An Introduction to Semidefinite Programming for Combinatorial Optimization (Lecture 2)

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Outline

1. How to Convexify Nonconvex QP
2. Continuous Case
3. Mixed Binary Case
4. Mixed Integer Case
5. Final Thoughts

How to Convexify Nonconvex QP

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & x \in F \end{array}$$

$$F := \left\{ x \in \mathbb{R}^n : \begin{array}{l} Ax \leq b \\ x_j \in \mathbb{Z} \quad \forall\, j \in J \end{array} \right\}$$

What is $\overline{\text{conv}}(F)$?

$$\begin{array}{ll} \min & x^T Q x + 2\, c^T x \\ \text{s.t.} & x \in F \end{array}$$

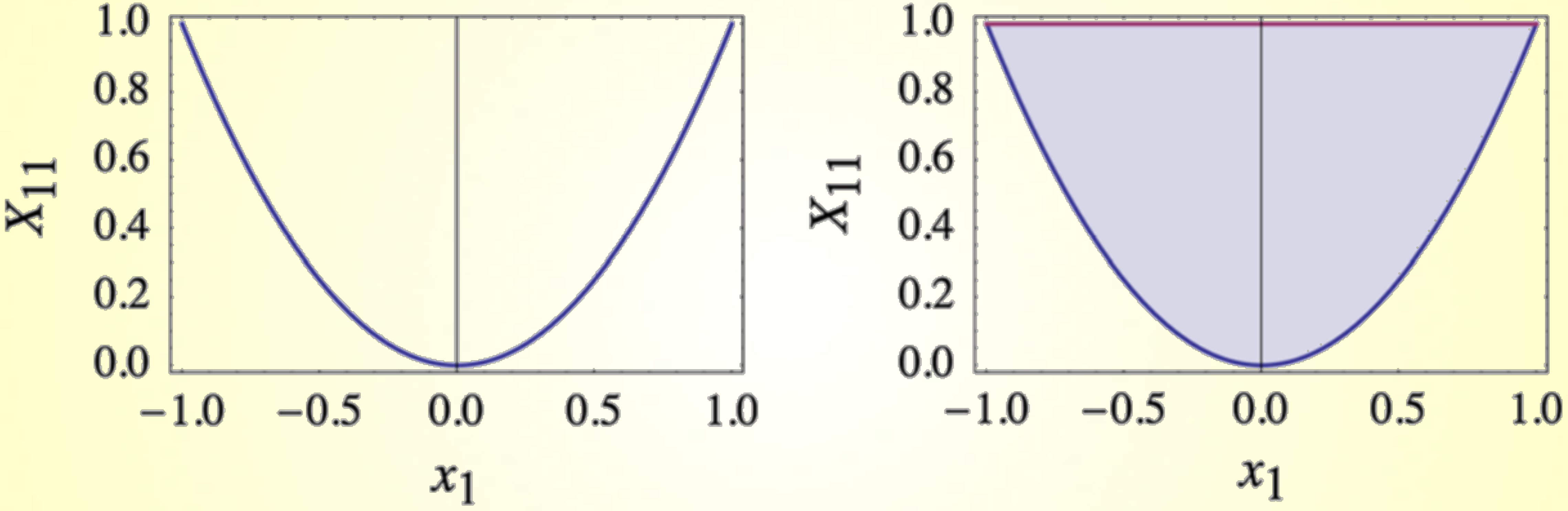
$$\downarrow$$

$$\begin{array}{ll} \min & \begin{pmatrix} 0 & c^T \\ c & Q \end{pmatrix} \bullet \begin{pmatrix} 1 & x^T \\ x & xx^T \end{pmatrix} \\ \text{s.t.} & x \in F \end{array}$$

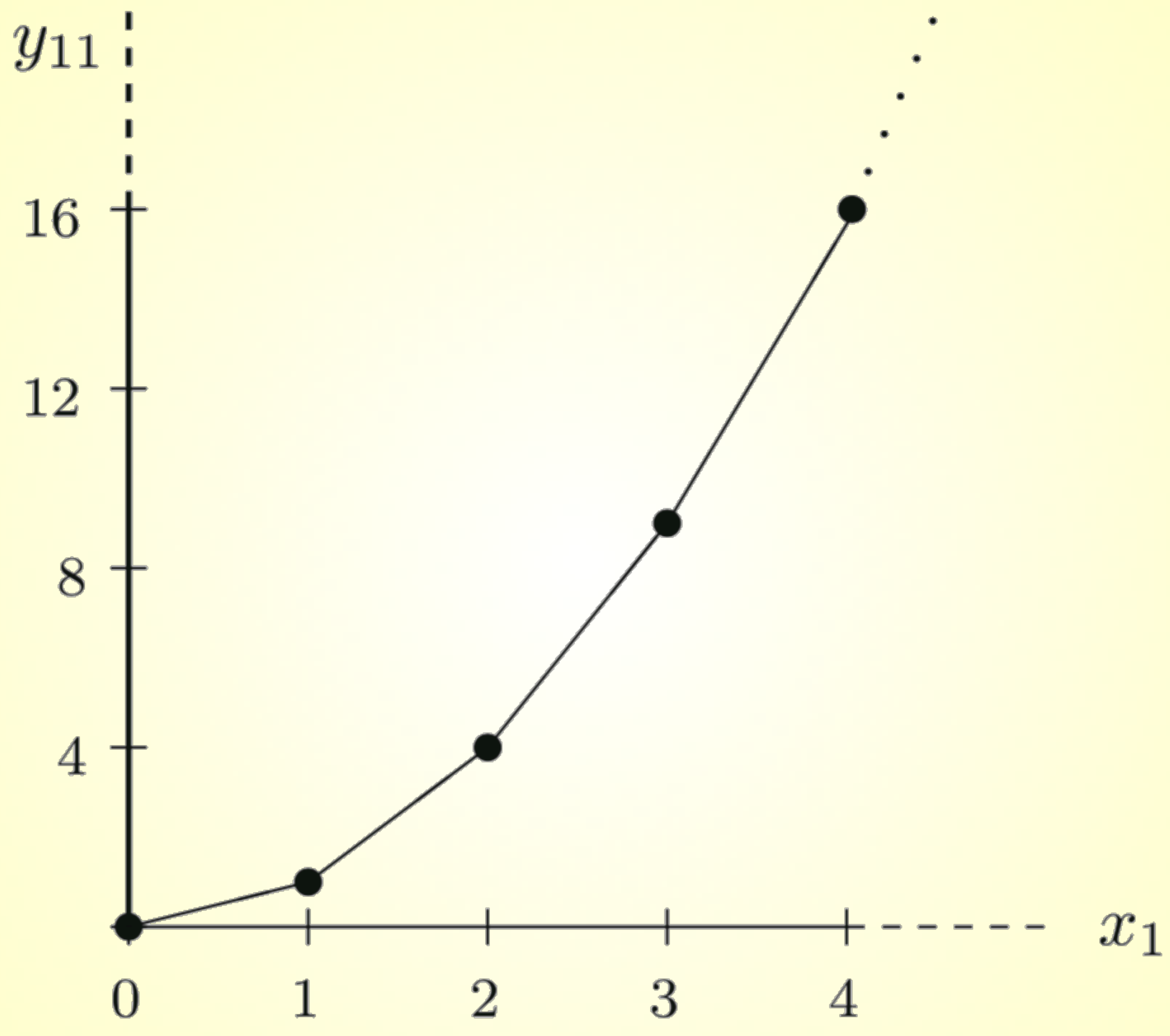
$$
\begin{aligned}
\min \quad & \begin{pmatrix} 0 & c^T \\ c & Q \end{pmatrix} \bullet Y \\
\text{s.t.} \quad & Y \in \widehat{F}
\end{aligned}
$$

$$
\widehat{F} := \left\{ \begin{pmatrix} 1 & x^T \\ x & xx^T \end{pmatrix} : x \in F \right\}
$$

What is $HG := \overline{\mathrm{conv}}(\widehat{F})$?



Source: B, "A Gentle, Geometric Introduction to Copositive Optimization", ***Mathematical Programming Series B*****, 151(1), 89–116, 2015**



Source: B and Letchford, "Unbounded Convex Sets for Non-Convex Mixed-Integer Quadratic Programming", *Mathematical Programming*, 143(1-2), 231-256, 2014

- HG is *not* polyhedral, and taking the closure is necessary
- This approach may not be the most practical approach
  - Certainly characterizing HG is a hard problem
  - If your QP doesn't contain all $O(n^2)$ variables, it may make sense to convexify just over your variables
  - In fact, it might be better to lift to an even larger set of variables
- But understanding $HG \subseteq \mathbb{S}^{n+1}$ is our goal today

Observation. Fully characterizing HG is equivalent to identifying all quadratics $x^T Q x + 2\, c^T x + \kappa$, which are nonnegative for all $x \in F$

Which is equivalent to identifying all matrices

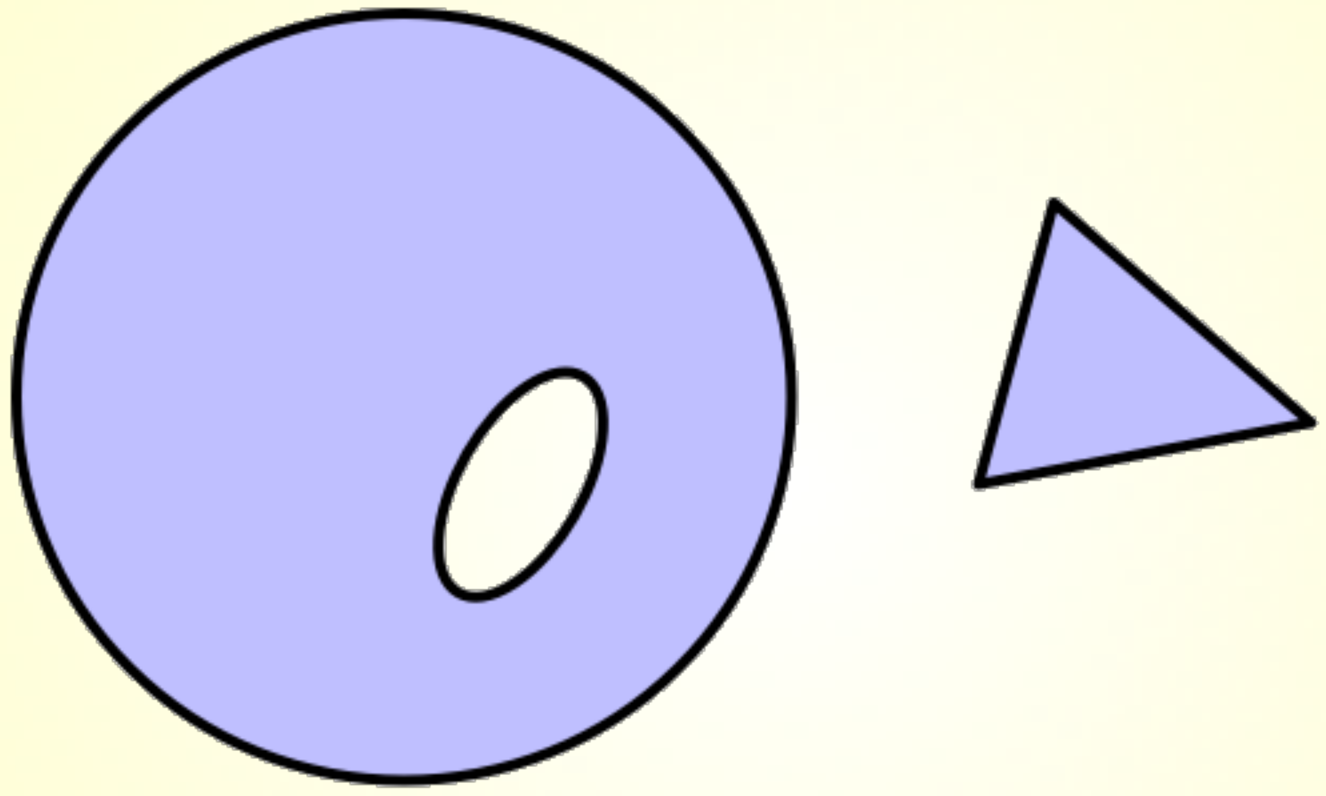
$$\begin{pmatrix} \kappa & c^T \\ c & Q \end{pmatrix} \in \mathbb{S}^{n+1}$$

whose associated quadratic is nonnegative over F. Such matrices are called *copositive over* F

In fact, HG is essentially the dual of "copositive over F"

Three Types of Valid Inequalities

- Given F, we will identify valid inequalities for HG
- Each will come from one of three classes...



Type 1: Explicit Quadratics

1

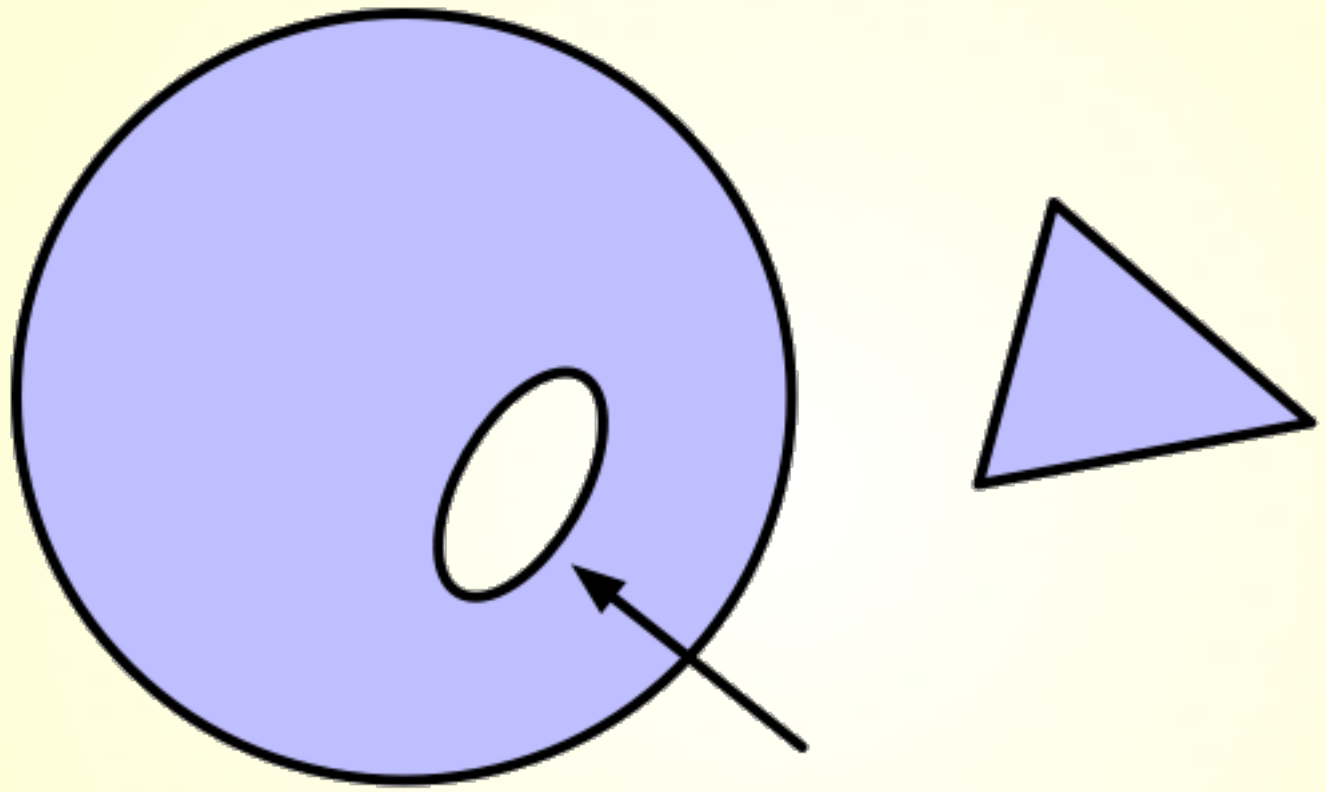
If

$$x^T Qx + 2c^T x + \kappa \geq 0$$

constrains F, then

$$Q \bullet X + 2c^T x + \kappa \geq 0$$

is a valid linear inequality for HG



Explicit quadratic for the
complement of an ellipsoid

Type 2: Linear Conjunctions

2

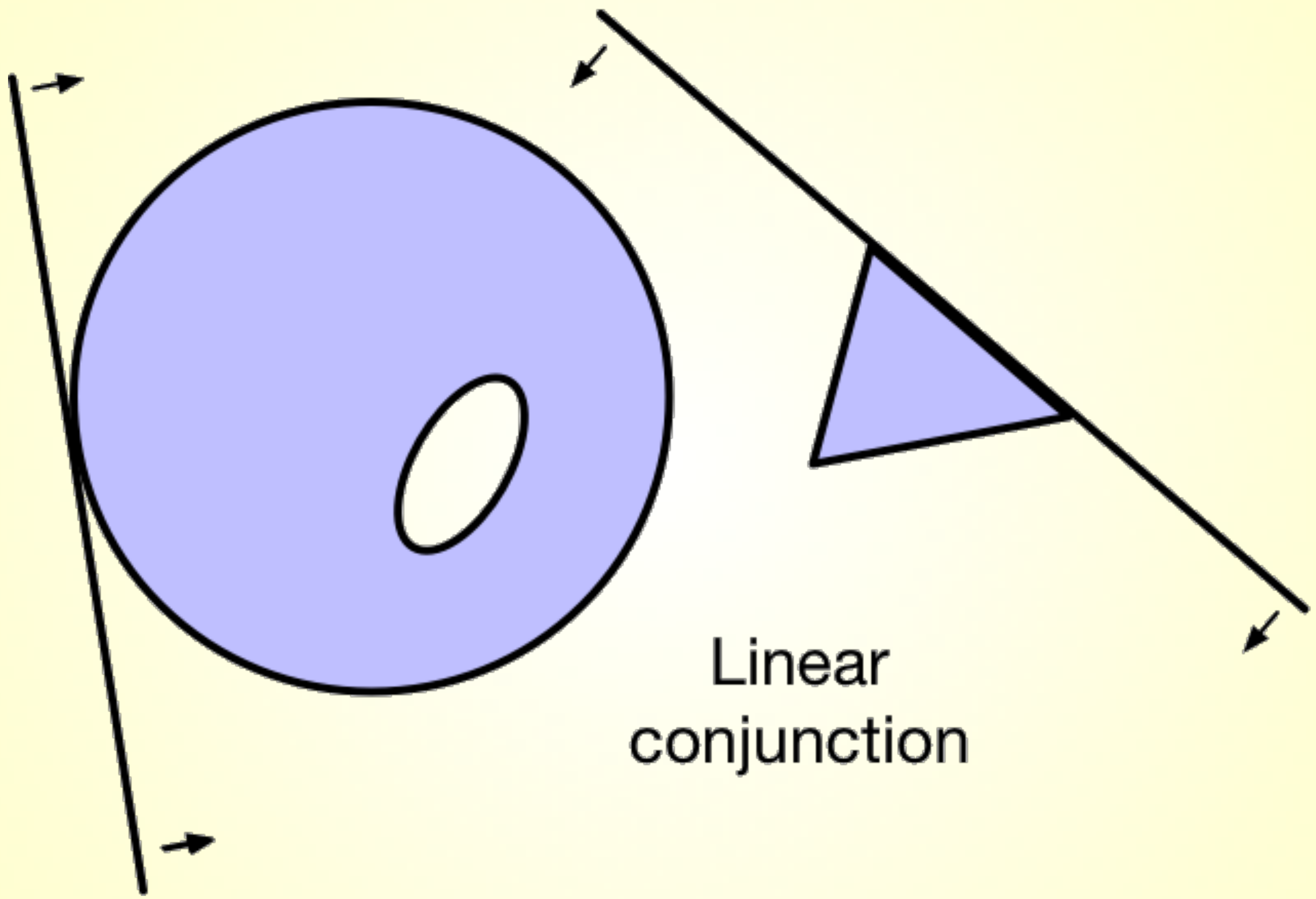
If

$$a_1^T x + b_1 \geq 0 \quad \text{and} \quad a_2^T x + b_2 \geq 0$$

are valid for F, then

$$\frac{1}{2}(a_1 a_2^T + a_2 a_1^T) \bullet X + (b_2 a_1 + b_1 a_2)^T x + b_1 b_2 \geq 0$$

is a valid linear inequality for HG


Linear
conjunction

3

Type 3: Linear Disjunctions

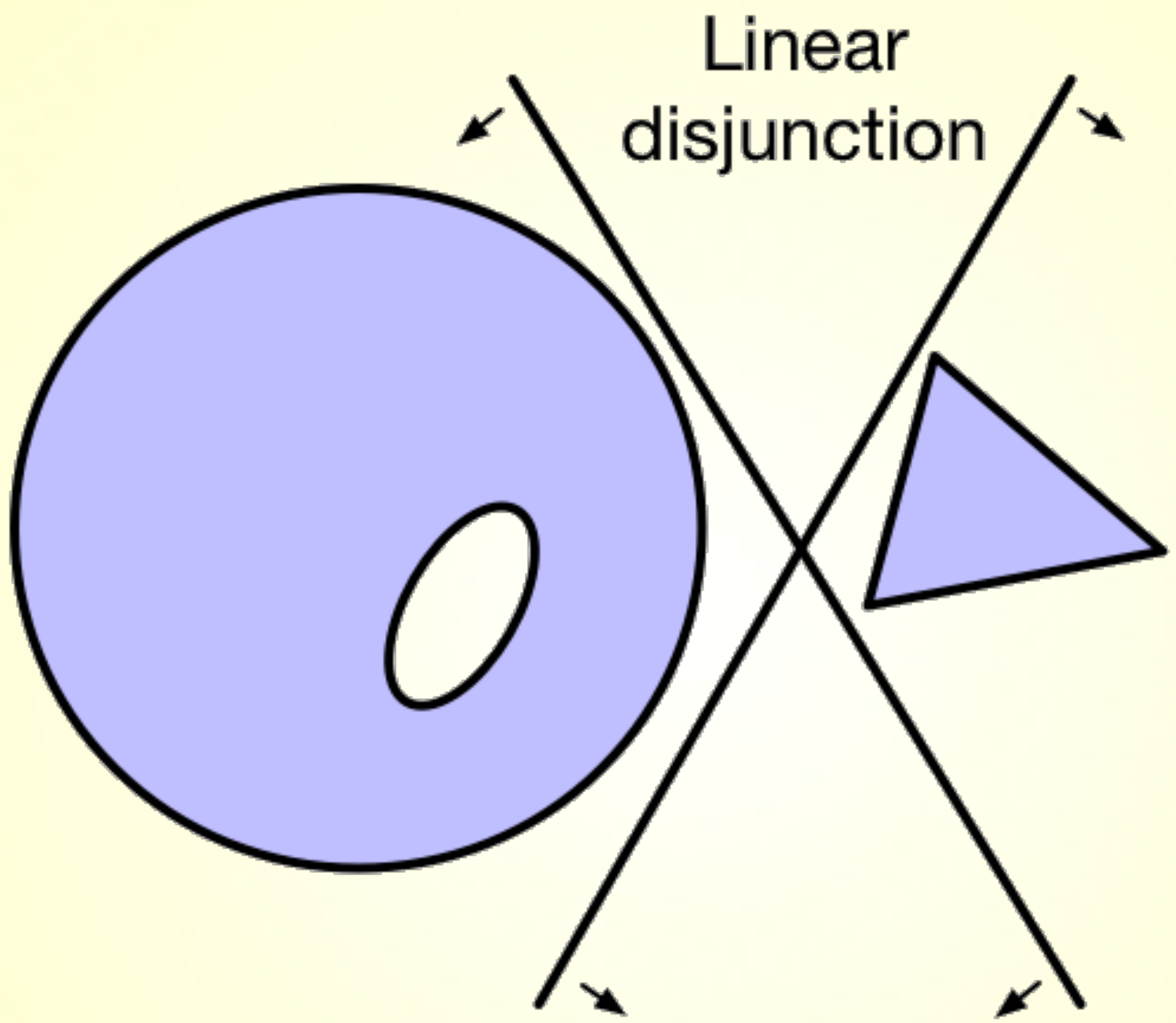
If every $x \in F$ satisfies

$$\text{either} \quad a_1^T x + b_1 \geq 0 \quad \text{or} \quad a_2^T x + b_2 \geq 0$$

then

$$\frac{1}{2}(a_1 a_2^T + a_2 a_1^T) \bullet X + (b_2 a_1 + b_1 a_2)^T x + b_1 b_2 \leq 0$$

is a valid linear inequality for HG


Linear
disjunction

Continuous Nonconvex QP

1. Unconstrained
2. Linear equations
3. Nonnegative orthant
4. Nonnegative orthant, linear equations, and complementarities
5. Linear inequalities
6. Half-ellipsoid
7. Swiss cheese

Unconstrained

3

$$F = \mathbb{R}^n$$

$$\Downarrow$$

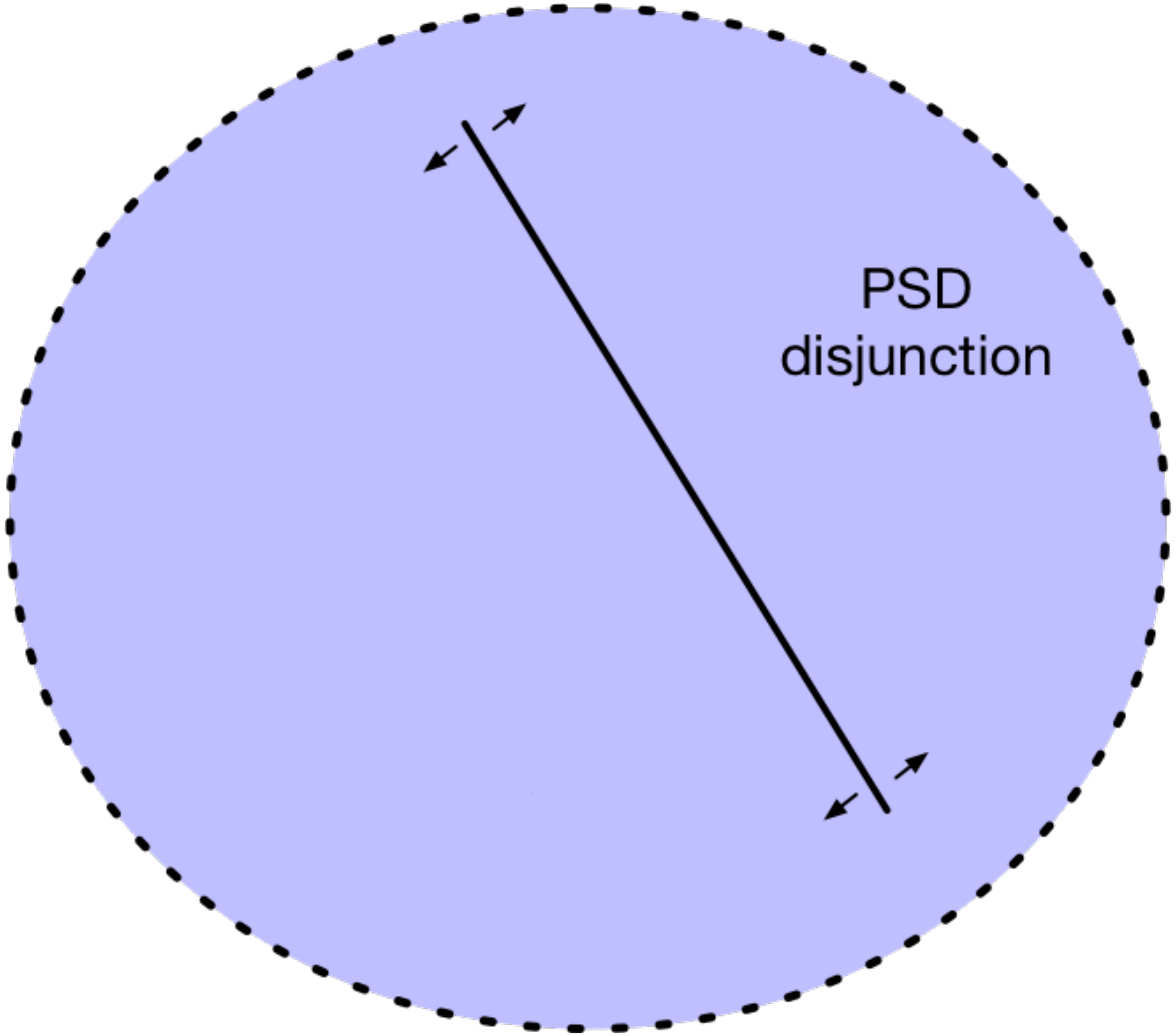
$$HG = \left\{ \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} : X \succeq xx^T \right\} =: PSD$$

Not hard to prove from first principles

Proposition. HG is generated by all disjunctions of the form

$$\text{either} \quad a^T x + b \geq 0 \quad \text{or} \quad a^T x + b \leq 0$$

where $(a, b) \in \mathbb{R}^n \times \mathbb{R}$.


PSD
disjunction

Linear Equations

2 3

$$F = \{x \in \mathbb{R}^n : Ax = b\}$$

$$\Downarrow$$

$$HG = PSD \cap \left\{ \begin{array}{c} Ax = b \\ \operatorname{diag}(AXA^T) = b \circ b \end{array} \right\}$$

Theorem (B 2009). HG is generated by

1. all PSD disjunctions
2. conjuctions of the form $(a_i^T x = b_i \text{ and } a_j^T x = b_j)$ for all pairs of constraints in $Ax = b$.

Nonnegative Orthant

2 3

$$F = \mathbb{R}^n_+$$

$$\Downarrow$$

$$HG \subseteq PSD \cap \left\{ \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \geq 0 \right\} =: DNN$$

Not hard to prove from first principles

Theorem (Maxfiled-Minc 1962). $HG = DNN$ if and only if $n \leq 3$.

- In the literature, "copositive over the nonnegative orthant" is typically just called "copositive"
- Regarding $F = \mathbb{R}^n_+$, HG is the dual cone of the copositive matrices, i.e., $HG = COP^*$
- Generating a valid inequality for $HG = COP^*$ is hard, but we can do better with a sums-of-squares approach

Nonnegative Orthant, Linear Equations, and Complementarities

1 2 3

$$F = \left\{ x \geq 0 : \begin{array}{c} Ax = b \\ x_j x_k = 0 \; \forall \, (j,k) \in E \end{array} \right\}$$

$$\Downarrow$$

$$HG = COP^* \cap \left\{ \begin{array}{c} Ax = b \\ \operatorname{diag}(AXA^T) = b \circ b \\ X_{jk} = 0 \; \forall \, (j,k) \in E \end{array} \right\}$$

Linear Inequalities

2 3

$$F = \{x \in \mathbb{R}^n : Ax \leq b\}$$

$$\Downarrow$$

$$HG \subseteq PSD \cap \left\{ \begin{pmatrix} b & -A \end{pmatrix} \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \begin{pmatrix} b^T \\ -A^T \end{pmatrix} \geq 0 \right\}$$

(equality when $\mathrm{length}(b) \leq 4$)

$$F = [0,1]^2$$

$$\Downarrow$$

$$HG = \left\{ \begin{pmatrix} 1 & x_1 & x_2 \\ x_1 & X_{11} & X_{12} \\ x_2 & X_{12} & X_{22} \end{pmatrix} \succeq 0 : \begin{array}{l} X_{11} \leq x_1,\ X_{22} \leq x_2 \\ X_{12} \leq x_1 \\ X_{12} \leq x_2 \\ X_{12} \geq 0 \\ X_{12} \geq x_1 + x_2 - 1 \end{array} \right\}$$

The linear constraints

$$\begin{pmatrix} b & -A \end{pmatrix} \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \begin{pmatrix} b^T \\ -A^T \end{pmatrix} \geq 0$$

are known as *RLT constraints*

Half-Ellipsoid

2 3

$$F = \{\|x\| \leq 1 : a^T x \leq b\}$$

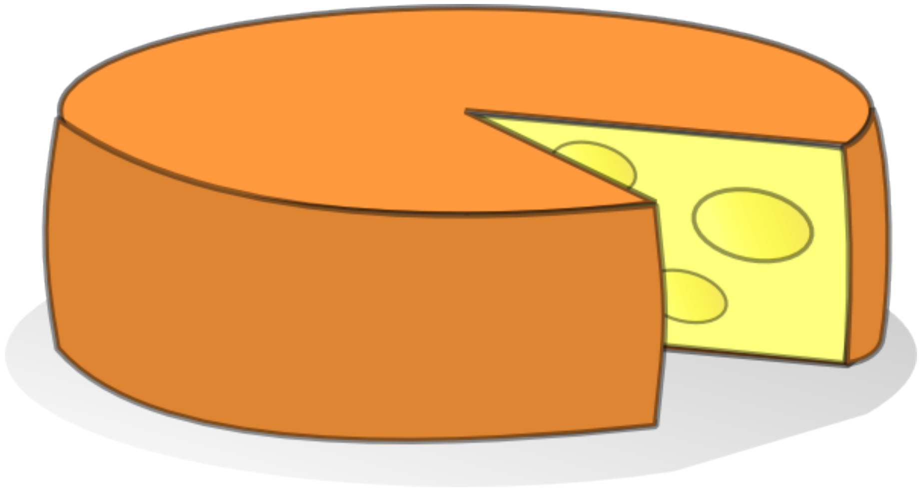
$$\Downarrow$$

$$HG = PSD \cap \{\|bx - Xa\| \leq b - a^T x\}$$

The constraint

$$\|bx - Xa\| \leq b - a^T x$$

is known as an *SOCRLT constraint*



1 2 3

Theorem (Yang-Anstreicher-B 2018). Consider the intersection F of

- ("ball") $\|x\| \leq 1$
- ("cuts") $Ax \leq b$
- ("holes") $x^T Q_k x + 2c_k^T x + \kappa_k \geq 0$ for all k, where each $Q_k \succ 0$

If none of the cuts and holes touch each other, then

$$HG = PSD \cap RLT \cap SOCRLT \cap \{Q_k \bullet X + 2c_k^T x + \kappa_k \geq 0\}$$

Mixed Binary QP

1 2 3

$$F = \left\{ x \geq 0 : \begin{array}{r} Ax = b \\ x_j x_k = 0 \; \forall \, (j,k) \in E \\ x_j \in \{0,1\} \; \forall \, j \in J \end{array} \right\}$$

$$\Downarrow^*$$

$$HG = COP^* \cap \left\{ \begin{array}{c} Ax = b \\ \operatorname{diag}(AXA^T) = b \circ b \\ X_{jk} = 0 \; \forall \, (j,k) \in E \\ X_{jj} = x_j \; \forall \, j \in J \end{array} \right\}$$

* As long as $\{x \geq 0 : Ax = b\}$ ensures

- $x_j \leq 1$ for all $j \in J$
- x_j, x_k bounded for all $(j, k) \in E$

$[0, 1]^n$ and $\{0, 1\}^n$

Proposition. Since $\{0,1\}^n \subset [0,1]^n$,

$$HG(\{0,1\}^n) \subset HG([0,1]^n)$$

I.e., any valid inequality for $HG([0,1]^n)$ is automatically valid for $HG(\{0,1\}^n)$

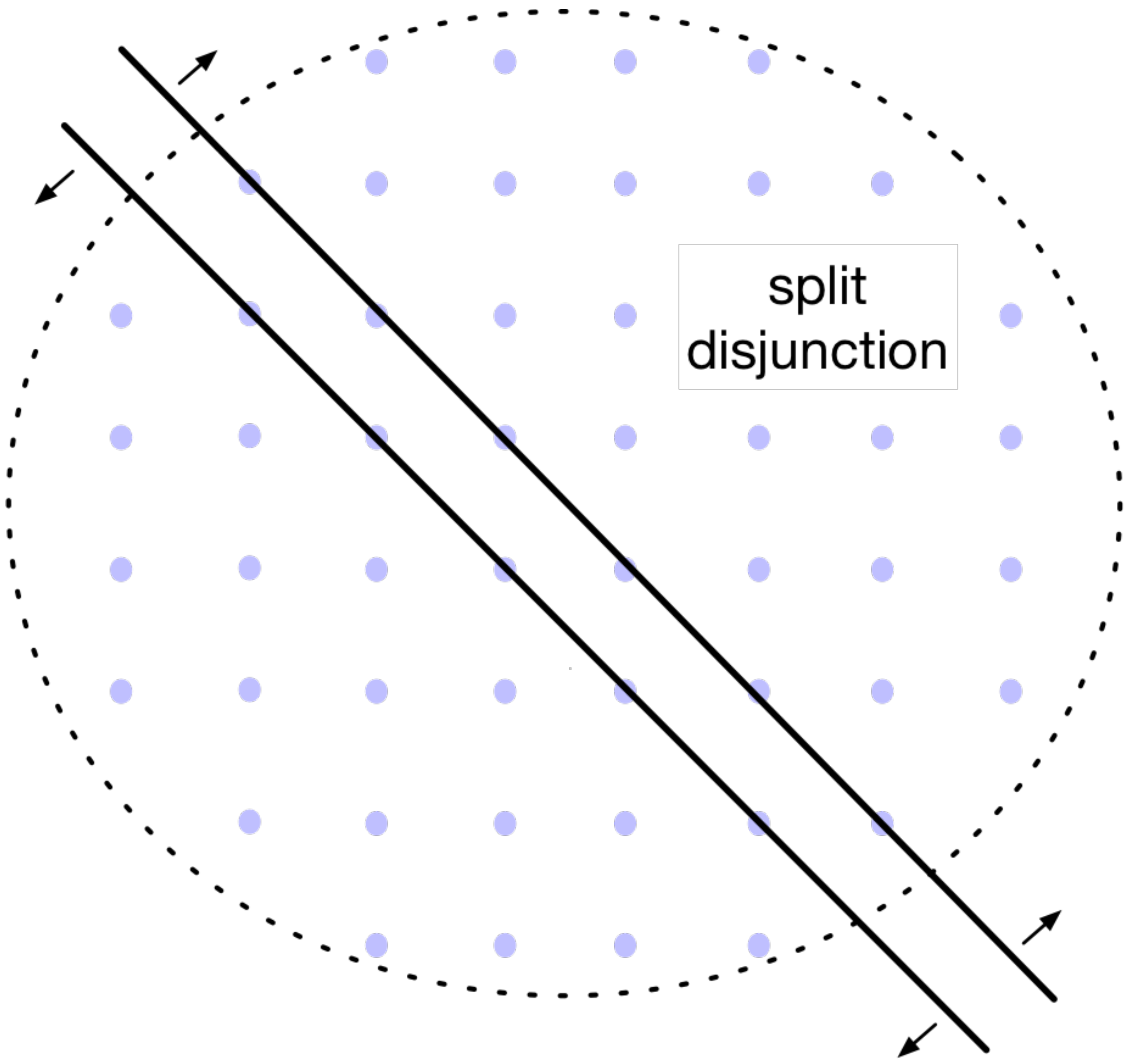
A partial converse holds...

Theorem (B-Letchford 2009). Adding $\mathrm{diag}(X) = x$ to $HG([0,1]^n)$ captures $HG(\{0,1\}^n)$

Corollary. Any valid inequality for $HG(\{0,1\}^n)$—which has no X_{jj} terms—is automatically valid for $HG([0,1]^n)$

Mixed Integer QP

Integer Lattice


split
disjunction

3

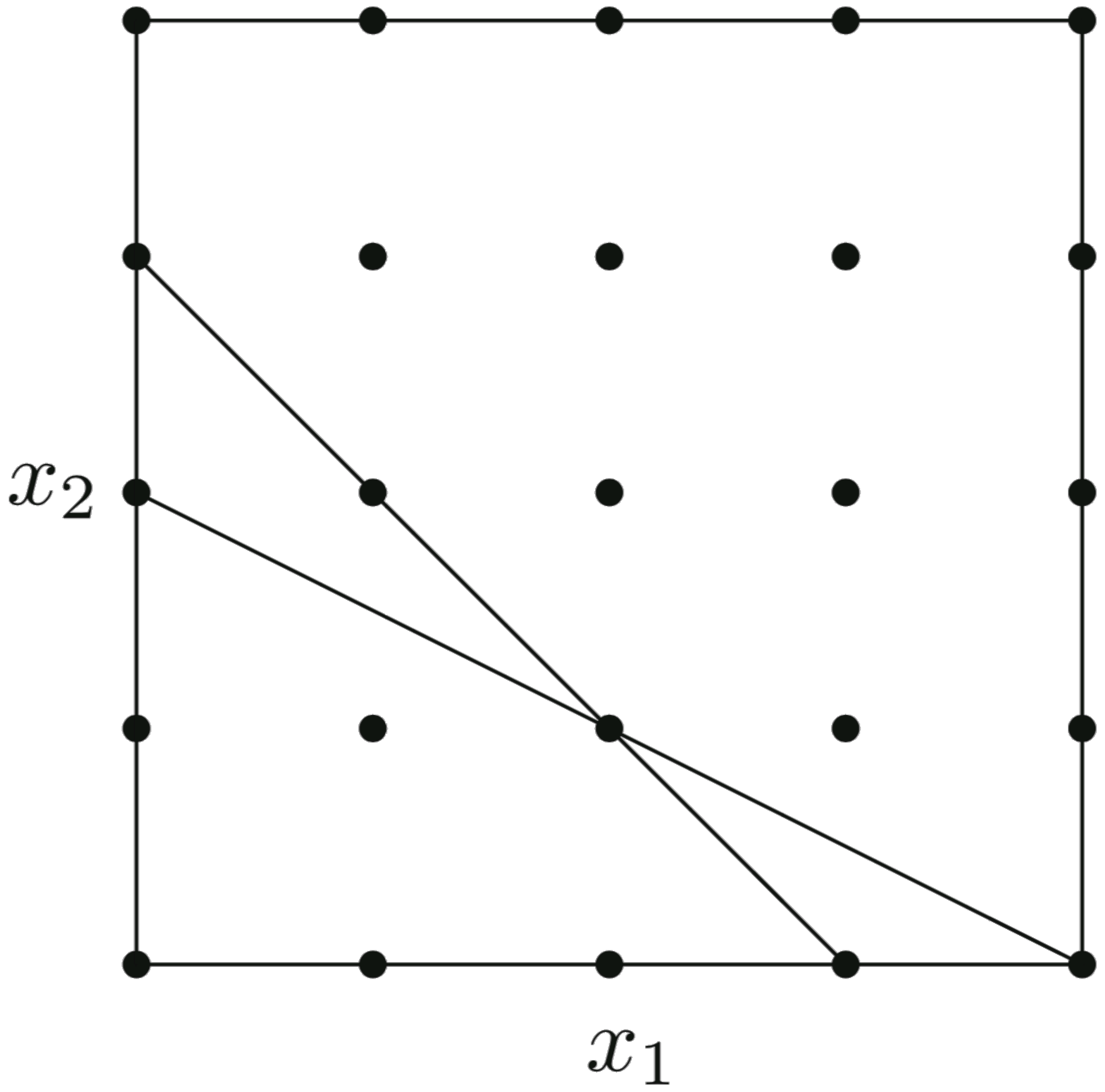
Theorem (B-Letchford 2014). For $F = \mathbb{Z}^n$,

$$HG \subseteq SPLIT \subsetneq PSD$$

The first inclusion holds with equality if $n \leq 2$ but is strict for $n \geq 6$

Note. The cases $n \in \{3, 4, 5\}$ are unresolved

Nonnegative Integer Lattice



Source: B and Letchford, "Unbounded Convex Sets for Non-Convex Mixed-Integer Quadratic Programming", *Mathematical Programming*, 143(1-2), 231-256, 2014

2 3

Theorem (B-Letchford 2014). For $F = \mathbb{Z}^n_+$,

$$HG \subseteq SPLIT \cap SPLIT_+ \cap RLT$$

But not much else is known 😦

Remark. Buchheim-Traversi showed how, in practice, to separate:

- $SPLIT$
- $SPLIT_+$ for the ternary case

They also demonstrated the effectiveness of these cuts in terms of closing the gap

Crazy Observation. For both cases $F = \mathbb{Z}^n$ and $F = \mathbb{Z}^n_+$, every extreme point of HG lies on a countably infinite number of facets

Final Thoughts

- For nonconvex QP, there is still lots to do, especially the integer case
- Convexification involves aspects of
  - convex analysis
  - polyhedral theory
  - SDP
  - polynomial optimization
- Hence, a very interesting area to study

Thank You

And good luck with your research!