Appendix F

Bounds for Bilinear Forms

F.1 The operator norm of a matrix

In various situations we are confronted with a problem of bounding a bilinear form—namely an expression of the general shape

$$\sum_{m=1}^{M}\sum_{n=1}^{N} a_{mn}x_n y_m.$$

In applications the x_n and y_m may have considerable arithmetic structure, but we can often obtain a serviceable estimate using only the mean square sizes of the variables. Thus we seek an inequality of the sort

$$\Big|\sum_{m,n} a_{mn}x_n y_m\Big| \le \Delta\Big(\sum_n |x_n|^2\Big)^{1/2}\Big(\sum_m |y_m|^2\Big)^{1/2}. \tag{F.1}$$

Here Δ depends on the coefficient matrix $A = [a_{mn}]$, but is independent of the vectors $\boldsymbol{x}, \boldsymbol{y}$.

Let $A = [a_{mn}]$ be an $M \times N$ matrix with complex entries. Then A determines a linear map $\boldsymbol{x} \mapsto \boldsymbol{y} = A\boldsymbol{x}$ from $\mathbb{C}^N$ to $\mathbb{C}^M$. The norm of A, as a linear operator, is the maximum of the ratio $\|\boldsymbol{y}\|/\|\boldsymbol{x}\|$ as $\boldsymbol{x}$ runs over all non-zero members of $\mathbb{C}^N$,

$$\|A\| = \max_{\boldsymbol{x}\ne\boldsymbol{0}} \frac{\|A\boldsymbol{x}\|}{\|\boldsymbol{x}\|}$$

where $\|\boldsymbol{x}\| = \big(\sum |x_n|^2\big)^{1/2}$ denotes the usual Euclidean norm. By homogeneity we may write instead

$$\|A\| = \max_{\|\boldsymbol{x}\|=1} \|A\boldsymbol{x}\|.$$

We now show that $\|A\|$ is the optimal constant in the inequality (F.1).

Theorem F.1 (Duality) *Let $A = [a_{mn}]$ be a fixed $M \times N$ matrix. The following three assertions concerning the positive constant Δ are equivalent*:

(a) *For any $\boldsymbol{x} \in \mathbb{C}^N$,*

$$\sum_{m=1}^{M} \Big| \sum_{n=1}^{N} a_{mn} x_n \Big|^2 \le \Delta^2 \sum_{n=1}^{N} |x_n|^2;$$

(b) *For any $\boldsymbol{x} \in \mathbb{C}^N$ and any $\boldsymbol{y} \in \mathbb{C}^M$,*

$$\Big| \sum_{m=1}^{M} \sum_{n=1}^{N} a_{mn} x_n y_m \Big| \le \Delta \Big(\sum_{n=1}^{N} |x_n|^2 \Big)^{1/2} \Big(\sum_{m=1}^{M} |y_m|^2 \Big)^{1/2};$$

(c) *For any $\boldsymbol{y} \in \mathbb{C}^M$,*

$$\sum_{n=1}^{N} \Big| \sum_{m=1}^{M} a_{mn} y_m \Big|^2 \le \Delta^2 \sum_{m=1}^{M} |y_m|^2.$$

In terms of linear maps and inner products, these inequalities assert that

$$\text{(a}'\text{)} \qquad \|A\boldsymbol{x}\| \le \Delta \|\boldsymbol{x}\|,$$

$$\text{(b}'\text{)} \qquad |(A\boldsymbol{x}, \boldsymbol{y})| \le \Delta \|\boldsymbol{x}\| \|\boldsymbol{y}\|,$$

and that

$$\text{(c}'\text{)} \qquad \|A^*\boldsymbol{y}\| \le \Delta \|\boldsymbol{y}\|.$$

Here A^* is the *adjoint* of A. That is, $A^* = (\overline{A})^{\mathrm{T}}$ is the $N \times M$ matrix whose entries are $\overline{a_{nm}}$. In terms of inner products, A^* is characterized by the property that $(A\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{x}, A^*\boldsymbol{y})$ for all $\boldsymbol{x}$ and $\boldsymbol{y}$. Since (a′) and (c′) are equivalent, we deduce that

$$\|A\| = \|A^*\|.$$

Proof We show that (a) and (b) are equivalent. Then by interchanging the roles of m and n it is clear that (b) and (c) are equivalent.

(a) $\Longrightarrow$ (b). By Cauchy's inequality

$$\Big| \sum_m \Big(\sum_n a_{mn} x_n \Big) y_m \Big| \le \Big(\sum_m \Big| \sum_n a_{mn} x_n \Big|^2 \Big)^{1/2} \Big(\sum_m |y_m|^2 \Big)^{1/2}.$$

In the first factor on the right we insert the bound provided by (a), and we obtain (b).

(b) $\Longrightarrow$ (a). Set

$$y_m = \sum_{n=1}^{N} a_{mn} x_n,$$

and let S denote the left and side of (a). Then $S = \sum_n a_{mn} x_n \overline{y_m}$, and by (b) we see that

$$S \le \Delta \Big(\sum_{n=1}^{N} |x_n|^2 \Big)^{1/2} \Big(\sum_{m=1}^{M} |y_m|^2 \Big)^{1/2} = \Delta \Big(\sum_{n=1}^{N} |x_n|^2 \Big)^{1/2} S^{1/2}.$$

If $S = 0$, then (a) is obviously satisfied. Otherwise $S > 0$, and we may square both sides above and divide by S to obtain (a). □

Corollary F.2 *For any $M \times N$ matrix A,*

$$\|A\| = \|A^*\| \le \|A^*A\|^{1/2}.$$

By using Corollary F.11 below it will become apparent that the inequality here may be replaced by equality.

Proof The identity represents the equivalence of (a) and (c). To obtain the inequality, let $\boldsymbol{x}$ be a unit vector for which $\|A\boldsymbol{x}\| = \|A\|$. Then

$$\|A\|^2 = \|A\boldsymbol{x}\|^2 = (A\boldsymbol{x}, A\boldsymbol{x}) = (A^*A\boldsymbol{x}, \boldsymbol{x}).$$

By (b) with $\boldsymbol{y} = \boldsymbol{x}$ we see that this last expression is $\le \|A^*A\|$. □

As a first upper bound for $\|A\|$ we establish

Theorem F.3 *Let A be an $M \times N$ matrix. Then*

$$\|A\| \le \Big(\max_m \sum_{n=1}^{N} |a_{mn}|\Big)^{1/2} \Big(\max_n \sum_{m=1}^{M} |a_{mn}|\Big)^{1/2}.$$

Proof By Cauchy's inequality

$$\Big|\sum_{m,n} a_{mn} x_n y_m\Big| \le \Big(\sum_{m,n} |a_{mn}||x_n|^2\Big)^{1/2} \Big(\sum_{m,n} |a_{mn}||y_m|^2\Big)^{1/2}.$$

The first sum on the right hand side is

$$\sum_n |x_n|^2 \sum_m |a_{mn}| \le \Big(\max_n \sum_m |a_{mn}|\Big) \sum_n |x_n|^2.$$

We treat the second sum similarly, and thus obtain the situation of Theorem F.1(b) with

$$\Delta = \Big(\max_n \sum_m |a_{mn}|\Big)^{1/2} \Big(\max_m \sum_n |a_{mn}|\Big)^{1/2}.$$

Thus $\|A\| \le \Delta$ by Theorem F.1. □

In general, Theorem F.3 provides a useful bound only if the a_{mn} are non-negative and approximately the same size, or if the matrix is nearly diagonal. Otherwise the bound for $\|A\|$ may be weak because it takes no account of possible cancellation. We apply this to the matrix A^*A and appeal to Corollary F.2 to obtain

Corollary F.4 *Let $A = [a_{mn}]$ be an $M \times N$ matrix. Then*

$$\|A\| \le \Big(\max_{n_1} \sum_{n_2=1}^{N} \Big|\sum_{m=1}^{M} \overline{a_{mn_1}} a_{mn_2}\Big|\Big)^{1/2}.$$

If the columns of A are nearly orthonormal, then A^*A is nearly the identity matrix, and by the above $\|A\|$ is not much more than 1. We may use columns rather than rows, by applying the above to A^{T} instead of A. If the columns are far from orthonormal, then the above bound will in general be weak. In some instances greater precision can be obtained by introducing a type of weighting factor.

Theorem F.5 *Let $A = [a_{mn}]$ be an $M \times N$ matrix, but suppose that the a_{mn} are defined for all integral values of m. Let w_m be non-negative and suppose that $w_m \ge 1$ for $1 \le m \le M$. Then*

$$\|A\| \le \Big(\max_{n_1} \sum_{n_2}^{N} \Big| \sum_{m=-\infty}^{\infty} \overline{a_{mn_1}} a_{mn_2} w_m \Big| \Big)^{1/2}$$

provided that the inner sum converges for all n_1, n_2.

Proof Let $\boldsymbol{x}$ be a unit vector for which $\|A\boldsymbol{x}\| = \|A\|$. Then by the properties of the w_m we see that

$$\|A\boldsymbol{x}\|^2 = \sum_{m=1}^{M} \Big| \sum_{n=1}^{N} a_{mn} x_n \Big|^2 \le \sum_{m=-\infty}^{\infty} w_m \Big| \sum_{n=1}^{N} a_{mn} x_n \Big|^2.$$

We square out and take the sum over m inside to see that this is

$$\sum_{n_1} \overline{x_{n_1}} \sum_{n_2} x_{n_2} \sum_{m} w_m \overline{a_{mn_1}} a_{mn_2} = (B\boldsymbol{x}, \boldsymbol{x})$$

where B is the matrix with entries

$$b_{n_1 n_2} = \sum_{m=-\infty}^{\infty} w_m \overline{a_{mn_1}} a_{mn_2}.$$

From Theorem F.1(b) we know that $|(B\boldsymbol{x}, \boldsymbol{x})| \le \|B\|$, so that $\|A\| \le \|B\|^{1/2}$. Then by applying Theorem F.3 to B we obtain the stated result. □

If $w_m = 1$ for $1 \le m \le M$ and $w_m = 0$ otherwise, then the argument above paraphrases the proofs of Corollary F.4. If the a_{mn} are oscillatory and random in appearance, then the upper bounds for $\|A\|$ that we might derive from the theorems above are likely to be much larger than the true order of magnitude. In such a situation, the following lower bound may be closer to the truth.

Theorem F.6 *Let A be an $M \times N$ matrix. Then*

$$\|A\|^2 \ge \frac{\sum_{m,n} |a_{mn}|^2}{\min(M, N)}.$$

Proof We consider the size of $\|A\boldsymbol{x}\|$ with $x_n = e(n\theta)$, and average over θ. By the orthogonality of the functions $e(n\theta)$ we see that

$$\int_0^1 \sum_{m=1}^{M} \Big| \sum_{n=1}^{N} a_{mn} e(n\theta) \Big|^2 \, d\theta = \sum_{m,n} |a_{mn}|^2.$$

We choose a θ for which the integrand is at least as large as the right hand side. Since $\|\boldsymbol{x}\| = N^{1/2}$ for any θ, we conclude that

$$\|A\| \ge \Big(\frac{1}{N} \sum_{m,n} |a_{mn}|^2 \Big)^{1/2}.$$

By applying this argument to A^{T} instead of A we obtain this lower bound with N replaced by M. Thus the proof is complete. □

F.1.1 Exercises

1. For $\boldsymbol{x} \in \mathbb{C}^N$ and real $p > 1$, put $\|\boldsymbol{x}\|_p = \left(\sum |x_n|^p\right)^{1/p}$. Similarly put $\|\boldsymbol{x}\|_\infty = \max |x_n|$. Suppose that p and q are real numbers, $1 \le p \le \infty$, $1 \le q \le \infty$, and that p' and q' are determined by the relations $1/p + 1/p' = 1$, $1/q + 1/q' = 1$. Let A be an $M \times N$ matrix. Show that the following assertions concerning the constant Δ are equivalent:

(a) $\|A\boldsymbol{x}\|_p \le \Delta \|\boldsymbol{x}\|_q$ for all $\boldsymbol{x} \in \mathbb{C}^N$;
(b) $\left|\sum a_{mn} x_n y_m\right| \le \Delta \|\boldsymbol{x}\|_q \|\boldsymbol{y}\|_{p'}$ for all $\boldsymbol{x} \in \mathbb{C}^N$ and $\boldsymbol{y} \in \mathbb{C}^M$;
(c) $\|A\boldsymbol{y}\|_{q'} \le \Delta \|\boldsymbol{y}\|_{p'}$ for all $\boldsymbol{y} \in \mathbb{C}^M$.

2. Let B and C be rectangular matrices, and put

$$A = \left[\begin{array}{c|c} B & 0 \\ \hline 0 & C \end{array}\right].$$

Show that $\|A\| = \max(\|B|, \|C\|)$.

3. Suppose that $|a_{mn}| \le b_{mn}$ for all m and n. Show that $\|A\| \le \|B\|$.

4. Let A be an $M \times N$ matrix, and suppose that there are positive numbers C, D, $u_1, \dots, u_M$, $v_1, \dots, v_N$ such that

$$\sum_{m=1}^{M} |a_{mn}| v_m \le C u_n$$

for $1 \le n \le N$, and also that

$$\sum_{n=1}^{N} |a_{mn}| u_n \le D v_m$$

for $1 \le m \le M$.
(a) Show that if $\|\boldsymbol{x}\| = \|\boldsymbol{y}\| = 1$, then

$$|(A\boldsymbol{x}, \boldsymbol{y})|^2 \le \Big(\sum_{m,n} |a_{mn}| v_m / u_n\Big)\Big(\sum_{m,n} |a_{mn}| u_n / v_m\Big).$$

(b) Deduce that $\|A\| \le (CD)^{1/2}$.

5. Let A be an $M \times N$ matrix with $a_{mn} = 1$ for all m and n. Show that $\|A\| = (MN)^{1/2}$.

6. Let A be an $N \times N$ matrix, and let C, $u_1, \dots, u_N$ be positive numbers such that

$$\sum_{n=1}^{N} |a_{mn}| u_n \le C u_m \tag{F.2}$$

for $1 \le m \le N$.
(a) Show that $\rho(A) \le C$. (Suggestion: Let $\boldsymbol{x}$ be an eigenvector, and consider that m for which $|x_m|/u_m$ is maximal.)
(b) Show that if $a_{mn} > 0$ for all m and n, and if C is chosen minimally, then equality holds in (F.2) for all m, so that $\rho(A)$ is an eigenvalue, and $\boldsymbol{u}$ is an associated eigenvector with positive coordinates.

7. Let Δ_N denote the least number such that

$$\Big|\sum_{pq\le N} x_p\overline{y_q}\Big| \le \Delta_N \sum_{p\le N}\frac{|x_p|^2}{p}$$

for all complex numbers x_p where p and q are to take only prime values. Show that $\Delta \asymp N(\log N)^{-1/2}$.

8. Let Δ_N denote the least number such that

$$\Big|\sum_{\substack{p\le N, q\le N\\ pq\ge N}} \frac{x_p\overline{y_q}}{pq}\Big| \le \Delta_N \sum_{p\le N}\frac{|x_p|^2}{p}$$

for all complex numbers x_p where p and q are to take only prime values. Show that $\Delta = 1 + O(1/\log N)$.

9. Let A be the $N \times N$ matrix with coefficients

$$a_{mn} = \begin{cases} \Lambda(n/m)(m/n)^{1/2} & \text{if } m|n,\\ \Lambda(m/n)(n/m)^{1/2} & \text{if } n|m,\\ 0 & \text{otherwise.}\end{cases}$$

Show that $\|A\| = \log N + O(1)$. (Suggestion: Consider the vector $\boldsymbol{x}$ with coordinates $x_n = n^{-1/2}$.)

F.2 Square matrices

The operator norm is defined for an arbitrary rectangular matrix, but if A is square, say $N \times N$, then further numbers can be associated with it. In the first place, A has N eigenvalues λ_n, which are the roots of the polynomial $\det(A - zI)$, and we define the *spectral radius* of A to be

$$\rho(A) = \max_n |\lambda_n|.$$

We also consider the *numerical radius* of A,

$$\nu(A) = \max_{\|\boldsymbol{x}\|=1}\Big|\sum_{m,n} a_{mn}x_n\overline{x_m}\Big| = \max_{\|\boldsymbol{x}\|=1}|(A\boldsymbol{x},\boldsymbol{x})|.$$

These quantities are related to the operator norm $\|A\|$ in the following simple manner.

Theorem F.7 *Let A be an arbitrary $N \times N$ matrix. Then*

$$\rho(A) \le \nu(A) \le \|A\|.$$

Proof Let λ be an eigenvalue of A, and let $\boldsymbol{x} \ne \boldsymbol{0}$ be an associated eigenvector, so that $A\boldsymbol{x} = \lambda\boldsymbol{x}$. Without loss of generality we may suppose that $\|\boldsymbol{x}\| = 1$. For this vector, $(A\boldsymbol{x}, \boldsymbol{x}) = (\lambda\boldsymbol{x}, \boldsymbol{x}) = \lambda$, so that $\nu(A) \ge |\lambda|$, and hence $\nu(A) \ge \rho(A)$.

By Theorem F.1(b),

$$\|A\| = \max_{\|\boldsymbol{x}\|=\|\boldsymbol{y}\|=1} |(A\boldsymbol{x}, \boldsymbol{y})|.$$

Thus $\nu(A) \le \|A\|$, and the proof is complete. □

The first inequality above can not be reversed in general, since $\nu(A)$ may be large even when all the eigenvalues vanish. (Consider a matrix A for which $a_{mn} = 0$ whenever $m \ge n$.) However, $\nu(A)$ and $\|A\|$ are always comparable.

Theorem F.8 *Let A be an $N \times N$ matrix. Then*

$$\frac{1}{2}\|A\| \le \nu(A) \le \|A\|,$$

and if A is hermitian (i.e., if $A^* = A$), then $\nu(A) = \|A\|$.

In Corollary F.11 below it will also be established that if A is hermitian, then also $\rho(A) = \|A\|$.

Proof We establish the last assertion first. The hypothesis that A is hermitian is equivalent to saying that $(A\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{x}, A\boldsymbol{y})$ for all $\boldsymbol{x}$ and $\boldsymbol{y}$. Put $\boldsymbol{u} = \boldsymbol{x} + \boldsymbol{y}$ and $\boldsymbol{v} = \boldsymbol{x} - \boldsymbol{y}$. It is easily verified that if A is hermitian, then

$$4\Re(A\boldsymbol{x}, \boldsymbol{y}) = (A\boldsymbol{u}, \boldsymbol{u}) - (A\boldsymbol{v}, \boldsymbol{v}).$$

By Theorem F.1(b) we can choose unit vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ so that $(A\boldsymbol{x}, \boldsymbol{y}) = \|A\|$. Then

$$4\|A\| = (A\boldsymbol{u}, \boldsymbol{u}) - (A\boldsymbol{v}, \boldsymbol{v}) \le \nu(A)(\|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2).$$

But $\|\boldsymbol{u}\|^2 = 2 + 2\Re(\boldsymbol{x}, \boldsymbol{y})$ and $\|\boldsymbol{v}\|^2 = 2 - 2\Re(\boldsymbol{x}, \boldsymbol{y})$, so that $\|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2 = 4$, and hence $\|A\| \le \nu(A)$.

The second of the displayed inequalities follows trivially from Theorem F.1(b) and the definition of $\nu(A)$. To establish an inequality in the reverse direction, suppose that A is an arbitrary $N \times N$ matrix. Write $A = B + iC$ where $B = (A + A^*)/2$ and $C = (A - A^*)/(2i)$. The triangle inequality holds for the operator norm $\|\cdot\|$, so that $\|A\| \le \|B\| + \|C\|$. But B and C are hermitian, so that this latter quantity is $\nu(B) + \nu(C)$. For any $\boldsymbol{x} \in \boldsymbol{C}^N$ we see that $(B\boldsymbol{x}, \boldsymbol{x}) = \Re(A\boldsymbol{x}, \boldsymbol{x})$, and $(C\boldsymbol{x}, \boldsymbol{x}) = \Im(A\boldsymbol{x}, \boldsymbol{x})$. Hence $\nu(B) \le \nu(A)$, $\nu(C) \le \nu(A)$, and we conclude that $\|A\| \le 2\nu(A)$. □

We now consider the possibility that a square matrix A might be converted to a diagonal matrix by means of a suitable change of basis. In general, if S is non-singular, so that

$\boldsymbol{x} = S\boldsymbol{u}$ expresses a linear change of variables, then the linear transformation $\boldsymbol{x} \mapsto A\boldsymbol{x}$ is computed as $\boldsymbol{u} \mapsto B\boldsymbol{u}$ in the new coordinate system, where $B = S^{-1}AS$. In this case we say that A and B are *similar*. An easy calculation reveals that if A and B are similar, then $\operatorname{tr} A = \operatorname{tr} B$, $\det A = \det B$, and indeed A and B have the same characteristic polynomial. Hence A and B have the same eigenvalues, so that $\rho(A) = \rho(B)$. On the other hand, the norm of a matrix is a metric quantity, and in general $\|A\| \neq \|B\|$. In order that $\|A\|$ should be invariant we restrict our attention to those similarity transformations that preserve distances. Let U be an $N \times N$ matrix. Then it is easy to verify that the following assertions are equivalent:

(i) U is unitary (i.e. $U^* = U^{-1}$);
(ii) The columns of U form orthonormal vectors;
(iii) The rows of U form orthonormal vectors;
(iv) The map $\boldsymbol{x} \mapsto U\boldsymbol{x}$ is an isometry of $\mathbb{C}^N$ (i.e. $\|U\boldsymbol{x}\| = \|\boldsymbol{x}\|$ for all $\boldsymbol{x}$);
(v) $(U\boldsymbol{x}, U\boldsymbol{y}) = (\boldsymbol{x}, \boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^N$.

Thus a unitary transformation maps one orthonormal basis to another, and conversely, if two orthonormal bases are given, then there is a unitary transformation that takes one to the other. In the analogous situation of linear maps from $\mathbb{R}^N$ to itself, we would find that the orthogonal matrices have corresponding properties. (A matrix X is *orthogonal* if $X^{\mathrm{T}} = X^{-1}$). If $A = U^{-1}BU$ where U is unitary, then we say that A and B are *unitarily similar*. In this case it is clear that $\|A\| = \|B\|$, and that $\nu(A) = \nu(B)$. Moreover, we note that A is hermitian ($A^* = A$) if and only if B is, that A is normal ($AA^* = A^*A$) if and only if B is, and that A is unitary ($A^* = A^{-1}$) if and only if B is. We now produce a unitarily similar canonical form for A.

Theorem F.9 (Schur's triangularization theorem) *For any $N \times N$ matrix A there is a unitarily similar upper triangular matrix $T = U^{-1}AU$. The diagonal entries of T are the eigenvalues of A.*

Proof We prove the first assertion by induction on N. For $N = 1$ there is nothing to show. Suppose we have the result for $N - 1$. Let λ_1 be an eigenvalue of A, and that $\boldsymbol{v}_1$ is an associated unit eigenvector. Choose $\boldsymbol{v}_2, \ldots, \boldsymbol{v}_N$ so that the $\boldsymbol{v}_n$ form an orthonormal basis for $\mathbb{C}^N$, and let V be the matrix whose columns are the $\boldsymbol{v}_n$. Then V is unitary, and V^*AV has the form

$$V^*AV = \left[\begin{array}{c|c} \lambda_1 & * \\ \hline \mathbf{0} & B \end{array}\right].$$

By the inductive hypothesis there is a unitary matrix W such that $W^{-1}BW$ is upper-triangular. Put

$$X = \left[\begin{array}{c|c} 1 & \mathbf{0} \\ \hline \mathbf{0} & W \end{array}\right].$$

Then

$$X^*V^*AVX = \left[\begin{array}{c|c} \lambda_1 & * \\ \hline \mathbf{0} & W^*BW \end{array}\right]$$

is upper-triangular, and we take $U = VX$. The second assertion is obvious, since

$$\operatorname{char\,poly} A = \operatorname{char\,poly} T = \prod_{n=1}^{N} (x - t_{nn}).$$

□

If D is a diagonal matrix, then clearly $D^*D = DD^*$, so that D is normal. Conversely, suppose that T is a normal upper-triangular matrix. On comparing the diagonal entries of T^*T with those of TT^*, we see that

$$\sum_{m=1}^{n} |t_{mn}|^2 = \sum_{m=n}^{N} |t_{nm}|^2$$

for $1 \leq n \leq N$. On taking $n = 1$, we deduce that $t_{1m} = 0$ for $m > 1$. Then we set $n = 2$ to show that $t_{2m} = 0$ for $m > 2$. Hence by induction we find that $t_{mn} = 0$ for $m \neq n$, so that T is diagonal. Thus we have

Corollary F.10 *A square matrix A is unitarily similar to a diagonal matrix, $U^*AU = D$, if and only if A is normal.*

If D is diagonal, then clearly $\rho(D) = \|D\|$. Thus we deduce

Corollary F.11 *If A is normal, then $\rho(A) = \nu(A) = \|A\|$.*

We note that if A is hermitian or unitary, then A is normal, and the above applies. We consider again Corollary F.2, whose proof amounted to observing that

$$\|A\|^2 = \nu(A^*A) \leq \|A^*A\|.$$

Since A^*A is hermitian, we know by Theorem F.8 that equality holds here. By Corollary F.11 we can add the further observation that

$$\|A\|^2 = \rho(A^*A).$$

F.2.1 Exercises

1. Let A be an $M \times N$ matrix, and let $\lambda_1, \ldots, \lambda_N$ be the eigenvalues of A^*A. Show that the λ_n are non-negative, and that

$$\sum_{n=1}^{N} \lambda_n = \sum_{m,n} |a_{mn}|^2.$$

Use this to give a second proof of Theorem F.6.

2. Let A be an $M \times N$ matrix.
(a) Show that $A(A^*A - zI)^{-1}A^* = I + z(AA^* - zI)^{-1}$ for any complex number z for which either of the inverses exists.
(b) Show that the non-zero eigenvalues of A^*A coincide with those of AA^*.

3. Show that an $N \times N$ matrix A is normal if and only if its eigenvectors form an orthogonal basis for $\mathbb{C}^N$.

4. Show that the following are equivalent:
(i) U is unitary;
(ii) U is normal and all its eigenvalues are unimodular.

5. Show that the following are equivalent:
(i) X is hermitian;
(ii) X is normal and all its eigenvalues are real.

6. Let A be an $N \times N$ matrix. The *field of values* of A is the set of complex numbers $\{(A\boldsymbol{x}, \boldsymbol{x}) : \|\boldsymbol{x}\| = 1\}$.
(a) Show that if A and B are unitarily similar, then they have the same field of values.
(b) Show that if A is normal, then its field of values is the convex hull of its eigenvalues.
(c) Show that the field of values of A is an interval on the real line if and only if A is hermitian.
(d) The field of values is a convex set that contains the eigenvalues of A.
(e) If B is an $M \times N$ matrix, then the field of values of B^*B is the same as the field of values of BB^*.

7. Let A be a hermitian matrix for which $(A\boldsymbol{x}, \boldsymbol{x}) \ge 0$ for all $\boldsymbol{x}$. Show that $|(A\boldsymbol{x}, \boldsymbol{y})| \le (A\boldsymbol{x}, \boldsymbol{x})(A\boldsymbol{y}, \boldsymbol{y})$. (Suggestion: Consider $(A(\lambda\boldsymbol{x} + \mu\boldsymbol{y}), \lambda\boldsymbol{x} + \mu\boldsymbol{y})$.)

8. Suppose that $A_1, \ldots, A_K$ are commuting normal matrices. Show that there is a unitary matrix U such that all the matrices U^*A_kU are diagonal.

9. (Watkins 1980) Suppose that A and B are real square matrices that are similar over $\mathbb{C}$. Show that they are similar over $\mathbb{R}$.

10. Let A be a real symmetric matrix. Show that any number of the form $(A\boldsymbol{x}, \boldsymbol{x})$ where $\boldsymbol{x}$ is a unit vector in $\mathbb{C}^N$ can also be written in this form with $\boldsymbol{x}$ a unit vector in $\mathbb{R}^N$.

11. (a) Let

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Show that the eigenvalues of A are $0, 1, 2$, that the eigenvalues of A^*A are $0, 2, 4$, that A is not normal, and that $\|A\| = \rho(A) = 2$.
(b) Show that the converse of Corollary F11 is true for $N \le 2$, but false for $N > 2$.

12. Let A be a normal matrix, λ a complex number, and $\boldsymbol{x}$ a vector. Put $\boldsymbol{e} = A\boldsymbol{x} - \lambda\boldsymbol{x}$. Show that A has an eigenvalue in the disk $|z - \lambda| \le \|\boldsymbol{e}\|/\|\boldsymbol{x}\|$. (Hint: If $A - \lambda I$ is singular, then this is obvious. Otherwise, argue that $\rho((A - \lambda I)^{-1}) = \|(A - \lambda I)^{-1}\| \ge \|\boldsymbol{x}\|/\|\boldsymbol{e}\|$.)

13. (a) Let C be an $N \times N$ hermitian matrix such that $(C\boldsymbol{x}, \boldsymbol{x}) \ge 0$ for all $\boldsymbol{x} \in \mathbb{C}^N$. Show that there is an $N \times N$ matrix B such that $B^*B = C$.
(b) Suppose that A is an $M \times N$ matrix, and put $\Delta = \|A^*A\|^{1/2}$. Show that there is an $N \times N$ matrix B such that $A^*A + B^*B = \Delta^2 I$.
(c) Suppose that A is an $M \times N$ matrix for which condition (a) of Theorem F.1 holds. Show that there is an $N \times N$ matrix B such that

$$\sum_{m=1}^{M} \left| \sum_{n=1}^{N} a_{mn} x_n \right|^2 + \sum_{m=1}^{N} \left| \sum_{n=1}^{N} b_{mn} x_n \right|^2 = \Delta^2 \sum_{n=1}^{N} |x_n|^2$$

for all $\boldsymbol{x} \in \mathbb{C}^N$.

14. (a) Suppose that $f \in L^\infty(\mathbb{T})$ has Fourier coefficients $\widehat{f}(k) = \int_0^1 f(x) e(-kx)\, dx$. Show that

$$\left| \sum_{m=1}^{N} \sum_{n=1}^{N} \widehat{f}(m-n) x_n y_m \right| \le \|f\|_\infty \left(\sum_{n=1}^{N} |x_n|^2 \right)^{1/2} \left(\sum_{m=1}^{N} |y_m|^2 \right)^{1/2}.$$

(b) Deduce in particular that

$$\left| \sum_{\substack{m=1 \\ m \ne n}}^{N} \sum_{n=1}^{N} \frac{x_m \overline{x_n}}{m-n} \right| \le \pi \sum_{n=1}^{N} |x_n|^2.$$

(c) Similarly, show that

$$\left| \sum_{m=1}^{N} \sum_{n=1}^{N} \frac{x_m \overline{x_n}}{m+n-1} \right| \le \pi \sum_{n=1}^{N} |x_n|^2.$$

15. (a) Let U be the $q \times q$ matrix with coefficients $u_{mn} = e(mn/q)/\sqrt{q}$. Show that U is unitary.
(b) Let $f(n)$ be an arithmetic function that is periodic with period q, and let C be the $q \times q$ matrix with coefficients $c_{mn} = f(m-n)$. (Such a matrix is called a circulant.) Show that U^*CU is diagonal.
(c) Let

$$\widehat{f}(k) = \frac{1}{q}\sum_{h=1}^{q} f(h)e(-hk/q)$$

be the finite Fourier transform of f, as discussed in §4.1. Show that

$$\sum_{m=1}^{q}\sum_{n=1}^{q} f(m-n)x_m\overline{x_n} = \sum_{k=1}^{q}\widehat{f}(k)\Big|\sum_{n=1}^{q} x_n e(kn/q)\Big|^2$$

16. (Schur?, Morton) Let U be as in the preceding exercise, and suppose that q is an odd prime. Let $\boldsymbol{x}$ be the vector with coordinates $x_n = \left(\frac{n}{q}\right)$. Show that $\boldsymbol{x}$ is an eigenvector of U.

17. Let A be the $\varphi(q) \times \varphi(q)$ matrix $A = [\tau(\chi\overline{\psi})/\varphi(q)]$, where the rows are indexed by the Dirichlet character χ (mod q) and the columns are indexed by the Dirichlet character ψ (mod q).
(a) Show that A is unitary.
(b) Show that the vector $\boldsymbol{x}$ with coordinates $x_\psi = \psi(a)$ is an eigenvector $e(a/q)$ is an eigenvalue of A with eigenvalue $e(a/q)$.
(c) Show that

$$\sum_{\chi,\psi}\tau(\chi\overline{\psi})x_\chi\overline{x_\psi} = \sum_{\substack{a=1\\(a,q)=1}}^{q}\Big|\sum_{\chi}\chi(a)x_\chi\Big|^2 e(a/q).$$

(d) Show that

$$\Big|\sum_{\chi,\psi}\tau(\chi\overline{\psi})x_\chi\overline{x_\psi}\Big| \le \varphi(q)\sum_{\chi}|x_\chi|^2$$

for arbitrary complex numbers x_χ, and that the constant is best possible.

18. Let $f(n)$ be an arithmetic function with period q, and let $A = [f(mn)]$ be the $\varphi(q) \times \varphi(q)$ matrix whose rows m and columns n are indexed by the reduced residue classes (mod q).
(a) Show that $\|A\| = \Delta$ where

$$\Delta = \max_{\chi}\Big|\sum_{n=1}^{q} f(n)\chi(n)\Big|.$$

(b) Show that for arbitrary complex numbers x_n, y_m,

$$\Bigg|\sum_{\substack{m=1 \\ (mn,q)=1}}^{q} \sum_{n=1}^{q} f(mn) x_n \overline{y_m}\Bigg| \le \Delta \Bigg(\sum_{\substack{n=1 \\ (n,q)=1}}^{q} |x_n|^2 \Bigg)^{1/2} \Bigg(\sum_{\substack{m=1 \\ (m,q)=1}}^{q} |y_m|^2 \Bigg)^{1/2},$$

and that the constant Δ is best possible.

19. Let $g(n)$ be an arithmetic function, and let A be the $N \times N$ matrix with entries $a_{mn} = g(n)$ if $n|m$, $a_{mn} = 0$ otherwise. Let B be the $N \times N$ matrix with entries $b_{mn} = 1$ if $m|n$, $b_{mn} = 0$ otherwise. Let C be the $N \times N$ matrix with entries $f((m,n))$ where $f(n) = \sum_{d|n} g(d)$. Show that $C = AB$, that $\det B = 1$, and that

$$\det C = \det A = \prod_{n=1}^{N} g(n).$$