www.mat.uc.pt/∼jgouveia/ipco2019problems.pdf

[Yesterday slides: www.mat.uc.pt/∼jgouveia/ipco2019day1handout.pdf]

1. Show that the global infimum of a polynomial in $\mathbb{R}^2$ might not be reached.
2. Write $x^4 + 2x^3 + 6x^2 - 22x + 13$ as a sum of two squares by computing roots.
3. Find for which a and b is the polynomial $x^4 + ax + b$ nonnegative in $\mathbb{R}$.
4. Prove that the following polynomials are nonnegative but not sos:
   - a $x^2y^2 + x^2z^2 + y^2z^2 + 1 - 4xyz$ [Choi-Lam 1976]
   - b $x^4y^2 + y^4 + x^2 - 3x^2y^2$ [Choi-Lam 1976]
   - c $\sum_{i=1}^{5} \prod_{i \neq j}(x_i - x_j)$ [Lax-Lax 1978] see also IMO 1971
5. Show that 3SAT can be formulated as checking if a degree 6 polynomial has minimum 0.
6. Write an explicit SDP to verify that the following polynomials are sos:
   - a $x^4 + 4x^3 + 6x^2 + 4x + 5$
   - b $2x^4 + 5y^4 - x^2y^2 + 2x^3y + 2x + 2$

   Bonus: Prove exactly that such a decomposition exists.
7. Write an SDP to minimize/maximize the rational function $\frac{x^3 - 8x + 1}{x^4 + x^2 + 12}$
8. Give a sums of squares certificate of nonnegativity with multipliers for the polynomials in problem 4. **You can use a computer if you have one**

   Bonus: Write actual exact certificates

Check *Semidefinite Optimization and Convex Algebraic Geometry* edited by Blekherman, Parrilo, and Thomas for some of these and much more

9. A trignometric polynomial of degree d is a function of the form

$$p(\theta) = a_0 + \sum_{k=1}^{d} (a_k \cos(k\theta) + b_k \sin(k\theta)).$$

   a Show that if d is even $p(\theta) \geq 0$ for all θ if and only if $p(\theta) = p_1(\theta)^2 + p_2(\theta)^2$ for some trignometric polynomials p_1 and p_2.

   b Write a semidefinite program to certify the nonnegativity of

      i $p(\theta) = 4 - \sin(\theta) + \sin(2\theta) - 3\cos(2\theta)$

      ii $p(\theta) = 5 - \sin(\theta) + \sin(2\theta) - 3\cos(3\theta)$

   Compute their sos certificates.