Topic Proposal
Applying Representation Stability to Arithmetic Statistics

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1 Introduction

The classical Grothendieck-Lefschetz fixed point formula relates the number of points on a variety over a finite field to the trace of the Frobenius automorphism on its étale cohomology. By considering a more refined result for cohomology with local coefficients, we can compute more subtle statistics other than just the number of points on the variety. Thus we can translate questions about the arithmetic of $\mathbb{F}_q$ into questions about cohomology and vice versa.

In many natural examples, the cohomology of sequences of spaces exhibits a form of stability called *representation stability*. Translating these stability results into the world of arithmetic using the Lefschetz formula yields asymptotic stabilization in arithmetic counts, and conversely, arithmetic stabilization implies results about representation-stable cohomology. In this topic proposal I will introduce the archetypal example of this phenomenon and the tools involved in the proof.

1.1 The space of square free polynomials

For a field $k = \mathbb{C}$ or $\mathbb{F}_q$, define

$$\begin{aligned}\mathrm{Poly}_n(k) &= \left\{(a_0,\ldots,a_{n-1}) \in k^n \mid \mathrm{P}(t) = t^n + \sum_{i=0}^{n-1} a_i t^i \text{ has no repeated roots in } \overline{k}\right\} \\ &= k^n - \{\Delta = 0\}\end{aligned}$$

Here, Δ denotes the discriminant.

This is a quasi-projective variety defined over $\mathbb{Z}$, and if $k = \mathbb{C}$, it is also a complex analytic manifold. We will use representation stability to study its cohomology with local coefficients, and see how this information implies the stabilization of a large class of statistics for polynomials over finite fields. In §1.2 we start by considering the untwisted case to illustrate how topology and arithmetic are reflected in each other.

1.2 Point counts and the cohomology of Poly_n

On the arithmetic side, we consider the varieties $\mathrm{Poly}_n(\mathbb{F}_q)$ of polynomials over finite fields. The main invariant of these spaces is their cardinality $\#\mathrm{Poly}_n(\mathbb{F}_q)$. This can be computed combinatorially, but we will use étale cohomology and the Grothendieck-Lefschetz fix point formula, since this will later allow us to apply representation stability.

In an attempt to solve the Weil conjectures, Grothendieck constructed the étale cohomology theory for schemes, giving a bridge between arithmetic and topology. On the one hand, this theory is geometric enough, as demonstrated by the following.

Fact 1 (Artin's comparison theorem). If X is a smooth nonsingular algebraic variety defined over $\mathbb{C}$, then for every finite group Λ there exists an natural isomorphism

$$\mathrm{H}^i(X_{\text{ét}};\Lambda) \xrightarrow{\sim} \mathrm{H}^i(X(\mathbb{C});\Lambda)$$

where the codomain is singular cohomology with coefficients in Λ.

By taking the direct limit over $\Lambda = \mathbb{Z}/n\mathbb{Z}$ and tensoring with $\mathbb{Q}_\ell$ we get an isomorphism of the cohomology groups with coefficients in $\mathbb{Q}_\ell$. Thus étale cohomology captures the same geometric information as singular cohomology.

On the other hand, étale cohomology is defined by purely algebraic means. Thus it can be applied to finite fields. Let q be some prime power, $\mathbb{F}_q$ the field of q elements and $\overline{\mathbb{F}}_q$ the algebraic closure. The map $\mathrm{Frob}_q(x) = x^q$ is the Frobenius automorphism on $\overline{\mathbb{F}}_q$, and its fixed points are precisely the elements of $\mathbb{F}_q$. Similarly, for any variety X defined over $\mathbb{F}_q$ there is an induced Frobenius automorphism, which on every affine chart is given by

$$\mathrm{Frob}_q(x_1, \ldots, x_n) = ({x_1}^q, \ldots, {x_n}^q).$$

Since X is defined using equations with coefficients in $\mathbb{F}_q$ and those are fixed by Frob_q, this map is an algebraic automorphism of X whose fixed points are precisely $X(\mathbb{F}_q)$.

To compute the number of these fixed points, we will apply the Grothendieck-Lefschetz fix point formula for the étale cohomology

$$\# X(\mathbb{F}_q) = \# \operatorname{fix}(\mathrm{Frob}_q) = \sum_{i \geq 0} (-1)^i \operatorname{tr}\left({\mathrm{Frob}_q}^* |_{\mathrm{H}^i_{ét}(X;\mathbb{Q}_\ell)} \right). \tag{1}$$

However, this formula applies only to projective schemes, and Poly_n is affine, so we must replace $\mathrm{H}^i_{ét}$ with compactly supported étale cohomology. In many important examples, $X = \mathrm{Poly}_n$ included, the map ${\mathrm{Frob}_q}^*$ acts on H^i_c by the scalar $q^{-(n-i)}$ where $n = dim(X)$. By Poincaré duality, $dim_{\mathbb{Q}_\ell}\left(\mathrm{H}^{2n-i}_c(X;\mathbb{Q}_\ell)\right) = dim_{\mathbb{Q}_\ell}\left(\mathrm{H}^i_{ét}(X;\mathbb{Q}_\ell)\right)$, which, together with Artin's comparison, gives us a precise formula for $\# \mathrm{Poly}_n(\mathbb{F}_q)$ whose input is purely topological:

$$\# X(\mathbb{F}_q) = \sum_{i \geq 0} (-1)^i q^{n-i} dim\left(\mathrm{H}^i(X(\mathbb{C}))\right) \tag{2}$$

Now we consider topology. Arnol'd studied the space $\mathrm{Poly}_n(\mathbb{C})$ as a smooth manifold and proved in [1] that its rational cohomology is $\mathbb{Q}$ in dimensions $i = 0$ and 1 and vanishes otherwise.

Plugging this in, we get the well-known point count $\# \mathrm{Poly}_n(\mathbb{F}_q) = q^n - q^{n-1}$.

1.3 The S_n-cover $\mathrm{Conf}_n(\mathbb{C})$

Definition 2. Let $\mathrm{Conf}_n(\mathbb{C})$ be the space of n-ordered tuples of distinct points in the plane $\mathbb{C}$, topologized as a subspace of $\mathbb{C}^n$.

This space is the complement of the hyperplanes $z_i = z_j$ for $1 \leq i \neq j \leq n$, and thus it is also an algebraic variety defined over $\mathbb{Z}$. The group S_n acts topologically freely on Conf_n by permuting the coordinates, and the quotient Conf_n / S_n is the configuration space of n unordered points in the place. There is a natural identification between this quotient Conf_n/S_n and Poly_n given by

$$\{z_1, \ldots, z_n\} \mapsto P(t) = t^n - (z_1 + \ldots + z_n)t^{n-1} + \ldots + (-1)^n (z_1 \cdot \ldots \cdot z_n) = \prod_{i=1}^{n} (t - z_i)$$

Thus Conf_n is a regular S_n-covering space of Poly_n. The fiber above a polynomial $P(t) \sim \{z_1, \ldots, z_n\}$ is the set of all orderings of these roots $\{(z_{\sigma(1)}, \ldots z_{\sigma(n)}) \mid \sigma \in S_n\}$. Any two such orderings can be connected by a path in Conf_n, whose image down in Poly_n is a loop based at P. Thus any permutation of the roots is realized by some based loop in Poly_n.

The same construction works over any algebraically closed field k. In particular we consider $\overline{\mathbb{F}}_q$ and the action of Frob_q on this cover. A polynomial $P(t) \in \overline{\mathbb{F}}_q[t]$ is fixed by Frob_q if and only if all its coefficients are in $\mathbb{F}_q$, in which case Frob_q acts on the set of it's roots by a permutation σ_P. Every orbit is the set of roots of an irreducible factor of P. Thus the number of k-cycles in σ_P is equal to the number of degree-k irreducible factors of P.

As we shall see below, this information together with the Grothendieck-Lefschetz fixed point formula reveals many arithmetic statistics of polynomials over $\mathbb{F}_q$. For example, we will be able to compute the expected number of degree k irreducible factors of a random polynomial $P \in \mathbb{F}_q[t]$ using cohomological information about $\mathrm{Conf}_n(\mathbb{C})$.

2 Twisted coefficients and the Lefschetz fixed point formula

In order to compute the more subtle statistics we must replace the cohomology theory used above by cohomology with twisted coefficients.

2.1 Local systems and cohomology with twisted coefficients

Let B be a locally path-connected space and V a finite dimensional vector space over some field k.

Definition 3. A *local coefficient system with fiber V* is a fiber bundle whose fiber is V, on which trivialization along paths is invariant under homotopy with fixed endpoints.

Suppose $p : \widetilde{B} \to B$ is a finite regular G-cover and E a local system over B with fiber V. The pullback $\widetilde{E} = p^*(E)$ is again a local system over $\widetilde{B}$. If E pulls-back to the trivial V-bundle, then the deck group G acts on the fiber V via the identification

$$\widetilde{E}_{\tilde{b}} = E_b = \widetilde{E}_{g(\tilde{b})}$$

for all $b \in B$, $\tilde{b} \in p^{-1}(b)$ and $g \in G$.

Given a regular G-cover $p : \widetilde{B} \to B$ as above, every G-representation V arises in this way from some local system $E \to B$ by defining $E = \left(\widetilde{B} \times V\right)/_G$ where G acts diagonally on the two factors, and the projection map to B is obvious one.

Example 4. Let B be the space of polynomials $\mathrm{Poly}_n(\mathbb{C})$ discussed above. For any polynomial $f \in \mathrm{Poly}_n$ let $R(f)$ denote the set of its roots $\{z_1, \ldots, z_n\} \subset \mathbb{C}$. For some field k define the following k-bundle R over $B = \mathrm{Poly}_n$: the fiber above the point $f \in B$ is the n-dimensional k-vector space of functions $s : R(f) \to k$ with pointwise operations. This bundle is flat, since polynomials $g \in B$ very close to f will have roots very close to those of f and thus we can identify $R(f)$ with $R(g)$ canonically near f and trivialize the bundle accordingly.

This is a nontrivial bundle, since going around a loop γ that permutes the roots by some permutation $\sigma : R(f) \to R(F)$ will take a function $s : R(f) \to k$ to the permuted function $s \circ \sigma$.

Since the action of $\pi_1(\mathrm{Poly}_n) = B_n$ on the fiber of R factors through the the quotient $B_n \to S_n$, the pullback of R to a bundle over Conf_n is trivial: the fiber over $(z_1, \ldots, z_n)$ is the set of k-valued functions on $z_1, \ldots, z_n$ and thus canonically identified with functions on the set $\{1, \ldots, n\}$ via the bijection $i \leftrightarrow z_i$.

A deck transformation $\sigma \in S_n$ takes the basepoint $(z_1, \ldots, z_n)$ to the point $(z_{\sigma(1)}, \ldots, z_{\sigma(n)})$ and the fibers over these points are identified via the identification with the fiber over the unordered set $\{z_1, \ldots, z_n\}$. Thus the identification sends the function $s : i \mapsto z_i \mapsto s(z_i)$ to the function $s' : i \mapsto z_{\sigma(i)} \mapsto s(z_{\sigma(i)})$, i.e. it send s to $s \circ \sigma$.

Definition 5. Let B be a space and E a local system over B with fiber V. The *cohomology of B with coefficients in the local system E* is the cohomology whose n-cochains $C^n(B; E)$ are functions on $C_n(B)$ that for every singular simplex σ in B assign a constant section $\widetilde{\sigma}$ of E over σ. Define addition and scalar multiplication on these sections by the pointwise operations in every fiber.

For a trivial bundle this reduces to the standard singular cohomology with coefficients in V.

Assume that for the G-cover $p : \widetilde{B} \to B$, the pullback bundle $\widetilde{E}$ is trivial and there is an induced G-action on V. We will now show that the dimension of $\mathrm{H}^i(B;E)$ can be computed from the characters of the G-reps V and $\mathrm{H}^i(\widetilde{B};V)$, thus representation stability results will apply.

If $\mathrm{char}(K) = 0$, we have a transfer map which identifies $\mathrm{H}^i(B;E)$ with the G-invariant part of $\mathrm{H}^i(\widetilde{B};\widetilde{E}) = \mathrm{H}^i(\widetilde{B};V)$. However, unlike the untwisted case, the deck group G acts on the coefficients V as well as on the space $\widetilde{B}$. Thus we get the following presentation for the cohomology of B.

Proposition 6. $\mathrm{H}^i(B;E) \cong \mathrm{H}^i(\widetilde{B};V)^G \cong \left(\mathrm{H}^i(\widetilde{B};\mathbb{C}) \otimes_{\mathbb{C}} V\right)^G \cong \mathrm{H}^i(\widetilde{B};\mathbb{C}) \otimes_{\mathbb{C}[G]} V,$
where the middle isomorphism comes from the universal coefficient theorem.

The dimension of these spaces can thus be expressed in terms of their characters as G-representations.

Corollary 7. $dim_{\mathbb{C}}\ \mathrm{H}^i(\widetilde{B};\mathbb{C}) \otimes_G V = dim_{\mathbb{C}}\ \mathrm{Hom}_G\left(V^*, \mathrm{H}^i(\widetilde{B};\mathbb{C})\right) = \langle V^*, \mathrm{H}^i(\widetilde{B};\mathbb{C})\rangle_G$
where V^* is the dual representation to V and the inner product is the standard one.

Remark 8. We can replace the bundle $E \to B$ with the sheaf of it's locally constant sections, denoted by $\mathcal{F}_{\mathcal{E}}$. This sheaf is clearly locally constant, and the cohomology of B with coefficients in $\mathcal{F}$ is the same as the previously defined cohomology with twisted coefficients $\mathrm{H}^i(B;E)$.

This formulation of cohomology with twisted coefficients makes sense in the setting of schemes, and will allow us to apply the results to étale cohomology.

2.2 Twisted Lefschetz fixed point formula

To find a formula for the arithmetic statistics, we wish to apply the Lefschetz formula to twisted cohomology.

Let $p : \widetilde{B} \to B$ be a finite regular G-cover, and $E \to B$ a local system that pulls back to a trivial V-bundle. Let $\widetilde{B} \xrightarrow{\widetilde{f}} \widetilde{B}$ be a map that commutes with the deck action. Then it induces a map $B \xrightarrow{f} B$ such that $f \circ p = p \circ \widetilde{f}$. Similarly, $\widetilde{F} = \widetilde{f} \times \mathrm{Id}_V$ is an bundle map of $\widetilde{E} = \widetilde{B} \times V$ that commutes with the deck action and induces a bundle map $E \xrightarrow{F} E$. These bundle maps induce pullbacks on cochains which descend to endomorphisms on cohomology with twisted coefficients.

Consider a fixed point $b \in B$ of f. The map $\widetilde{f}$ permutes the fiber $p^{-1}(b)$ say by taking a point $\tilde{b}$ to $g_b.\tilde{b}$. The bundle map F then induces an automorphism of the E-fiber $E_b \simeq V$ which we will denote by F_b. This automorphism coincides with g_b^{-1} since

$$F\left([b',v]\right) = [f(b'),v] = [g.b',v] = [b', g^{-1}v]$$

The Lefschetz fix point formula relates the local information about the behavior of F at fixed points of f to the global information about the action of F on cohomology.

Theorem 9 (Twisted Lefschetz fix point formula)**.** *Let B be a closed oriented manifold, E a local system over B and $\widetilde{B}$ a finite regular G-cover of B over which E pulls-back to a trivial bundle. Suppose $\widetilde{f} : \widetilde{B} \to \widetilde{B}$ commutes with the deck action and the maps f, $\widetilde{F}$ and F are defined as above. Moreover, suppose that the fiber V is finite dimensional and that the graph of f intersects the diagonal $\Delta \subset B \times B$ transversely at finitely many points. Then the following equality holds:*

$$\sum_{f(b)=b} \mathrm{tr}\left(g_b^{-1} \curvearrowright V\right) = \sum_{f(b)=b} \underbrace{\mathrm{tr}\left(F_b \curvearrowright E_b\right)}_{local\ information} = \sum_{i\geq 0} (-1)^i \underbrace{\mathrm{tr}\left(F^* \curvearrowright \mathrm{H}^i(B;E)\right)}_{global\ information}. \tag{3}$$

When working with non-closed manifolds, one must use compactly supported cohomology with twisted coefficients, defined in the obvious way.

This formula comes from computing the intersection number of the graph of f and the diagonal in $B \times B$, in two different ways. The proof follows the same lines as that of the classical Lefschetz formula for constant coefficients.

Since the diagonal class has the same form in the setting of étale cohomology (see [7]), the same formula will hold, as we now explain. Let X be a scheme defined over $\mathbb{F}_q$ with a Galois G-cover $\widetilde{X} \to X$ (analogous to a regular G-cover of a manifold). Every finite dimensional G-representation V gives rise to a locally constant sheaf $\mathcal{F}$ on X whose pullback to $\widetilde{X}$ is the constant sheaf $\overline{V}$ and the deck action of G the stalks coincides with the action on V. The Frobenius automorphism commutes with the deck action and thus we can construct endomorphisms on cohomology, as we did in the case of manifolds.

Suppose $x \in X$ is a closed point fixed by $Frob_q$. Denote by $g_x \in G$ the deck transformation by which $Frob_q$ acts on the fiber over x.

Theorem 10 (Twisted Grothendieck-Lefschetz fix point formula)**.** *If X be a nonsingular projective variety defined over $\mathbb{F}_q$ and $\mathcal{F}$ a locally constant sheaf of finite dimensional $\mathbb{Q}_\ell$-vector spaces. Then*

$$\sum_{\mathrm{Frob}_q(x)=x} \mathrm{tr}\left(g_x^{-1} \curvearrowright V\right) = \sum_{\mathrm{Frob}_q(x)=x} \mathrm{tr}\left(\mathrm{Frob}_q^* \curvearrowright \mathcal{F}_x\right) = \sum_{i\geq 0} (-1)^i \,\mathrm{tr}\left(\mathrm{Frob}_q^* \curvearrowright \mathrm{H}^i_{\acute{e}t}(X/_{\overline{\mathbb{F}}_q}; \mathcal{F})\right). \qquad (4)$$

The sum at the left is over the closed fixed points of Frob_q *in X, i.e. precisely over $X(\mathbb{F}_q)$.*

If X is not projective, the cohomology must be replaced with compactly supported cohomology.

Example 11. Consider $X = \mathrm{Poly}_n$ and $\widetilde{X} = \mathrm{Conf}_n$ as schemes defined over $\mathbb{Z}$. By the above, for every S_n-representation $\rho : S_n \to GL(V)$ we can associate a locally constant sheaf $\mathcal{F}$ on Poly_n that pulls back to the constant sheaf on Conf_n on which S_n acts by ρ. If P is a fixed point of the Frobenius map, the induced action on $\mathcal{F}_P$ is the inverse of the deck transformation $\sigma_P \in S_n$ by which Frob_q acts on the fiber above P. This is precisely the permutation σ_P defined in §1.3. Thus we get a formula for the local terms

Corollary 12.

$$\mathrm{tr}\left(\mathrm{Frob}_q^* \curvearrowright \mathcal{F}_P\right) = \mathrm{tr}\left(\sigma_P^{-1} \curvearrowright V\right) = \chi_V(\sigma_P^{-1}) = \chi_V(\sigma_P) \qquad (5)$$

where χ_V is the character of the S_n-representation V.

The rightmost equality follows since every permutation is conjugate to its inverse.

2.3 Hyperplane complements and Poincaré duality

In the case of hyperplane complements, such as Conf_n, we will derive an explicit formula for statistics over finite fields. Let the hyperplane $\mathrm{H}_j \subset \mathbb{A}^n$ be the kernel of a linear functional L_j. Then the restriction $L_j : (\mathbb{A}^n - \{\mathrm{H}_j = 0\}) \to (\mathbb{A} - 0)$ induces a map on étale cohomology.

Fact 13. It is a theorem of étale cohomology theory that $\mathrm{H}^i_{\acute{e}t}(\mathbb{A} - 0; \mathbb{Q}_\ell) \simeq \mathbb{Q}_\ell$ if $i = 0{,}1$ and vanishes in all other dimensions. The Frobenius map acts on $\mathrm{H}^1_{\acute{e}t}$ by multiplication by q. Moreover, the cohomology of hyperplane complements is generated as an algebra by the pullbacks of $\mathrm{H}^1_{\acute{e}t}(\mathbb{A} - 0)$ along the maps L_j (see [6]).

Thus, by the naturality of the cup product, the Frobenius map acts on $\mathrm{H}^i_{\acute{e}t}$ of a hyperplane complement by multiplication by q^i.

The final ingredient involves getting rid of the compactly supported cohomology. Poincaré duality asserts that when $\widetilde{X}$ is a nonsingular variety of dimension n defined over $\overline{\mathbb{F}}_q$, then $\mathrm{H}_c^{2n}(\widetilde{X};\mathbb{Q}_\ell) \simeq \mathbb{Q}_\ell$ on which Frob_q acts by q^n. Moreover, the cup product is a perfect pairing that induces an isomorphism $(\mathrm{H}^i_{\acute{e}t})^* \simeq \mathrm{H}_c^{2n-i}$.

If Frob_q acts on $\mathrm{H}^i_{\acute{e}t}$ by q^i, as is the case for hyperplane complements, the naturality of the cup product implies that Frob_q^* acts on H_c^{2n-i} by q^{n-i}.

Plugging this into the global side of the fixed point formula we get

$$\mathrm{tr}\left(\mathrm{Frob}_q^* \curvearrowright \mathrm{H}_c^{2n-i}\right) = q^{n-i} \cdot dim_{\mathbb{Q}_\ell} \mathrm{H}^i_{\acute{e}t}\,. \tag{6}$$

Example 14. We apply all the above conclusions to the S_n-cover $\mathrm{Conf}_n \to \mathrm{Poly}_n$, seeing as Conf_n is a hyperplane complement, in order to find a formula for statistics of square-free monic polynomials over $\mathbb{F}_q$ in terms of certain characters of S_n.

The naturality of Artin's comparison theorem shows that $\mathrm{H}^i_{\acute{e}t}(Poly_n;\mathbb{Q}_\ell)$ is in fact isomorphic to $\mathrm{H}^i(Poly_n(\mathbb{C}))$ as an S_n-rep. Combined with the results on transfer, we get the equality

$$dim\,\mathrm{H}^i_{\acute{e}t}(Poly_n;\mathcal{F}) = \langle V^*, \mathrm{H}^i_{\acute{e}t}(Conf_n;\mathbb{Q}_\ell)\rangle_G = \langle V^*, \mathrm{H}^i(Conf_n(\mathbb{C}))\rangle_G. \tag{7}$$

Using the Grothendieck-Lefschetz formula, and the fact the any S_n-representation can be used as the system of local coefficients, we arrive at the following result.

Theorem 15. *For any character χ of an S_n-representation,*

$$\sum_{P\in \mathrm{Poly}_n(\mathbb{F}_q)} \chi(\sigma_P) = \sum_{i\geq 0} (-1)^i q^{n-i} \langle \chi, \mathrm{H}^i(\mathrm{Conf}_n(\mathbb{C}))\rangle_{S_n}. \tag{8}$$

Note that the characters of S_n-representations span the space of class functions on S_n, and that the two sides of the equation are linear in χ.

Corollary 16. *We can replace χ in equation 8 by any class function on S_n.*

Consider the class function χ_k that for every permutation σ returns the number of k-cycles in σ. Since the number of k-cycles of σ_P is the number of degree-k irreducible factors in P, we get an expression for the total number of degree-k irreducible factors over $\mathbb{F}_q$, and similarly for all other statistics of this form.

3 Representation stability and FI-modules

The other half of the story is that the numbers $\langle \chi, \mathrm{H}^i(\mathrm{Conf}_n(\mathbb{C}))\rangle_{S_n}$ stabilize as $n \to \infty$. To prove this, the key observation is that the spaces Conf_n form a single object $\mathrm{Conf}_\bullet$ which we now describe.

3.1 FI-Modules

Definition 17. Let FI be the category of finite sets with injections between them. A (co)FI-module is a (contravariant) covariant functor from the category FI to the category k-Mod. Similarly a (co)FI-space is a similar functor into the category Top. Morphisms of FI-modules and spaces are natural transformations.

Example 18. The ordered configuration space $\mathrm{Conf}_\bullet(X)$ is a coFI-space as follows: for every finite set S let $\mathrm{Conf}_S(X)$ be the space of injections $S \hookrightarrow X$, topologized as a subspace of the cartesian product. For every injection $T \hookrightarrow S$ the map $\mathrm{Conf}_S(X) \to \mathrm{Conf}_T(X)$ is given by precomposing an injection $S \hookrightarrow X$ by the injection $T \hookrightarrow S$.

Composing the functor $\mathrm{Conf}_\bullet(X) : FI \to Top$ with the cohomology functor $\mathrm{H}^i\left((\bullet);k\right)$ we naturally get an FI-module.

The category FI has a skeleton whose objects are the finite sets $\overline{n} = \{1, \ldots, n\}$ where $n \in \mathbb{N}$, and their endomorphisms are the groups $S_n = Aut(\overline{n})$. Thus an FI-module (space) $M_\bullet$ is essentially a sequence of modules (spaces) M_n together with maps between them, compatible with the S_n-actions. The fact that k-Mod is an abelian category turns FI-Mod into an abelian category as well with respect to pointwise kernels and cokernels. In particular we can talk about FI-submodules and quotient objects.

Definition 19. An FI-module $M_\bullet$ is said to be finitely generated if there are finitely many elements $a_i \in M_{n_i}$ that are not contained in any proper FI-submodule. This is equivalent to saying that every M_n is generated as a k-module by the images of these elements. If $N = \max\{n_i\}$, we say that $M_\bullet$ is generated in degree N.

3.2 Representation stability

FI-modules prove very useful and give a uniform treatment of representation stability as defined in [3]. This is illustrated by the following theorem.

Theorem 20. *Suppose k is a field of characteristic 0 and $M_\bullet$ is an FI-module over k finitely generated in degree N. Then the M_n's form a sequence of S_n-representations with maps $f_n : M_n \to M_{n+1}$ which satisfy uniform representation stability with stable range $2N$. This means that the following conditions hold of all $n \geq 2N$:*

1. *(Injectivity) The map $f_n : M_n \to M_{n+1}$ is injective.*
2. *(Surjectivity) The S_{n+1}-orbit of $Im(f_n)$ is the whole degree $n+1$ part of M.*
3. *(Multiplicity) For every partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ s.t. $n - |\lambda| \geq \lambda_1 \geq \ldots \geq \lambda_n$, let $\lambda(n)$ be the padded partition $(n - |\lambda|, \lambda_1, \ldots, \lambda_n)$ of n. Decomposing M_n into irreducible S_n-representation $V_{\lambda(n)}$ indexed by partitions λ, the multiplicity $c_{n,\lambda}$ of $V_{\lambda(n)}$ is independent of n.*

Moreover, the partitions λ for which $V_{\lambda(n)}$ appears in the decomposition of M_n all satisfy $|\lambda| \leq N$.

This follows from the branching rules for S_n-representations (see [5]), which are:

Theorem 21. *Let λ be a partition of n and V_λ the irreducible representation corresponding to λ. We identify partitions with the corresponding Young diagrams.*

1. *$Ind_{S_n \times S_k}^{S_{n+k}} V_\lambda \boxtimes Triv = \oplus V_\mu$, where the μ's are all the Young diagrams obtained from λ by adding k boxes to distinct columns.*
2. *$\left(Res_{S_{n-k} \times S_k}^{S_n} V_\lambda \right)_{S_k} = \oplus V_\mu$ where the μ's are all the Young diagrams obtained from λ by removing k boxes from distinct columns.*

It is clear that after considering all the ways of adding or removing more than λ_1 many boxes, there are no new combinations other than adding or removing boxes from the rightmost end of the top row. Thus the multiplicities of irreducibles appearing in these sequences of inductions or restrictions stabilize.

Theorem 20 also describes the characters of the S_n representations M_n. The function $\chi_k : \coprod S_n \to \mathbb{N}$ that assigns to every permutation $\sigma \in S_n$ the number of k-cycles appearing in σ defines a class function on S_n simultaneously for all n. We call a polynomial $P \in \mathbb{Q}[\chi_1, \chi_2, \ldots]$ a *character polynomial*, and these again define class functions on all the S_n's. Declare the degree of χ_k to be k for every k and extend this degree to all polynomials. It is a fact that for every partition λ there exists a single character polynomial P_λ of degree $|\lambda|$ such that the characters of the S_n-representations $V_{\lambda(n)}$ all coincide with P_λ.

Corollary 22. *If $M_\bullet$ is an FI-module over $\mathbb{Q}$ finitely generated in degree N, then there exists a character polynomial P of degree $\leq N$ s.t. the characters of the S_n-representation M_n coincide with P for all large n.*

3.3 Stabilization of arithmetic statistics

In the previous section we found formulas for statistics over finite fields in terms of $\langle \chi_k, \mathrm{H}^i(\widetilde{X}_n)\rangle_{S_n}$. If the modules $\mathrm{H}^i(\widetilde{X}_\bullet)$ turn out to form an FI-module finitely generated in degree N_i then the above results apply and the characters of $\mathrm{H}^i(\widetilde{X}_n)$ are given by a single character polynomial P_i for all $n \geq 2N_i$. Thus we find that the coefficients appearing in equation 8 are the inner products $\langle \chi_k, P_i\rangle_{S_n}$.

A combinatorial counting argument on S_n shows that the inner product of two character polynomials $\langle P, Q\rangle_{S_n}$ is independent of n for all $n \geq \deg(P) + \deg(Q)$. Thus the coefficient $\langle \chi_k, \mathrm{H}^i(\widetilde{X}_n(\mathbb{C}); \mathbb{Q})\rangle_{S_n}$ is independent of n for all $n \geq 2N_i + k$. Denote the stable coefficient $\lim_{n\to\infty}\langle P, \mathrm{H}^i(\widetilde{X}_n)\rangle_{S_n}$ by $\langle P, \mathrm{H}^i(\widetilde{X})\rangle$.

Example 23 (Hyperplane complement)**.** If $L_1, \ldots, L_k$ are linear functionals on $\mathbb{A}^N$, the FI-hyperplane complement $\widetilde{X}_\bullet$ is defined to be the complement in $\mathbb{A}^\bullet$ of the hyperplanes $L_j \circ \sigma = 0$ for all $1 \leq j \leq k$ and $\sigma \in \mathrm{Hom}_{FI}(\bullet, N)$. As stated above, the cohomology ring of a hyperplane complement is generated by the pullbacks $\omega_j = L_j^*(\omega)$ for $\omega \in \mathrm{H}^1(\mathbb{A}^1 - 0)$. Thus the FI-module $\mathrm{H}^1(\widetilde{X}_\bullet)$ is finitely generated in degree N. Since higher dimensional cohomology is generated by the products of these w_j's, they are also finitely generated and stability follows.

Note however, that for each i the cohomology $\mathrm{H}^i(\widetilde{X}_\bullet)$ is a different FI-module and they are possibly generated in different degrees. Even though the coefficient of q^{n-i} in equation 8 eventually stabilizes for large values of n, more terms appear in the sum and there are contributions from higher dimensional cohomology.

In some cases there is control over the growth of the coefficients $\langle P, \mathrm{H}^i(\widetilde{X})\rangle$ for all P. If these coefficients are bounded by some sub-exponential function $F_P(i)$ then the limit

$$\lim_{n\to\infty} \sum_{i\geq 0} (-1)^i q^{-i} \langle P, \mathrm{H}^i(\widetilde{X}_n)\rangle_{S_n}$$

exists. Thus we get an equality in the limit

$$\lim_{n\to\infty} q^{-n} \sum_{x \in X_n(\mathbb{F}_q)} P(\sigma_x) = \sum_{i\geq 0} (-1)^i q^{-i} \langle P, \mathrm{H}^i(\widetilde{X})\rangle \tag{9}$$

and in particular the limit at the left hand side exists for all q.

Knowing that we have such stabilization, we can now reverse the process and compute the stable cohomology coefficients $\langle P, \mathrm{H}^i(\widetilde{X})\rangle$ from explicit expressions for the left hand side of 9 on a sequence of q's diverging to ∞. The coefficients are determined uniquely by these values by the uniqueness theorem for power series.

4 Applications

We apply the results summarized above to two example.

1. *The space of monic square-free polynomials.* Let χ_k denote the function on $\mathrm{Poly}_\bullet$ that for every polynomial returns the number of it's degree-k irreducible factors.

   Theorem 24. *For every prime power q and a polynomial $P \in \mathbb{Q}[\chi_1, \chi_2, \ldots]$ there is an equality*

   $$\lim_{n\to\infty} q^{-n} \sum_{f \in \mathrm{Poly}_n(\mathbb{F}_q)} P(f) = \sum_{i\geq 0} (-1)^i q^{-i} \langle P, \mathrm{H}^i(\mathrm{Conf})\rangle$$

   and in particular the limits on both sides exist.

The rational cohomology ring of $\mathrm{Conf}_n(\mathbb{C})$ is the alternating algebra generated by the classes $\omega_{ij} \in \mathrm{H}^1$ which count the winding number around the hyperplane $z_i = z_j$, subject only to the relations

$$\omega_{ij} \wedge \omega_{jk} + \omega_{jk} \wedge \omega_{ki} + \omega_{ki} \wedge \omega_{ij} = 0.$$

The group S_n acts on these generators by permuting the indices. We apply the theorem to the character polynomial χ_1, counting the number of linear factors in a square-free polynomial, and compute the two leading terms of this series expansion corresponding to H^0 and H^1.

Clearly H^0 is the trivial representation $V_{(0)}$ and H^1 is the permutation representation on the set of unordered pairs $\{i, j\} \subset \{1, \ldots, n\}$. Decomposing into irreducibles we find $\mathrm{H}^1 = V_{(0)} \oplus V_{(1)} \oplus V_{(2)}$. Using the orthogonality of irreducible characters and the fact that $\chi_1 = \chi_{V_0} + \chi_{V_{(1)}}$,

$$\langle \chi_1, \mathrm{H}^0 \rangle_{S_n} = 1\ , \quad \langle \chi_1, \mathrm{H}^1 \rangle_{S_n} = 2$$

independently of n.

Corollary 25. *The total number of linear factors of square-free polynomials over $\mathbb{F}_q$ is $q^n - 2q^{n-1} + O(q^{n-2})$.*

2. *The space of maximal tori.* Let $\mathcal{T}_n'$ be the space of n linear independent lines in $\mathbb{A}^n$. The quotient $\mathcal{T}_n = \mathcal{T}_n'/S_n$ in the space parameterizing maximal tori in GL_n via associating to every maximal torus it's set of eigenspaces in $\mathbb{A}^n$. Denote by χ_k the function on $M_\bullet$ that for every maximal torus returns the number of it's degree-k irreducible tori.

   The cohomology of $\mathcal{T}_n'$ is known to be concentrated in even dimensions and $H^{2\bullet} = \oplus_{2i \geq 0} \mathrm{H}^{2i}$ isomorphic to the co-invariant algebra $R[z_1, \ldots, z_n] = \oplus_{i \geq 0} R_i$. Moreover, every homogeneous component R_i is a finitely generated FI-module, and the multiplicities $\langle \chi, R_i \rangle$ are known to be sub-exponential. Over $\overline{\mathbb{F}}_q$ the Frobenius automorphism acts on $H^{2i}_{\acute{e}t}$ by q^i, so formula 8 applies under the appropriate substitutions.

   Theorem 26. *For every polynomial $P \in \mathbb{Q}[\chi_1, \chi_2, \ldots]$ we have an equality*

   $$\lim_{n \to \infty} q^{-(n^2-n)} \sum_{\tau \in \mathcal{T}_n(\mathbb{F}_q)} P(\tau) = \sum_{i \geq 0} (-1)^i q^{-i} \langle P, R_i \rangle$$

   here $n^2 - n$ is the dimension of $\mathcal{T}_n$. In particular the limits exist.

   [8] proves a formula for the multiplicity of V_λ in R_i. In particular, it follows that for all $i > 0$ the trivial representation does not occur in R_i.

   Corollary 27. *The total number of maximal tori in $Gl_n(\mathbb{F}_q)$ is precisely*

   $$q^{n^2-n} \left(1 - 0 \cdot q^{-1} + 0 \cdot q^{-2} - \ldots \right) = q^{n^2-n}.$$

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