An Introduction to Semidefinite Programming for Combinatorial Optimization (Problem Session)

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Exercise 1. Given $X \in \mathbb{S}^p_+$, prove there exists $Y \in \mathbb{S}^p_+$ such that $Y^2 = X$.

Comments and Hints: Y is called the *square root* of X, and is usually written as $X^{1/2}$. To prove its existence, as a warm-up, first consider the case when X is diagonal. Then prove the general case by using the spectral decomposition VDV^T of X, and remember that *orthogonal* means $V^TV = VV^T = I$.

Exercise 2. Prove $X, S \in \mathbb{S}^p_+$ implies $X \bullet S \geq 0$.

Comments and Hints: This will establish that $\mathbb{S}^p_+$ is self-dual. To prove it, use the previous exercise along with the identities

$$\begin{aligned} &\text{trace}(M^T N) = M \bullet N = N^T \bullet M^T = \text{trace}(N M^T), \\ &\|M\|_F = \sqrt{M \bullet M} \end{aligned}$$

to show that $X \bullet S = \|X^{1/2} S^{1/2}\|_F^2$.

Exercise 3. For $X, S \in \mathbb{S}^p_+$, prove $X \bullet S = 0 \Leftrightarrow XS = 0$.

Comments and Hints: Prove this using the prior exercise. Note that, when X and S are diagonal, this result reduces to the well-known vector condition $x^T s = 0 \Leftrightarrow x \circ s = 0$ for $x, s \geq 0$.

Exercise 4. What is the dimension of the feasible set of the primal problem

$$\begin{array}{ll} \inf & C \bullet X \\ \text{s.t.} & A_i \bullet X = b_i \;\; \forall\, i = 1, \ldots, m \\ & X \succeq 0 \end{array} \tag{P}$$

assuming that the data matrices $\{A_1, \ldots, A_m\}$ are linearly independent in $\mathbb{S}^n$ and that (P) is interior feasible? Assuming also that the dual problem

$$\begin{array}{ll} \sup & b^T y \\ \text{s.t.} & C - \sum_{i=1}^m y_i A_i \succeq 0. \end{array} \tag{D}$$

has an interior feasible solution, determine the dimension of the feasible set of (D). Here, *dimension* refers to the number of degrees of freedom of the (relative) interior of the feasible set.

Exercise 5. The following primal-dual example shows that strong duality does not hold for SDP in general:

$$\begin{aligned} 1 = \inf \quad & X_{33} \\ \text{s.t.} \quad & X_{11} = 0 \\ & X_{12} + X_{21} + 2X_{33} = 2 \\ & X \succeq 0; \end{aligned}$$

$$\begin{aligned} 0 = \sup \quad & 2\,y_2 \\ \text{s.t.} \quad & \begin{pmatrix} -y_1 & -y_2 & 0 \\ -y_2 & 0 & 0 \\ 0 & 0 & 1-2\,y_2 \end{pmatrix} \succeq 0. \end{aligned}$$

Argue that 1 and 0 are in fact the respective optimal values. Also, discuss the violation(s) of the strong-duality theorem.

Comments and Hints. One argument uses the following 2×2 determinant property of $X \in \mathbb{S}^p_+$: $X_{jk}^2 \leq X_{jj}X_{kk}$ for all j, k. In particular, consider how $X_{jj} = 0$ affects the j-th row and column of X.

Exercise 6. For a column vector x and nonnegative scalar t, prove

$$\|x\| \leq t \qquad \Longleftrightarrow \qquad t^2 I - xx^T \succeq 0.$$

Comments and Hints. This is the first step in proving that any SOCP can be modeled as an SDP; the next step (which is not part of this exercise) is to apply the *Schur complement theorem*, which ensures

$$t^2 I - xx^T \succeq 0 \qquad \Longleftrightarrow \qquad \begin{pmatrix} t & x^T \\ x & tI \end{pmatrix} \succeq 0.$$

To prove the exercise, apply the first definition of positive semidefiniteness: $X \in \mathbb{S}^p$ if and only if $v^T X v \geq 0$ for all $v \in \mathbb{R}^p$. To make the proof a little easier, you can assume without loss of generality that $\|v\| = 1$.

Exercise 7. For any $\mu > 0$, the point on the primal-dual central path corresponding to μ is the unique solution (X_μ, y_μ, S_μ) of the following system of equations (assuming $X, S \succ 0$):

$$\begin{aligned} A_i \bullet X &= b_i \quad \forall\, i = 1, \ldots, m \\ \sum_{i=1}^{m} y_i A_i + S &= C \\ XS &= \mu I. \end{aligned}$$

Derive a simple, closed-form expression for $X_\mu \bullet S_\mu$.

Exercise 8. An alternate form of the ϑ-number SDP relaxation is

$$\begin{aligned} \max \quad & ee^T \bullet X \\ \text{s.t.} \quad & \operatorname{trace}(X) = 1 \\ & X_{ij} = 0 \;\; \forall \text{ edges } (i,j) \\ & X \succeq 0. \end{aligned}$$

The dual of this form is

$$\begin{aligned} \min \quad & \lambda \\ \text{s.t.} \quad & \lambda I + \sum_{\text{edges } (i,j)} y_{ij} E_{ij} - ee^T \succeq 0. \end{aligned}$$

Prove that strong duality holds between this primal-dual pair by exhibiting positive definite feasible solutions in each problem.

Exercise 9. Given a MaxCut instance with adjacency matrix A, a compact way to write the SDP relaxation is

$$\max\{L \bullet X : \mathrm{diag}(X) = e, X \succeq 0\},$$

where:

- $L := \frac{1}{4}\left(\mathrm{Diag}(Ae) - A\right)$ is the *Laplacian matrix* of the graph;
- the operator $\mathrm{diag}(\cdot)$ extracts the diagonal of its matrix input;
- the operator $\mathrm{Diag}(\cdot)$ makes a diagonal matrix out of its vector input.

A compact way to write the dual is

$$\min\{e^T y : \mathrm{Diag}(y) - L = S, S \succeq 0\}.$$

Using these compact forms, prove weak duality between the primal and dual.

Comments and Hints. As with LP, sometimes it's easier to work with a specific form of your problem rather than a standard form. This particular form for the MaxCut SDP relaxation highlights the importance of the operators $\mathrm{diag}(\cdot)$ and $\mathrm{Diag}(\cdot)$, which are in fact adjoint operators, i.e., no matter the inputs X and y, it holds that $\mathrm{diag}(X)^T y = X \bullet \mathrm{Diag}(y)$.

Exercise 10. Referring to the previous exercise, the low-rank approach for solving the MaxCut SDP solves instead

$$\max\{L \bullet (RR^T) : \operatorname{diag}(RR^T) = e\},$$

where the number of columns p in $R \in \mathbb{R}^{n \times p}$ is approximately $\sqrt{2n}$. In terms of the n rows of R, describe the geometric interpretation of the constraint $\operatorname{diag}(RR^T) = e$.