell}}{\ell}\right)^{\mu_\ell}$$

$$= \prod_{\ell=1}^{\infty} \exp\left(\frac{\epsilon^\ell t_{d\ell}}{\ell}\right).$$

To get the stated form of this expression, use the multiplicative property of the exponential series. $\square$

Apply this observation to the calculation of $\mathbb{E}_{H_f}[F^{(\tau)}]$: introduce n nilpotent elements ϵ_i with respective order $r_i + 1$ and let ϕ be the map $\mathbb{Q}[\mathbf{t}]$-linear map that sends $(\epsilon_i)^j \mapsto 1$ whenever $j \leq r_i$. Then

$$\mathbb{E}_{H_f}\left[F^{(\tau)}(\mathbf{t})\right] = \prod_{i=1}^{n} \mathbb{E}_{S_{r_i}}[\prod_{\ell=1}^{\infty}(1 + t_{d_i\ell})^{X_\ell}] = \phi\left[\prod_{i=1}^{n} \exp\left(\sum_\ell \epsilon_i^\ell \frac{t_{d_i\ell}}{\ell}\right)\right]$$

$$= \phi\left[\exp\left(\sum_{i=1}^{n}\sum_\ell \epsilon_i^\ell \frac{t_{d_i\ell}}{\ell}\right)\right].$$

Relabel the terms $d_i\ell = k$ back by summing only over $\{i : d_i | k\}$ and replacing $\ell = k/d_i$. The resulting expression becomes

$$\mathbb{E}_{H_f}\left[F^{(\tau)}(\mathbf{t})\right] = \phi\left[\exp\left(\sum_{k=1}^{\infty} \sum_{\{i:d_i|k\}} \epsilon_i^{k/d_i} d_i \frac{t_k}{k}\right)\right]. \qquad (4.3.10)$$

To bring the expression to the desired form, replace the ϵ_i's with the symbols $\epsilon_p \in R_q$ introduced above. Recall that the $(S_{r_i})^{\times d_i}$-factor of H_f corresponds to the irreducible factor p_i of degree d_i that divides f precisely r_i times. Thus, using the symbol ϵ_{p_i} and its evaluation of f as defined in Equation 4.3.1, there is an equality

$$\phi(\epsilon_i^d) = \epsilon_{p_i}^d(f) = \begin{cases} 1 & d \leq r_i \\ 0 & \text{otherwise} \end{cases}.$$

More generally, $\phi\left[\epsilon_1^{j_1} \ldots \epsilon_n^{j_n}\right] = 1$ if $j_i \leq r_i$ for all i and is 0 otherwise. This is precisely the value of $\epsilon_{p_1}^{j_1} \ldots \epsilon_{p_n}^{j_n} = \epsilon_{p_1^{j_1} \ldots p_n^{j_n}}$ evaluated on f, since the p_i's are coprime. It follows that Equation 4.3.10 can be written with ϵ_{p_i} in the place of ϵ_i everywhere.

For every other irreducible polynomial $p \neq p_1, \ldots, p_n$ the evaluation $\epsilon_p(f) = 0$, so adding all such symbols into Equation 4.3.10 does not change the resulting evaluation.

It does, however, allow us to write the sum uniformly without making any reference to the divisors of f. This is the sought after form of the generating function.

Lastly, to get the individual expectations $\binom{X}{\mu}(f) = \mathbb{E}_{H_f}\left[\binom{X}{\mu}^{(\tau)}\right]$ one needs only to look compute the μ-th partial derivative with respect to $\mathbf{t}$. This resulting expression is as stated. □

4.3.2   The case of all monic polynomials

We now complete the calculation in the case where one does not impose any restrictions on roots, i.e. we are considering the S^N action on the whole of $\mathbb{A}^N$. The cohomology groups of $\mathbb{A}^N$ are very simple: they are known to be $\mathbb{Q}_\ell(0)$ in dimension 0 and vanish in all higher degrees. Moreover, the induced S_N-action on these cohomology groups is trivial. Thus Theorem 4.1.2 reduces to the surprising Corollary 4.1.5.

Proof of Corollary 4.1.5. For every S_N-class function χ Theorem 4.1.2 gives an equality

$$\frac{1}{q^N} \sum_{f \in \mathrm{Poly}^N(\mathbb{F}_q)} \chi(\sigma_f) = \langle H^0(\mathbb{A}^N; \mathbb{Q}_\ell), \chi \rangle_{S_N} = \frac{1}{N!} \sum_{g \in S_N} 1 \cdot \chi(g).$$

□

Remark 4.3.11. In Corollary 4.1.5, there is no reason to restrict attention to the group action $S_N \curvearrowright \mathbb{A}^N$: one can more generally consider any subgroup of $\mathrm{Aut}(\mathbb{A}^N)(\mathbb{F}_q) = \mathrm{Aff}_N(\mathbb{F}_q)$ and get a similar result:

Theorem 4.3.12 (**Equal expectation for** $G \leq \mathrm{Aff}_N(\mathbb{F}_q)$). *Let a subgroup $G \leq \mathrm{Aff}_N(\mathbb{F}_q)$ act on $\mathbb{A}^N$ naturally, and denote the resulting quotient $\mathbb{A}^N/G$ by X_G. Then every G-class function χ induces random variables on the two uniform probability*

spaces $X_G(\mathbb{F}_q)$ and G, and these satisfy

$$\frac{\left|X_G(\mathbb{F}_q)\right|}{q^N} \mathbb{E}_{X_G(\mathbb{F}_q)}[\chi] = \mathbb{E}_G[\chi].$$

We will not pursue this idea any further.

Example 4.3.13 (Case $\chi = \binom{X}{\mu}$). The explicit form of $\binom{X}{\mu}(f)$ given by Theorem 4.1.7 can be used to unpack Equation 4.1.5 and get definite facts regarding polynomials.

Claim 4. *Equation 4.1.5 is equivalent to the well known Necklace relations: for every degree d let N_d be the number of degree d monic irreducible polynomials in $\mathbb{F}_q[t]$, then for every $k \in \mathbb{N}$ one has*

$$\sum_{d|k} dN_d = q^k. \tag{4.3.11}$$

More specifically, for every multi-index $\mu = (\mu_1, \mu_2, \ldots)$ the following equality holds

$$\mathbb{E}_{\mathrm{Poly}^N(\mathbb{F}_q)}\left[\binom{X}{\mu}\right] = \mathbb{E}_{S_N}\left[\binom{X}{\mu}\right] \prod_{k=1}^{\infty} \left(\frac{1}{q^k} \sum_{d|k} dN_d\right)^{\mu_k}. \tag{4.3.12}$$

Proof. The major challenge in computing the evaluation $\binom{X}{\mu}(f)$ directly is that one has to deal with the irreducible decomposition of f, and these can take many forms. It turns out that this challenge disappears completely when one in only interested in the average value over all $f \in \mathrm{Poly}^N(\mathbb{F}_q)$, as the following observation shows.

Lemma 4.3.14 (Average of ϵ_g). *Let $\mathbb{E}$ denote the expectation over the set $\mathrm{Poly}^N(\mathbb{F}_q)$ of monic polynomials of degree N as a uniform probability space. Then for every polynomial g, the symbol ϵ_g satisfies*

$$\mathbb{E}[\epsilon_g] = \begin{cases} \frac{1}{q^{\deg(g)}} & \deg(g) \leq N \\ 0 & Otherwise. \end{cases} \tag{4.3.13}$$

Stated equivalently, let ϵ be a formal nilpotent element of order $N + 1$. Then the $\mathbb{Q}$-algebra homomorphisms $R_q \xrightarrow{\Lambda} \mathbb{Q}[\epsilon]/(\epsilon^{N+1})$ defined by linearly extending

$$\Lambda : \epsilon_g \mapsto \left(\frac{\epsilon}{q}\right)^{\deg(g)} \qquad \forall g \in \mathbb{F}_q[t]$$

and the $\mathbb{Q}$-module map $\phi_\epsilon : \mathbb{Q}[\epsilon]/(\epsilon^{N+1}) \longrightarrow \mathbb{Q}$ defined by sending $\epsilon^j \mapsto 1$ for all $j \leq N$ satisfy the relation

$$\phi_\epsilon \circ \Lambda = \mathbb{E}.$$

Thus the expectation factors through the homomorphism Λ, which sends all ϵ_g symbols to a single expression involving ϵ.

Proof. First, if $\deg(g) > N$ then $\epsilon_g \equiv 0$ on Poly^N and thus $\mathbb{E}[\epsilon_g] = 0$. Next, if $\deg(g) \leq N$, then since $\mathbb{F}_q[t]$ is a UFD, division gives a bijection

$$\left\{ f \in \mathrm{Poly}^N(\mathbb{F}_q) \text{ s.t. } g | f \right\} \longleftrightarrow \mathrm{Poly}^{(N-\deg(g))}(\mathbb{F}_q) .$$

Thus the number of monic polynomial divisible by g is precisely $q^{N-\deg(g)}$. Evaluating the expectation amounts to dividing this count by the cardinality of $\mathrm{Poly}^N(\mathbb{F}_q)$, which is q^N. $\square$

Let $F(\mathbf{t})$ be the generating function of the $\binom{X}{\mu}$'s as introduced in the previous section. Recall that by Theorem 4.1.7

$$F(\mathbf{t}) = \exp\left(\sum_{k=1}^{\infty} \sum_{p \in \mathrm{Irr}_{|k}} \deg(p) \epsilon_p^{\frac{k}{\deg(p)}} \frac{t_k}{k} \right)$$

as functions on $\mathrm{Poly}^N(\mathbb{F}_q)$. Using Lemma 4.3.14 proved above, one can readily compute $\mathbb{E}[F(\mathbf{t})]$ by first applying Λ: Since the evaluation $\Lambda : \epsilon_g \mapsto \left(\frac{\epsilon}{q}\right)^{\deg(g)}$ is a $\mathbb{Q}$-algebra

homomorphism, it commutes with taking the exponential series

$$\Lambda(F) = \exp\left(\sum_{k=1}^{\infty} \sum_{p \in \mathrm{Irr}_{|k}} \deg(p)\Lambda(\epsilon_p)^{\frac{k}{\deg(p)}} \frac{t_k}{k}\right) = \exp\left(\sum_{k=1}^{\infty} \sum_{p \in \mathrm{Irr}_{|k}} \deg(p)\left(\frac{\epsilon}{q}\right)^k \frac{t_k}{k}\right).$$

Collect all terms that depend only on k

$$\exp\left(\sum_{k=1}^{\infty} \frac{\epsilon^k t_k}{k} \cdot \frac{1}{q^k} \sum_{p \in \mathrm{Irr}_{|k}} \deg(p)\right).$$

Reindex the sum $\displaystyle\sum_{p \in \mathrm{Irr}_{|k}} \deg(p)$ based on the degrees $d = \deg(p)$, and observe that it is precisely the sum $\sum_{d|k} dN_d$ that appears in the Necklace relations.

By differentiation with respect to $\mathbf{t}$ one finds that the μ-th coefficient of this generating function is

$$\Lambda\binom{X}{\mu} = \prod_{k=1}^{\infty} \frac{\epsilon^{k\mu_k}}{k^{\mu_k}\mu_k!} \cdot \left(\frac{1}{q^k}\sum_{d|k} dN_d\right)^{\mu_k} = \epsilon^{\|\mu\|}\prod_{k=1}^{\infty} \frac{1}{k^{\mu_k}\mu_k!} \cdot \left(\frac{1}{q^k}\sum_{d|k} dN_d\right)^{\mu_k}.$$

Furthermore, Fact 4.3.10 produced an equality

$$\phi_\epsilon\left(\epsilon^{\|\mu\|}\prod_{k=1}^{\infty} \frac{1}{k^{\mu_k}\mu_k!}\right) = \mathbb{E}_{S_N}\left[\binom{X}{\mu}\right]$$

which produces the desired result using $\phi_\epsilon \circ \Lambda = \mathbb{E}_{\mathrm{Poly}^N(\mathbb{F}_q)}$. $\qquad\square$

4.3.3   Statistics on cosets of Young subgroups

One can rephrase Theorem 4.1.7 as a purely combinatorial statement regarding the cycle-decomposition statistics of cosets of Young subgroups. This could be stated as follows.

As stated above, for every multi-index $\mu = (\mu_1, \mu_2, \ldots)$, the character polynomial $\binom{X}{\mu}$ counts the number of ways to arrange cycles into sets of μ_k many k-cycles for every k.

Theorem 4.3.15 (Statistics on Young cosets). *Let $H = S_{\lambda_1} \times \ldots \times S_{\lambda_m}$ be a Young subgroup of S_N. Consider a coset gH where $g \in S_N$ is in the normalizer of H (i.e. $gH = Hg$). Then conjugation by g induces a permutation τ of the S_{λ_i} factors, say*

$$H = S_{r_1}^{d_1} \times \ldots \times S_{r_n}^{d_n}$$

and τ cyclically permutes the factors in each $S_{r_i}^{d_i}$.

Then for every multi-index $\mu = (\mu_1, \mu_2, \ldots)$, the expected value of $\binom{X}{\mu}$ on the coset gH is given by the expression

$$\mathbb{E}_{S_N}\left[\binom{X}{\mu}\right] \Phi \left[\prod_{k=1}^{\infty} \left(\sum_{\{i : d_i | k\}} \epsilon_i^{\frac{k}{d_i}} d_i \right)^{\mu_k} \right] \tag{4.3.14}$$

where the symbol ϵ_i is a formal nilpotent element of order $r_i + 1$, and one eliminates these applying the $\mathbb{Q}$-linear transformation

$$\Phi : \mathbb{Q}[\epsilon_1, \ldots, \epsilon_n]/(\epsilon_i^{r_i+1}) \longrightarrow \mathbb{Q} \quad , \qquad \Phi : (\epsilon_i)^j \mapsto 1 \quad \forall j \leq r_i.$$

In particular, for every cycle type $\mu = (\mu_1, \mu_2, \ldots, \mu_N)$, the number of elements in gH with cycle-type μ is given by the expression

$$\frac{|H| \cdot |C_\mu|}{N!} \Phi \left[\prod_{k=1}^{N} \left(\sum_{\{i : d_i | k\}} \epsilon_i^{\frac{k}{d_i}} d_i \right)^{\mu_k} \right] \tag{4.3.15}$$

where C_μ is the conjugacy class of all elements with cycle-type μ, and ϵ_i and Φ are as above.

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Example 4.3.16 (Expected number of k-cycles in $gH \subseteq S_N$). Let H and g be as in the statement of Theorem 4.3.15. Compute the expected number of k-cycles in an element $\sigma \in gH$, i.e. the average of X_k over gH, using Equation 4.3.14 with $\mu_k = 1$ and $\mu_j = 0$ for all $j \neq k$. It takes the form

$$\frac{1}{k} \Phi \left(\sum_{\{i:d_i | k\}} \epsilon_i^{k/d_i} d_i \right) = \frac{1}{k} \sum_{\substack{\{i:d_i | k\} \\ d_i r_i \geq k}} d_i.$$

Proof of Theorem 4.3.15. The proof given for Theorem 4.1.7 applies more generally in this case: replacing the set of roots $\{\alpha_{i,k:1 \leq i \leq n, k \in \mathbb{Z}/d_i\mathbb{Z}}\}$ by a formal set of pairs $\{(i,k) : 1 \leq i \leq n, k \in \mathbb{Z}/d_i\mathbb{Z}\}$, and the Frobenius permutation by the formal permutation $\tau : (i,k) \mapsto (i, k+1)$, the proof proceeds as presented above.

This shows that our derivation leading up to Equation 4.3.10 applies and the expected values of $\binom{X}{\mu}$ on gH are given by the generating function

$$\Phi \left[\exp \left(\sum_{k=1}^{\infty} \sum_{\{i:d_i | k\}} \epsilon_i^{k/d_i} d_i \frac{t_k}{k} \right) \right].$$

Extract the μ-th coefficient by derivation with respect to $\mathbf{t}$. The resulting Taylor coefficient is

$$\Phi \prod_{k=1}^{\infty} \frac{1}{k^{\mu_k} \mu_k!} \left(\sum_{\{i:d_i | k\}} \epsilon_i^{k/d_i} d_i \right)^{\mu_k}.$$

Now use Fact 4.3.10 to substitute $\prod_{k=1}^{\infty} \frac{1}{k^{\mu_k} \mu_k!} = \mathbb{E}_{S_N} \left[\binom{X}{\mu} \right]$ and arrive at the final form of our equation.

Lastly, the statement regarding the case $\|\mu\| = N$ follows from the observation that $\binom{X}{\mu}$ is the indicator function of C_μ. $\qquad \square$

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