1. **Steinitz Problem.** Use the partial coloring lemma to show an $O(d^{1/2}\log n)$ discrepancy bound for the discrepancy for prefix sums problem, in the ℓ_∞ norm.

   [Hint: Suppose $n = 2^k$ and for $i = 0, 1, \ldots, k$, partition $[n]$ into $n/2^i$ intervals of size 2^i. Each prefix $[j]$ is a disjoint union of $O(\log n)$ such *canonical* intervals, at most one of each type.

   Consider such intervals for each coordinate $q \in [d]$. Choose Δ_i for sets of size 2^i, so that (i) $|\sum_i \Delta_i = O(d^{1/2})$. (ii) Partial coloring condition $\sum_i n_i g(\Delta_i/2^{i/2}) \leq n/5$ holds, where $n_i = nd2^{-i}$ is the number of sets of size 2^i. The calculation is easier if set sizes are considered relative to d.]

2. **Vector Discrepancy.** Let A be a $m \times n$ matrix with rows a_i. We say A has a vector discrepancy at most t, if there exists unit vectors v (in arbitrary dimensional space) satisfying

   $$\|\sum_j a_{ij} v_j\|_2 \leq t \qquad \forall i \in [m].$$

   Given a target discrepancy t, write an SDP to determine if A has vector discrepancy $\leq t$.

3. **Vector Discrepancy for arbitrary set systems.** Given any set system on n elements and m sets, show that it has vector discrepancy $\sqrt{n}$.

   [Hint: Look for a very simple feasible vector solution.]

4. **SDPs and Gaussian random variables.** Let $g \in \mathbb{R}^n$ be a gaussian random vector with iid $N(0,1)$ coordinates. For any vector $u \in \mathbb{R}^n$, show that the random variable $X = \langle g, u \rangle$ is distributed as $N(0, \|u\|^2)$. For any two vectors u and v, the random variables $X = \langle g, u \rangle$ and $Y = \langle g, v \rangle$ (the g is the same for X and Y) satisfy $\mathbb{E}[XY] = u \cdot v$.

   For a PSD matrix A, let $u_1, \ldots, u_n \in \mathbb{R}^n$ be the vectors in its Cholesky decomposition (i.e. $A_{ij} = u_i \cdot u_j$ for all $i, j \in [n]$). Show that the random variables $X_1, \ldots, X_n$ with $X_i = \langle g, u_i \rangle$ for $i \in [n]$ are jointly gaussian (i.e. every linear combination $\sum_i c_i X_i$ is a Gaussian) with covariance matrix A (i.e. $A_{ij} = \mathbb{E}[X_i X_j]$).

5. **Distance of random walks.** Let $S = X_1 + \ldots + X_n$ where X_i are independent mean-zero random variables satisfying $|X_i| \leq 1$. Show that $\mathbb{E}[|S|] \leq \sqrt{n}$.

   [Hint: Consider $\mathbb{E}[S^2]$.]

6. (Optional). Use the probabilistic method to show that there exists a set system on n elements and m sets with discrepancy $\Omega(\sqrt{n\log(m/n)})$, and hence Spencer's bound is tight.

   [Hint: Consider a random set system. For a fixed coloring x, show that the probability that each set has discrepancy $\ll \sqrt{n\log(m/n)}$ is $\ll 2^{-n}$. Why does this suffice?]

7. (Optional). Use the Lovasz Local Lemma to show a $O(\sqrt{k\log k})$ discrepancy bound for the Beck-Fiala problem, if the set sizes are also bounded by k (and the degree is also k).