COMPUTER SIMULATIONS OF SOLAR PLASMAS

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Abstract. Plasma dynamics has been investigated intensively for toroidal magnetic confinement in tokamaks with the aim to develop a controlled thermonuclear energy source. On the other hand, it is known that more than 90% of visible matter in the universe consists of plasma, so that the discipline of plasma-astrophysics has an enormous scope. Magnetohydrodynamics (MHD) provides a common theoretical description of these two research areas where the hugely different scales do not play a role. It describes the interaction of electrically conducting fluids with magnetic fields that are, in turn, produced by the dynamics of the plasma itself. Since this theory is scale invariant with respect to lengths, times, and magnetic field strengths, for the nonlinear dynamics it makes no difference whether tokamaks, solar coronal magnetic loops, magnetospheres of neutron stars, or galactic plasmas are described. Important is the magnetic geometry determined by the magnetic field lines lying on magnetic surfaces where also the flows are concentrated.

Yet, transfer of methods and results obtained in tokamak research to solar coronal plasma dynamics immediately runs into severe problems with trans'sonic' (surpassing any one of the three critical MHD speeds) stationary flows. For those flows, the standard paradigm for the analysis of waves and instabilities, viz. a split of the dynamics in equilibrium and perturbations, appears to break down. This problem is resolved by a detailed analysis of the singularities and discontinuities that appear in the trans'sonic' transitions, resulting in a unique characterization of the permissible flow regimes. It then becomes possible to initiate *MHD spectroscopy of axi-symmetric transonic astrophysical plasmas*, like accretion disks or solar magnetic loops, by computing the complete wave and instability spectra by means of the same methods (with unprecedented accuracy) exploited for tokamak plasmas. These large-scale linear programs are executed in tandem with the non-linear (shock-capturing, massively parallel) Versatile Advection Code to describe both the linear and the nonlinear phases of the instabilities.

Key words: MHD waves, transonic plasmas

1. Introduction and Outline

Magnetized plasmas are essentially extended structures because magnetic field lines do not have a beginning or end ($\nabla \cdot \mathbf{B} = 0$). This implies that regions of space are connected that have very different physical properties. For example, solar magnetism arises due to nuclear fusion powering in the extremely dense core of the Sun, radiation transport establishing a convectively stable temperature profile up to the convection zone ($R \sim 0.7\ R_{\odot}$) where dynamo action by convective instability and differential rotation produces concentrated magnetic field bundles that

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are expelled from the sun proper, giving rise to tremendously complex magnetic field structuring and dynamics in the photosphere and corona (see any SOHO or TRACE web site). Along the magnetic field lines that escape, the solar wind carries a tenuous plasma that is accelerated to transonic speeds and exhibits discontinuous flow (shocks) when the magnetospheres of the planets are encountered and when the heliosphere is finally terminated beyond the solar system.

Obviously, such a complex system (with the huge variety of relevant spatial and temporal scales) cannot be described by a single analytical or computational model. Instead, we here present an approach to some of the plasma dynamical problems encountered in astrophysical plasmas (encompassing solar and space plasmas as well) that is motivated by an attempt to exploit methods that have proved their power for laboratory plasmas. We start by confronting the basic facts of solar magnetism with the main constituents of magnetohydrodynamics (MHD), viz. the description by conservation laws (the most important one being magnetic flux conservation), the occurrence of specific waves and instabilities (in particular, Alfvén wave dynamics), the distinct stationary flow patterns with *trans'sonic' transitions* (apostrophes indicating three, rather than one, critical speeds for magnetized plasmas), and the different types of nonlinear dynamics (e.g. shocks).

Whereas subsonic MHD has been highly developed in the context of laboratory plasma fusion research (where plasmas are basically in static equilibrium and perturbations are controlled to avoid the occurrence of sudden disruptions), in transonic MHD models of astrophysical plasmas the basic equilibrium consists of stationary flows admitting a much larger variety of waves and instabilities whereas sudden transitions by shocks are a rule rather than exception. Evidently, since the construction of the dynamical picture for the much simpler static laboratory plasmas took 40 years of intensive research, a similar description for transonic plasmas is still far from completion. Hence, to appreciate the immense theoretical problems associated with trans'sonic' plasma flows, we first recapitulate the results of the simpler static laboratory plasmas (where spectral analysis yields detailed information about the underlying equilibria: MHD spectroscopy), then generalize this method to plasmas with background equilibrium flows (where rotations and outflows produce new types of waves and instabilities), and then try to generalize the obtained picture to trans'sonic' background flows (i.e. construct two-dimensional equilibrium flow patterns, their waves and instabilities, and find the associated shock solutions).

The first part [2. MHD Modeling; 3. MHD Waves; 4. Spectral Theory; 5. Waves in Tokamaks] recapitulates the basic facts of subsonic MHD based on material in *Principles of magnetohydrodynamics* by J. P. Goedbloed and S. Poedts (Cambridge University Press, to appear). The second part [6. Waves in Astrophysical Objects; 7. Transonic Flow: Singularities; 8. Large-Scale Nonlinear Computing; 9. Waves in Astrophysical Objects Revisited] then formulates effective methods and results for transonic flows that are relevant not only for solar plasmas but for astrophysical plasmas in general.

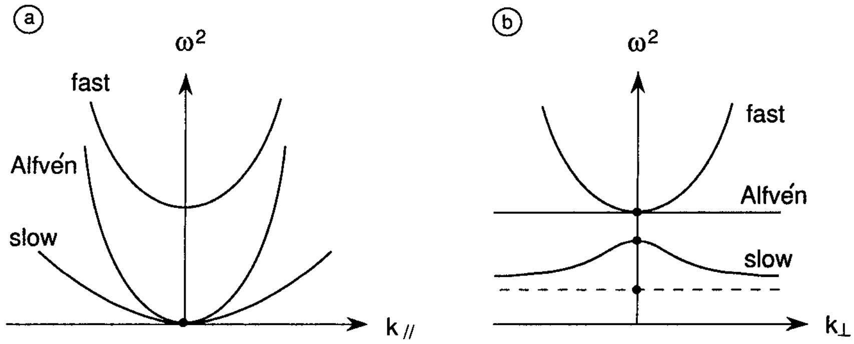

Figure 1. Parallel and perpendicular wavenumber dependence of the frequencies of the three MHD waves.

2. MHD modeling

MHD modeling consists of prescribing: (1) The nonlinear partial differential equations for the motion of *a (perfectly) conducting fluid interacting with a magnetic field* (a perfect transposition of the laws of gas dynamics and electrodynamics); (2) A particular plasma confinement structure, i.e. a generic *magnetic geometry*, fixing the boundary conditions to be imposed. Examples of the latter are the toroidal magnetic confinement geometry of a tokamak (closed in itself), coronal magnetic loops 'closed' onto the photosphere, and magnetic flux bundles emanating from the Sun with 'open' ends associated with the solar wind and the heliosphere.

The strength of the MHD model is that it is *scale invariant*: The MHD equations are unchanged by changing the scales of *length* L_0, *magnetic field* B_0, and density ρ_0, or Alfvén speed $v_A \equiv B_0/\sqrt{\mu_0\rho_0}$, i.e. *time scale* $\tau_A \equiv L_0/v_A$. Thus, MHD is an excellent tool for global analysis of magnetized plasmas on all scales, which justifies the transfer of methods and results from laboratory to astrophysical plasmas.

3. MHD Waves

The three MHD waves (Alfvén, slow, and fast magnetosonic) permit a complete description of the response to arbitrary excitations of a magnetized plasma. However, in the analysis of confined plasmas, the Alfvén waves are the most prominent ones since (1) they may propagate as point disturbances along the magnetic field lines, so that *Alfvén waves 'sample' the magnetic geometry*, (2) their frequency vanishes for $k_\parallel \to 0$ which marks the condition for *marginal stability of tokamaks as well as coronal magnetic flux tubes* (Figure 1(a)). On the other hand, the unique (anisotropic) properties of the three MHD waves are best appreciated by considering the asymptotic dependence of their frequency on the wave number perpendicular to the magnetic field (Figure 1(b)):

$$\begin{cases} \partial\omega/\partial k_\perp > 0\,, & \omega_f^2 \to \infty \qquad \text{(fast)}, \\ \partial\omega/\partial k_\perp = 0\,, & \omega_A^2 \to k_\parallel^2 b^2 \qquad \text{(Alfven)}, \\ \partial\omega/\partial k_\perp < 0\,, & \omega_s^2 \to k_\parallel^2 \dfrac{b^2c^2}{b^2+c^2} \qquad \text{(slow)}, \end{cases} \tag{1}$$

where b is the Alfvén speed and c is the sound speed. Hence, the asymptotic spectra behave distinctly different for the three waves. In inhomogeneous plasmas, they give rise to three *continuous spectra*: $\omega_F^2 \equiv \infty$, $\{\omega_A^2\}$, and $\{\omega_S^2\}$.

4. Spectral Theory

Analogous to quantum mechanics, spectral theory of MHD waves and instabilities revolves about the two equivalent view points of *force* and *energy*, respectively leading to a spectral differential equation in terms of the plasma displacement vector field $\boldsymbol{\xi}$ (Bernstein *et al.*, 1958):

$$\mathbf{F}(\boldsymbol{\xi}) = \rho\frac{\partial^2\boldsymbol{\xi}}{\partial t^2} = -\rho\omega^2\boldsymbol{\xi}\,, \tag{2}$$

and a variational principle for the eigenfrequencies ω^2 of the modes:

$$\delta(W/I) = 0\,, \quad W \equiv -\tfrac{1}{2}\int \boldsymbol{\xi}^*\cdot\mathbf{F}(\boldsymbol{\xi})\,dV\,, \quad I \equiv \tfrac{1}{2}\int |\boldsymbol{\xi}|^2\,dV\,, \tag{3}$$

which involves the quadratic forms $W[\boldsymbol{\xi}]$ for the potential energy and $I[\boldsymbol{\xi}]$ related to the kinetic energy of the perturbations. Whereas quantum mechanical spectral theory has led to a deep understanding of atomic and subatomic structures (occupying much of 20th century physics), the analogous theory for fluids and plasmas is still in its infancy. Yet, the observation of a *classical spectrum* of oscillations and comparison with computed eigenvalues may lead to a firm knowledge of the internal characteristics of fluids and plasmas, which we have called *MHD spectroscopy* (Goedbloed *et al.*, 1993). Relevant examples are helioseismology, sunspot seismology, MHD spectroscopy of tokamaks, and magnetoseismology of accretion disks (Keppens *et al.*, 2002).

In Figure 2, we recall the principle of helioseismology: Comparison of computed frequencies for the p and g modes of a solar model with the observed ones led to the validation of the standard solar model, and may lead to improvements with respect to 2D extensions such as the influence of differential rotation and magnetic fields. The three boxed activities together show what is involved in MHD spectroscopy. For the present purpose, we concentrate on two of them, viz. *analysis* to reveal the structure of spectra and *numerical tools* to compute them. Once these issues are resolved, we will have obtained a very powerful instrument to analyze magnetically confined plasmas.

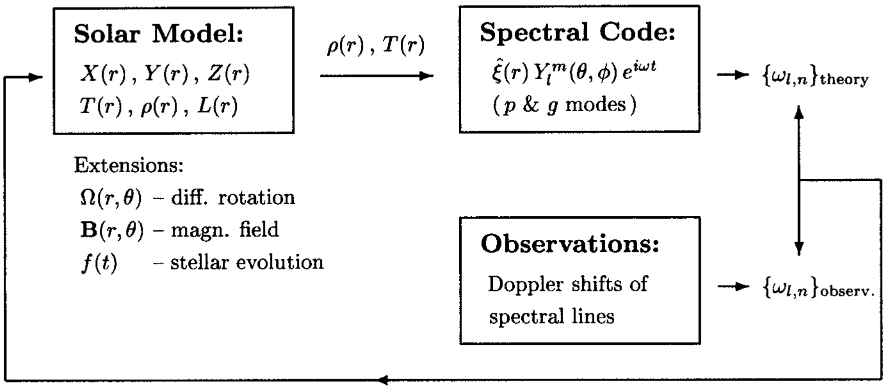

Figure 2. Systematics of helioseismology.

5. Waves in Tokamaks

Let us see how this program is carried out in toroidal fusion experiments of the tokamak type. First, consider the standard case of a static axi-symmetric equilibrium. The basic approach here is to split the problem in a study of the *static equilibrium*, basically described by the force balance equations $\nabla p = \mathbf{j} \times \mathbf{B}$, $\mathbf{j} = \nabla \times \mathbf{B}$, $\nabla \cdot \mathbf{B} = 0$, and the *linear waves and instabilities* described by Equations (2) or (3). The most important property of these equilibria is that they consist of nested *magnetic surfaces* of the magnetic field $\mathbf{B}$ and the current density $\mathbf{j}$, producing confinement of the pressure gradient ∇p through the Lorentz force. We recall from the introduction that the systematic analysis of the spectra (with top priority on the practical issue of improving overall stability for higher values of $\beta \equiv 2\mu_0 p/B^2$) has taken about 40 years of intensive research. This has led to steady increase of confinement, and concomitant understanding of the processes involved, from μseconds in the early days to minutes at present. For our present purpose (a similar effort for astrophysical plasmas with sizeable background flow), this implies: a lot of work ahead and great promise for understanding in the end!

One of the intriguing aspects of wave dynamics in toroidal plasmas is the occurrence of singular perturbations and *continuous spectra* which manifest the preference of the waves and instabilities to localize inside the magnetic surfaces. In Figure 3 we show the schematic structure of the MHD spectrum which clearly demonstrates this (Goedblood, 1975). Most important: through this singular asymptotics, the three MHD wave spectra maintain the essential features of Equation (1) shown by Figure 1(b) and, thus, make them suitable to be used in MHD spectroscopy. Techniques to accurately compute the static equilibria of tokamaks (Huysmans *et al.*, 1991) and a large-scale spectral code to compute the spectra of these 2D equilibria (Kerner *et al.*, 1998) were developed. Recently, the necessary accurate MHD spectra of ideal and resistive waves in static tokamak equilibria could be com-

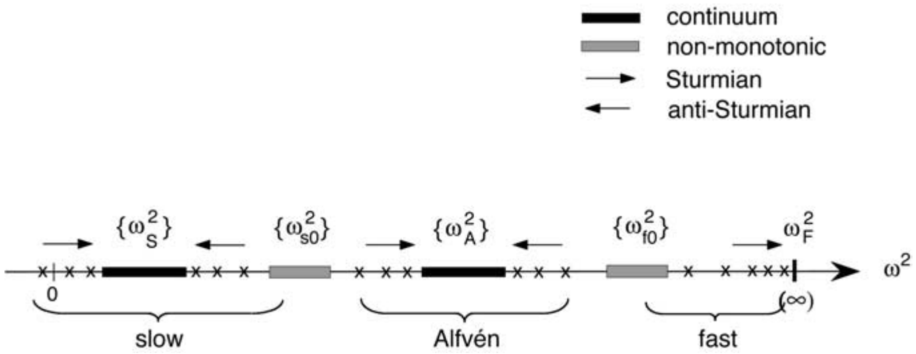

Figure 3. Schematic structure of the spectrum of MHD waves for a static equilibrium. Three sub-spectra of fast, Alfvén, and slow modes concentrate about continua $\omega_F^2 \equiv \infty$, $\{\omega_A^2\}$, and $\{\omega_S^2\}$, separated by regions with non-monotonic discrete modes. The inhomogeneity is chosen to be small so that sub-spectra are well separated.

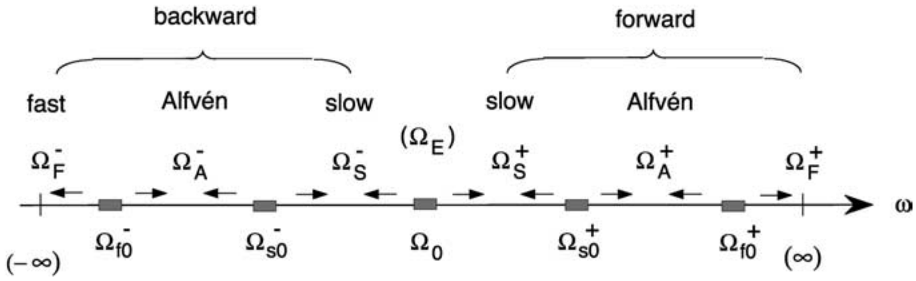

Figure 4. Schematic structure of the spectrum of MHD waves for an equilibrium with flow: The three sub-spectra split into six sub-spectra of forward and backward propagating fast, Alfvén, and slow modes, concentrated about the continua $\Omega_F^{\pm} \equiv \pm\infty$, $\{\Omega_A^{\pm}\}$, $\{\Omega_S^{\pm}\}$, where $\Omega_\ell^{\pm} \equiv \Omega_0 \pm \omega_l$ $(\ell = F, A, S)$ with Doppler shift $\Omega_0 \equiv \mathbf{k} \cdot \mathbf{v}$. The picture should be asymmetric with respect to $\omega = 0$ (not indicated).

puted in full detail (Van der Holst *et al.*, 1999) with the powerful Jacobi–Davidson method (Sleijpen and Van der Vorst, 1996).

An exciting new development in tokamak research is the realization that static equilibria are actually not quite adequate since heating by neutral beams causes sizeable toroidal flows and divertor operation for exhaust removal causes (supersonic!) poloidal flows in the outer layers. Hence, the *paradigm of static equilibrium breaks down*. However, since astrophysical plasmas are all dominated by flow, the good side about this development is that the subject of plasmas with background flow now becomes a common research theme for laboratory and astrophysical plasmas.

In order to enter this common field, all spectral calculations have to be redone with proper incorporation of the background flow of the equilibrium. Again exploiting the standard approach, with a split in equilibrium and perturbations, this first involves construction of a stationary state (where $\mathbf{v} \neq 0$ so that all MHD equations contribute now) and, next, computation of the waves and instabilities by means of a quadratic eigenvalue equation (Frieman and Rotenberg, 1960):

$$\mathbf{F}(\boldsymbol{\xi}) + \nabla \cdot \left[\rho(\mathbf{v} \cdot \nabla \mathbf{v})\boldsymbol{\xi} - \rho \mathbf{v}\mathbf{v} \cdot \nabla \boldsymbol{\xi}\right] + 2i\rho\omega \mathbf{v} \cdot \nabla \boldsymbol{\xi} + \rho\omega^2 \boldsymbol{\xi} = 0\,. \tag{4}$$

We note in passing that, in contrast to the widely used spectral Equation (2) for static equilibria, the Frieman and Rotenberg spectral Equation (4) has rarely been applied to realistic stationary states. The obvious reason is that the equilibria are much more complicated and that the eigenvalues are complex (admitting overstable modes). Another, even more fundamental, problem will be faced in Section 6.

The schematic spectral structure of stationary equilibria (Figure 4) is again concentrated about the continuous spectra, which now split into six due to the Doppler shift. [An additional, somewhat esoteric, Eulerian entropy continuum $\{\Omega_E\}$ is not found from the Lagrangian Equation (4), but only when the primitive, Eulerian, variables are exploited.] On the road to a systematic MHD spectroscopy of moving plasmas, with precise input of tokamak equilibria, these continua turn out to contain large gaps where new global Alfvén waves driven by the toroidal flow (called TFAEs) were discovered (Van der Holst, 2000). This appears to open up a new chapter in MHD spectroscopy which, obviously, calls for a generalization admitting poloidal flows as well. To do this, we constructed the necessary numerical tools FINESSE (Beliën *et al.*, 2002) to compute the stationary axi-symmetric equilibria and PHOENIX Van der Holst *et al.*, 2003) for the perturbations, and started to apply them to tokamaks. This worked well as long as the poloidal velocities were restricted to sub‘sonic’ flows. In this manner, we have contributed to MHD spectroscopy as a highly developed tool to investigate the dynamics of plasmas in future fusion machines.

6. Waves in Astrophysical Objects: A Hair in the Soup

So far, so good. But why did we have to restrict the poloidal velocities to sub‘sonic’ speeds? [Recall our use of apostrophes to indicate the occurrence of three (slow/ Alfvén/ fast), rather than one, critical MHD speeds.] Obviously, such a restriction is prohibitive if we wish to exploit the same tools for astrophysically relevant flows, which are usually trans‘sonic’. What happens precisely when the critical MHD speeds are surpassed?

Consider the stationary equilibrium equations for rotating and gravitating magnetized plasmas:

$$\begin{aligned}
&\nabla \cdot (\rho \mathbf{v}) = 0\,,\\
&\rho \mathbf{v} \cdot \nabla \mathbf{v} + \nabla p = \mathbf{j} \times \mathbf{B} - \rho \mathbf{g}\,, \qquad \mathbf{j} = \nabla \times \mathbf{B}\,,\\
&\mathbf{v} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{v} = 0\,,\\
&\nabla \times (\mathbf{v} \times \mathbf{B}) = 0\,, \qquad \nabla \cdot \mathbf{B} = 0\,.
\end{aligned} \tag{5}$$

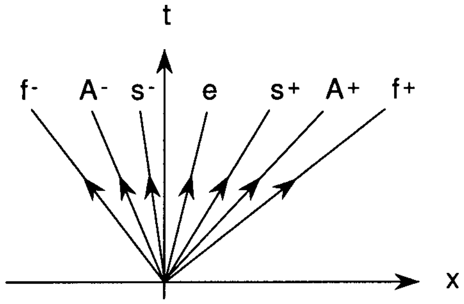

Figure 5. Space-time characteristics of the three MHD waves ($s^{\pm}$, $A^{\pm}$, $f^{\pm}$), travelling in forward and backward directions, and the entropy disturbances (e), which are just carried with the plasma flow.

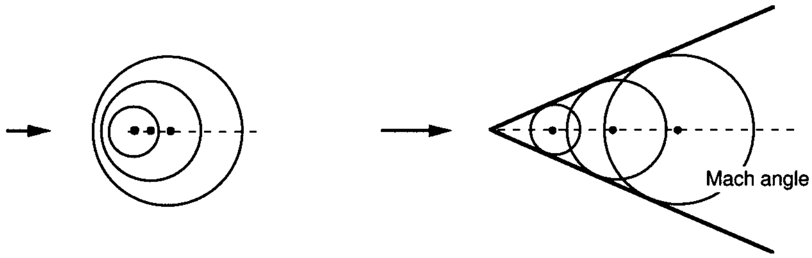

Figure 6. Sound in (a) subsonic and (b) supersonic gas flow about a point source.

For axi-symmetric geometries, like tokamaks and magnetic flux loops in the solar convection zone, or even complete accretion disks, these equations have solutions that basically correspond to nested surfaces of the magnetic field **B** (not of the current density **j** anymore) and of the plasma velocity **v**. Hence, the stronghold of magnetic confinement (related to the existence of magnetic surfaces) remains intact in the presence of arbitrary toroidal and poloidal plasma flows: *magnetic and flow surfaces coincide*! Before we start to indicate some blemishes on this beautiful edifice, let us first spell out the portent of this statement: Together with scale-invariance of the MHD equations, this implies that we can transfer the techniques and results on MHD spectroscopy, developed in laboratory tokamak research, directly to astrophysical problems like axi-symmetric winds, accretion flows, jets, etc.

Yet, we get stuck immediately if we try to do this. Invariably, when the plasma velocities are increased, the equilibrium solvers stop converging before relevant velocities are obtained. The hair in the soup comes from the velocity component in the symmetry-breaking direction, i.e. the poloidal direction. This is so because the equilibrium Equations (5) have a property, entering with the poloidal flow, that is completely lacking in their static counterparts (obtained from them in the limit $\mathbf{v} \to 0$). To appreciate it, we need to make a small detour in the topic of transonic flow.

A tacit assumption in the construction of the equilibria, including the ones with toroidal flows, has been that the governing Grad-Shafranov (nonlinear partial differential) equation is *elliptic*. The numerical techniques exploited need this property. In fact, all of the standard methods in use in MHD spectral analysis are based on the assumption that the equilibria are described by elliptic equations and the perturbations by hyperbolic ones. However, when the poloidal flow velocity increases beyond certain critical values, to be computed yet, the stationary equilibrium Equations (5) become hyperbolic (Zehrfeld and Green, 1972; Hameiri, 1983) and both the classical paradigm of a split in equilibrium and perturbations and the numerical techniques based on it break down. As a result, the standard equilibrium solvers, as used in tokamak computations, diverge and we need to rethink the problem completely.

Clearly, we have to go back to basics, in particular to the meaning of hyperbolicity. This concept is associated with the *characteristics* of the flow, which are the space-time manifolds along which perturbations propagate. For MHD, there are seven of such characteristics, as shown in Figure 5 for the case of one spatial dimension. Permitting two spatial dimensions, the temporal snapshots of the three MHD perturbations become the well-known figures of the Friedrichs group diagram. In two dimensions, these figures may exhibit an interesting new feature, depending on the magnitude of the background flow. This is illustrated in Figure 6 for the case of sound waves in ordinary fluids: When the flow velocity becomes supersonic, the spatial part of the characteristics forms envelopes where information accumulates and discontinuous solutions (shocks) are formed. Whereas in elliptic flows the solutions propagate everywhere in space, in hyperbolic flows these discontinuities separate space in regions where the solutions propagate and regions where they do not propagate. Unfortunately, although magnetic/flow surfaces exist in axi-symmetric MHD flows, the transitions from ellipticity to hyperbolicity occur somewhere, at a-priori unknown locations, on these surfaces and the elliptic solvers become useless.

The fundamental reason of the bankruptcy of the classical paradigm of equilibrium and perturbations is associated with the Lagrangian time derivative $D/Dt \equiv \partial/\partial t + \mathbf{v} \cdot \nabla$ in the MHD equations. Whereas, the Eulerian time derivative $\partial/\partial t$ produces the eigenfrequencies ω of the waves, the spatial derivative $\mathbf{v} \cdot \nabla$ not only produces the Doppler shifts of the perturbations but also the possibility of spatial discontinuities of the equilibria. [Note that this occurs through the poloidal, symmetry-breaking, part only since the toroidal derivative operator vanishes by assumption of axi-symmetry.] However, the two pieces of the Lagrangian time derivative really belong together so that *the waves and the stationary equilibria, with transitions from ellipticity to hyperbolicity, are no longer separate issues* (Goedbloed, 2002).

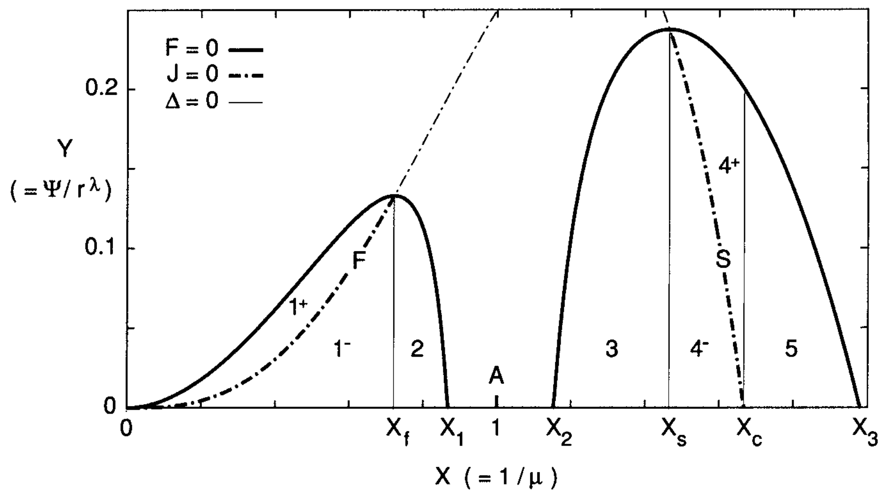

Figure 7. Four main flow regimes due to Alfvén gap (A) and fast (F) and slow (S) magnetoacoustic limiting lines.

7. Transonic Flow: Singularities

Since the transonic transitions present the basic problem, let us analyze some specific stationary equilibria in detail to see what is going on. For the present purpose, it is sufficient to consider 2D equilibria that are translation symmetric (Goedbloed and Lifschitz, 1997) so that the physical quantities are functions of the Cartesian x,y coordinates of the poloidal cross-section of the plasma. The stationary equilibrium states are then characterized by the poloidal magnetic flux $\psi(x, y)$ and by the square of the poloidal Alfvén Mach number, $M^2 \equiv \rho v_p^2/B_p^2 = \mu(x, y)$. The flux ψ is determined by a partial differential equation (a generalization of the Grad–Shafranov equation) that is elliptic or hyperbolic depending on the value of μ, which is in turn determined by an algebraic equation (the Bernoulli equation). This pair of highly non-linear equations for ψ and μ admits solutions only for certain values of the parameters involved: The distinguishing feature of transonic flows is that *there are distinct flow regimes that cannot be connected by continuous flows when the speed is increased or decreased.*

A specific example is shown in Figure 7, obtained by imposing the following self-similarity in terms of the polar coordinates r, θ in the polidal plane:

$$M^2 \equiv \mu = [X(\theta)]^{-1}\,, \qquad \psi = r^\lambda Y(\theta)\,. \tag{6}$$

This reduces the problem to its bare essentials, viz. the solution of a pair of autonomous differential equations for X and Y:

$$\frac{dX}{d\theta} = \pm\frac{H}{J}\sqrt{2F}\,, \qquad \frac{dY}{d\theta} = \pm\sqrt{2F}\,, \tag{7}$$

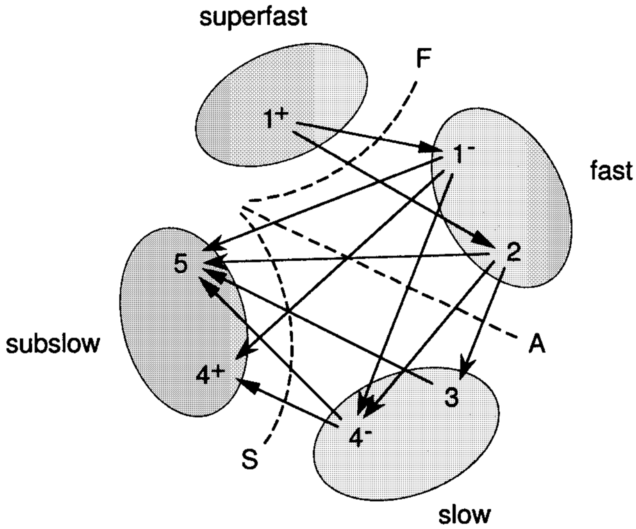

Figure 8. Connecting the four flow regimes: Fast, Alfvén, slow shocks.

where H, J, and F are explicit functions of X and Y. In this case, the different flow regimes show up as regions in the X-Y phase diagram that may be constructed without actually solving Equations (7). First of all, the condition $F(X, Y) = 0$ (the *Bernoulli boundary*) delineates two permissible flow regimes, viz. a *slow* ($X > 1$) and a *fast* ($X < 1$) one, where the poloidal field is real ($F > 0$). These two islands in phase space imply that there is no continuous path from static equilibria ($X = \infty$) to slow stationary equilibria, and also not from the slow to the fast equilibria since a gap at $X = 1$ (the Alfvén gap) interferes. Next, another algebraic condition, $\Delta(X) = 0$, separates the regions of *ellipticity* ($\Delta < 0$: no real characteristics) and *hyperbolicity* ($\Delta > 0$: two real characteristics). Finally, the most dramatic separation of flow regimes is due to the singularity $J(X, Y) = 0$, where the characteristics of the hyperbolic solutions exhibit *limiting line* behavior, i.e. both characteristics are 'reflected' there so that solutions can not propagate beyond the limiting line. Consequently, four *smooth* types of stationary 2D equilibrium solutions are obtained, viz. superfast (1^+), fast (1^-, 2), slow (3, 4^-), and subslow (4^+, 5) ones.

Of course, the explicit solutions of Equation (7) and the corresponding flow patterns have been investigated in detail (Goedbloed and Lifschitz, 1997). However, for our present discussion on the possibility of constructing spectral codes for transonic MHD flows, we just focus on one particular aspect of those solutions: their trajectories $dY/dX = J/H$ in phase space either cross or do not cross the limiting lines. In the latter case, smooth stationary flow solutions are obtained that have the requisite property of globally nested magnetic/flow surfaces. On the other hand, when trajectories cross the limiting lines, multiple solutions are obtained within a sector cutting through the magnetic/flow surfaces. More precisely, magnetic/flow surfaces are exclusively obtained within that sector. Could one reflect the solutions obtained at the sector boundaries (the limiting lines) so as to get periodic *discontinuous* stationary flows involving both super- and sub-critical regimes? Extensive study of the MHD jump conditions, including the requirement that entropy should increase across the discontinuity, has shown that this possibility must be excluded: Discontinuous solutions, satisfying the appropriate jump conditions, can only be found for solutions that stay away from the limiting lines.

We then finally come to the following state of affairs: Four types of smooth periodic stationary MHD equilibria with nested magnetic surfaces are obtained that strictly remain within the main flow regimes. For these equilibria, MHD spectral codes can be constructed with the existing tools. However, stationary *trans'sonic'* MHD flows, connecting two flow regimes, necessarily involve shock-type discontinuities, as illustrated in Figure 8. This picture shows that limiting lines and Alfvén gap are quite genuine obstacles in transonic stationary flows, but it also highlights the fascinating connection between linear waves and stationary states: In analogy to the three types of linear MHD waves, with their local singular asymptotics, in transonic flows also three types of MHD discontinuities appear that locally exhibit slow, Alfvén, and fast character at the singularity.

To sum up: For static or toroidally rotating tokamaks, the equilibria are complicated but essentially computable. When trans'sonic' poloidal flows are admitted, the determination of the stationary states becomes a fundamentally different and difficult problem because *discontinuities and singularities* appear manifesting that the waves and stationary states are entangled in a deep sense. An obvious way out is to drop the idea of a split in equilibrium and perturbations altogether and to employ a nonlinear time stepping code, e.g. the Versatile Advection Code (VAC), which we will discuss in Section 8. This should be considered as an aside though since we do not really wish to abandon the equilibrium–wave dichotomy because it has proved too useful. Therefore, in Section 9, we will return to it and show how to exploit the Frieman–Rotenberg formalism with the knowledge of the present section.

8. Large-Scale Nonlinear Computing

The development of a general set of state-of-the-art spectral codes for the analysis of MHD waves and instabilities for realistic laboratory experiments and astrophysical objects has been stimulated by our studies of resonant absorption in solar coronal flux tubes with inclusion of the geometric influence of line-tying (Halberstadt and Goedbloed, 1993–1995) and loop expansion (Beliën *et al.*, 1996–97). Visualization of coronal heating mechanisms proved to be instrumental for our transition to nonlinear MHD simulations of wave dissipation in flux tubes (Poedts *et al.*, 1996–97; Keppens *et al.*, 1997–98) and, finally, to simulations of SOHO observations (Beliën *et al.*, 1999). In the latter phase, operation of the VAC code was already in full swing.

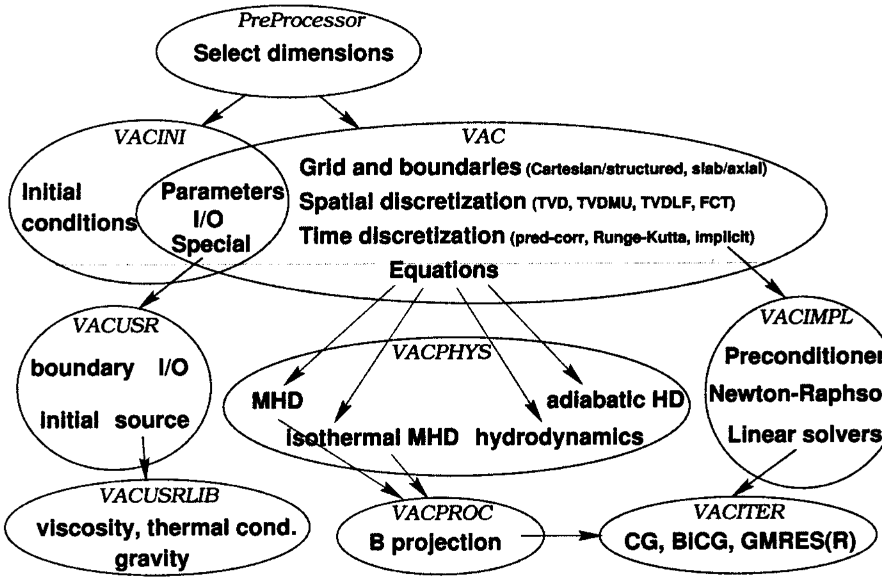

Figure 9. Structure of the Versatile Advection Code (VAC): A modular approach ensures the compatibility between the different code segments. As a result, several spatial and temporal (explicit, semi-implicit, and fully implicit) discretizations are applicable to all physics modules.

The Versatile Advection Code (Tóth, 1996) was developed as part of a Massively Parallel Computing project of NWO (Poedts, Keppens and Goedbloed, 1996–2000). It is a massively parallel MHD solver which is shock-capturing (through the use of conservative variables) and can bridge the huge time-scale disparities (from Alfvénic to dissipative) encountered in realistic astrophysical simulations by means of implicit time integration. Designed to permit inclusion of almost all present discretization methods, with a modular structure (Figure 9), it became an extremely versatile research instrument used by a rapidly increasing number of scientists. The code was steadily developed (Keppens and Tóth, 1999–2000), and applied to basic plasma dynamics like the Kelvin–Helmholtz instability and jets (Keppens *et al.*, 1999). Application to solar and stellar winds from axi-symmetric, rotating and gravitating, stars (Keppens and Goedbloed, 1999–2000) produced continuous acceleration from sub-slow flow at the surface to super-fast flow at large distances. Adding a ‘dead’ zone at the equator, anisotropy as observed by the Ulysses spacecraft was obtained (Figure 10). The recent extension with adaptive mesh refinement (AMR-VAC; Keppens *et al.*, 2002) is yet another step towards simulating realistic astrophysical plasma flows with small-scale structures.

In conclusion: With the new powerful tool VAC to compute the non-linear MHD evolution, we have a completely independent entry into the exciting field of transonic plasma dynamics.

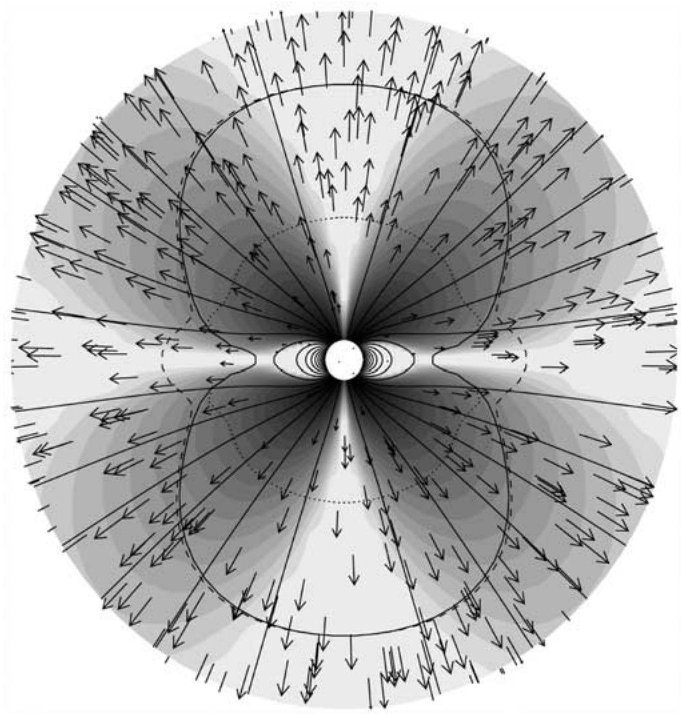

Figure 10. Axisymmetric magnetized wind with a 'wind' and 'dead' zone. Shown are the poloidal magnetic field lines and the poloidal flow field as vectors (parallel to the magnetic field, as they should). Also indicated are the slow (dotted), Alfvén (solid), and fast (dashed) critical surfaces. Shading indicates the toroidal field strength.

9. Waves in Astrophysical Objects Revisited

Returning now to our subject of spectral analysis of trans'sonic' astrophysical plasmas, where the intrinsic difficulty of lack of precise stationary equilibria in the hyperbolic regions appears to be near insurmountable, one might be inclined to settle for a cheap solution: Why not abandon the spectral approach altogether and exclusively exploit nonlinear MHD solvers like VAC? That would be an inferior solution indeed since it would amount to giving up the incredible precise and detailed information that spectral theory delivers on all 3D waves and instabilities *and* their dependence on the relevant physical parameters characterizing the stationary states. Clearly, the royal road is to keep both approaches operational, each in their respective domain of validity, and to try to approach the physical phenomena from the linear as well as from the nonlinear angle. For example, the prediction by a spectral code of exponential instability for a well-described equilibrium is already invaluable for the prescription of initial data for a nonlinear evolution code. However, there is more . . ..

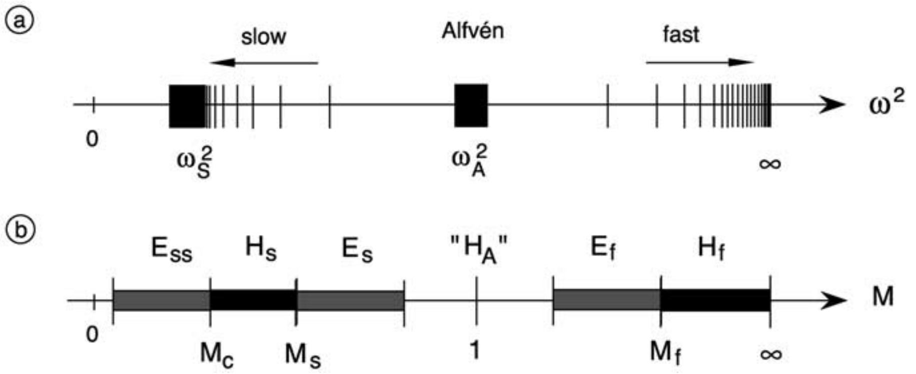

Figure 11. (a) Schematic spectrum of the three MHD waves for a *static* background equilibrium. For large wave numbers, the discrete eigenvalues accumulate at the continua $\{\omega_S^2\}$, $\{\omega_A^2\}$, and $\omega_F^2 \equiv \infty$; (b) Flow regimes characterized by the value of the poloidal Alfvén Mach Number $M \equiv v_p/v_{A,p}$ of a *stationary* equilibrium flow. The flow turns from elliptic to hyperbolic at the boundaries of the hatched regions H_s and H_f, whereas the Alfvén region 'H_A' has collapsed into the point $M \equiv 1$.

Consider again the phase space of our model trans'sonic' stationary states depicted in Figure 7: Clearly, to study the consequences of transition through the hyperbolic regions on the waves and instabilities one does not have to restrict the analysis to the sub-slow elliptic regime 5 (E_{ss}), since there are two more elliptic regimes, viz. the slow regime 3 (E_s), and the fast regime 2 (E_f). Hence, one may study the qualitative change of the spectra due to transition through the critical poloidal Alfvén Mach numbers by comparing the spectra *after* the transition through the slow or through the Alfvén critical value has been made. This may be done on the basis of the standard paradigm of a split in elliptic equilibrium and hyperbolic perturbations, and exploiting the numerical tools based on it. One essential complication must then be faced: The transition speeds M_c, M_s, and M_f depend on the local values of the physical variables, i.e. they are not known beforehand but are to be determined together with the solutions. Hence, staying in the elliptic flow regimes is a delicate numerical problem. This problem has been addressed and satisfactorily solved in the numerical equilibrium solver FINESSE (Beliën *et al.*, 2002). Hence, we can proceed now with the computation of waves and instabilities of trans'sonic' astrophysical plasmas with precise prescriptions of background flows.

Finally, we have argued in Section (5) that linear waves and nonlinear stationary states are not independent issues in trans'sonic' MHD flows. One may turn the coin and notice that this also implies that there is an incredibly beautiful connection between the two. As illustrated in Figure 11, somehow the asymptotic 'concentration' points of the wave spectra correspond to the hyperbolic regions of the equilibrium states, and their embedded singularities. Hence, studying the spectra by approaching the hyperbolic regimes while staying in the elliptic regimes,

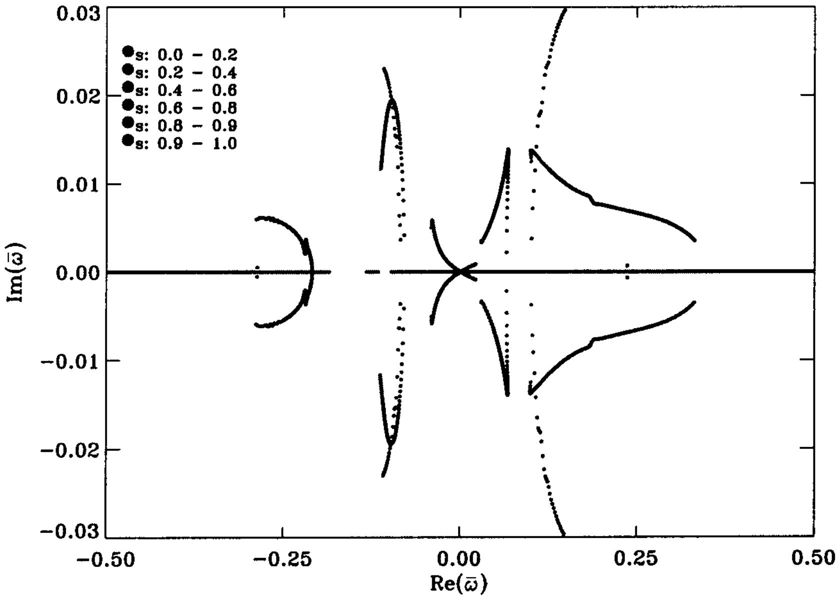

Figure 12. Instabilities in the 2nd elliptic flow regime E_S (slow and sub-Alfvénic: $M_S < M < 1$): Continuous spectrum of waves (real ω) and overstable modes (with an additional imaginary part of ω) of a thick accretion disk; the continuous distribution of the eigenvalue parameter ω is shown in the complex ω-plane with the radial location $s \equiv \sqrt{\psi}$ as a parameter.

undoubtedly will reveal important clues on the physical mechanisms of transonic flows. We will now give just one example to demonstrate this point.

The first spectral results for localized modes of a gravitating torus (a thick accretion disk or any closed flux loop) with both poloidal and toroidal magnetic fields and flows (Beliën *et al.*, 2001) are shown in Figure 12. [This is a corrected version of Figure 4 of Beliën *et al.*, 2001) and of Figure 9 of Goedbloed, 2002)).] The eigenvalues of the stable waves are located along the Re ω-axis, the curves in the complex ω-plane correspond to forward and backward propagating instabilities driven by the poloidal flow and gravity. The spectrum is quite characteristic for flows in the second elliptic flow regime and instability will generally occur when the value of the poloidal Alfvén Mach number for the flow has surpassed the critical value M_c. The instabilities are localized on magnetic/flow surfaces and occupy a large fraction of the outer part of the torus so that they may be considered as suitable candidates for anomalous dissipation by MHD turbulence, e.g. in accretion disks.

In conclusion: We have analyzed the waves and instabilities of tokamaks and toroidal astrophysical plasmas (like thick accretion disks or parts of solar magnetic loops) in the second elliptic flow regime (E_s, i.e. region 3 of Figure 7) and found that significant instabilities operate there that are absent in the first elliptic flow regime (E_{ss}, i.e. region 5). These instabilities should be ascribed to the transonic transition at $M = M_c$. Hence, there appears to be a strong correlation between the singularities and discontinuities that occur in the background nonlinear stationary states when the critical values of the poloidal Alfvén Mach number (lying in the hyperbolic flow regimes, which are as yet inaccessible for spectral studies) are surpassed and the instabilities that are found in the next elliptic flow regime.

The persistent development of the stationary equilibrium program FINESSE and the spectral code PHOENIX, and the accompanying in-depth analysis, have produced a new angle on the study of waves and instabilities in trans'sonic' plasma flows. Presently, the linear codes are operated in tandem with the nonlinear time-stepping code VAC to investigate both the linear and the nonlinear phases of the dynamics in the different flow regimes. They exhibit an abundance of new instabilities of interest for the different kinds of MHD turbulence operating in solar and astrophysical plasmas. Will be continued!

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References

Beliën, A. J. C., Botchev, M. A., Goedbloed, J. P., van der Holst, B. and Keppens, R.: 2002, *J. Comp. Phys.* **182**, 91.

Beliën, A. J. C., Goedbloed, J. P. and Van der Holst, B.: 2001, Proc. 28th Eur. Conf. on *Controlled Fusion and Plasma Physics*, Madeira, p. 1309.

Beliën, A. J. C., Martens, P. C. H. and Keppens, R.: 1999, in *Plasma Dynamics and Diagnostics in the Solar Transition Region and Corona*, 8th SOHO Workshop, ESA **SP-446**, 167–172.

Beliën, A. J. C., Poedts, S. and Goedbloed, J. P.: 1997, *Phys. Rev. Lett.* **76**, 567–570; in *Magnetodynamic Phenomena in the Solar Atmosphere–Prototypes of stellar magnetic activity*, IAU Coll. **153**, 423–424 (Kluwer Dordrecht, 1996); *Astron. Astrophys.* **322**, 995–1006 (1997); *Comp. Phys. Comm.* **106**, 21–38 (1997); in *The Corona and Solar Wind near Minimum Activity*, 5th SOHO Workshop, ESA **SP-404**, 193–197 (1997).

Bernstein, I. B., Frieman, E. A., Kruskal, M. D. and Kulsrud, R. M.: 1958, *Proc. Roy. Soc. London* **A244**, 17–40.

Frieman, E. and Rotenberg, M.: 1960, *Rev. Mod. Phys.* **32**, 898–902.

Goedbloed, J. P.: 2002, *Physica Scripta* **T98**, 43–47.

Goedbloed, J. P.: 1975, *Phys. Fluids* **18**, 1258–1268.

Goedbloed, J. P., Huysmans, G. T. A., Holties, H., Kerner, W. and Poedts, S.: 1993, *Plasma Phys. Contr. Fusion* **35**, B277–292.

Goedbloed, J. P. and Lifschitz, A.: 1997, *Phys. Plasmas* **4**, 3544.

Halberstadt, G. and Goedbloed, J. P.: 1995, *Astron. Plasmas* **301**, 577–592; **280**, 647–660 (1993); **286**, 265–301 (1994); **301**, 559–576 (1995).

Hameiri, E.: 1983, *Phys. Fluids* **26**, 230–237.

Huysmans, G. T. A., Goedbloed, J. P. and Kerner, W.: 1991, in *Comput. Phys.* **371** (World Scientific, 1991).

Keppens, R. and Goedbloed, J. P.: 2000, *Astron. Astrophys.* **343**, 251–260 (1999); *Space Sci. Rev.* **87**, 223–226 (1999); *Astrophys. J* **530**, 1036–1048.

Keppens, R. and Tóth, G.: 2000, in *Third International Conference for Vector and Parallel Processing*, Lecture Notes in Computer Science **1573**, 680–690 (Springer, 1999); Keppens, R., Tóth, G. and Goedbloed, J.P.: 2000, in *Pallallel Computing: Fundamentals and Applications*, ParCo99, 160–167, Imperial College Press, London.

Keppens, R., Casse, F. and Goedbloed, J. P.: 2002, *Astrophys. J.* **569**, L121–L126.

Keppens, R., Nool, M. and Goedbloed, J. P.: 2002, in *Pallallel Computational Fluid Dynamics – Practice and Theory*, 215–223 (Elsevier Science, 2002); Nool, M. and Keppens, R.: 2002, *Comp. Math. Appl. Math* **2**, 92–109.

Keppens, R., Tóth, G., Westermann, R. H. J. and Goedbloed, J. P.: 1999, *J. Plasma Phys.* **61**, 1–19 (1999); Keppens, R. and Tóth, G.: 1999, *Phys. of Plasmas* **6**, 1461–1469.

Keppens, R., Poedts, S., Meijer, P. M. and Goedbloed, J. P., in *High Performance Computing and Networking*, Lecture Notes in Computer Science **1225**, 190–199 (Springer, 1997); Keppens, R., Poedts, S. and Goedbloed, J. P., in *High Performance Computing and Networking*, Lecture Notes in Computer Science **1401**, 233–241 (Springer, 1998).

Kerner, W., Goedbloed, J. P., Huysmans, G. T. A., Poedts, S. and Schwarz, E.: 1998, *J. Comp. Phys.* **142**, 271–303.

MPP-CMFD team, ed. Poedts, S., Keppens, R. and Goedbloed, J. P.: 2000, 1st, 2nd, 3rd, Final Annual Report of the Massively Parallel Computing Cluster Project 95MPR04, Rijnhuizen Reports 96–231 (1996); 97–233 (1997); 98–234 (1998); 00–235 (2000).

Poedts, S. and Goedbloed, J. P., in *Magnetodynamic Phenomena in the Solar Atmosphere–Prototypes of Stellar Magnetic Activity*, IAU Coll. **153**, 425–426 (Kluwer Dordrecht, 1996); *Astron. Astrophys.* **321**, 935–944 (1997); Poedts, S., Tóth, G., Beliën, A. J. C. and Goedbloed, J.P., *Solar Physics* **172**, 45–52 (1997).

Sleijpen, G. L. G. and Van der Vorst, H. A.: 1996, *SIAM J. Matrix Anal. Appl.* **17**, 401.

Tóth, G.: 1996, *Astrophys. Lett. Comm.* **34**, 245.

Van der Holst, B., Beliën, A. J. C. and Goedbloed, J. P.: 2003, (in press).

Van der Holst, B., Beliën, A. J. C. and Goedbloed, J. P.: 2000, *Phys. Rev. Lett.* **84**, 2865; *Phys. Plasmas* **7**, 4208.

Van der Holst, B., Beliën, A. J. C., Goedbloed, J. P., Nool, M. and Van der Ploeg, A.: 1999, *Phys. Plasmas* **6**, 1554–1561.

Zehrfeld, H. P. and Green, B. J.: 1972, *Nuclear Fusion* **12**, 569–575.