Moduli of Tropical Curves

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Abstract

An (abstract) tropical curve is, roughly, a combinatorial graph with a real length assigned to each edge. They serve as 1-dimensional analogues of closed oriented surfaces. We can study moduli spaces of such objects, just as we do for surfaces. In this minicourse we will study topological and combinatorial properties of these moduli spaces, and explain their strong connections to other areas of math; i.e.

– Teichmueller space and its boundary
– automorphisms of free groups.

There will be no prerequisites for this course, but it may help to have seen Teichmueller space for motivation, and to have seen the definition of a colimit for some important constructions.

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We will mainly be using [CGP16] and [Vog15] as references.

*Notes were taken by Takumi Murayama, who is responsible for any and all errors. Please e-mail takumim@umich.edu with any more corrections. Compiled on May 11, 2018.

0 Why care?

Lecture 1

We first start by answering the question: Why we should care about the moduli space of tropical curves? There are three overarching reasons:

1. (Combinatorics) Tropical curves have inherent interest because they generalize graphs, and also have connections to polyhedral geometry.
2. (Differential geometry and geometric group theory) The moduli space of tropical curves is a tool to study the (outer) automorphism group of the free group F_n, in a similar fashion to how Teichmueller space is used to study the mapping class group.
3. (Algebraic geometery) The moduli space of tropical curves can be used to explicitly describe the boundary complex of the algebraic moduli space $\mathscr{M}_{g,n}$.

Note that for (2), there is no precise relationship between the theory for tropical curves and Teichmueller theory; it is merely an analogy. On the other hand, (3) gives an *exact* connection between the two objects. This is a bit strange, since Teichmueller space and $\mathscr{M}_{g,n}$ are very related.

We will now give some more details about motivations (2) and (3).

0.1 Differential geometry

To be a bit more precise about the differential geometric motivation (2), we list some analogies:

topological space	π_1	outer automorphisms	act on
closed surface Σ_g	$\pi_1(\Sigma_g)$	$\mathrm{MCG}^\pm(\Sigma_g)$	Teichmueller space $\mathscr{T}_g$
wedge of circles $\bigvee^g S^1$	free group F_g	$\mathrm{Out}(F_g)$	outer space $\mathscr{O}_g$
torus $\mathbf{T}^g$	free abelian group $\mathbf{Z}^g$	$\mathrm{GL}_g(\mathbf{Z})$	symmetric space

Here, $\mathrm{Out}(G) := \mathrm{Aut}(G)/\,\mathrm{Inn}(G)$ is the group of *outer automorphisms* of a group G, where the *inner automorphisms* $\mathrm{Inn}(G)$ are those coming from conjugation by elements in G. The superscript $^\pm$ in $\mathrm{MCG}^\pm(\Sigma_g)$ denotes that the mapping class group is for unoriented maps. If one restricts to the mapping class group $\mathrm{MCG}(\Sigma_g)$ for oriented maps in the third column, then the analogous objects in the other rows would be $\mathrm{SOut}(F_g)$ and $\mathrm{SL}_g(\mathbf{Z})$, the special outer automorphism group of F_g and the special linear group of $\mathbf{Z}^g$, respectively.

There are some cases in which different entries in a column coincide, which suggests that properties in one row may also hold for other rows. For example,

- $\bigvee^1 S^1 = \mathbf{T}^1$ and $\Sigma_1 = \mathbf{T}^2$;
- $\pi_1(\Sigma_{g,1}) = F_{2g}$, where $\Sigma_{g,1}$ is the surface Σ_g with one puncture;
- $\mathrm{Out}(F_2) = \mathrm{MCG}^\pm(\Sigma_{1,1}) = \mathrm{GL}_2(\mathbf{Z})$.

In general, we have the following relations between rows:

- Since $\mathbf{Z}^g$ is the abelianization of F_g, we have a group homomorphism
$$\mathrm{Out}(F_g) \longrightarrow \mathrm{GL}_g(\mathbf{Z})$$
which happens to be surjective. The kernel is not well-understood.
- Since the punctured surface $\Sigma_{g,1+n}$ contracts onto the wedge of circles $\bigvee^{2g+n} S^1$, we have a group homomorphism
$$\mathrm{MCG}^\pm(\Sigma_{g,1+n}) \longrightarrow \mathrm{Out}(F_{2g+n})$$
which happens to be injective.

0.2 Algebraic geometry

From algebraic geometry (3), the motivating questions are the following:

1. What kind of space is $\mathscr{M}_{g,n}$, the moduli space of smooth algebric curves of genus g with n marked points?
2. What kind of space is the *Deligne–Mumford compactification* $\bar{\mathscr{M}}_{g,n}$ of this moduli space, and in particular, what does the boundary $\bar{\mathscr{M}}_{g,n} \smallsetminus \mathscr{M}_{g,n}$ look like?

For question (2), the answer is that the dual complex of the boundary divisor $\bar{\mathscr{M}}_{g,n} \smallsetminus \mathscr{M}_{g,n}$ is isomorphic to a certain moduli space $\Delta M_{g,n}^{\mathrm{trop}}$ of tropical curves. This notion of isomorphism depends on the exact definition of $\Delta M_{g,n}^{\mathrm{trop}}$; one subtlety is that you need to deal with automorphisms in addition to the topology of the space to make this relationship precise. Once made precise, however, one can get information about $\mathscr{M}_{g,n}$ by looking at the cohomology of $\Delta M_{g,n}^{\mathrm{trop}}$.

This concludes our description of why you would care about tropical curves.

0.3 Overview of the minicourse

For the rest of the week, we will talk about the differential geometric perspective for the first few days. Then, we will talk about the algebraic geometric perspective. Even though a precise description of the algebraic geometric story would require some knowledge of schemes and stacks, we hope that this section of the minicourse would still be useful.

We pause to give some motivation for the stacky point of view. When studying manifolds with automorphisms, one is often led to thinking about orbifolds; similarly, the study of schemes with automorphisms leads to thinking about stacks. The corresponding analogy on the combinatorial side is that when studying Δ-complexes, one is led to thinking about what are called *generalized* Δ*-complexes*. This construction will not rely on any scheme theory, and its definition just requires knowledge of colimits. We will present examples so that the full abstract knowledge is not necessary.

Part of the usefulness of this course is that many people have more familiarity with one side more than the other, and it is useful to hear and learn things about the other perspectives.

1 Tropical curves

We return to the inherently combinatorial viewpoint for tropical curves. We note that our definition for tropical curves in this section is not always consistent with definitions in the literature.

1.1 Basic definitions

We first recall the following:

Definition 1.1. A *CW-complex* of dimension 1 is a topological space X constructed (up to homeomorphism) as a quotient space

$$X = (V \sqcup E)/\sim,$$

where $V = \{v_1, v_2, \ldots\}$ is a discrete set of points, and $E = e_1 \sqcup e_2 \sqcup \cdots$ is a disjoint union of edges, each of which homeomorphic to a closed interval, together with (continuous) gluing maps $\varphi_i \colon \partial e_i \to V$.

For all of our constructions, the set V will be finite and E will be a finite disjoint union of intervals. The analogue for metric spaces is the following:

Definition 1.2. An *(unweighted) tropical curve* Γ is a connected metric space constructed (up to isometry) as a quotient space

$$\Gamma = (V \sqcup E)/\sim,$$

where $V = \{v_1, \ldots\}$ is a discrete set of points and $E = e_1 \sqcup e_2 \sqcup \cdots$ is a disjoint union of edges $e_i = [0, \ell_i]$ with lengths $\ell_i > 0$, together with (continuous) gluing maps $\varphi_i \colon \partial e_i \to V$.

In the above constructions, the data (V, E, φ_i) is *not* part of the data of the object X or Γ.

Example 1.3. The two unweighted tropical curves in Figure 1.1 are constructed with different sets of combinatorial data, but are considered the same.

Just as for closed oriented surfaces Σ_g, we can have pointed versions of these definitions:

Definition 1.4. An *n-pointed* tropical curve (Γ, m) is an unweighted tropical curve Γ together with a *marking function*

$$m \colon \{1, 2, \ldots, n\} \longrightarrow \Gamma.$$

The points $m(i)$ are the *marked points* or *base points*.

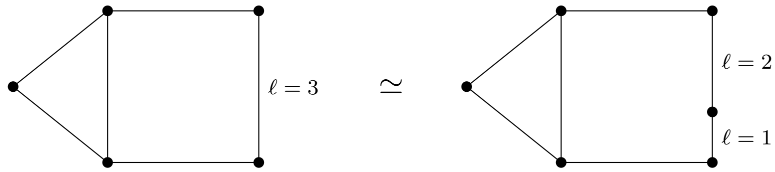


Figure 1.1: Different combinatorial data can define the same unweighted tropical curve

We will prefer the terminology "base points" since there will be another notion called "marking."

Example 1.5. We can visualize base points by adding half edges as in Figure 1.2.

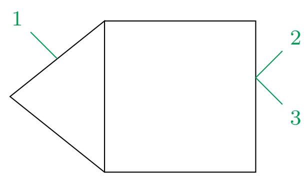


Figure 1.2: Half edges represent base points.

Definition 1.6. Let (Γ, m) be a pointed tropical curve. The *valence* of a point $x \in \Gamma$ is

$$\mathrm{val}(x) \coloneqq \#\{\text{half edges at } x\} + \# m^{-1}(x).$$

Note that in the definition of valence, we count half edges because loops are allowed in Γ. We note that most points in a curve Γ have valence 2, except for finitely many exceptions.

Example 1.7. We give some situations in which valences differ from 2 in Figure 1.3; note that this happens at only finitely many points.

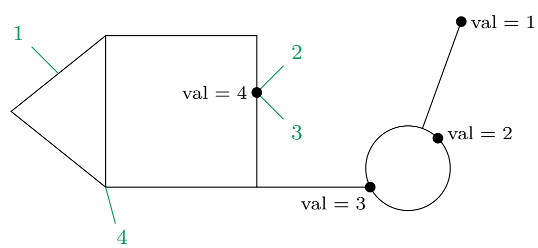


Figure 1.3: Valences of different points on a pointed tropical curve.

1.2 Weights, stability, and genus

To construct a nice moduli space (in particular, one that is closed), we have to answer the following question: What happens when edge lengths go to zero?

Example 1.8. Perturbing edge lengths preserves the homotopy class of a tropical curve. The limit as an edge length goes to zero, however, can possibly change the homotopy class, as seen in Figure 1.4.

We therefore see that the data we have kept track of so far does not contain the information needed to keep track of such shrinking phenomena. To keep track of these changes in homotopy type, we introduce the following definition:

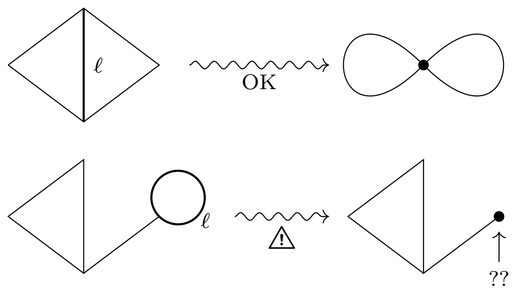


Figure 1.4: The limit as an edge length goes to zero can change the homotopy class.

Definition 1.9. A *(genus-)weighted* tropical curve (Γ, w) is an unweighted tropical curve Γ with a weight function

$$w \colon \Gamma \longrightarrow \mathbf{Z}_{\geq 0}$$

with $w(x) = 0$ except at finitely many points.

We can think of the weight as telling us how many loops shrunk down at that point.

Example 1.10. In Figure 1.5, the weight 1 keeps track of a loop that has been shrunk down. We note that this point with weight 1 still has valence 1; one can define a notion of *virtual valence* to keep track of these vanishing loops. In this case, the virtual valence of this point is 3.

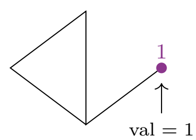


Figure 1.5: Weights keep track of loops that have been shrunk down to a point.

We give a name to special sets of combinatorial data that give rise to tropical curves.

Definition 1.11. A combinatorial model $(V, E, \sim)$ for a tropical curve (Γ, w, m) is *compatible* if the vertex set V contains

- all points with valence ≥ 3;
- all points with weight ≥ 1;
- all basepoints $m(i)$.

This notion of compatibility should be interpreted as saying there are "enough" vertices.

We can also now define stability:

Definition 1.12. A tropical curve (Γ, w, m) is *stable* if there exists a combinatorial model such that

- every weight 0 point has valence ≥ 2;
- every weight 1 vertex has valence ≥ 1.

A combinatorial model $(V, E, \sim)$ is *stable* if it satisfies the properties above.

The first condition is analogous to the condition that makes a stable curve have finite automorphism group. The second condition does not matter too much since all of our tropical curves are connected.

This notion of stability eliminates the "leaves" we had before that terminate in valence one points, and any extraneous vertices.

Example 1.13. Note that the notion of stability depends on both the weights and markings, since valence does. See Figure 1.6.

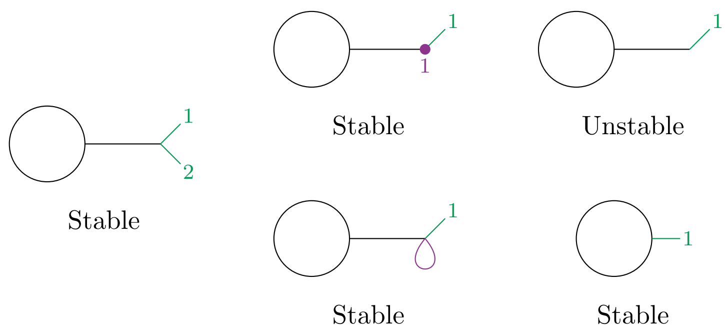


Figure 1.6: Stability depends on weights and markings, and is preserved by shrinking of loops but not edges.

Definition 1.14. The *genus* of a weighted tropical curve (Γ, w) is

$$g(\Gamma) \coloneqq b_1(\Gamma) + \sum_{x \in \Gamma} w(x)$$

where $b_1(\Gamma) = \dim H_1(\Gamma, \mathbf{R})$.

Example 1.15. The curve in Figure 1.3 has genus 3, and the curve in Figure 1.5 has genus 2.

Requiring a curve to be stable has some non-trivial consequences: Lecture 2

Fact 1.16. A stable curve has either

- $g \geq 2$;
- $g = 1$ and $n \geq 1$;
- $g = 0$ and $n \geq 3$.

Here, n is the number of basepoints.

However, there is a precise way in which not much generality is lost when only considering stable curves:

Proposition 1.17. *A tropical curve (Γ, w, m) has a canonical subspace Γ_{stable} containing*

- *all points with valence ≥ 3, and*
- *all points with weight ≥ 1,*

which is nonempty if

- *$g \geq 2$,*
- *$g = 1$ and $n \geq 1$, or*
- *$g = 0$ and $n \geq 3$.*

Moreover, there is a deformation retraction $\Gamma \to \Gamma_{\text{stable}}$ if $g \geq 1$.

We can therefore usually contract out the bad edges without changing the homotopy type.

1.3 Moduli space

We now want to describe all different types of tropical curves with some fixed data to form a moduli space. We first start with the following observation:

Fact 1.18. For a fixed genus g and number of basepoints n, there are finitely many stable combinatorial models (with markings and weights, and forgetting edge lengths).

Fact 1.19. These stable combinatorial models form a poset under contracting edges.

We can therefore list all possibilities in some simple examples. The lines between models depicts the partial ordering induced by contracting edges.

Example 1.20. If $g = 0$ and $n = 4$, then we have four stable combinatorial models; if $g = 2$ and $n = 0$, then we have seven stable combinatorial models. See Figure 1.7.

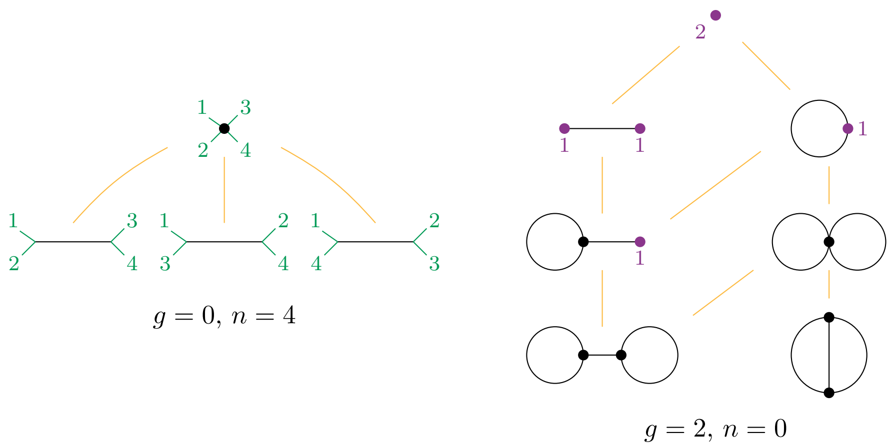


Figure 1.7: All stable combinatorial models in two simple cases. Yellow lines depict the poset structure.

Definition 1.21. A curve (Γ, w, m) is *smooth* if each point has $\operatorname{val} + 2 \cdot \operatorname{wt} \leq 3$.

Example 1.22. The minimal elements in the posets depicted in Figure 1.7 are smooth.

We can now define moduli spaces of tropical curves. We first describe them purely set-theoretically.

Definition 1.23.

- The *moduli space* $M_{g,n}$ is the set of all stable tropical curves of genus g with n marked points. In other words, it is similar to the poset we had before, except lengths are allowed to vary.
- The *link* $\Delta M_{g,n}$ is the set of all stable curves of genus g with n marked points of total length 1.

The topology on $M_{g,n}$ and $\Delta M_{g,n}$ comes from gluing cones then taking the quotient by a finite group action, and from gluing simplices then taking the quotient by a finite group action, respectively.

Definition 1.24. A(n open) *real cone* is

$$C_n = \mathbf{R}^n_{>0}$$

for some $n \geq 0$. A(n open) *simplex* is

$$\Delta_n = \left\{ \sum x_i = 1 \right\} \subset \mathbf{R}^{n+1}_{>0}.$$

The topology on $M_{g,n}$ and $\Delta M_{g,n}$ are then given by

$$M_{g,n} \overset{\text{top}}{=} \left(\bigsqcup_{\substack{\text{stab. comb.} \\ \text{models G}}} C_e / \operatorname{Aut}(G) \right) \Big/ \sim$$

$$\Delta M_{g,n} \overset{\text{top}}{=} \left(\bigsqcup_{\substack{\text{stab. comb.} \\ \text{models G}}} \Delta_{e-1} / \operatorname{Aut}(G) \right) \Big/ \sim$$

where the equivalence relation comes from contracting things, e is the number of edges in G, and automorphisms must respect (w, m).

Example 1.25. We can visualize $M_{0,4}$ and $\Delta M_{2,0}$ as in Figure 1.8. Note that for $\Delta M_{2,0}$, two combinatorial types have extra automorphisms, which we remember by drawing a half-triangle and sixth-triangle.

We list some facts about the topological types of these spaces $\Delta M_{g,n}$.

Theorem 1.26 (see [CGP16])**.**

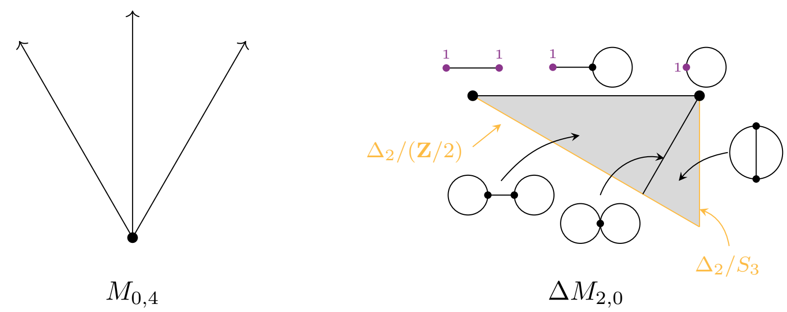


Figure 1.8: The topology of $M_{0,4}$ and $\Delta M_{2,0}$. Yellow lines and text keep track of "extra automorphisms".

(a) [Vog90] *For* $g = 0$,

$$\dim \widetilde{H}^i(\Delta M_{0,n}; \mathbf{Q}) = \begin{cases} (n-2)! & \text{if } i = n-4 \\ 0 & \text{otherwise} \end{cases}$$

and in fact, $\Delta M_{0,n} \simeq \bigvee^{(n-2)!} S^{n-4}$.

(b) *For* $g = 1$, $\Delta M_{1,n}$ *is contracible for* $n = 1, 2$, *and for* $n \geq 3$, *we have*

$$\dim \widetilde{H}^i(\Delta M_{1,n}; \mathbf{Q}) = \begin{cases} \frac{1}{2}(n-1)! & \text{if } i = n-1 \\ 0 & \text{otherwise} \end{cases}$$

and in fact, $\Delta M_{1,n} \simeq \bigvee^{\frac{1}{2}(n-1)!} S^{n-1}$.

(c) *For* $g \geq 1$ *and* $n \geq 1$, *rational homology* $\widetilde{H}_i(\Delta M_{g,n}, \mathbf{Q})$ *vanishes in degrees* $i \leq 2g+n-4$, *and homotopy groups* $\pi_k(\Delta M_{g,n})$ *vanish in degrees* $1 \leq k \leq n-3$.

Lecture 3

Exercises 1.27.

1. What is the dimension of $\Delta M_{g,n}$?
2. (a) What is $\Delta M_{1,2}$?
   (b) What is the outer space $\mathscr{O}_{1,2}$?

1.4 Maps between $\Delta M_{g,n}$'s

It is useful to have relationships between different moduli spaces when g and n vary.

There are forgetful maps

$$f_{g,n} \colon \Delta M_{g,n} \longrightarrow \Delta M_{g,n-1}$$

which forget the last marked point, where we have to be a bit careful to ensure the result is still stable; see Figure 1.9.

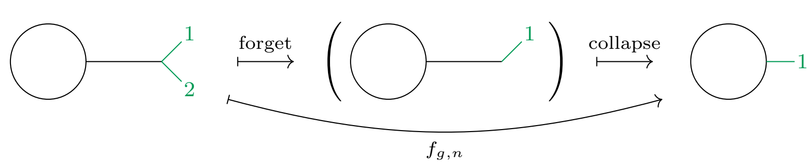


Figure 1.9: Forgetful maps require collapsing edges.

There are also *clutching maps*

$$\Delta M_{g_1,n_1+1} \times \Delta M_{g_2,n_2+1} \longrightarrow \Delta M_{g_1+g_2,n_1+n_2}$$

by wedging at the last base point, and halving edge lengths. The result is often not smooth, but it is still stable. Weights are added. The map is not surjective, but varying over all genus maps and by also considering unshrinking edges, you can obtain everything in $\Delta M_{g_1+g_2,n_1+n_2}$.

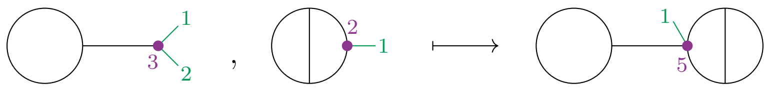


Figure 1.10: Clutching maps attach two curves at their last base points.

2 Outer space

Now that we have defined the link $\Delta M_{g,n}$, we would like to ask: What is its "orbifold" universal covering space? For example, for $\Delta M_{2,0}$ (see Figure 1.8), the naïve guess is that there is a $6:1$ cover from a larger triangle that is six copies of the one depicted in Figure 1.8. However, this is not quite correct, since we would like the covering group to be the group of outer automorphisms of the free group according to the analogy we setup in §0.1.

It turns out that outer space is the appropriate "orbifold" universal cover.

2.1 Teichmüller space

We first start by recalling the analogous construction in differential geometry. Let Σ_g be a closed, oriented, genus g surface. We then have a covering map

$$\begin{array}{c} \mathscr{T}_g = \left\{ (S_g, m, r) \;\middle|\; \begin{array}{l} S_g \text{ a closed, oriented, genus } g \text{ surface} \\ m \text{ a constant curvature metric on } S_g \\ r\colon \Sigma_g \to S_g \text{ a homeomorphism} \end{array} \right\} \Big/ \begin{array}{c}\text{diffeomorphisms} \\ \text{fixing } r\end{array} \\ \Big\downarrow \\ \mathscr{M}_g = \left\{ (S_g, m) \;\middle|\; \begin{array}{l} S_g \text{ a closed, oriented, genus } g \text{ surface} \\ m \text{ a constant curvature metric on } S_g \end{array} \right\} \Big/ \text{diffeomorphisms} \end{array}$$

which is a ramified cover with finite stabilizer and covering group $\mathrm{MCG}(\Sigma_g)$. While the covering space is smooth, the target is not.

For tropical curves, the analogous covering map is

$$\begin{array}{c} \mathscr{O}_g = \left\{ (\Gamma, r) \;\middle|\; \begin{array}{l} \Gamma \text{ a constant length, unweighted, stable} \\ \quad \text{genus } g \text{ tropical curve} \\ r\colon \Sigma_g \to S_g \text{ a homotopy equivalence} \end{array} \right\} \\ \Big\downarrow \\ \Delta M_g^\circ = \left\{ \begin{array}{c} \text{constant length, unweighted, stable} \\ \text{genus } g \text{ tropical curves} \end{array} \right\} \end{array}$$

which has covering group $\mathrm{Out}(F_g)$. The superscript $\circ$ corresponds to restricting to unweighted curves. We call $\mathscr{O}_g$ the *outer space* of genus g. It is also known as the *Culler–Vogtmann space* [CV86]; they defined the space outside of the context of tropical curves in order to study $\mathrm{Out}(F_g)$.

Example 2.1. Let $g = 2$. Recall that $\mathrm{Out}(F_2) \cong \mathrm{GL}_2(\mathbf{Z})$, and so the covering map $\mathscr{O}_2 \to \Delta M_2^\circ$ should have covering group $\mathrm{GL}_2(\mathbf{Z})$. A visualization of $\mathscr{O}_2$ is in Figure 2.1. The "extra information" compared to the naïve $6:1$ cover comes from the orientation of the loops in the curves.

2.2 Distance in outer space

Outer space has a natural metric-like pairing on it, called the *Lipschitz distance*. It is defined using the following notion:

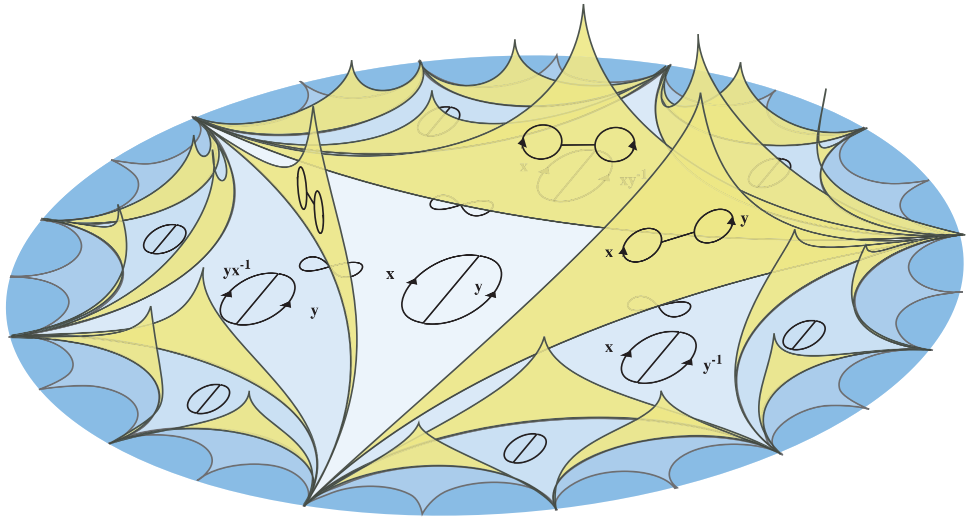


Figure 2.1: A visualization of the outer space $\mathscr{O}_2$ taken from [Vog15, Fig. 2].

Definition 2.2. The *Lipschitz constant* of a map of metric spaces $f\colon X \to Y$ is

$$L(f) := \sup_{x,y\in X}\left\{\frac{d_Y\big(f(x),f(y)\big)}{d_X(x,y)}\right\}.$$

Definition 2.3. The *Lipschitz distance* between (Γ, r) and (Γ', r') in $\mathscr{O}_g$ is

$$d\big((\Gamma,r),(\Gamma',r')\big) := \inf_f\{\log L(f)\}$$

taken over all continuous maps $f\colon \Gamma \to \Gamma'$ which preserve the marking, i.e., all continuous maps f such that the triangle

$$\begin{tikzcd} \Gamma \arrow[rr, "f"] & \arrow[d, phantom, "\simeq"] & \Gamma' \\ & \bigvee^g S^1 \arrow[ul, "r"] \arrow[ur, "r'"'] & \end{tikzcd}$$

commutes up to homotopy.

Proposition 2.4. *The Lipschitz distance satisfies the following properties:*

- $d(-,-) = 0 \iff (\Gamma, r) = (\Gamma', r')$*;*
- *the triangle inequality*

but is not *symmetric (but is quasi-symmetric, that is, symmetric up to a fixed constant).*

Example 2.5. We give an example where the Lipschitz distance is not symmetric in Figure 2.2. In general, paths back and forth may not even be the same; see [FM11, §6] and [Vog15, §5.5].

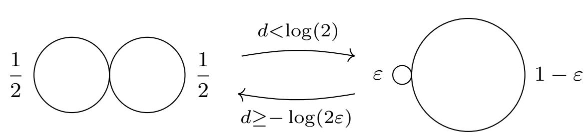


Figure 2.2: "It is fast to get to edge of $\mathscr{O}_g$, but slow getting back."

We note that the construction of outer space also gives an interpretation of the classifying space $\mathrm{B}\,\mathrm{Out}(F_n)$:

$$\begin{tikzcd} \mathscr{O}_g \arrow[r, "\sim"] \arrow[d] & \mathrm{E}\,\mathrm{Out}(F_n) \arrow[d] \\ \Delta M_{g,0} \arrow[r, "\sim"] & \mathrm{B}\,\mathrm{Out}(F_n) \end{tikzcd}$$

If there is one basepoint, then this generalizes to

$$\begin{tikzcd} \mathscr{O}_{g,1} \arrow[r, "\sim"] \arrow[d] & \mathrm{E}\,\mathrm{Aut}(F_n) \arrow[d] \\ \Delta M_{g,1} \arrow[r, "\sim"] & \mathrm{B}\,\mathrm{Aut}(F_n) \end{tikzcd}$$

Problem 2.6. What about $n \geq 2$ basepoints?

3 Boundary complex of $\bar{\mathscr{M}}_{g,n}$

We now turn to the algebraic geometric interpretation for $\Delta M_{g,n}$ from §0.2. Consider the Deligne–Mumford compactification

$$\mathscr{M}_{g,n} \subset \bar{\mathscr{M}}_{g,n}$$

of the moduli space $\mathscr{M}_{g,n}$ of algebraic curves of genus g with n base points. The *boundary complex* is the dual complex $\Delta(D)$ of the boundary divisor

$$D = \bar{\mathscr{M}}_{g,n} \smallsetminus \mathscr{M}_{g,n}.$$

The connection with tropical curves is the following:

Theorem 3.1. $\Delta(D) \cong \Delta M_{g,n}$.

The rigorous definition of these objects involves a lot of machinery from algebraic geometry, but we hope that we can give some visual description of them without being too technical.

3.1 What is a boundary complex?

Let X be a compact (proper) variety, and let $D \subset X$ be a divisor (a codimension one algebraic set).

Definition 3.2. A divisor $D \subset X$ (a codimension one algebraic set) has *normal crossings* if singularities of D look (analytically locally) like (at worst) intersecting hyperplanes (see Figure 3.1). We say D has *simple normal crossings* if moreover, no component has self-intersections.

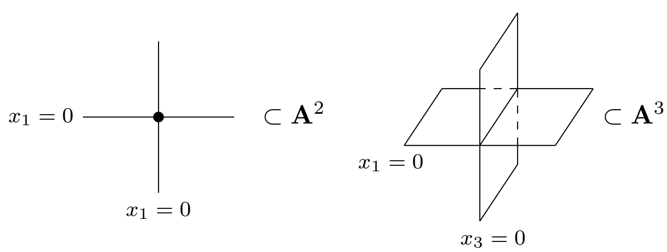


Figure 3.1: Intersecting axes in $\mathbf{A}^2$ and $\mathbf{A}^3$.

Example 3.3. The main difference between normal crossings and simple normal crossings divisors is that normal crossings divisors may have irreducible components that intersect themselves; see Figure 3.2.

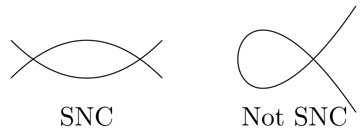


Figure 3.2: Normal crossings and simple normal crossings divisors.

Definition 3.4. The *dual complex* $\Delta(D)$ of a simple normal crossings divisor has
- a vertex for every irreducible component $\Delta_i \subset D$;
- a p-simplex for every codimension p intersection $\subset D$ (codimension $p+1$ intersection $\subset X$).

Example 3.5. We give an example of a simple normal crossings divisor and its dual complex in Figure 3.3.

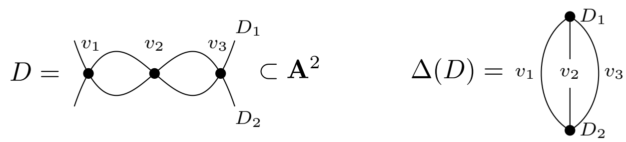


Figure 3.3: A simple normal crossings divisor and its dual complex.

One reason to care about the dual complex is that it gives topological information about its complement:

Fact 3.6. For any smooth variety U and any compactification $X \supset U$ such that $D = X \smallsetminus U$ has simple normal crossings, there is an isomorphism

$$\widetilde{H}_i\big(\Delta(D); \mathbf{Q}\big) \simeq \mathrm{Gr}^W_{2d} H^{2d-i-1}(U, \mathbf{Q}),$$

where $\mathrm{Gr}^W_k H^*$ is the kth graded piece of Deligne's weight filtration.

For our application, we will need to work with normal crossings instead of simple normal crossings divisors. We therefore have the following:

Goal 3.7. Define a notion of *generalized* Δ*-complex* that fits into the correspondence below:

$$\begin{aligned} \text{generalized } \Delta\text{-complex} &\longleftrightarrow \text{normal crossings} \\ \Delta\text{-complex} &\longleftrightarrow \text{simple normal crossings} \end{aligned}$$

These will be combinatorial objects X with a topological realization $|X|$. There will be more data in the combinatorial object than just the resulting space $|X|$. In particular, there is a notion of cellular homology for X that does *not* agree with the singular homology of $|X|$ over $\mathbf{Z}$, but *does* agree with coefficients in $\mathbf{Q}$. This might be what you expect since generalized Δ-complexes have torsion group actions, which might disappear when extending scalars to $\mathbf{Q}$.

3.2 Δ-complexes and generalized Δ-complexes

Recall that

$$\Delta^p = \Big\{(x_0, x_1, \dots, x_p) \;\Big|\; \sum x_i = 1\Big\} \subset \mathbf{R}^{p+1}_{\geq 0}$$

is a standard p-simplex for $p \geq 0$. If S is a finite set, then

$$\Delta^S = \Big\{(x_i)_{i \in S} \;\Big|\; \sum_S x_i = 1\Big\} \subset \mathbf{R}^{|S|}_{\geq 0}.$$

Any function $f\colon S \to T$ induces a continuous map

$$f_*\colon \Delta^S \longrightarrow \Delta^T$$

by linearly interpolating between vertices.

Definition 3.8.

- The category $\Delta_{\text{inj},\leq}$ is the category with objects
$$\big\{[p] = \{0, 1, \ldots, p\} \;\big|\; \text{for every } p \geq 0\big\},$$
and morphisms that are injective, order-preserving functions.
- The category Δ_{inj} is the category with the same objects as $\Delta_{\text{inj},\leq}$, but with morphisms that are just injective functions.

Definition 3.9.

- A(n *ordered*) Δ*-complex* is a functor
$$X\colon (\Delta_{\text{inj},\leq})^{\mathsf{op}} \longrightarrow \mathsf{Sets}.$$
- A *generalized* Δ*-complex* is a functor
$$X\colon (\Delta_{\text{inj}})^{\mathsf{op}} \longrightarrow \mathsf{Sets}.$$

These functors make the assignments

$$\begin{aligned} X([p]) &= \text{set of } p\text{-simplices } \{\Delta^p\} \\ X([p] \hookrightarrow [q]) &= \text{face maps } (q\text{-simplices}) \to (p\text{-simplices}) \end{aligned}$$

Example 3.10. We give an example of a face map in Figure 3.4.

$$\left(\{0,1\} \overset{\substack{0\mapsto 0\\ 1\mapsto 1}}{\hookrightarrow} \{0,1,2\}\right) \longmapsto$$

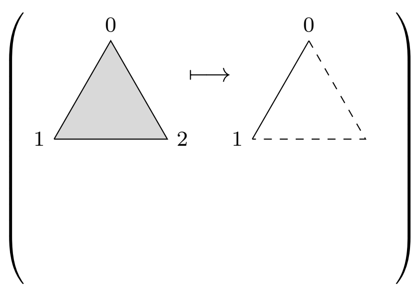


Figure 3.4: A face map where the "removed" part of the 2-simplex is contracted onto the segment $\overline{01}$.

We now describe how to think of these objects as topological spaces.

Definition 3.11. A *geometric* or *topological realization* $|X|$ of a (generalized) Δ-complex is

$$|X| = \left(\coprod_{p\geq 0} X\big([p]\big) \times \Delta^p\right) \Big/ \sim,$$

where $\sim$ glues along face maps $f\colon [p] \hookrightarrow [q]$ (ranging over all maps $[p] \hookrightarrow [q]$ in the appropriate category) via

$$\begin{array}{ccc} X([q]) \times \Delta^q & \longrightarrow & X([p]) \times \Delta^p \\ (x \;,\; f_* a) & \sim & (f^* x \;,\; a) \end{array}$$

We close with examples of generalized Δ-complexes.

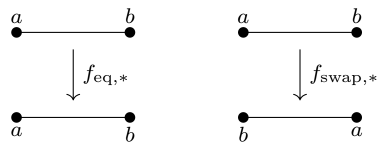
Example 3.12 (Half-segment)**.** Consider

$$\begin{aligned} X\colon (\Delta_{\text{inj}})^{\mathsf{op}} &\longrightarrow \mathsf{Sets} \\ [0] &\longmapsto \{\text{pt}\} \\ [1] &\longmapsto \{\text{line}\} \end{aligned}$$

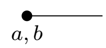
with topological realization

$$|X| = \left(\bullet \sqcup \bullet\!\!-\!\!\!-\!\!\bullet\right)\Big/\sim$$

where the equivalence relation $\sim$ comes from all injective maps $f\colon [0] \hookrightarrow [1]$ and $f\colon [1] \hookrightarrow [1]$. There are two inclusions $[0] \hookrightarrow [1]$, and two inclusions $[1] \hookrightarrow [1]$, given by



Thus, the topological realization is

$|X| =$ 

where the right end point is (what used to be) the midpoint. Topologically, $|X|$ is homeomorphic to the 1-simplex Δ^1, but $H_1(X, \mathbf{Z}) = \mathbf{Z}/2 \neq 0$.

We next give an example of a normal crossings divisor D and its associated generalized Δ-complex $\Delta(D)$, although we will not carefully give the general construction for $\Delta(D)$.

Example 3.13 (Whitney umbrella)**.** Let $X = \mathbf{A}^3 \smallsetminus \{y = 0\}$ with the divisor $D = \{x^2 y = z^2\}$. This divisor has singularities and intersects itself, but there is only one component; see Figure 3.5.

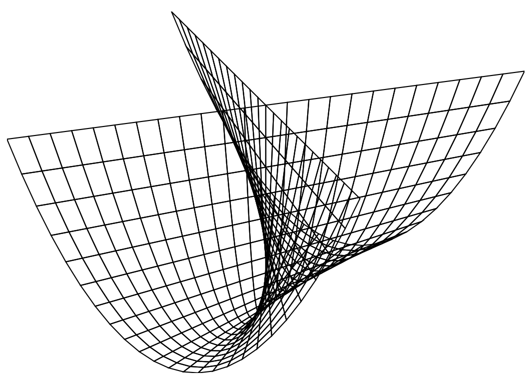


Figure 3.5: The Whitney umbrella.

We give a sketch of how to construct $\Delta(D)$ in general. First, consider

$$n\colon \widetilde{D} \xrightarrow{\nu} D \hookrightarrow X,$$

where ν is the normalization map, and then consider the $(p+1)$-fold fiber product (as topological spaces)

$$\widetilde{D}^{(p)} \coloneqq \widetilde{D} \times_X \cdots \times_X \widetilde{D}.$$

Note that $\widetilde{D} \simeq \mathbf{A}^2 \smallsetminus \{y = 0\}$. Let $\widetilde{D}_p$ be the open subset of $\widetilde{D}^{(p)}$ corresponding to *distinct* $(p+1)$-tuples mapping to codimension p strata in D. Then,

$$\Delta(D)([p]) = \{\text{irreducible components of } \widetilde{D}_p\}.$$

For the Whitney umbrella, only the cases $p = 0, 1$ matter, and we have

$$\Delta(D)([0]) = \{\bullet\}, \qquad \Delta(D)([1]) = \{\bullet\}.$$

Note that $\Delta(D)([0])$ corresponds to all of $\widetilde{D}$. Then, in the cartesian diagram

$$\begin{tikzcd} \widetilde{D}^{(0)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{D} \arrow[d, "n"] \\ \widetilde{D} \arrow[r, "n"] & X \end{tikzcd}$$

one can identify

$$\widetilde{D}^{(0)} \simeq \mathbf{A}^2 \sqcup \left(\mathbf{A}^1 \smallsetminus \{0\}\right).$$

After some work, one can show $\Delta(D)$ is the half-segment from Example 3.12.

There is an alternative construction of $\Delta(D)$ using a choice of étale cover.

Exercise 3.14. Consider when D is the nodal cubic (the non-simple normal crossings example in Figure 3.2).

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